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# ON VALUES TAKEN BY THE LARGEST PRIME FACTOR OF SHIFTED PRIMES (II)

BIN CHEN & JIE WU

ABSTRACT. Denote by  $\mathbb{P}$  the set of all primes and by  $P(n)$  the largest prime factor of integer  $n \geq 1$  with the convention  $P(1) = 1$ . Let  $\eta_0 \approx 2.1426$  be the unique positive zero of the equation  $\eta - 1 - 4\eta \log(\eta - 1) = 0$  in  $(1, \infty)$ . Very recently Wu proved that for  $\eta \in (\frac{32}{17}, \eta_0)$  there is a constant  $c(\eta) > 1$  such that for each fixed non-zero integer  $a \in \mathbb{Z}^*$  the set

$$(0.1) \quad \mathcal{P}_{a,c,\eta} := \{p \in \mathbb{P} : p = P(q - a) \text{ for some prime } q \text{ with } p^\eta < q \leq c(\eta)p^\eta\}$$

has relative density 1 in  $\mathbb{P}$ . In this short note, we shall further extend the domain of  $\eta$  at the cost of obtaining a lower bound in place of an asymptotic formula, by showing that for each  $\eta \in [\eta_0, 1 + \sqrt[4]{e})$  the set  $\mathcal{P}_{a,c,\eta}$  has relative positive density in  $\mathbb{P}$ .

## 1. INTRODUCTION

Denote by  $\mathbb{P}$  the set of all prime numbers and by  $P(n)$  the largest prime factor of the positive integer  $n \geq 1$  with the convention  $P(1) = 1$ . Banks & Shparlinski [1] proposed to estimate the number of primes  $p$  that occur as the largest prime factor of a shifted prime  $q - a$  when  $q \in \mathbb{P}$  lies in a certain interval determined by  $p$ . This question has applications in theoretical computer science and has been considered by Vishnoi [8].

For  $a \in \mathbb{Z}^*$ ,  $c > 1$  and  $\eta > 0$ , we put

$$\mathcal{P}_{a,c,\eta} := \{r \in \mathbb{P} : r = P(q - a) \text{ for some prime } q \text{ with } r^\eta < q \leq cr^\eta\}$$

and

$$\pi_{a,c,\eta}(x) := |\{r \leq x : r \in \mathcal{P}_{a,c,\eta}\}|, \quad \pi(x) := |\{r \leq x : r \in \mathbb{P}\}|.$$

Banks & Shparlinski [1, Theorem 1.1] proved that for each  $\eta \in (\frac{32}{17}, 1 + \frac{3}{4}\sqrt{2})$ , there exists a constant  $c = c(\eta) > 1$  such that the asymptotic formula

$$(1.1) \quad \pi_{a,c,\eta}(x) = \pi(x) + O_{A,a,c,\eta}\left(\frac{x}{(\log x)^A}\right) \quad (x \rightarrow \infty)$$

holds for every fixed non-zero integer  $a \in \mathbb{Z}^*$  and any constant  $A > 1$ . Moreover for  $2 \leq \eta < 1 + \frac{3}{4}\sqrt{2}$ , this estimate holds for any constant  $c > 1$ . Very recently, Wu [9] extended Banks-Shparlinski's interval  $(\frac{32}{17}, 1 + \frac{3}{4}\sqrt{2})$  to  $(\frac{32}{17}, \eta_0)$ , where  $\eta_0$  is the unique positive zero of the equation  $\eta - 1 - 4\eta \log(\eta - 1) = 0$  in  $(1, \infty)$ .

In this short note, we shall further extend domain of  $\eta$  at the cost of obtaining a positive proportion in place of the density 1.

Our result is as follows.

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**Theorem 1.** *Let  $\eta \in (2, 1 + \sqrt[4]{e})$  and  $c > 1$ . There is a small positive number  $\delta = \delta(\eta, c)$  such that for every fixed non-zero integer  $a \in \mathbb{Z}^*$ , we have*

$$(1.2) \quad \pi_{a,c,\eta}(x) \geq \frac{(\log 2)(\eta - 1)}{4\eta} \left( 1 - 4 \log(\eta - 1) - 8\delta \frac{\log(\eta - 1)}{c - 1} \right) \pi(x),$$

as  $x \rightarrow \infty$ .

For comparison, we have

$$1 + \frac{3}{4}\sqrt{2} \approx 2.0606, \quad \eta_0 \approx 2.142 \quad \text{and} \quad 1 + \sqrt[4]{e} \approx 2.284.$$

The improvement of Wu [9] comes from the following two simple observations:

- (i) In many arithmetic applications, the linear sieve is more powerful than the sieve of dimension 2;
- (ii) With the help of the Chen-Iwaniec switching principle [2, 3] and his theorem of Bombieri-Vinogradov type (see [9, Proposition 3.2] or (2.7) below), Wu can sieve the sequence of convolution:

$$\{k\ell r + a : k \leq 2cr^{\eta-2}, 2 \nmid (a+k), c_1y/(kr) < \ell \leq c_2y/(kr)\}$$

for each fixed  $r \in (\frac{1}{2}x, x] \cap \mathbb{P}$  by the linear sieve, instead of fixing  $(k, r)$  and sieving  $\{n(knr + a)\}_n$  by the sieve of dimension 2 as in [1].

For every prime  $r \in [\frac{1}{2}x, x]$ , we write

$$(1.3) \quad \mathcal{Q}_1(r) := \sum_{\substack{y < q \leq cy \\ q \equiv a \pmod{r}}} 1 = \sum_{\substack{y < q \leq cy \\ P(q-a)=r}} 1 - \mathcal{Q}_2(r),$$

where

$$(1.4) \quad \mathcal{Q}_2(r) := \sum_{\substack{y < q \leq cy \\ q \equiv a \pmod{r}, P(q-a) > r}} 1.$$

The observations above allowed Wu to obtain a better upper bound for  $\mathcal{Q}_2(r)$  than [1, Formula (9) or Page 143, line 2] of Banks & Shparlinski.

We shall prove Theorem 1 by introducing a new idea into Wu's refinement on Banks-Shparlinski's argument. Our key point is Proposition 3.1 below, which gives a good upper bound for the mean value of the counting function  $\mathcal{Q}_2(r)$ :

$$\sum_{x/2 < r \leq x} \mathcal{Q}_2(r).$$

For this, we shall sieve the sequence of convolution defined as in (4.1) below by the Rosser-Iwaniec linear sieve. The lower bound in Theorem 1 can be increased with more work, but this does not improve the point when the bound becomes trivial.

## 2. SOME PRELIMINARY LEMMAS

In this section, we cite three lemmas, which will be useful later.

## 2.1. The Rosser-Iwaniec linear sieve.

The first lemma is due to Iwaniec [4, 5].

**Lemma 2.1.** *Let  $D \geq 2$  and let  $\mu(n)$  be the Möbius function. Then there are two sequences  $\{\lambda_d^\pm\}_{d \geq 1}$ , vanishing for  $d > D$  or  $\mu(d) = 0$ , verifying  $|\lambda_d^\pm| \leq 1$ , such that*

$$(2.1) \quad \sum_{d|n} \lambda_d^- \leq \sum_{d|n} \mu(d) \leq \sum_{d|n} \lambda_d^+ \quad (n \geq 1)$$

and

$$(2.2) \quad \sum_{d|P_{\mathcal{P}}(z)} \lambda_d^+ \frac{w(d)}{d} \leq \prod_{\substack{p \leq z \\ p \in \mathcal{P}}} \left(1 - \frac{w(p)}{p}\right) \left\{ F(s) + O\left(\frac{e^{\sqrt{L-s}}}{\sqrt[3]{\log D}}\right) \right\}$$

$$(2.3) \quad \sum_{d|P_{\mathcal{P}}(z)} \lambda_d^- \frac{w(d)}{d} \geq \prod_{\substack{p \leq z \\ p \in \mathcal{P}}} \left(1 - \frac{w(p)}{p}\right) \left\{ f(s) + O\left(\frac{e^{\sqrt{L-s}}}{\sqrt[3]{\log D}}\right) \right\}$$

for any  $z \in [2, D]$ ,  $s = (\log D)/\log z$ , set of prime numbers  $\mathcal{P}$  and multiplicative function  $w$  satisfying

$$(2.4) \quad 0 < w(p) < p \quad (p \in \mathcal{P}),$$

$$(2.5) \quad \prod_{u < p \leq v, p \in \mathcal{P}} \left(1 - \frac{w(p)}{p}\right)^{-1} \leq \frac{\log v}{\log u} \left(1 + \frac{L}{\log u}\right) \quad (2 \leq u \leq v),$$

where  $P_{\mathcal{P}}(z) := \prod_{p \leq z, p \in \mathcal{P}} p$  and the implied  $O$ -constants are absolute. Here  $F, f$  are defined by the continuous solutions to the system

$$\begin{cases} (sF(s))' = f(s-1) & (s > 2) \\ (sf(s))' = F(s-1) & (s > 2) \end{cases}$$

with the initial condition

$$\begin{cases} sF(s) = 2e^\gamma & (1 \leq s \leq 2) \\ sf(s) = 0 & (0 < s \leq 2) \end{cases}$$

where  $\gamma$  is the Euler constant.

## 2.2. A mean value theorem of Bombieri-Vinogradov type.

As usual, for  $(a, d) = 1$  define

$$\pi(x; d, a) := \sum_{\substack{p \leq x \\ p \equiv a \pmod{d}}} 1.$$

The Bombieri-Vinogradov theorem can be stated as follows: For any  $A > 0$ , there exists a constant  $B = B(A) > 0$  such that

$$(2.6) \quad \sum_{d \leq x^{1/2}(\log x)^{-B}} \max_{z \leq x} \max_{(a,d)=1} \left| \pi(z; d, a) - \frac{\pi(z)}{\varphi(d)} \right| \ll_A \frac{x}{(\log x)^A}$$

for all  $x \geq 2$ , where  $\varphi(n)$  denotes the Euler totient function and the implied constant depends on  $A$  only.

The following proposition is a mean value theorem of Bombieri-Vinogradov type and will play a key role in the proof of Proposition 3.1 below.

**Proposition 2.2.** *Let  $\kappa_1(m)$  and  $\kappa_2(m)$  be the characteristic functions of the odd integers and of even integers, respectively. Then for any  $A > 0$ , there is a constant  $B = B(A) > 0$  such that the inequalities*

$$(2.7) \quad \sum_{\substack{d \leq x^{1/2}(\log x)^{-B} \\ 2 \nmid d}} \max_{z \leq x} \max_{(a,d)=1} \left| \sum_{\substack{mp \leq z \\ mp \equiv a \pmod{d}}} \kappa_i(m) - \frac{1}{\varphi(d)} \sum_{\substack{mp \leq z \\ (d,mp)=1}} \kappa_i(m) \right| \ll_A \frac{x}{(\log x)^A}$$

$$(2.8) \quad \sum_{\substack{d \leq x^{1/2}(\log x)^{-B} \\ 2 \nmid d}} \max_{z \leq x} \max_{(a,d)=1} \left| \sum_{\substack{mp_1 p_2 \leq z \\ mp_1 p_2 \equiv a \pmod{d}}} \kappa_i(m) - \frac{1}{\varphi(d)} \sum_{\substack{mp_1 p_2 \leq z \\ (d,mp_1 p_2)=1}} \kappa_i(m) \right| \ll_A \frac{x}{(\log x)^A}$$

hold for all  $x \geq 3$ .

*Proof.* The first inequality is Proposition 3.2 of [9]. With the help of Motohashi's convolution argument [7], (2.8) follows from (2.7) and the classical Bombieri-Vinogradov theorem (see (2.6) below).  $\square$

### 2.3. An asymptotic formula.

The next lemma is [9, Lemma 3.3].

**Lemma 2.3.** *For each positive integer  $n \geq 1$ , define*

$$(2.9) \quad \psi(n) := \prod_{2 < p \mid n} \frac{p-1}{p-2}.$$

Then for  $x \geq 2$ , we have

$$\sum_{n \leq x} \psi(n) = \frac{x}{\Xi} \left\{ 1 + O\left(\frac{1}{\log x}\right) \right\},$$

where

$$(2.10) \quad \Xi := \prod_{p > 2} \left( 1 - \frac{1}{(p-1)^2} \right).$$

## 3. BANKS-SHPARLINSKI'S ARGUMENT AND SKETCH OF THE PROOF OF THEOREM 1

As in [9], the letters  $p, q, r$  and  $\ell$  are always used to denote prime numbers, and  $d, m$ , and  $n$  always denote positive integers. In what follows, let  $a \in \mathbb{Z}^*$  and  $\eta \in [\eta_0, 1 + \sqrt[4]{e}]$ . Let  $\delta$  be a sufficiently small positive constant and let  $c > 1$  be a parameter to be chosen later. Let  $x_0(A, a, c, \eta, \delta)$  be a large constant depending on  $A, a, c, \eta, \delta$  at most. For  $x \geq x_0(A, a, c, \eta, \delta)$  and  $r \in (\frac{1}{2}x, x]$ , put  $y := r^\eta$ .

For every prime  $r \in [\frac{1}{2}x, x]$ , recall the definition of

$$(3.1) \quad \mathcal{Q}_1(r) := \sum_{\substack{y < q \leq cy \\ P(q-a)=r}} 1 \quad \text{and} \quad \mathcal{Q}_2(r) := \sum_{\substack{y < q \leq cy \\ q \equiv a \pmod{r}, P(q-a) > r}} 1.$$

Since  $\eta > 2$ , we have  $x \leq 2r = 2y^{1/\eta} \leq y^{1/2}(\log y)^{-B}$ . Thus the Bombieri-Vinogradov theorem (2.6) allows us to write

$$(3.2) \quad \begin{aligned} \sum_{x/2 < r \leq x} \mathcal{Q}_1(r) &= \sum_{x/2 < r \leq x} \left( \sum_{\substack{y < q \leq cy \\ q \equiv a \pmod{r}}} 1 - \mathcal{Q}_2(r) \right) \\ &= (c-1) \sum_{x/2 < r \leq x} \frac{\pi(y)}{\varphi(r)} + O\left(\frac{y}{(\log y)^A}\right) - \sum_{x/2 < r \leq x} \mathcal{Q}_2(r). \end{aligned}$$

The following result constitutes the key to prove Theorem 1.

**Proposition 3.1.** *Under the previous notation, we have*

$$(3.3) \quad \sum_{x/2 < r \leq x} \mathcal{Q}_2(r) \leq 4(c-1+2\delta) \log(\eta-1) \sum_{x/2 < r \leq x} \frac{\pi(y)}{\varphi(r)} \left\{ 1 + O\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\}$$

for  $x \rightarrow \infty$ , where the implied constant depends on  $c, \eta$  and  $\delta$ .

In Section 4, we shall prove this proposition. Now we suppose this proposition and complete the proof of Theorem 1.

From (3.2), (3.3) and the following simple asymptotic formula

$$\sum_{x/2 < r \leq x} \frac{1}{\varphi(r)} = \frac{\log 2}{\log x} \left\{ 1 + O\left(\frac{1}{\log x}\right) \right\},$$

we deduce that

$$(3.4) \quad \sum_{x/2 < r \leq x} \mathcal{Q}_1(r) \geq C_1(c, \eta, \delta) \frac{\pi(y)}{\log x} \left\{ 1 + O\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\},$$

where

$$(3.5) \quad C_1(c, \eta, \delta) := (\log 2)(c-1) \left( 1 - 4 \log(\eta-1) - \delta \frac{\log(\eta-1)^8}{c-1} \right).$$

On the other hand, the Brun-Titchmarsh inequality (see [6]) give us

$$\begin{aligned} \mathcal{Q}_1(r) &\leq \sum_{\substack{y < q \leq cy \\ q \equiv a \pmod{r}}} 1 \leq \frac{2(c-1)y}{\varphi(r) \log((c-1)y/r)} \\ &\leq \frac{4(c-1)y}{(\eta-1)x \log x} \left\{ 1 + O\left(\frac{1}{\log x}\right) \right\} \end{aligned}$$

for all primes  $r \in (\frac{1}{2}x, x]$ . This implies that

$$(3.6) \quad \sum_{x/2 < r \leq x} \mathcal{Q}_1(r) \leq \frac{4(c-1)y}{(\eta-1)x \log x} \left\{ 1 + O\left(\frac{1}{\log x}\right) \right\} \sum_{\substack{x/2 < r \leq x \\ \mathcal{Q}_1(r) \neq 0}} 1.$$

Combining (3.4) and (3.6), it follows that

$$\sum_{\substack{x/2 < r \leq x \\ \mathcal{Q}_1(r) \neq 0}} 1 \geq C_2(c, \eta, \delta) \pi(x) \left\{ 1 + O\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\}$$

with

$$C_2(c, \eta, \delta) := \frac{(\log 2)(\eta - 1)}{4\eta} \left( 1 - 4 \log(\eta - 1) - \delta \frac{\log(\eta - 1)^8}{c - 1} \right).$$

This implies the required inequality (1.2). The proof of Theorem 1 is completed assuming Proposition 3.1.

#### 4. PROOF OF PROPOSITION 3.1

For simplicity of notation, we put

$$c_1 := 1 - \delta \quad \text{and} \quad c_2 := c + \delta.$$

If a prime number  $q$  is counted in  $\mathcal{Q}_2(r)$ , then we can write  $q - a = k\ell r$ , where  $\ell$  is the largest prime factor of  $q - a$ . Since  $\ell > r \in (\frac{1}{2}x, x]$  and  $y < q \leq cy$ , we have  $k \leq (cy - a)/(\ell r) \leq 2cr^{\eta-2}$ . On the other hand, noticing that  $\ell, r$  and  $q = k\ell r + a$  are odd, we must have  $2 \nmid (a + k)$ . By the Chen-Iwaniec switching principle (see [2, 3]), we see that  $\sum_{x/2 < r \leq x} \mathcal{Q}_2(r)$  does not exceed the number of primes in the sequence

$$(4.1) \quad \left\{ k\ell r + a : \frac{1}{2}x < r \leq x, k \leq 2cr^{\eta-2}, 2 \nmid (a + k), c_1 y / (kr) < \ell \leq c_2 y / (kr) \right\}.$$

We shall sieve this sequence by the set of primes  $\mathcal{P}_2 := \{p \in \mathbb{P} : p > 2\}$ . Define

$$P_m(z) := \prod_{p < z, p \nmid m} p \quad \text{with} \quad z := y^{1/4} (\log y)^{-B(3)/2} < r.$$

The inversion formula of Möbius allows us to write that

$$\begin{aligned} \sum_{x/2 < r \leq x} \mathcal{Q}_2(r) &\leq \sum_{x/2 < r \leq x} \sum_{\substack{k \leq 2cr^{\eta-2}, 2 \nmid (a+k) \\ k\ell r + a \text{ is prime}}} \sum_{c_1 y / (kr) < \ell \leq c_2 y / (kr)} 1 \\ &\leq \sum_{x/2 < r \leq x} \sum_{\substack{k \leq 2cr^{\eta-2}, 2 \nmid (a+k) \\ (k\ell r + a, P_2(z)) = 1}} \sum_{c_1 y / (kr) < \ell \leq c_2 y / (kr)} 1 \\ &= \sum_{x/2 < r \leq x} \sum_{k \leq 2cr^{\eta-2}, 2 \nmid (a+k)} \sum_{c_1 y / (kr) < \ell \leq c_2 y / (kr)} \sum_{d | (k\ell r + a, P_2(z))} \mu(d). \end{aligned}$$

Using Lemma 2.1, it follows that

$$\sum_{x/2 < r \leq x} \mathcal{Q}_2(r) \leq \sum_{x/2 < r \leq x} \sum_{k \leq 2cr^{\eta-2}, 2 \nmid (a+k)} \sum_{c_1 y / (kr) < \ell \leq c_2 y / (kr)} \sum_{d | (k\ell r + a, P_2(z))} \lambda_d^+.$$

By inversion of summations, we can write

$$\sum_{x/2 < r \leq x} \mathcal{Q}_2(r) \leq \sum_{d | P_2(z)} \lambda_d^+ \mathcal{K}(y; d, -a),$$

where

$$\mathcal{K}(y; d, b) := \sum_{x/2 < r \leq x} \sum_{\substack{k \leq 2cr^{\eta-2}, 2 \nmid (a+k) \\ k\ell r \equiv b \pmod{d}}} \sum_{c_1 y / (kr) < \ell \leq c_2 y / (kr)} 1.$$

Introducing the notation

$$\mathcal{K}_d(y) := \sum_{x/2 < r \leq x} \sum_{\substack{k \leq 2cr^{\eta-2}, 2 \nmid (a+k) \\ (k\ell r, d)=1}} \sum_{c_1 y / (kr) < \ell \leq c_2 y / (kr)} 1$$

and

$$E(t; d, b) := \sum_{x/2 < r \leq x} \sum_{\substack{k \leq 2cr^{\eta-2}, 2 \nmid (a+k) \\ k\ell r \equiv b \pmod{d}}} \sum_{\ell \leq t / (kr)} 1 - \frac{1}{\varphi(d)} \sum_{x/2 < r \leq x} \sum_{\substack{k \leq 2cr^{\eta-2}, 2 \nmid (a+k) \\ (k\ell r, d)=1}} \sum_{\ell \leq t / (kr)} 1,$$

we write

$$\mathcal{K}(y; d, -a) = \frac{1}{\varphi(d)} \mathcal{K}_d(y) + E(c_2 y; d, -a) - E(c_1 y; d, -a).$$

Inserting into the preceding formula, it follows that

$$(4.2) \quad \sum_{x/2 < r \leq x} \mathcal{Q}_2(r) \leq \mathcal{M} + \mathcal{E},$$

where

$$\mathcal{M} := \sum_{d|P_2(z)} \frac{\lambda_d^+}{\varphi(d)} \mathcal{K}_d(y), \quad \mathcal{E} := \sum_{d|P_2(z)} \lambda_d^+ (E(c_2 y, d, -a) - E(c_1 y, d, -a)).$$

With the help of (2.8) of Proposition 2.2 with  $D := z^2 = y^{1/2}(\log y)^{-B(3)}$ , we can derive

$$(4.3) \quad |\mathcal{E}| \leq \sum_{d \leq y^{1/2}(\log y)^{-B(3)}} (|E(c_2 y, d, -a)| + |E(c_1 y, d, -a)|) \ll \frac{y}{(\log y)^3}.$$

It remains to evaluate  $\mathcal{M}$ . By inversion of summations, we deduce that

$$\begin{aligned} \mathcal{M} &= \sum_{d|P_2(z)} \frac{\lambda_d^+}{\varphi(d)} \sum_{x/2 < r \leq x} \sum_{\substack{k \leq 2cr^{\eta-2}, 2 \nmid (a+k) \\ (k\ell r, d)=1}} \sum_{c_1 y / (kr) < \ell \leq c_2 y / (kr)} 1 \\ &= \sum_{x/2 < r \leq x} \sum_{\substack{k \leq 2cr^{\eta-2}, 2 \nmid (a+k) \\ c_1 y / (kr) < \ell \leq c_2 y / (kr)}} \sum_{d|P_2 k \ell r(z)} \sum_{\varphi(d)} \frac{\lambda_d^+}{\varphi(d)}. \end{aligned}$$

Using Lemma 2.1 with  $w(d) = d/\varphi(d)$ ,  $D = z^2$  and  $\mathcal{P} = \{p \in \mathbb{P} : p \nmid 2k\ell r\}$ , it follows that

$$\mathcal{M} \leq \left\{ F(2) + O\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\} \sum_{x/2 < r \leq x} \sum_{\substack{k \leq 2cr^{\eta-2} \\ 2 \nmid (a+k)}} \sum_{c_1 y / (kr) < \ell \leq c_2 y / (kr)} \prod_{\substack{p \leq z \\ p \nmid 2k\ell r}} \left(1 - \frac{1}{p-1}\right).$$

On the other hand, the Mertens formula allows us to deduce that

$$\begin{aligned} \prod_{p \leq z, p \nmid 2m} \left(1 - \frac{1}{p-1}\right) &= \prod_{2 < p \leq z} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{2 < p|m} \frac{p-1}{p-2} \prod_{2 < p \leq z} \left(1 - \frac{1}{p}\right) \\ &= \frac{2\Xi\psi(m)e^{-\gamma}}{\log z} \left\{ 1 + O\left(\frac{1}{\log z}\right) \right\}, \end{aligned}$$



where  $\Xi$  and  $\psi(m)$  are defined as in (2.10) and (2.9), respectively. Inserting this into the preceding relation and using the fact that  $F(2) = e^\gamma$ , we find

$$\mathcal{M} \leq \left\{ 1 + O\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\} \frac{2\Xi}{\log z} \sum_{x/2 < r \leq x} \sum_{\substack{k \leq 2cr^{\eta-2} \\ 2 \nmid (a+k)}} \sum_{c_1 y/(kr) < \ell \leq c_2 y/(kr)} \psi(k\ell r).$$

Noticing that  $\ell > r > \frac{1}{2}x$  and that  $\ell$  and  $r$  are primes, we have

$$\psi(k\ell r) \leq \psi(k)\psi(\ell)\psi(r) = \{1 + O(x^{-1})\}\psi(k).$$

Thus

$$\begin{aligned} \mathcal{M} &\leq \left\{ 1 + O\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\} \frac{2\Xi}{\log z} \sum_{x/2 < r \leq x} \sum_{\substack{k \leq 2cr^{\eta-2} \\ 2 \nmid (a+k)}} \psi(k) \sum_{c_1 y/(kr) < \ell \leq c_2 y/(kr)} 1 \\ &= \left\{ 1 + O\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\} \frac{2\Xi(c_2 - c_1)y}{\log z} \sum_{x/2 < r \leq x} \frac{\mathcal{S}(r)}{r \log(y/r)}, \end{aligned}$$

where

$$\mathcal{S}(r) := \sum_{\substack{k \leq 2cr^{\eta-2} \\ 2 \nmid (a+k)}} \frac{\psi(k)}{k(1 - (\log k)/\log(y/r))}.$$

With the help of Lemma 2.3, a simple partial integration leads to

$$\begin{aligned} \mathcal{S}(r) &= \left\{ 1 + O\left(\frac{1}{\log r}\right) \right\} \frac{1}{2} \sum_{k \leq cr^{\eta-2}} \frac{\psi(k)}{k(1 - (\log k)/\log(y/r))} \\ &= \left\{ 1 + O\left(\frac{1}{\log r}\right) \right\} \frac{1}{2\Xi} \int_1^{cr^{\eta-2}} \frac{dt}{t(1 - (\log t)/\log(y/r))} \\ &= \left\{ 1 + O\left(\frac{1}{\log r}\right) \right\} \frac{\log(\eta - 1)}{2\Xi} \log(y/r), \end{aligned}$$

where  $\Xi$  is defined as in (2.10). Combining it with the preceding formula, it follows that

$$\begin{aligned} (4.4) \quad \mathcal{M} &\leq \left\{ 1 + O\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\} \sum_{x/2 < r \leq x} \frac{(c_2 - c_1)y}{r \log z} \log(\eta - 1) \\ &\leq \left\{ 1 + O\left(\frac{1}{\sqrt[3]{\log x}}\right) \right\} 4(c - 1 + 2\delta) \log(\eta - 1) \sum_{x/2 < r \leq x} \frac{\pi(y)}{\varphi(r)}. \end{aligned}$$

Inserting (4.4) and (4.3) into (4.2), we obtain the required inequality (3.3).

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