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ON A COMBINATORIAL PROBLEM OF ANTENNAS IN RADIOASTRONOMY

J.-C. BERMOND — A. KOTZIG — J. TURGEON

The following problem arises in radioastronomy. A few movable antennas are used in several successive configurations to measure various spatial frequencies relative to some area of the sky. The distances between antennas determine which frequencies are obtained. A repetition of the same spacing between antennas gives useless redundant information. But if a frequency is missing, it has to be recovered by an interpolation which is always risky. An array is thus optimal if the distances between antennas are successive multiples of a given increment a , without hole or redundancy. Practically, the increment a is of the order of the radius of the antennas and the spacing a (and sometimes $2a$) cannot be obtained, due to overlapping of antennas. When no optimal solution exists, configurations of antennas with repetitions of some spacings are used. The reader interested in the radioastronomy aspect of this problem can see the articles [2] and [3] of Biraud, Blum and Ribes, where the problem is presented in essentially the same combinatorial form as we are using it here.

In the present paper we restrict ourselves to the question of existence or non-existence of optimal arrays for a given minimal spacing and a given number of antennas and of days of observation. Most results obtained so far were found by an exhaustive computer program but, as noted in [3], "the time required becomes too long for more than 4 antennas and 5 configurations". Our non-existence results can thus save a lot of computer time!

1. DEFINITIONS, NOTATION

Let m, n_1, \dots, n_m and c be positive integers. Let

$$A_i = (a_{i1} < a_{i2} < \dots < a_{in_i}); \quad i = 1, \dots, m$$

be sequences of integers and let

$$D_i = \{a_{ij} - a_{ih} \mid 1 \leq h < j \leq n_i\}$$

be their difference sets. We shall say that the system

$$D = (D_1, \dots, D_m)$$

is perfect for c if

$$(1.1) \quad D = D_1 \cup D_2 \cup \dots \cup D_m = \{c, c+1, \dots, c-1 + \sum_{i=1}^m \binom{n_i}{2}\}.$$

Each D_i is called a *component* of the system. A perfect system of difference sets for c is called *regular* if $n_1 = n_2 = \dots = n_m = n$. We shall then speak of a *perfect (m, n, c) -system*.

A perfect (m, n, c) -system yields a perfect array of antennas with n antennas, m configurations or days of observation, where ca is the minimal spacing wanted. In the applications, we have $c = 2$ or 3 , but we shall consider here the general theory.

If $n = 2$ then D_i is reduced to a single element and a perfect $(2, m, c)$ -system exists trivially for every choice of m and c . We shall thus assume in the sequel that $n \geq 3$.

Examples

A perfect $(1, 3, 1)$ -system:

$$A = \{0, 1, 3\}, \quad D = \{1, 2, 3\}.$$

A perfect $(1, 4, 1)$ -system:

$$A = \{0, 1, 4, 6\}, \quad D = \{1, 2, 3, 4, 5, 6\}.$$

A perfect $(3, 3, 2)$ -system:

$$A_1 = \{0, 5, 9\}, \quad D_1 = \{4, 5, 9\};$$

$$A_2 = \{0, 6, 8\}, \quad D_2 = \{2, 6, 8\};$$

$$A_3 = \{0, 7, 10\}, \quad D_3 = \{3, 7, 10\}.$$

A perfect $(4, 4, 1)$ -system:

$$A_1 = \{0, 1, 7, 23\}, \quad D_1 = \{1, 6, 7, 16, 22, 23\};$$

$$A_2 = \{0, 2, 14, 19\}, \quad D_2 = \{2, 5, 12, 14, 17, 19\};$$

$$A_3 = \{0, 3, 13, 21\}, \quad D_3 = \{3, 8, 10, 13, 18, 21\};$$

$$A_4 = \{0, 4, 15, 24\}, \quad D_4 = \{4, 9, 11, 15, 20, 24\}.$$

A perfect $(5, 4, 3)$ -system:

$$A_1 = \{0, 3, 20, 29\}, \quad D_1 = \{3, 9, 17, 20, 26, 29\};$$

$$A_2 = \{0, 4, 18, 28\}, \quad D_2 = \{4, 10, 14, 18, 24, 28\};$$

$$A_3 = \{0, 5, 21, 32\}, \quad D_3 = \{5, 11, 16, 21, 27, 32\};$$

$$A_4 = \{0, 6, 19, 31\}, \quad D_4 = \{6, 12, 13, 19, 25, 31\};$$

$$A_5 = \{0, 7, 22, 30\}, \quad D_5 = \{7, 8, 15, 22, 23, 30\}.$$

Case $c = 1$

In this case the problem is related to the graceful numbering of some graphs. Let G be a graph with e edges, then a graceful numbering of G [4] (or β -valuation [7] or restricted difference basis [5], [6]) is an assign-

ment of a subset of the numbers $\{0, 1, 2, \dots, e\}$ to the vertices of G in such a way that the values of the edges are all the numbers from 1 to e , where the value of an edge is the absolute value of the difference between the numbers attributed to its endpoints.

Perfect $(1, 3, 1)$ and $(1, 4, 1)$ -systems give graceful numbering of the complete graphs K_3 and K_4 (see Fig. 1) and the perfect $(4, 4, 1)$ -system gives a graceful numbering of the graph consisting of four K_4 having exactly one vertex in common.

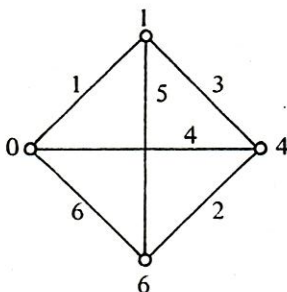


Fig. 1. Graceful numbering of K_4

In the case $c = 1$, the existence of a perfect $(m, n, 1)$ -system corresponds to a graceful numbering of the graph consisting of m complete graphs on a vertices having exactly one vertex in common.

For example it is known that K_n admits a graceful numbering if and only if $n \leq 4$; thus there exists a perfect $(1, n, 1)$ -system if and only if $n \leq 4$. The "isolated example" for ungraceful graph [4, Fig. 8] corresponds to the non-existence of a perfect $(2, 4, 1)$ -system.

Remarks. The characterization of the graphs admitting a graceful numbering is an open problem; in particular it has been conjectured by Ringel that any tree admits a graceful numbering (see [7]). Partial results have been obtained concerning the maximum number $f(n)$ of edges of a graceful graph. This gives, for the original problem of radioastronomy with one configuration of n antennas, the maximum number of consecutive

spacings (starting at spacing one) available. In fact we need in this case the solution of a less restrictive problem, called "unrestricted different bases" which differ from the graceful numbering problem by the fact that the numbers assigned to the vertices do not need to belong to $\{1, \dots, e\}$; without this restriction we can obtain for some values of n more consecutive spacings. (For details and for the best known values see [6]).

We may always suppose, as in the examples, that the first element a_{i1} of A_i is zero. The difference set D_i of A_i remains the same if we add the same integer to all the elements of A_i .

We shall use the notation $d_{ij}^k = a_{i,j+k} - a_{ij}$; $k = 1, 2, \dots, n-1$; $j = 1, 2, \dots, n-k$; $i = 1, 2, \dots, m$.

Proposition 1.1. Let $\lfloor \frac{n-1}{2} \rfloor$ denote the largest integer $\leq \frac{n-1}{2}$. Then, for $k = 1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor$, we have

$$\sum_{j=1}^k d_{ij}^{n-k} = \sum_{j=1}^{n-k} d_{ij}^k, \quad i = 1, 2, \dots, m.$$

Proof. We have

$$d_{ij}^k = \sum_{h=j}^{j+k-1} d_{ih}^1; \quad k = 1, 2, \dots, n; \quad j = 1, 2, \dots, n-k.$$

Hence

$$(1.2) \quad \sum_{j=1}^k d_{ij}^{n-k} = \sum_{j=1}^k \sum_{h=j}^{j+k-1} d_{ih}^1$$

and

$$(1.3) \quad \sum_{j=1}^{n-k} d_{ij}^k = \sum_{j=1}^{n-k} \sum_{h=j}^{j+k-1} d_{ih}^1.$$

Let $k \in \{1, 2, \dots, \lfloor \frac{n-1}{2} \rfloor\}$. Then both in (1.2) and (1.3) the coefficient of d_{ih}^1 is

$$\begin{cases} h & \text{if } 1 \leq h \leq k \\ k & \text{if } k \leq h \leq n-k \\ n-h & \text{if } n-k \leq h \leq n-1. \end{cases}$$

This proves 1.1.

2. GENERAL RELATIONS AMONG m, n AND c IN A PERFECT (m, n, c) -SYSTEM. THE CASE $n \geq 6$.

Proposition 2.1. *In a perfect (m, n, c) -system, if $n = 3$ or 4 , then $m \geq 2c - 1$. If $n = 5$, then $m \geq 4c - 2$.*

Proposition 2.2. *No perfect (m, n, c) -system exists with $n \geq 6$.*

Proof of 2.1 and 2.2. 1.1 implies that, in every component D_i of a perfect (m, n, c) -system,

$$\sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{j=1}^k d_{ij}^{n-k} = \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{j=1}^{n-k} d_{ij}^k.$$

Let

$$(2.1) \quad \alpha = \sum_{i=1}^m \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{j=1}^k d_{ij}^{n-k} = \sum_{i=1}^m \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{j=1}^{n-k} d_{ij}^k.$$

We shall obtain an upper and a lower bound for α . The first expression for α in (2.1) contains N distinct terms, where

$$N = m \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} k = \frac{m}{2} \left[\frac{n-1}{2} \right] \left(\left[\frac{n-1}{2} \right] + 1 \right).$$

Each of these terms is $\leq M$, where

$$M = c - 1 + \frac{m}{2} n(n-1).$$

Hence

$$(2.2) \quad \alpha \leq \sum_{q=M-N+1}^M q = \frac{1}{2} N(2M - N + 1).$$

The second expression for α in (2.1) contains P distinct terms, all greater than or equal to c , with

$$P = m \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} (n-k) = \frac{m}{2} \left[\frac{n-1}{2} \right] (2n-1 - \left[\frac{n-1}{2} \right]).$$

Hence

$$(2.3) \quad \alpha \geq \sum_{q=c}^{c-1+P} q = \frac{1}{2} P(P+2c-1).$$

Case I. Suppose n is even, say $n = 2p + 2$. Then $\left[\frac{n-1}{2} \right] = p$,

$$N = \frac{1}{2} mp(p+1)$$

and

$$M = c - 1 + m(2p^2 + 3p + 1),$$

so that (2.2) becomes

$$(2.4) \quad \alpha \leq \frac{1}{8} mp(p+1)(7mp^2 + 11mp + 4m + 4c - 2).$$

Also, $P = \frac{3}{2} mp(p+1)$ and (2.3) becomes

$$(2.5) \quad \alpha \geq \frac{3}{8} mp(p+1)(3mp^2 + 3mp + 4c - 2).$$

Combining (2.4) and (2.5), we obtain

$$(2.6) \quad 2mp^2 - 2mp - 4m + 8c - 4 \leq 0.$$

If $p = 1$ ($n = 4$), then (2.6) reduces to $m \geq 2c - 1$. If $p \geq 2$ (n even, $n \geq 6$), then $p^2 - p \geq 2$ and (2.6) implies

$$8c - 4 \leq 0,$$

which is impossible.

Case II. Suppose n is odd, say $n = 2p + 1$. Then again $\left[\frac{n-1}{2} \right] = p$,

$$N = \frac{m}{2} p(p+1)$$

$$M = c - i + mp(2p+1)$$

and

$$P = \frac{m}{2} p(3p+1).$$

Therefore (2.2) becomes

$$(2.7) \quad \alpha < \frac{m}{8} p(7mp^3 + 10mp^2 + 3mp + 4pc + 4c - 2p - 2)$$

and (2.3) becomes

$$(2.8) \quad \alpha \geq \frac{m}{8} p(9mp^3 + 6mp^2 + mp + 12pc + 4c - 6p - 2).$$

From (2.7) and (2.8) we deduce

$$(2.9) \quad mp^2 - 2mp - m + 4c - 2 \leq 0.$$

If $p = 1$ ($n = 3$), then (2.9) becomes $m \geq 2c - 1$. If $p = 2$ ($n = 5$), we obtain $m \geq 4c - 2$. Finally, if $p \geq 3$ (n odd, $n \geq 7$), then $p^2 - 2p \geq 3$ and (2.9) implies

$$2m + 4c - 2 \leq 0,$$

which is impossible.

Proposition 2.3. *If a perfect $(m, 5, c)$ -system exists, then m is even.*

Proposition 2.4. *In a perfect $(m, 3, c)$ -system, if c is odd, then $m \equiv 0$ or $1 \pmod{4}$. If c is even, then $m \equiv 0$ or $3 \pmod{4}$.*

These two results and proofs were also found independently by Biraud, Blum and Ribes (private communication).

Proof of 2.3 and 2.4. Let n be odd, $n = 2p + 1$. For $i = 1, \dots, m$, the sum S_i of all differences in D_i is

$$S_i = \sum_{k=1}^{2p} \sum_{j=1}^{n-k} d_{ij}^k = 2 \sum_{k=1}^p \sum_{j=1}^{n-k} d_{ij}^k;$$

cf. 1.1. Hence $S = \sum_{i=1}^m S_i$ is even. But, by (1.1),

$$(2.10) \quad S = \sum_{i=c}^{c-1 + \frac{m}{2}n(n-1)} i = \frac{1}{8} [4c - 2 + mn(n-1)][mn(n-1)].$$

Let $n = 5$. Then (2.10) becomes

$$S = m(10c - 5 + 50m)$$

and m must be even.

If $n = 3$, then (2.10) becomes

$$2S = m(9m + 6c - 3).$$

Since S is even, we must have

$$m(9m + 6c - 3) \equiv 0 \pmod{4},$$

which yields 2.4.

Proposition 2.5. *Let D be a perfect (m_1, n, c) -system and D' be a perfect $(m_2, n, m_1 \frac{n(n-1)}{2} + c)$ -system. Then $D \cup D'$ is a perfect $(m_1 + m_2, n, c)$ -system.*

Proof. By the hypothesis,

$$D = \left\{ c, c+1, \dots, c-1 + m_1 \frac{n(n-1)}{2} \right\}$$

and

$$\begin{aligned} D' &= \left\{ c + m_1 \frac{n(n-1)}{2}, \dots, \left[c + m_1 \frac{n(n-1)}{2} \right] - \right. \\ &\quad \left. - 1 + m_2 \frac{n(n-1)}{2} \right\} = \\ &= \left\{ c + m_1 \frac{n(n-1)}{2}, \dots, c-1 + (m_1 + m_2) \frac{n(n-1)}{2} \right\}. \end{aligned}$$

Hence $D \cup D'$ is a perfect $(m_1 + m_2, n, c)$ -system.

$$(4.2) \quad 6\alpha + 3\beta + 2\gamma = \frac{1}{2}(6m) = 3m$$

are even. From (4.1) and (4.2) we obtain that $\gamma = 3\alpha$. If $m \in \{2, 3\}$ then $\alpha = \gamma = 0$, by (4.1), and $\beta = m$, i.e. each component has exactly three even differences and is of one of the following four types:

$$\begin{array}{cccc} \omega & & \omega & \epsilon \\ \epsilon & \omega & \omega & \epsilon \\ \epsilon & \epsilon & \omega & \omega \end{array} \quad \begin{array}{cccc} \epsilon & & \epsilon & \omega \\ \omega & \epsilon & \omega & \omega \\ \epsilon & \omega & \omega & \omega \end{array} \quad \begin{array}{cccc} \epsilon & & \omega & \omega \\ \omega & \epsilon & \omega & \omega \\ \epsilon & \omega & \omega & \omega \end{array} \quad \begin{array}{cccc} \epsilon & & \omega & \omega \\ \omega & \epsilon & \omega & \omega \\ \epsilon & \omega & \omega & \omega \end{array}$$

In each of these types, the largest even difference is the sum of the other two. If we consider only the even differences and divide them by 2, we obtain a perfect $(m, 3, 1)$ -system. But, by 2.4, no such system exists for $m \in \{2, 3\}$.

Proposition 4.2. *Let D be a perfect $(m, 4, 3)$ -system. Then there exists a perfect $(m+1, 4, 1)$ -system.*

Proof. Let

$$A_0 = \{0, 1, 6m+4, 6m+6\}.$$

The set of differences of A_0 is

$$D_0 = \{1, 2, 6m+3, 6m+4, 6m+5, 6m+6\}.$$

Since $D = \{3, 4, \dots, 6m+2\}$, we obtain a perfect $(m+1, 4, 1)$ -system by joining D_0 to D .

In Table 2, indicating the known results for $n=4$, the indication ≥ 12 for $m=6$ and $c=1$ is a corollary of 4.2, since 12 distinct perfect $(5, 4, 3)$ -systems were found by Biraud, Blum and Ribes. An X indicates the results of 4.1.

J.C. Bermond has made the conjecture that a perfect $(m, 4, 1)$ -system exists for every $m \geq 4$.

	$m=1$	2	3	4	5	6	7
$c=1$	1	X	X	12	174	≥ 12	
2			0	0	22		
3					12		
4	Eliminated by 2.1						
5							

Table 2. The case $n=4$

5. THE CASE $n=5$

Proposition 5.1. *There is no perfect $(2, 5, 1)$ -system.*

Proof. Suppose D is a perfect $(2, 5, 1)$ -system. Then

$$2 \sum_{i=1}^2 d_{i1}^4 = \sum_{i=1}^2 [d_{i1}^2 + d_{i3}^2 + \sum_{j=1}^4 d_{ij}^1] \geq \sum_{i=1}^{12} i = \frac{12}{2} \cdot 13 = 78.$$

Hence $d_{11}^4 + d_{22}^4 \geq 39$. Since d_{11}^4 and d_{22}^4 are two distinct numbers in $\{1, \dots, 20\}$, we may assume that

$$(5.1) \quad d_{11}^4 = 20 \quad \text{and} \quad d_{22}^4 = 19.$$

By 1.1,

$$(5.2) \quad \sum_{i=1}^2 (d_{i1}^4 + d_{i1}^3 + d_{i2}^3) = \sum_{i=1}^2 \left[\sum_{j=1}^4 d_{ij}^1 + \sum_{j=1}^3 d_{ij}^2 \right].$$

But the sum of the two sides of (5.2) is the sum of all the differences of D , which is

$$\sum_{i=1}^{20} i = \frac{20}{2} \cdot 21 = 210.$$

So each side of (5.2) is equal to 105. We thus have

$$(5.3) \quad \bigcup_{i=1}^2 \{d_{i1}^4, d_{i1}^3, d_{i2}^3\} = \{20, 19, 18, 17, 16, 15\}.$$

In view of assumption (5.1), we shall now consider the possible values of d_{21}^3 and d_{22}^3 .

If $d_{22}^3 = 18$ then $d_{21}^1 = 1$ and $d_{21}^3 = 15$ (for if $d_{21}^3 > 15$, then $d_{22}^2 > 14$, contrary to (5.3)). Hence $d_{24}^1 = 4$. But then $\{d_{11}^3, d_{12}^3\} = \{17, 16\}$, so that one of d_{11}^1 and d_{14}^1 is also 4, a contradiction.

So we may assume that none of d_{21}^3 and d_{22}^3 is 18. If $d_{22}^3 = 17$ then $d_{21}^1 = 2$. But one of d_{11}^3 and d_{12}^3 is then 18 so one of d_{11}^1 and d_{14}^1 is also 2, a contradiction.

Since none of d_{21}^3 and d_{22}^3 is 18 or 17, we have $\{d_{21}^3, d_{22}^3\} = \{16, 15\}$ and thus $\{d_{21}^1, d_{24}^1\} = \{3, 4\}$. But then $\{d_{11}^3, d_{12}^3\} = \{18, 17\}$ and $\{d_{11}^1, d_{14}^1\} = \{3, 2\}$, a contradiction again. This completes the proof of 5.1.

No perfect $(m, 5, c)$ -system is known. (See Propositions 2.3 and 5.1 for non-existence results.)

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