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## A proof of the Conjecture of Lehmer

Jean-Louis Verger-Gaugry

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# A PROOF OF THE CONJECTURE OF LEHMER

JEAN-LOUIS VERGER-GAUGRY

ABSTRACT. The Conjecture of Lehmer is proved to be true. The proof mainly relies upon: (i) the properties of the Parry Upper functions  $f_{|\alpha|}(z)$  associated with the dynamical zeta functions  $\zeta_{|\alpha|}(z)$  of the Rényi–Parry arithmetical dynamical systems ( $\beta$ -shift), for  $\alpha$  a reciprocal algebraic integer of house  $|\alpha|$  greater than 1, (ii) the discovery of lenticuli of poles of  $\zeta_{|\alpha|}(z)$  which uniformly equidistribute at the limit on a limit “lenticular” arc of the unit circle, when  $|\alpha|$  tends to  $1^+$ , giving rise to a continuous lenticular minorant  $M_r(|\alpha|)$  of the Mahler measure  $M(\alpha)$ , (iii) the Poincaré asymptotic expansions of these poles and of this minorant  $M_r(|\alpha|)$  as a function of the dynamical degree. The Conjecture of Schinzel-Zassenhaus is proved to be true. A Dobrowolski type minoration of the Mahler measure  $M(\alpha)$  is obtained. The universal minorant of  $M(\alpha)$  obtained is  $\theta_\eta^{-1} > 1$ , for some integer  $\eta \geq 259$ , where  $\theta_\eta$  is the positive real root of  $-1 + x + x^\eta$ . The set of Salem numbers is shown to be bounded from below by the Perron number  $\theta_{31}^{-1} = 1.08545\dots$ , dominant root of the trinomial  $-1 - z^{30} + z^{31}$ . Whether Lehmer’s number is the smallest Salem number remains open. For sequences of algebraic integers of Mahler measure smaller than the smallest Pisot number  $\Theta = 1.3247\dots$ , whose houses have a dynamical degree tending to infinity, the Galois orbit measures of conjugates are proved to converge towards the Haar measure on  $|z| = 1$  (limit equidistribution). The dynamical zeta function is used to investigate the domain of very small Mahler measures of algebraic integers in the range  $(1, 1.176280\dots]$ , if any.

Keywords: Lehmer conjecture, Schinzel-Zassenhaus conjecture, Mahler measure, minoration, Dobrowolski inequality, asymptotic expansion, transfer operator, dynamical zeta function, Rényi-Parry  $\beta$ -shift, Parry Upper function, Perron number, Pisot number, Salem number, Parry number, limit equidistribution.

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## 1. INTRODUCTION

The question asked by Lehmer in [123] (1933) about the existence of integer univariate polynomials of Mahler measure arbitrarily close to one became a conjecture. Lehmer's Conjecture is stated as follows:

**Conjecture 1** (Lehmer's Conjecture). *There exists an universal constant  $c > 0$  such that the Mahler measure  $M(\alpha)$  satisfies  $M(\alpha) \geq 1 + c$  for all nonzero algebraic numbers  $\alpha$ , not being a root of unity.*

Many works attempted to solve it, e.g. by Amoroso [8] [9], Blansky and Montgomery [28], Boyd [32] [33], Cantor and Strauss [50], David and Hindry [56], Dobrowolski [63], Dubickas [68] [69], Hindry and Silverman [104], Langevin [117], Laurent [122], Louboutin [132], Masser [138], Mossinghoff, Rhin and Wu [146], Schinzel [181], Silverman [187], Smyth [191] [192], Stewart [195] [196], Waldschmidt [217] [218].

If  $\alpha$  is a nonzero algebraic integer,  $M(\alpha) = 1$  if and only if  $\alpha = 1$  or is a root of unity by Kronecker's Theorem (1857) [114]. Lehmer's Conjecture asserts a discontinuity of the value of  $M(\alpha)$ ,  $\alpha \in \mathcal{O}_{\overline{\mathbb{Q}}}$ , at 1. In the panorama [215] the meaning of this discontinuity is evoked in number theory and in several domains by analogy where it admits different reformulations.

In this note a proof of the Conjecture of Lehmer is proposed, which is based on the dynamics of algebraic numbers (cf Sections §2 and §3): more precisely on the dynamical system of numeration of the beta-shift in the sense of Rényi and Parry and on the generalized Fredholm Theory which is associated to it by the transfer operators of the  $\beta$ -transformation. It brings the dynamical zeta functions  $\zeta_{\beta}(z)$  of the  $\beta$ -shift into play. An ad-hoc theory of divergent series (Poincaré asymptotic expansions) is introduced for formulating the *lenticular poles* of the dynamical zeta functions  $\zeta_{\beta}(z)$  as functions of the dynamical degree of  $\beta$ . It allows to establish an universal minoration of the Mahler measure  $M(\alpha)$  for any nonzero algebraic integer  $\alpha$  which is not a root of unity, by a new Dobrowolski type minoration.

Let us reduce the problem. If  $\alpha$  is an algebraic number which is not an algebraic integer, then  $M(\alpha) \geq 2$ . If  $\alpha$  is an algebraic integer for which the minimal polynomial is not reciprocal, then  $M(\alpha) \geq \Theta$  the smallest Pisot number, by a Theorem of C. Smyth [189]. For every reciprocal algebraic integer  $\alpha$ , such that  $|\alpha| \geq c$ , where  $c > 1$  is a (fixed) constant, then  $M(\alpha) \geq c$ . Therefore the problem of Lehmer amounts to find an universal minoration of  $M(\alpha)$  when  $|\alpha|$  tends to  $1^+$ . It is a limit problem. The problem is strengthened when the condition "when  $|\alpha|$  tends to  $1^+$ " is replaced by "when  $|\overline{\alpha}|$  tends to  $1^+$ ", taking into account all the conjugates at the same time. This limit problem constitutes the statement of the Conjecture of Schinzel-Zassenhaus [54], as follows:

**Conjecture 2** (Schinzel - Zassenhaus's Conjecture). *Denote by  $m_h(n)$  the minimum of the houses  $|\overline{\alpha}|$  of the algebraic integers  $\alpha$  of degree  $n$  which are not a root of unity. There exists a (universal) constant  $C > 0$  such that*

$$(1.0.1) \quad m_h(n) \geq 1 + \frac{C}{n}, \quad n \geq 2.$$

Schinzel - Zassenhaus's Conjecture is a consequence of Lehmer's conjecture: if  $r$  is the number of conjugates  $\alpha^{(i)}$  of  $\alpha$  satisfying  $|\alpha^{(i)}| > 1$ , then  $M(\alpha) \leq |\overline{\alpha}|^r$ . Thus

$$|\overline{\alpha}| \geq M(\alpha)^{1/r} \geq M(\alpha)^{1/\deg(\alpha)} \geq (1+c)^{1/\deg(\alpha)} \geq 1 + \frac{C}{\deg(\alpha)}.$$

For Lehmer's Conjecture, due to the invariance of the Mahler measure  $M(\alpha)$  by the transformations  $z \rightarrow \pm z^{\pm 1}$  and  $z \rightarrow \pm \bar{z}^{\pm 1}$ , it is sufficient to consider the two following cases:

- (i)  $\alpha$  real reciprocal algebraic integer  $> 1$ , in which case  $\alpha$  is generically named  $\beta$ ,
- (ii)  $\alpha$  complex reciprocal algebraic integer,  $|\alpha| > 1$ , with  $\arg(\alpha) \in (0, \pi/2]$ ,

with  $|\alpha| > 1$  sufficiently close to 1 in both cases. In Section § 6 we show how the nonreal complex case (ii) can be deduced from the real case (i) by considering the Rényi-Parry dynamics of the houses  $|\overline{\alpha}|$ .

Let us consider case (i). By the Northcott property, the degree  $\deg(\beta)$ , valued in  $\mathbb{N} \setminus \{0, 1\}$ , is necessarily not bounded when  $\beta > 1$  tends to  $1^+$ . To compensate the absence of an integer function of  $\beta$  which "measures" the proximity of  $\beta$  with 1, we introduce the natural integer function of  $\beta$ , that we call the *dynamical degree of  $\beta$* , denoted by  $\text{dyg}(\beta)$ , which is defined by the relation: for  $1 < \beta \leq \frac{1+\sqrt{5}}{2}$  any real number,  $\text{dyg}(\beta)$  is the unique integer  $n \geq 3$  such that

$$(1.0.2) \quad \theta_n^{-1} \leq \beta < \theta_{n-1}^{-1}$$

where  $\theta_n$  is the unique root in  $(0, 1)$  of the trinomial  $G_n(z) = -1 + z + z^n$ .

**Important Nota (N):** (A) An infinite subcollection of reciprocal algebraic integers  $\beta > 1$  is excluded from the present study. Let us explain what is this subcollection and why. We refer to Section 5.9.1 for more details.

The minimal polynomial  $P_\beta(X)$  of the generic reciprocal algebraic integer  $\beta > 1$  can be written

$$(1.0.3) \quad P_\beta(X) = \widetilde{P}_\beta(X^r)$$

for some integer  $r \geq 1$  and some  $\mathbb{Z}$ -minimal integer polynomial  $\widetilde{P}_\beta(X)$ . The integer  $r$  is the largest one such that (1.0.3) holds. There are two cases.

The case  $r = 1$  is called the *Main Case*; we say that " $\beta$  belongs to the Main Case". All the statements and the proofs below till Section 7 assume " $r = 1$ ", ie are only concerned with  $\beta$ s belonging to the Main Case, in particular Proposition 5.29.

The case " $r \geq 2$ " is called the *Second Case*; we say that " $\beta$  belongs to the Second Case". This case corresponds to the action of the  $r$ th roots of unity. The actions of the roots of unity produce infinite families of reciprocal algebraic integers  $\beta$  tending to 1, on which the Mahler measure remains constant, as explained in Section 5.9.1. Therefore the Second Case does not bring any further complementary insight into the study of the minoration of the Mahler measure  $M$  with respect to the Main Case. The reason of this is that in the Second Case, the dynamical degree  $\text{dyg}(\beta)$ , as defined above, has to be replaced by  $\text{dyg}(\beta^r)$  and the lenticular roots (cf below) are relative to  $f_{\beta^r}(z)$ , not of  $f_\beta(z)$ , and that  $\beta^r > \beta$ .

Only the statements relative to the reciprocal algebraic integers  $\beta$  belonging to the Main Case are considered below.

(B) The dynamical zeta function  $\zeta_\beta(z)$  is used to investigate to domain of very small Mahler measures  $M(\beta)$  of reciprocal algebraic integers  $\beta$ , in the range  $(1, 1.176280\dots]$ , in the eventual case where such  $\beta$ s exist. The problem of the existence of such  $\beta$ s, apart from Lehmer's number  $1.176280\dots$ , is another problem.

The (unique) simple zero  $> 1$  of the trinomial  $G_n^*(z) := 1 + z^{n-1} - z^n$ ,  $n \geq 2$ , is the Perron number  $\theta_n^{-1}$ . The set of dominant roots  $(\theta_n^{-1})_{n \geq 2}$  of the nonreciprocal trinomials  $(G_n^*(z))_{n \geq 2}$  constitute a strictly decreasing sequence of Perron numbers, tending to one. Section § 4 summarizes the properties of these trinomials. The sequence  $(\theta_n^{-1})_{n \geq 2}$  will be extensively used in the sequel. It is a fundamental set of Perron numbers of the interval

$(1, \theta_2^{-1})$  simply indexed by the integer  $n$ , and this indexation is extended to any real number  $\beta$  lying between two successive Perron numbers of this family by (1.0.2). Let us note that  $\text{dyg}(\beta)$  is well-defined for algebraic integers  $\beta$  and also for transcendental numbers  $\beta$ . Let  $\kappa := \kappa(1, a_{\max}) = 0.171573\dots$  (cf Sections §4, §5 and §6 for the proofs).

**Theorem 1.1.** (i) For  $n \geq 2$ ,

$$(1.0.4) \quad \text{dyg}(\theta_n^{-1}) = n = \begin{cases} \deg(\theta_n^{-1}) & \text{if } n \not\equiv 5 \pmod{6}, \\ \deg(\theta_n^{-1}) + 2 & \text{if } n \equiv 5 \pmod{6}, \end{cases}$$

(ii) if  $\beta$  is a real number which satisfies  $\theta_n^{-1} \leq \beta < \theta_{n-1}^{-1}$ ,  $n \geq 2$ , then the asymptotic expansion of the dynamical degree  $\text{dyg}(\beta) = \text{dyg}(\theta_n^{-1}) = n$  of  $\beta$  is:

$$(1.0.5) \quad \text{dyg}(\beta) = -\frac{\text{Log}(\beta - 1)}{\beta - 1} \left[ 1 + O\left( \left( \frac{\text{Log}(-\text{Log}(\beta - 1))}{\text{Log}(\beta - 1)} \right)^2 \right) \right],$$

with the constant 1 in the Big O; moreover, if  $\beta \in (\theta_n^{-1}, \theta_{n-1}^{-1})$ ,  $n \geq 260$ , is a reciprocal algebraic integer of degree  $\deg(\beta)$ , with  $M(\alpha) < 1.176280\dots$ , then

$$(1.0.6) \quad \text{dyg}(\beta) \left( \frac{2 \arcsin\left(\frac{\kappa}{2}\right)}{\pi} \right) + \left( \frac{2\kappa \text{Log} \kappa}{\pi \sqrt{4 - \kappa^2}} \right) \leq \deg(\beta).$$

The Poincaré asymptotic expansion method has been introduced for the roots of  $(G_n)$  of modulus  $< 1$  in [214]. Then the Conjecture of Lehmer for the family  $(\theta_n^{-1})_{n \geq 2}$  was solved (in [214]) by using this method and the resulting asymptotic expansions of the Mahler measures  $(M(\theta_n^{-1}))_{n \geq 2}$ , as functions of  $n$ . In the present note this method is extended to any reciprocal algebraic integer  $\beta$  of dynamical degree  $\text{dyg}(\beta)$  large enough, where now “ $\text{dyg}(\beta)$ ” replaces “ $n$ ”.

The choice of the sequence of trinomials  $(G_n)$  is natural in the context of the Rényi-Parry dynamical systems (Section § 2) and leads to a theory of *lexicographical perturbation* of the trinomials  $G_n$  compatible with the dynamics. This is at variance with other attacks of the Conjecture of Lehmer by perturbed cyclotomic polynomials or polynomials having all their roots on the unit circle ([7], Ray [163], Doche [65], Sinclair [188], Mossinghoff, Pinner and Vaaler [145], Toledano [208]). Taking the integer function  $\text{dyg}(\beta)$  as an integer variable tending to infinity when  $\beta > 1$  tends to  $1^+$  is natural. All the asymptotic expansions, for the roots of modulus  $< 1$  of the minimal polynomials  $P_\beta(z)$ , for the lower bounds of the lenticular Mahler measures  $M_r(\beta)$ , will be obtained as a function of the integer  $\text{dyg}(\beta)$ , when  $\beta > 1$  tends to  $1^+$ .

To the  $\beta$ -shift, to the Rényi-Parry dynamical system associated with an algebraic integer  $\beta > 1$ , are attached several analytic functions: (i) the minimal polynomial function  $P_\beta(z)$  which is reciprocal by Smyth’s Theorem [189] as soon as  $M(\beta) < \Theta = 1.3247\dots$ ; (ii) the (Artin-Mazur) dynamical zeta function of the  $\beta$ -shift [10], the generalized Fredholm determinant of the transfer operator associated with the  $\beta$ -transformation  $T_\beta$  [15], the Perron-Frobenius operator associated to  $T_\beta$  [109] [141] [142] [199]. The main theorems below are obtained using the Parry Upper function  $f_\beta(z)$ , which is the opposite of the inverse of the dynamical zeta function  $\zeta_\beta(z)$ . The Parry Upper function at  $\beta$  is the generalized Fredholm determinant associated with the transfer operator of the  $\beta$ -transformation (Baladi [13]).

Using ergodic theory Takahaski [199] [200], Ito and Takahashi [109], Flatto, Lagarias and Poonen [84] have given an explicit expression (reformulation) of the Parry Upper function  $f_\beta(z)$  of the  $\beta$ -shift. This simplified expression is extensively used in the sequel.

The Parry Upper upper function at  $\beta$  takes the general form, with a lacunarity controlled by the dynamical degree (Theorem 2.7, Proposition 3.2):

$$(1.0.7) \quad \begin{aligned} f_\beta(z) &= -1 + z + z^{\text{dyg}(\beta)} + z^{m_1} + z^{m_2} + z^{m_3} + \dots \\ &= G_{\text{dyg}(\beta)} + z^{m_1} + z^{m_2} + z^{m_3} + \dots \end{aligned}$$

with  $m_1 \geq 2 \text{dyg}(\beta) - 1, m_{q+1} - m_q \geq \text{dyg}(\beta) - 1, q \geq 1$ . For  $\theta_{\text{dyg}(\beta)}^{-1} \leq \beta < \theta_{\text{dyg}(\beta)-1}^{-1}$ , the lenticulus  $\mathcal{L}_\beta$  of zeroes of  $f_\beta(z)$  relevant for the Mahler measure is obtained by a deformation of the lenticulus of zeroes  $\mathcal{L}_{\theta_{\text{dyg}(\beta)}^{-1}}$  of  $G_{\text{dyg}(\beta)}$  due to the tail  $z^{m_1} + z^{m_2} + \dots$  itself. For instance, for Lehmer's number  $\beta = 1.17628\dots$ , the dynamical degree  $\text{dyg}(\beta)$  is equal to 12 and

$$f_\beta(z) = -1 + z + z^{12} + z^{31} + z^{44} + z^{63} + z^{86} + z^{105} + z^{118} + \dots$$

is sparse with gaps of length  $\geq 10 = \text{dyg}(\beta) - 2$ . The lenticulus  $\mathcal{L}_\beta$  is close to  $\mathcal{L}_{\theta_{12}^{-1}}$  represented in Figure 1.

The passage from the Parry Upper function  $f_\beta(z)$  to the Mahler measure  $M(\beta)$  (when  $\beta > 1$  is a reciprocal algebraic integer) constitutes the main discoveries of the author, and relies upon two facts:

- (i) the discovery of lenticular distributions of zeroes of  $f_\beta(z)$  in the annular region  $e^{-\text{Log} \beta} = \frac{1}{\beta} \leq |z| < 1$  which are very close to the lenticular sets of zeroes of the trinomials  $G_{\text{dyg}(\beta)}(z)$  of modulus  $< 1$ ;
- (ii) the identification of these zeroes as conjugates of  $\beta$ . The quantity  $\text{Log} \beta$  is the topological entropy of the  $\beta$ -shift. These lenticular distributions of zeroes lie in the cusp of the fractal of Solomyak of the  $\beta$ -shift [193] (recalled in Section § 3.2). The key ingredient for obtaining the Dobrowolski type minoration of the Mahler measure  $M(\beta)$  in Theorem 1.4 relies upon the best possible evaluation of the deformation of these lenticuli of zeroes by the method of Rouché (in Section § 5) and the coupling between the Rouché conditions and the asymptotic expansions of the lenticular zeroes.

The identification of the complete set of the conjugates  $\beta^{(i)}$  of  $\beta$ ,  $|\beta^{(i)}| < 1$  ( $\beta > 1$  being a reciprocal algebraic integer), seems to be unreachable by the present method. The identification of the lenticular conjugates of modulus  $< 1$  can only be done in an angular subsector of  $\arg(z) \in [-\frac{\pi}{3}, \frac{\pi}{3}]$  (Proposition 5.36 and Theorem 6.1). Consequently the present method only gives access to a “*part*” of the Mahler measure itself. We denote by  $\mathcal{L}_\beta, \mathcal{L}_{\theta_{\text{dyg}(\beta)}^{-1}}$  the lenticular sets of zeroes of  $f_\beta(z)$ , resp. of  $G_{\text{dyg}(\beta)}(z)$ . We call

$$M_r(\beta) = \prod_{\omega \in \mathcal{L}_\beta} |\omega|^{-1}$$

the lenticular Mahler measure of  $\beta$ . It satisfies  $M_r(\beta) \leq M(\beta)$ . We show that  $\beta \rightarrow M_r(\beta)$  is continuous on each open interval  $(\theta_n^{-1}, \theta_{n-1}^{-1})$  for the usual topology, and that it admits a lower bound which can be expanded as an asymptotic expansion of  $\text{dyg}(\beta)$  (Theorem

6.3). The general minorant of  $M(\beta)$  for solving the problem of Lehmer comes from the asymptotic expansion of the lower bound of  $M_r(\beta)$ ; it is given by (1.0.19).

Case (ii) is now an extension of case (i). The minoration of the Mahler measure of  $P_\alpha$  is deduced from the Rényi-Parry *dynamics* of the house  $|\overline{\alpha}|$ . For  $\alpha$  a nonreal complex reciprocal algebraic integer,  $|\alpha| > 1$ , such that  $1 < |\overline{\alpha}| \leq \frac{1+\sqrt{5}}{2}$ , the dynamical degree of  $\alpha$  is defined by  $\text{dyg}(\alpha) := \text{dyg}(|\overline{\alpha}|)$ . Once  $|\overline{\alpha}| > 1$  is close enough to  $1^+$ , three new notions appear:

- (i) the equality  $P_\alpha = P_{|\overline{\alpha}|}$  between the minimal polynomials, resp. of  $\alpha$  and  $|\overline{\alpha}|$ ,
- (ii) the identification of the lenticular zeroes of  $f_{|\overline{\alpha}|}$  with the lenticular zeroes of  $P_\alpha$ , and the *continuity* of the minorant of the lenticular Mahler measure with the house  $|\overline{\alpha}|$  of  $\alpha$  (Theorem 5.37 and Remark 5.38),
- (iii) the *fracturability* of the minimal polynomial  $P_\alpha(z) = (-\zeta_{|\overline{\alpha}|}(z)P_\alpha(z)) \times f_{|\overline{\alpha}|}(z)$  as a product of the two integer arithmetic series:  $-\zeta_{|\overline{\alpha}|}(z)P_\alpha(z), f_{|\overline{\alpha}|}(z) \in \mathbb{Z}[[z]]$  (Theorem 5.36 and Theorem 6.1). The fracturability of the minimal polynomial  $P_\alpha(z)$  obeys the *Carlson-Polya dichotomy* [52] [161] as:

$$(1.0.8) \quad P_\alpha(z) = -\zeta_{|\overline{\alpha}|}(z)P_\alpha(z) \times f_{|\overline{\alpha}|}(z) \quad \left\{ \begin{array}{l} \text{on } \mathbb{C} \quad \text{if } |\overline{\alpha}| \text{ is a Parry number,} \\ \quad \quad \quad \text{with } -\zeta_{|\overline{\alpha}|}(z)P_\alpha(z) \text{ and } f_{|\overline{\alpha}|} \\ \quad \quad \quad \text{as meromorphic functions,} \\ \\ \text{on } |z| < 1 \quad \text{if } \beta \text{ is a nonParry number,} \\ \quad \quad \quad \text{with } |z| = 1 \text{ as natural boundary} \\ \quad \quad \quad \text{for both } -\zeta_{|\overline{\alpha}|}(z)P_\alpha(z) \text{ and } f_{|\overline{\alpha}|}, \end{array} \right.$$

The *domain of fracturability* of  $P_\alpha$  is the open subset of the open unit disk on which  $-\zeta_{|\overline{\alpha}|}(z)P_\alpha(z)$  is not constant, does not vanish, is holomorphic. The domain of fracturability of  $P_\alpha$  contains the lenticular zeroes of  $P_\alpha$ .

Lehmer's number, say  $\beta_0$ , is the smallest Mahler measure ( $> 1$ ) of algebraic integers known and the smallest Salem number known [144] [146]. It is the dominant root of Lehmer's polynomial, of degree 10,

$$(1.0.9) \quad P_{\beta_0}(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1.$$

The above general equality  $P_\alpha = P_{|\overline{\alpha}|}$  in (i) is the generalization of the identity:  $P_{\beta_0}(X) = P_{|\overline{\beta_0}|}(X)$ .

The main theorems are the following.

**Theorem 1.2** (ex-Lehmer conjecture). *For any nonzero algebraic integer  $\alpha$  which is not a root of unity,*

$$M(\alpha) \geq \theta_\eta^{-1} > 1 \quad \text{for some integer } \eta \geq 259.$$

**Theorem 1.3** (ex-Schinzel-Zassenhaus conjecture). *Let  $\alpha$  be a nonzero reciprocal algebraic integer which is not a root of unity. Then*

$$(1.0.10) \quad |\overline{\alpha}| \geq 1 + \frac{c}{\text{deg}(\alpha)}$$

with  $c = \theta_\eta^{-1} - 1$  with  $\eta \geq 259$ .



The following definitions are given in Section § 5. We just report them here for stating Theorem 1.4. Denote by  $a_{\max} = 5.87433\dots$  the abscissa of the maximum of the function  $a \rightarrow \kappa(1, a) := \frac{1 - \exp(\frac{-\pi}{a})}{2 \exp(\frac{\pi}{a}) - 1}$  on  $(0, \infty)$  (Figure 2). Let  $\kappa := \kappa(1, a_{\max}) = 0.171573\dots$  be the value of the maximum. Let  $S := 2 \arcsin(\kappa/2) = 0.171784\dots$ . Denote

$$\Lambda_r \mu_r := \exp\left(\frac{-1}{\pi} \int_0^S \text{Log} \left[ \frac{1 + 2 \sin(\frac{x}{2}) - \sqrt{1 - 12 \sin(\frac{x}{2}) + 4(\sin(\frac{x}{2}))^2}}{4} \right] dx\right)$$

$$(1.0.11) \quad = 1.15411\dots, \quad \text{a value slightly below Lehmer's number } 1.17628\dots$$

**Theorem 1.4** (Dobrowolski type minoration). *Let  $\alpha$  be a nonzero reciprocal algebraic integer which is not a root of unity such that  $\text{dyg}(\alpha) \geq 260$ , with  $M(\alpha) < 1.176280\dots$ . Then*

$$(1.0.12) \quad M(\alpha) \geq \Lambda_r \mu_r - \Lambda_r \mu_r \frac{S}{2\pi} \left( \frac{1}{\text{Log}(\text{dyg}(\alpha))} \right)$$

In terms of the Weil height  $h$ , using Theorem 5.3, the asymptotics of the minoration (1.0.12) takes the following form:

$$(1.0.13) \quad \text{deg}(\alpha)h(\alpha) \geq \text{Log}(\Lambda_r \mu_r) + \frac{S}{2\pi} \frac{1}{\text{Log}(|\bar{\alpha}| - 1)}.$$

The minoration (1.0.12) can also be restated in terms of the usual degree. Let  $B > 0$ . Let us consider the subset  $\mathcal{F}_B$  of all nonzero reciprocal algebraic integers  $\alpha$  not being a root of unity such that  $|\bar{\alpha}| < \theta_{259}^{-1}$  satisfying  $n := \text{deg}(\alpha) \leq (\text{dyg}(\alpha))^B$ . Then

$$(1.0.14) \quad M(\alpha) \geq \Lambda_r \mu_r - \Lambda_r \mu_r \frac{SB}{2\pi} \left( \frac{1}{\text{Log } n} \right), \quad \alpha \in \mathcal{F}_B.$$

Comparatively, in 1979, Dobrowolski [63], using an auxiliary function, obtained the asymptotic minoration, with  $n = \text{deg}(\alpha)$ ,

$$(1.0.15) \quad M(\alpha) > 1 + (1 - \varepsilon) \left( \frac{\text{Log } \text{Log } n}{\text{Log } n} \right)^3, \quad n > n_0,$$

with  $1 - \varepsilon$  replaced by  $1/1200$  for  $n \geq 2$ , for an effective version of the minoration. In (1.0.12) or (1.0.14), the constant in the minorant is not any more 1 but  $1.15411\dots$  and the sign of the  $n$ -dependent term is negative, with an appreciable gain of  $(\text{Log } n)^2$  in the denominator.

The minoration (1.0.12) is general and admits a better lower bound, in a similar formulation, when  $\alpha$  only runs over the set of Perron numbers  $(\theta_n^{-1})_{n \geq 2}$ . Indeed, in [214], it is shown that

$$(1.0.16) \quad M(\theta_n^{-1}) > \Lambda - \frac{\Lambda}{6} \left( \frac{1}{\text{Log } n} \right), \quad n \geq 2,$$

holds with the following constant of the minorant

$$(1.0.17) \quad \Lambda := \exp\left(\frac{3\sqrt{3}}{4\pi} L(2, \chi_3)\right) = \exp\left(\frac{-1}{\pi} \int_0^{\pi/3} \text{Log}\left(2 \sin\left(\frac{x}{2}\right)\right) dx\right) = 1.38135\dots,$$

higher than  $1.1541\dots$ , and  $L(s, \chi_3) := \sum_{m \geq 1} \frac{\chi_3(m)}{m^s}$  the Dirichlet L-series for the character  $\chi_3$ , with  $\chi_3$  the uniquely specified odd character of conductor 3 ( $\chi_3(m) = 0, 1$  or  $-1$  according to whether  $m \equiv 0, 1$  or  $2 \pmod{3}$ ), equivalently  $\chi_3(m) = \left(\frac{m}{3}\right)$  the Jacobi symbol). The two constants in (1.0.12) and (1.0.16) are:  $\Lambda_r \mu_r S / (2\pi) = 0.0315536\dots$  in (1.0.12) and  $\Lambda/6 = 0.230225\dots$  in (1.0.16).

The Mahler measure  $M(G_n)$  of the trinomial  $G_n$  is equal to the lenticular Mahler measure  $M_r(G_n)$  itself, with limit  $\lim_{n \rightarrow +\infty} M(G_n) = \lim_{n \rightarrow +\infty} M_r(G_n) = \Lambda$ , having asymptotic expansion

$$(1.0.18) \quad M(G_n) = \Lambda \left( 1 + r(n) \frac{1}{\text{Log } n} + O\left(\frac{(\text{Log Log } n)^2}{\text{Log } n}\right) \right)$$

with  $r(n)$  real,  $|r(n)| \leq 1/6$ . In the case of the trinomials  $G_n$  the characterization of the roots of modulus  $< 1$  can be readily obtained (Section §4) and does not require the detection method of Rouché. In the general case, with  $\beta \in (\theta_n^{-1}, \theta_{n-1}^{-1})$ ,  $n = \text{dyg}(\beta)$  large enough, applying the method of Rouché only leads to the following asymptotic lower bound of the lenticular minorant, similarly as in (1.0.18), as (Section §6.2):

$$(1.0.19) \quad M_r(\beta) \geq \Lambda_r \mu_r \left( 1 + \frac{\mathcal{R}}{\text{Log } n} + O\left(\frac{(\text{Log Log } n)^2}{\text{Log } n}\right) \right),$$

with  $\left| \mathcal{R} + O\left(\frac{(\text{Log Log } n)^2}{\text{Log } n}\right) \right| < \frac{\arcsin(\kappa/2)}{\pi}$ .

Denote by  $M_{\text{inf}} := \liminf_{|\alpha| \rightarrow 1^+} M(\alpha)$  the limit infimum of the Mahler measures  $M(\alpha)$ ,  $\alpha \in \mathcal{O}_{\overline{\mathbb{Q}}}$ , when  $|\alpha| > 1$  tends to  $1^+$ . Then

$$(1.0.20) \quad \Lambda_r \mu_r \leq M_{\text{inf}} \leq \Lambda.$$

Because the  $\beta$ -shift is compact, it seems reasonable to formulate the following conjecture on the possible intermediate values between  $M_{\text{inf}}$  and  $\Lambda$ .

**Conjecture 3.** *For any  $\nu_0 \in [M_{\text{inf}}, \Lambda)$  there exists a sequence of integer monic irreducible polynomials  $(H_m(z))_m$  such that  $\lim_{m \rightarrow +\infty} M(H_m) = \nu_0$ .*

Lenticuli of conjugates lie in the cusp of Solomyak's fractal (Section § 3.2) [73]. The number of elements of a lenticulus  $\mathcal{L}_\alpha$  is an increasing function of the dynamical degree  $\text{dyg}(\alpha)$  as soon as  $\text{dyg}(\alpha)$  is large enough. The existence of lenticuli composed of three elements only (one real, a pair of nonreal complex-conjugated conjugates) is studied in Section § 7.1. Such lenticuli appear at small dynamical degrees. Since Salem numbers have no nonreal complex conjugate of modulus  $< 1$ , they should not possess 3-elements lenticuli of conjugates, therefore they should possess a small dynamical degree bounded from above. We obtain 31 as an upper bound as follows.

**Theorem 1.5** (ex-Lehmer conjecture for Salem numbers). *Let  $T$  denote the set of Salem numbers. Then  $T$  is bounded from below:*

$$\beta \in T \quad \implies \quad \beta > \theta_{31}^{-1} = 1.08544\dots$$

Lehmer's number  $1.17628\dots$  belongs to the interval  $(\theta_{12}^{-1}, \theta_{11}^{-1})$  (Table 1). This interval does not contain any other known Salem number. If there is another one, its degree should be greater than 44 [143] [146].

**Conjecture 4.** *There is no Salem number in the interval*

$$(\theta_{31}^{-1}, \theta_{12}^{-1}) = (1.08544\dots, 1.17295\dots).$$

Parry Upper functions  $f_\beta(z)$ , with  $\beta$  being an algebraic integer of dynamical degree  $\text{dyg}(\beta) = 12$  to  $16$ , do possess 3-elements lenticuli of zeroes in the open unit disc (as in Figure 1).

The main obstruction in Lehmer’s problem arises from the existence of lenticuli of conjugates in a small angular sector containing 1 in the complex plane. These lenticuli come from the type of factorization of the polynomial sections of  $f_{|\overline{\alpha}|}$ . The importance of the angular sectors containing 1 has already been guessed by Langevin in [117] [118] [119] [139], by Dubickas and Smyth [72], by Rhin and Smyth [165] and Rhin and Wu [166]. These lenticuli cannot be described by these classical approaches, but become visible by the present method. Though the lenticuli of roots lie inside and off the unit circle, the complete set of conjugates remains fairly regularly distributed in the sense that it equidistributes on the unit circle at the limit, once the Mahler measures are small enough, as follows.

**Theorem 1.6.** *Let  $(\alpha_q)_{q \geq 1}$  be a sequence of algebraic integers such that  $|\alpha_q| > 1$ ,  $M(\alpha_q) < 1.176280\dots$ ,  $\text{dyg}(\alpha_q) \geq 260$ , for  $q \geq 1$ , with  $\lim_{q \rightarrow +\infty} |\overline{\alpha}_q| = 1$ . Then the sequence  $(\alpha_q)_{q \geq 1}$  is strict and*

$$(1.0.21) \quad \mu_{\alpha_q} \rightarrow \mu_{\mathbb{T}}, \quad \text{dyg}(\alpha_q) \rightarrow +\infty, \quad \text{weakly,}$$

*i.e. for all bounded, continuous functions  $f : \mathbb{C}^\times \rightarrow \mathbb{C}$ ,*

$$(1.0.22) \quad \int f d\mu_{\alpha_q} \rightarrow \int f d\mu_{\mathbb{T}}, \quad \text{dyg}(\alpha_q) \rightarrow +\infty.$$

Parry numbers are Perron numbers (Adler and Marcus [2]); the characterization of the set of Parry numbers (Definition 2.1) is a deep question addressed to the dynamics of Perron numbers (Bertrand-Mathis [24], Boyd [34], Boyle and Handelman [42] [43] [44], Brunotte [45], Calegari and Huang [49], Dubickas and Sha [71], Lind [128] [129], Lind and Marcus [130], Thurston [207], Verger-Gaugry [210] [211]), associated with the rationality of the dynamical zeta function of the  $\beta$ -shift

$$(1.0.23) \quad \zeta_\beta(z) := \exp \left( \sum_{n=1}^{\infty} \frac{\mathcal{P}_n}{n} z^n \right), \quad \mathcal{P}_n := \#\{x \in [0, 1] \mid T_\beta^n(x) = x\}$$

counting the number of periodic points of period dividing  $n$ . For  $\alpha$  a nonzero algebraic integer which is not a root of unity, with  $\beta = |\overline{\alpha}|$ , by Theorem 3.4,

$\beta$  is a Parry number if and only if  $\zeta_\beta(z)$  is a rational function;

and  $|z| = 1$  is the natural boundary of the domain of fracturability of the minimal polynomial  $P_\alpha$ , in the sense of Theorem 6.1, if and only if  $\beta$  is not a Parry number, as soon as  $|\alpha|$  is close enough to 1, in the Carlson-Polya dichotomy. Comparatively, for complete nonsingular projective algebraic varieties  $X$  over the field of  $q$  elements,  $q$  a prime power, the zeta function  $\zeta_X(t)$  introduced by Weil [219] is only a rational function (Dwork [76], Tao [205]): the first Weil’s conjecture, for which there exists a set of characteristic values was proved by Dwork using  $p$ -adic methods (Dwork [76]), and “Weil II”, the Riemann hypothesis, proved by Deligne using  $l$ -adic étale cohomology in characteristic  $p \neq l$  (Deligne [57]). It is defined as a dynamical zeta function with the action of the Frobenius. The

purely  $p$ -adic methods of Dwork (Dwork [76]), continued by Kedlaya [111] for “Weil II” in the need of numerically computing zeta functions by explicit equations, allow an intrinsic computability, as in Lauder and Wan [121], towards a  $p$ -adic cohomology theory, are linked to “extrinsic geometry”, to the defining equations of the variety itself. They are in contrast with the relative version of crystalline cohomology developed by Faltings [80], or the Monsky-Washnitzer constructions used by Lubkin [134]. We refer the reader to Robba [168], Kedlaya [111], Tao [205], for a short survey on other developments.

After Weil [219], and introduced in general terms by Artin and Mazur in [10], the theory of dynamical zeta functions  $\zeta(z)$  associated with dynamical systems, based on an analogy with the number theory zeta functions, developed under the impulsion of Ruelle [173] in the direction of the thermodynamic formalism and with Pollicott, Baladi and Keller [15] towards transfer operators and counting orbits [154]. The determination and the existence of meromorphic extensions or/and natural boundaries of dynamical zeta functions is a deep problem.

In the present proof of the conjecture of Lehmer, the analytic extension of the dynamical zeta function of the  $\beta$ -shift behaves as an analogue of Weil’s zeta function (in the sense that both are dynamical zeta functions). But it generates questions beyond the analogues of Weil’s conjectures since not only the rational case of  $\zeta_\beta$  contributes to the minoration of the Mahler measure, but also the nonrational case with the unit circle as natural boundary and lenticular poles arbitrarily close to it. For instance, a part of the analogue of “Weil II” (Riemann Hypothesis) would be the determination of the geometry of the beta-conjugates in the rationality case. Beta-conjugates are zeroes of Parry polynomials, whose factorization has been studied in the context of the theory of Pinner and Vaaler [157] in [211].

An apparent difficulty for the computation of the minorant of  $M(\alpha)$  comes from the absence of complete characterization of the set of Parry numbers  $\mathbb{P}_P$ , when  $\beta = |\alpha|$  is close to one, since we never know whether  $\beta$  is a Parry number or not. But the Mahler measure  $M(\alpha)$  is independent of the Carlson-Polya dichotomy. Indeed, the two domains of definitions of  $\zeta_\beta$ , “ $\mathbb{C}$ ” and “ $|z| < 1$ ”, together with the corresponding splitting (1.0.8), may occur fairly “randomly” when  $\beta$  tends to one. But  $\{|z| < 1\}$  is a domain included in both,  $M(\alpha)$  “reading” only the roots in it and not taking care of the “status” of the unit circle. Whether  $f_{|\alpha|}(z)$  can be continued analytically or not beyond the unit circle has no effect on the value of the Mahler measure  $M(\alpha)$ .

The paper is organized as follows: in Section §2 we recall the properties of the Rényi-Parry numeration system of the  $\beta$ -shift. The analytic functions, in particular the Parry Upper function  $f_\beta(z)$  and the dynamical zeta function  $\zeta_\beta(z)$ , associated to the dynamics of the  $\beta$ -shift, are introduced in Section §3. In Section §5 the peculiar consequences of the lexicographical ordering, induced by the numeration system, on the zeroes of  $f_\beta(z)$  are developed, in particular the lenticular zeroes and their identification as Galois conjugates of the base of numeration  $\beta$ . Coupling the knowledge of the geometry of the lenticular roots with the method of asymptotic expansions (recalled in Section §4 for trinomials) gives a continuous lenticular minorant of  $M(\beta)$  and a Dobrowolski type minoration. The proofs of Lehmer’s Conjecture and Schinzel-Zassenhaus’s Conjecture follow, for  $\beta > 1$  any reciprocal algebraic integer (case **(i)**). These results are extended in Section §6 (case **(ii)**) for any nonzero reciprocal algebraic integer  $\alpha$  which is not a root of unity,  $\arg(z) \in (0, \pi/2]$ . This case is shown to be deduced from the preceding case (given by Section §5) by taking

$\beta := \overline{|\alpha|}$  which is real and  $> 1$ . In Section §7 the Conjecture of Lehmer is proved for Salem numbers, using another regime of asymptotic expansions of the roots of the trinomials  $G_n$ , more adapted to the cusp in Solomyak's fractal. Concomitantly to the limit problem of Lehmer it is shown that the conjugates of the base of numeration  $\overline{|\alpha|}$  equidistribute on the unit circle in the complex plane. Using a Theorem of Belotserkovski a Theorem of limit equidistribution of the conjugates is formulated in Section §8, when the dynamical degree of  $\overline{|\alpha|}$  tends to infinity. A few consequences are mentioned in Section §9, in particular a Conjecture of Margulis.

The proof of Lehmer's Conjecture starts at Section §5. It is preferable that the reader be acquainted with the results of [214]. Lenticuli of roots are investigated in [73]. The method of identification of the lenticular poles of  $\zeta_\beta(z)$  with zeroes of the minimal polynomial of  $\beta$  is published in [74].

## 2. THE $\beta$ -SHIFT, THE RÉNYI-PARRY DYNAMICAL SYSTEM OF NUMERATION

**2.1. Towards the problem of Lehmer.** The direction which is followed is the following: it consists in using the analytic functions associated with the Rényi-Parry dynamical system of numeration, the  $\beta$ -shift (i.e. with the language in base  $\beta$ ) with  $1 < \beta < 2$ , first where  $\beta$  is fixed to formulate the properties of these functions, and then vary continuously the basis of numeration  $\beta$  taking the limit to  $1^+$ , to use the limit properties of these functions for solving the problem of Lehmer. This is a method of variable basis, where the variable  $\beta$  runs over  $\mathbb{Q} \cap (1, +\infty)$ , more precisely over the set of reciprocal algebraic integers. This mathematics appears when the base of numeration is not an integer. The analytic functions are the Parry Upper function  $f_\beta(z)$  and the dynamical zeta function  $\zeta_\beta(z)$ . The Parry Upper function  $f_\beta(z)$  is the generalized Freholm determinant of the transfer operator  $\mathcal{L}_{i\beta}$  of the  $\beta$ -transformation. Both analytic functions are presented in Section §3. In Section §2 are recalled the properties of the  $\beta$ -shift, in particular the conversion of the inequality “ $<$ ” on  $(1, 2)$  to the lexicographical inequality “ $<_{lex}$ ” on sequences of 0, 1 digits.

Let us say a few words on the  $\beta$ -shift. In 1957 Rényi [164] introduced new representations of a real number  $x$ , using a positive function  $y = f(x)$  and infinite iterations of it, in the form of an “ $f$ -expansion”, as

$$x = \varepsilon_0 + f(\varepsilon_1 + f(\varepsilon_2 + \dots))$$

with “digits”  $\varepsilon_i$  in some alphabet and remainders  $f(\varepsilon_n + f(\varepsilon_{n+1} + \dots))$ . This approach considerably enlarged the usual decimal numeration system, that is internationally used today, and also numeration systems in integer basis (recall that the bases 5, 10, 12, 20 and 60 were used in the Antiquity in several countries as natural counting systems), by allowing arbitrary real bases of numeration (Fraenkel [85], Lothaire [131], Chap. 7): let  $\beta > 1$  be not an integer and consider  $f(x) = x/\beta$  if  $0 \leq x \leq \beta$ , and  $f(x) = 1$  if  $\beta < x$ . Then the  $f$ -expansion of  $x$  is the representation of  $x$  in base  $\beta$  as

$$x = \varepsilon_0 + \frac{\varepsilon_1}{\beta} + \frac{\varepsilon_2}{\beta^2} + \dots + \frac{\varepsilon_n}{\beta^n} + \dots$$

In terms of dynamical systems, in 1960, Parry [151] [152] [153] has reconsidered and studied the ergodic properties of such representations of real numbers in base  $\beta$ , in particular the conditions of faithfulness of the map:  $x \longleftrightarrow (\varepsilon_i)_i$  and the complete set of admissible

sequences  $(\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots)$  for all real numbers. This complete set is called the language in base  $\beta$ . Let us note that  $\beta > 1$  will run over the set of reciprocal algebraic integers in Section §5 and §6, but that the  $\beta$ -shift is defined in general for any real number  $\beta$ , algebraic or transcendental.

**2.2. The  $\beta$ -shift,  $\beta$ -expansions, lacunarity and symbolic dynamics.** Let  $\beta > 1$  be a real number and let  $\mathcal{A}_\beta := \{0, 1, 2, \dots, \lceil \beta - 1 \rceil\}$ . If  $\beta$  is not an integer, then  $\lceil \beta - 1 \rceil = \lfloor \beta \rfloor$ . Let  $x$  be a real number in the interval  $[0, 1]$ . A representation in base  $\beta$  (or a  $\beta$ -representation; or a  $\beta$ -ary representation if  $\beta$  is an integer) of  $x$  is an infinite word  $(x_i)_{i \geq 1}$  of  $\mathcal{A}_\beta^{\mathbb{N}}$  such that

$$x = \sum_{i \geq 1} x_i \beta^{-i}.$$

The main difference with the case where  $\beta$  is an integer is that  $x$  may have several representations. A particular  $\beta$ -representation, called the  $\beta$ -expansion, or the greedy  $\beta$ -expansion, and denoted by  $d_\beta(x)$ , of  $x$  can be computed either by the greedy algorithm, or equivalently by the  $\beta$ -transformation

$$T_\beta : x \mapsto \beta x \pmod{1} = \{\beta x\}.$$

The dynamical system  $([0, 1], T_\beta)$  is called the Rényi-Parry numeration system in base  $\beta$ , the iterates of  $T_\beta$  providing the successive digits  $x_i$  of  $d_\beta(x)$  [127]. Denoting  $T_\beta^0 := \text{Id}$ ,  $T_\beta^1 := T_\beta$ ,  $T_\beta^i := T_\beta(T_\beta^{i-1})$  for all  $i \geq 1$ , we have:

$$d_\beta(x) = (x_i)_{i \geq 1} \quad \text{if and only if} \quad x_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor$$

and we write the  $\beta$ -expansion of  $x$  as

$$(2.2.1) \quad x = \cdot x_1 x_2 x_3 \dots \quad \text{instead of} \quad x = \frac{x_1}{\beta} + \frac{x_2}{\beta^2} + \frac{x_3}{\beta^3} + \dots$$

The digits are  $x_1 = \lfloor \beta x \rfloor$ ,  $x_2 = \lfloor \beta \{\beta x\} \rfloor$ ,  $x_3 = \lfloor \beta \{\beta \{\beta x\}\} \rfloor$ ,  $\dots$ , depend upon  $\beta$ .

The Rényi-Parry numeration dynamical system in base  $\beta$  allows the coding, as a (positional)  $\beta$ -expansion, of any real number  $x$ . Indeed, if  $x > 0$ , there exists  $k \in \mathbb{Z}$  such that  $\beta^k \leq x < \beta^{k+1}$ . Hence  $1/\beta \leq x/\beta^{k+1} < 1$ ; thus it is enough to deal with representations and  $\beta$ -expansions of numbers in the interval  $[1/\beta, 1]$ . In the case where  $k \geq 1$ , the  $\beta$ -expansion of  $x$  is

$$x = x_1 x_2 \dots x_k \cdot x_{k+1} x_{k+2} \dots,$$

with  $x_1 = \lfloor \beta(x/\beta^{k+1}) \rfloor$ ,  $x_2 = \lfloor \beta \{\beta(x/\beta^{k+1})\} \rfloor$ ,  $x_3 = \lfloor \beta \{\beta \{\beta(x/\beta^{k+1})\}\} \rfloor$ , etc. If  $x < 0$ , by definition:  $d_\beta(x) = -d_\beta(-x)$ . The part  $x_1 x_2 \dots x_k$  is called the  $\beta$ -integer part of the  $\beta$ -expansion of  $x$ , and the terminant  $\cdot x_{k+1} x_{k+2} \dots$  is called the  $\beta$ -fractional part of  $d_\beta(x)$ .

A  $\beta$ -integer is a real number  $x$  such that the  $\beta$ -integer part of  $d_\beta(x)$  is equal to  $d_\beta(x)$  itself (all the digits  $x_{k+j}$  being equal to 0 for  $j \geq 1$ ): in this case, if  $x > 0$  for instance,  $x$  is the polynomial

$$x = \sum_{i=1}^k x_i \beta^{k-i}, \quad 0 \leq x_i \leq \lceil \beta - 1 \rceil$$

and the set of  $\beta$ -integers is denoted by  $\mathbb{Z}_\beta$ . For all  $\beta > 1$   $\mathbb{Z}_\beta \subset \mathbb{R}$  is discrete and  $\mathbb{Z}_\beta = \mathbb{Z}$  if  $\beta$  is an integer  $\neq 0, 1$ .

The set  $\mathcal{A}_\beta^{\mathbb{N}}$  is endowed with the lexicographical order (not usual in number theory), and the product topology. The one-sided shift  $\sigma : (x_i)_{i \geq 1} \mapsto (x_{i+1})_{i \geq 1}$  leaves invariant the subset  $D_\beta$  of the  $\beta$ -expansions of real numbers in  $[0, 1)$ . The closure of  $D_\beta$  in  $\mathcal{A}_\beta^{\mathbb{N}}$  is called the  $\beta$ -shift, and is denoted by  $S_\beta$ . The  $\beta$ -shift is a subshift of  $\mathcal{A}_\beta^{\mathbb{N}}$ , for which

$$d_\beta \circ T_\beta = \sigma \circ d_\beta$$

holds on the interval  $[0, 1]$  (Lothaire [131], Lemma 7.2.7). In other terms,  $S_\beta$  is such that

$$(2.2.2) \quad x \in [0, 1] \quad \longleftrightarrow \quad (x_i)_{i \geq 1} \in S_\beta$$

is bijective. This one-to-one correspondence between the totally ordered interval  $[0, 1]$  and the totally lexicographically ordered  $\beta$ -shift  $S_\beta$  is fundamental. Parry ([151] Theorem 3) has shown that only one sequence of digits entirely controls the  $\beta$ -shift  $S_\beta$ , and that the ordering is preserved when dealing with the greedy  $\beta$ -expansions. Let us precise how the usual inequality “ $<$ ” on the real line is transformed into the inequality “ $<_{lex}$ ”, meaning “lexicographically smaller with all its shifts”.

*The greatest element of  $S_\beta$* : it comes from  $x = 1$  and is given either by the Rényi  $\beta$ -expansion of 1, or by a slight modification of it in case of finiteness. Let us precise it. The greedy  $\beta$ -expansion of 1 is by definition denoted by

$$(2.2.3) \quad d_\beta(1) = 0.t_1t_2t_3\dots \quad \text{and uniquely corresponds to} \quad 1 = \sum_{i=1}^{+\infty} t_i\beta^{-i},$$

where

$$(2.2.4) \quad t_1 = \lfloor \beta \rfloor, t_2 = \lfloor \beta \{ \beta \} \rfloor = \lfloor \beta T_\beta(1) \rfloor, t_3 = \lfloor \beta \{ \beta \{ \beta \} \} \rfloor = \lfloor \beta T_\beta^2(1) \rfloor, \dots$$

The sequence  $(t_i)_{i \geq 1}$  is given by the orbit of one  $(T_\beta^j(1))_{j \geq 0}$  by

$$(2.2.5) \quad T_\beta^0(1) = 1, T_\beta^j(1) = \beta^j - t_1\beta^{j-1} - t_2\beta^{j-2} - \dots - t_j \in \mathbb{Z}[\beta] \cap [0, 1]$$

for all  $j \geq 1$ . The digits  $t_i$  belong to  $\mathcal{A}_\beta$ . We say that  $d_\beta(1)$  is finite if it ends in infinitely many zeros.

**Definition 2.1.** If  $d_\beta(1)$  is finite or ultimately periodic (i.e. eventually periodic), then the real number  $\beta > 1$  is said to be a *Parry number*. In particular, a Parry number  $\beta$  is said to be *simple* if  $d_\beta(1)$  is finite.

The greedy  $\beta$ -expansion of  $1/\beta$  is

$$(2.2.6) \quad d_\beta\left(\frac{1}{\beta}\right) = 0.0t_1t_2t_3\dots \quad \text{and uniquely corresponds to} \quad \frac{1}{\beta} = \sum_{i=1}^{+\infty} t_i\beta^{-i-1}.$$

From  $(t_i)_{i \geq 1} \in \mathcal{A}_\beta^{\mathbb{N}}$  is built  $(c_i)_{i \geq 1} \in \mathcal{A}_\beta^{\mathbb{N}}$ , defined by

$$c_1c_2c_3\dots := \begin{cases} t_1t_2t_3\dots & \text{if } d_\beta(1) = 0.t_1t_2\dots \text{ is infinite,} \\ (t_1t_2\dots t_{q-1}(t_q - 1))^\omega & \text{if } d_\beta(1) \text{ is finite, } = 0.t_1t_2\dots t_q, \end{cases}$$

where  $( )^\omega$  means that the word within  $( )$  is indefinitely repeated. The sequence  $(c_i)_{i \geq 1}$  is the unique element of  $\mathcal{A}_\beta^{\mathbb{N}}$  which allows to obtain all the admissible  $\beta$ -expansions of all the elements of  $[0, 1)$ .

**Definition 2.2** (Conditions of Parry). A sequence  $(y_i)_{i \geq 0}$  of elements of  $\mathcal{A}_\beta$  (finite or not) is said *admissible* if

$$(2.2.7) \quad \sigma^j(y_0, y_1, y_2, \dots) = (y_j, y_{j+1}, y_{j+2}, \dots) <_{lex} (c_1, c_2, c_3, \dots) \quad \text{for all } j \geq 0,$$

where  $<_{lex}$  means *lexicographically smaller*.

**Definition 2.3.** A sequence  $(a_i)_{i \geq 0} \in \mathcal{A}_\beta^{\mathbb{N}}$  satisfying (2.2.8) is said to be *Lyndon (or self-admissible)*:

$$(2.2.8) \quad \sigma^n(a_0, a_1, a_2, \dots) = (a_n, a_{n+1}, a_{n+2}, \dots) <_{lex} (a_0, a_1, a_2, \dots) \quad \text{for all } n \geq 1.$$

The terminology comes from the introduction of such words by Lyndon in [135], in honour of his work. Other orderings are reviewed in Nguéma Ndong [147] [148]. The present Lyndon ordering is reported in [147], ex. 2 in subsection 1.2 and in [148], subsection 4.1, Theorem 5 and ex. 4 for applications to the dynamical zeta function of negative  $\beta$ -shift.

Any admissible representation  $(x_i)_{i \geq 1} \in \mathcal{A}_\beta^{\mathbb{N}}$  corresponds, by (2.2.1), to a real number  $x \in [0, 1)$  and conversely the greedy  $\beta$ -expansion of  $x$  is  $(x_i)_{i \geq 1}$  itself. For an infinite admissible sequence  $(y_i)_{i \geq 0}$  of elements of  $\mathcal{A}_\beta$  the (strict) lexicographical inequalities (2.2.7) constitute an infinite number of inequalities which are unusual in number theory [27] [86] [87] [131] [151].

In number theory, inequalities are often associated to collections of half-spaces in euclidean or adelic Geometry of Numbers (Minkowski's Theorem, etc). The conditions of Parry are of totally different nature since they refer to a reasonable control, order-preserving, of the gappiness (lacunarity) of the coefficient vectors of the power series which are the generalized Fredholm determinants of the transfer operators of the  $\beta$ -transformations (cf Section §3).

In the correspondence  $[0, 1] \longleftrightarrow S_\beta$ , the element  $x = 1$  admits the maximal element  $d_\beta(1)$  as counterpart. The uniqueness of the  $\beta$ -expansion  $d_\beta(1)$  and its property to be Lyndon characterize the base of numeration  $\beta$  as follows.

**Proposition 2.4.** *Let  $(a_0, a_1, a_2, \dots)$  be a sequence of non-negative integers where  $a_0 \geq 1$  and  $a_n \leq a_0$  for all  $n \geq 0$ . The unique solution  $\beta > 1$  of*

$$(2.2.9) \quad 1 = \frac{a_0}{x} + \frac{a_1}{x^2} + \frac{a_2}{x^3} + \dots$$

*is such that  $d_\beta(1) = 0.a_0a_1a_2\dots$  if and only if*

$$(2.2.10) \quad \sigma^n(a_0, a_1, a_2, \dots) = (a_n, a_{n+1}, a_{n+2}, \dots) <_{lex} (a_0, a_1, a_2, \dots) \quad \text{for all } n \geq 1.$$

*Proof.* Corollary 1 of Theorem 3 in Parry [151] (Corollary 7.2.10 in Frougny [87]).  $\square$

If  $1 < \beta < 2$ , then the condition “ $a_0 \geq 1$  and  $a_n \leq a_0$  for all  $n \geq 0$ ” amounts to “ $a_0 = 1$ ”; in this case the  $\beta$ -integer part of  $\beta$  is equal to  $a_0 = 1$  and its  $\beta$ -fractional part is  $a_1\beta^{-1} + a_2\beta^{-2} + a_3\beta^{-3} + \dots$ . The base of numeration  $\beta = 1$  would correspond to the sequence  $(1, 0, 0, 0, \dots)$  in (2.2.9) but this sequence has its first digit 1 outside the alphabet  $\mathcal{A}_1 = \{0\}$ : it cannot be considered as a 1-expansion. Fortunately numeration in base one is not often used. The base of numeration  $\beta = 2$  would correspond to the constant sequence  $(1, 1, 1, 1, \dots)$  in (2.2.9) but this sequence is not self-admissible. When  $\beta = 2$ , 2 being an integer, 2-ary representations differ and  $(2, 0, 0, 0, \dots)$  is taken instead of  $(1, 1, 1, 1, \dots)$  (Frougny and Sakarovitch [89], Lothaire [131]).



Infinitely many cases of lacunarity, between  $(1, 0, 0, 0, \dots)$  and  $(1, 1, 1, 1, \dots)$ , may occur in the sequence  $(a_0, a_1, a_2, \dots)$  in (2.2.9). If  $\beta \in (1, 2)$  is fixed, with  $d_\beta(1) = 0.t_1t_2t_3\dots$  then any  $x$ ,  $1/\beta < x < 1$ , admits a  $\beta$ -expansion  $d_\beta(x)$  which lies lexicographically (Parry [151], Lemma 1) between those of the extremities:

$$(2.2.11) \quad d_\beta\left(\frac{1}{\beta}\right) = 0.0t_1t_2t_3\dots <_{lex} d_\beta(x) <_{lex} d_\beta(1) = 0.t_1t_2t_3\dots$$

Let  $1 < \beta < 2$  be a real number, with  $d_\beta(1) = 0.t_1t_2t_3\dots$ . If  $\beta$  is a simple Parry number, then there exists  $n \geq 2$ , depending upon  $\beta$ , such that  $t_n \neq 0$  and  $t_j = 0, j \geq n + 1$ . Parry [151] has shown that the set of simple Parry numbers is dense in the half-line  $(1, +\infty)$ . If  $\beta$  is a Parry number which is not simple, the sequence  $(t_i)_{i \geq 1}$  is eventually periodic: there exists an integer  $m \geq 1$ , the preperiod length, and an integer  $p \geq 1$ , the period length, such that

$$d_\beta(1) = 0.t_1t_2\dots t_m(t_{m+1}t_{m+2}\dots t_{m+p})^\omega,$$

$m$  and  $p$  depending upon  $\beta$ , with at least one nonzero digit  $t_j$ , with  $j \in \{m + 1, m + 2, \dots, m + p\}$ . The gaps of successive zeroes in  $(t_i)_{i \geq 1}$  are those of the preperiod  $(t_1, t_2, \dots, t_m)$  then those of the period  $(t_{m+1}, t_{m+2}, \dots, t_{m+p})$ , then occur periodically up till infinity. The length of such gaps of zeroes is at most  $\max\{m - 2, p - 1\}$ . The asymptotic lacunarity is controlled by the periodicity in this case.

If  $1 < \beta < 2$  is an algebraic number which is not a Parry number, the sequences of gaps of zeroes in  $(t_i)_{i \geq 1}$  remain asymptotically moderate and controlled by the Mahler measure  $M(\beta)$  of  $\beta$ , as follows.

**Theorem 2.5** (Verger-Gaugry). *Let  $\beta > 1$  be an algebraic number such that  $d_\beta(1)$  is infinite and gappy in the sense that there exist two infinite sequences  $\{m_n\}_{n \geq 1}$  and  $\{s_n\}_{n \geq 0}$  such that*

$$1 = s_0 \leq m_1 < s_1 \leq m_2 < s_2 \leq \dots \leq m_n < s_n \leq m_{n+1} < s_{n+1} \leq \dots$$

with  $(s_n - m_n) \geq 2$ ,  $t_{m_n} \neq 0$ ,  $t_{s_n} \neq 0$  and  $t_i = 0$  if  $m_n < i < s_n$  for all  $n \geq 1$ . Then

$$(2.2.12) \quad \limsup_{n \rightarrow +\infty} \frac{s_n}{m_n} \leq \frac{\text{Log}(M(\beta))}{\text{Log } \beta}$$

*Proof.* [209], Theorem 1.1. □

Theorem 2.5 also became a consequence of Theorem 2 in [1]. In Theorem 2.5 the quotient  $s_n/m_n, n \geq 1$ , is called the  $n$ -th Ostrowski quotient of the sequence  $(t_i)_{i \geq 1}$ . For a given algebraic number  $\beta > 1$ , whether the upper bound (2.2.12) is exactly the limsup of the sequence of the Ostrowski quotient of  $(t_i)_{i \geq 1}$  is unknown. For Salem numbers, this equality always holds since  $M(\beta) = \beta$ , and the upper bound (2.2.12) is 1.

*Varying the base of numeration  $\beta$  in the interval  $(1, 2)$ :* for all  $\beta \in (1, 2)$ , being an algebraic number or a transcendental number, the alphabet  $\mathcal{A}_\beta$  of the  $\beta$ -shift is always the same:  $\{0, 1\}$ . All the digits of all  $\beta$ -expansions  $d_\beta(1)$  are zeroes or ones. Parry ([151]) has proved that the relation of order  $1 < \alpha < \beta < 2$  is preserved on the corresponding greedy  $\alpha$ - and  $\beta$ - expansions  $d_\alpha(1)$  and  $d_\beta(1)$  as follows.

**Proposition 2.6.** *Let  $\alpha > 1$  and  $\beta > 1$ . If the Rényi  $\alpha$ -expansion of 1 is*

$$d_\alpha(1) = 0.t'_1t'_2t'_3\dots, \quad \text{i.e.} \quad 1 = \frac{t'_1}{\alpha} + \frac{t'_2}{\alpha^2} + \frac{t'_3}{\alpha^3} + \dots$$

and the Rényi  $\beta$ -expansion of 1 is

$$d_\beta(1) = 0.t_1t_2t_3\dots, \quad \text{i.e.} \quad 1 = \frac{t_1}{\beta} + \frac{t_2}{\beta^2} + \frac{t_3}{\beta^3} + \dots,$$

then  $\alpha < \beta$  if and only if  $(t'_1, t'_2, t'_3, \dots) <_{\text{lex}} (t_1, t_2, t_3, \dots)$ .

*Proof.* Lemma 3 in Parry [151]. □

For any integer  $n \geq 1$  the sequence of digits  $10^{n-1}1$ , with  $n - 1$  times “0” between the two ones, is self-admissible. By Proposition 2.4 it defines an unique solution  $\beta \in (1, 2)$  of (2.2.9). Denote by  $\theta_{n+1}^{-1}$  this solution. From Proposition 2.6 we deduce that the sequence  $(\theta_n^{-1})_{n \geq 2}$  is (strictly) decreasing and tends to 1 when  $n$  tends to infinity.

From (2.2.9) the real number  $\theta_2^{-1}$  is the unique root  $> 1$  of the equation  $1 = 1/x + 1/x^2$ , that is of  $X^2 - X - 1$ . Therefore it is the Pisot number (golden mean)  $= \frac{1+\sqrt{5}}{2} = 1.618\dots$ . Being interested in bases  $\beta > 1$  close to 1 tending to  $1^+$ , we will focus on the interval  $(1, \frac{1+\sqrt{5}}{2}]$  in the sequel. This interval is partitioned by the decreasing sequence  $(\theta_n^{-1})_{n \geq 2}$  as

$$(2.2.13) \quad \left(1, \frac{1+\sqrt{5}}{2}\right] = \bigcup_{n=2}^{\infty} [\theta_{n+1}^{-1}, \theta_n^{-1}) \cup \{\theta_2^{-1}\}.$$

Theorem 2.5 gives an upper bound of the asymptotic behaviour of the Ostrowski quotients of the  $\beta$ -expansion  $(t_i)_{i \geq 1}$  of 1, due to the fact that  $\beta > 1$  is an algebraic number. The following theorem shows that the gappiness of  $(t_i)_{i \geq 1}$  also admits some uniform lower bound, for all gaps of zeroes. The condition of minimality on the length of the gaps of zeroes in  $(t_i)_{i \geq 1}$  is only a function of the interval  $[\theta_{n+1}^{-1}, \theta_n^{-1})$  to which  $\beta$  belongs, when  $\beta$  tends to 1.

**Theorem 2.7.** *Let  $n \geq 2$ . A real number  $\beta \in (1, \frac{1+\sqrt{5}}{2}]$  belongs to  $[\theta_{n+1}^{-1}, \theta_n^{-1})$  if and only if the Rényi  $\beta$ -expansion of unity is of the form*

$$(2.2.14) \quad d_\beta(1) = 0.10^{n-1}10^{n_1}10^{n_2}10^{n_3}\dots,$$

with  $n_k \geq n - 1$  for all  $k \geq 1$ .

*Proof.* Since  $d_{\theta_{n+1}^{-1}}(1) = 0.10^{n-1}1$  and  $d_{\theta_n^{-1}}(1) = 0.10^{n-2}1$ , Proposition 2.6 implies that the condition is sufficient. It is also necessary:  $d_\beta(1)$  begins as  $0.10^{n-1}1$  for all  $\beta$  such that  $\theta_{n+1}^{-1} \leq \beta < \theta_n^{-1}$ . For such  $\beta$ s we write  $d_\beta(1) = 0.10^{n-1}1u$  with digits in the alphabet  $\mathcal{A}_\beta = \{0, 1\}$  common to all  $\beta$ s, that is

$$u = 1^{h_0}0^{n_1}1^{h_1}0^{n_2}1^{h_2}\dots$$

and  $h_0, n_1, h_1, n_2, h_2, \dots$  integers  $\geq 0$ . The self-admissibility lexicographic condition (2.2.10) applied to the sequence  $(1, 0^{n-1}, 1^{1+h_0}, 0^{n_1}, 1^{h_1}, 0^{n_2}, 1^{h_2}, \dots)$ , which characterizes uniquely the base of numeration  $\beta$ , readily implies  $h_0 = 0$  and  $h_k = 1$  and  $n_k \geq n - 1$  for all  $k \geq 1$ . □

**Remark 2.8.** The case  $n_1 = +\infty$  in (2.2.14) corresponds to the simple Parry number  $\beta = \theta_{n+1}^{-1}$ . The value  $+\infty$  is not excluded from the set  $(n_k)_{k \geq 1}$  in the following sense: if there

exists  $j \geq 2$  such that  $n - 1 \leq n_k < +\infty$ ,  $k < j$ , with  $n_j = +\infty$ , then  $\beta$  is a simple Parry number in  $[\theta_{n+1}^{-1}, \theta_n^{-1})$  characterized by (cf Section §2):

$$d_\beta(1) = 0.10^{n-1}10^{n_1}10^{n_2}1 \dots 10^{n_{j-1}}1.$$

All the simple Parry numbers lying in the interval  $[\theta_{n+1}^{-1}, \theta_n^{-1})$  are obtained in this way. On the contrary, the transcendental numbers  $\beta$  in  $[\theta_{n+1}^{-1}, \theta_n^{-1})$  have all Rényi  $\beta$ -expansions  $d_\beta(1) = 0.t_1t_2t_3 \dots$  of 1 such that the sequence of exponents  $(n_k)_{k \geq 1}$  of the successive zeroes, corresponding to the sequence of the lengths of the gaps of zeroes, never takes the value  $+\infty$ .

**Definition 2.9.** Let  $\beta \in (1, \frac{1+\sqrt{5}}{2}]$  be a real number. The integer  $n \geq 3$  such that  $\theta_n^{-1} \leq \beta < \theta_{n-1}^{-1}$  is called the *dynamical degree* of  $\beta$ , and is denoted by  $\text{dyg}(\beta)$ . By convention we put:  $\text{dyg}(\frac{1+\sqrt{5}}{2}) = 2$ .

The function  $n = \text{dyg}(\beta)$  is locally constant on the interval  $(1, \frac{1+\sqrt{5}}{2}]$ , is decreasing, takes all values in  $\mathbb{N} \setminus \{0, 1\}$ , and satisfies:  $\lim_{\beta > 1, \beta \rightarrow 1} \text{dyg}(\beta) = +\infty$ . The relations between the dynamical degree  $\text{dyg}(\beta)$  and the (usual) degree  $\text{deg}(\beta)$  will be investigated later (Theorem 1.1; § 5, § 6.4). Let us observe that the equality  $\text{deg}(\beta) = \text{dyg}(\beta) = 2$  holds if  $\beta = \frac{1+\sqrt{5}}{2}$ , but the equality case is not the case in general.

**Definition 2.10.** A power series  $\sum_{j=0}^{+\infty} a_j z^j$ , with  $a_j \in \{0, 1\}$  for all  $j \geq 0$ ,  $z$  the complex variable, is said to be *Lyndon (or self-admissible)* if its coefficient vector  $(a_i)_{i \geq 0}$  is Lyndon.

### 3. GENERALIZED FREDHOLM THEORY, DYNAMICAL ZETA FUNCTION, PERRON-FROBENIUS OPERATOR, TRANSFER OPERATOR, PARRY UPPER FUNCTION, AND THE $\beta$ -SHIFT

#### 3.1. The Parry Upper function, the Parry polynomial.

**Definition 3.1.** Let  $\beta \in (1, (1 + \sqrt{5})/2]$  be a real number, and  $d_\beta(1) = 0.t_1t_2t_3 \dots$  its Rényi  $\beta$ -expansion of 1. The power series  $f_\beta(z) := -1 + \sum_{i \geq 1} t_i z^i$  of the complex variable  $z$  is called the *Parry Upper function* at  $\beta$ .

In this paragraph a presentation of the Parry Upper function is given from the side of generalized Fredholm Theory, to show the relations between Fredholm determinants, generalized Fredholm determinants, and (weighted) dynamical zeta functions (introduced by Ruelle [170] [171] [172] [173] [174] [175] [176] [177] [178] by analogy with the thermodynamic formalism of statistical mechanism [173], and recently developped e.g. by Baladi [12] [13] [14], Baladi and Keller [15], Hofbauer [105], Hofbauer and Keller [106], Milnor and Thurston [140], Parry and Pollicott [154], Pollicott [159] [160], Preston [162], Takahashi [201] [202]).

**Proposition 3.2.** For  $1 < \beta < (1 + \sqrt{5})/2$  any real number, with  $d_\beta(1) = 0.t_1t_2t_3 \dots$ , the Parry Upper function  $f_\beta(z)$  is such that  $f_\beta(1/\beta) = 0$ . It is such that  $f_\beta(z) + 1$  has coefficients in the alphabet  $\mathcal{A}_\beta = \{0, 1\}$  and is Lyndon. It takes the form

$$(3.1.1) \quad f_\beta(z) = G_{\text{dyg}(\beta)} + z^{m_1} + z^{m_2} + \dots + z^{m_q} + z^{m_{q+1}} + \dots$$

with  $m_1 - \text{dyg}(\beta) \geq \text{dyg}(\beta) - 1$ ,  $m_{q+1} - m_q \geq \text{dyg}(\beta) - 1$  for  $q \geq 1$ . Conversely, given a power series

$$(3.1.2) \quad -1 + z + z^n + z^{m_1} + z^{m_2} + \dots + z^{m_q} + z^{m_{q+1}} + \dots$$

with  $n \geq 3$ ,  $m_1 - n \geq n - 1$ ,  $m_{q+1} - m_q \geq n - 1$  for  $q \geq 1$ , then there exists an unique  $\beta \in (1, (1 + \sqrt{5})/2)$  for which  $n = \text{dyg}(\beta)$  with  $f_\beta(z)$  equal to (3.1.2).

Moreover, if  $\beta$ ,  $1 < \beta < (1 + \sqrt{5})/2$ , is a reciprocal algebraic integer, the power series (3.1.1) is never a polynomial.

*Proof.* The expression of  $f_\beta(z)$  readily comes from Theorem 2.7. Let us prove the last claim. Assume that  $\beta$  is a reciprocal algebraic integer and that  $f_\beta(z)$  is a polynomial. The polynomial  $f_\beta(z)$  would vanish at the two real zeroes  $\beta$  and  $1/\beta$ . But the sequence  $-1 t_1 t_2 t_3 \dots$  has only one sign change. By Descartes's rule we obtain a contradiction.  $\square$

The lacunarity of  $f_\beta(z)$  is moderate since the Ostrowki quotients have an asymptotic upper bound, by Theorem 2.5. By Theorem 2.7, any gap of missing monomials in  $f_\beta(z)$  has a length greater than or equal to  $\text{dyg}(\beta) - 1$  what controls the lacunarity a minima.

The definition of  $f_\beta(z)$  seems simple since the vector coefficient of  $f_\beta(z) + 1$  is only a sequence of integers deduced from the orbit of 1 under the iterates of the  $\beta$ -transformation  $T_\beta$ , by (2.2.4) and (2.2.5) [83][84]; nevertheless it is deeply related to the Artin-Mazur dynamical zeta function  $\zeta_\beta(z)$  (given by (3.1.5)) of the Rényi-Parry dynamical system  $([0, 1], T_\beta)$ , to the Perron-Frobenius operator  $P_{T_\beta}$  associated with  $T_\beta$ , to the transfer operator of  $T_\beta$  and to the generalized "Fredholm determinant" (3.1.4) of this operator. In the kneading theory of Milnor and Thurston [140] it is a kneading determinant. Let us recall these links, knowing that the theory of Fredholm (Grothendieck [93] [94], Riesz and Nagy [167] Chap. IV) is done for compact operators while the Perron-Frobenius operators associated with the  $\beta$ -transformations  $T_\beta$  are noncompact by nature (Mori [141] [142], Takahashi [199] [200] [203]).

Let  $(X, \Sigma, \mu)$  be a  $\sigma$ -finite measure space and let  $T : X \rightarrow X$  be a nonsingular transformation, i.e.  $T$  is measurable and satisfies: for all  $A \in \Sigma$ ,  $\mu(A) = 0 \implies \mu(T^{-1}(A)) = 0$ . In ergodic theory, by the Radon-Nikodym theorem, the operator  $P_T : L^1(X, \Sigma, \mu) \rightarrow L^1(X, \Sigma, \mu)$  defined by

$$(3.1.3) \quad \int_A P_T f d\mu = \int_{T^{-1}(A)} f d\mu$$

is called the Perron-Frobenius operator associated with  $T$ . Let  $\beta \in (1, \theta_2^{-1})$ ,  $X = [0, 1]$ ,  $\Sigma$  the Borel  $\sigma$ -algebra and  $T_\beta$  the  $\beta$ -transformation. The  $T_\beta$ -invariant probability measure  $\mu = \mu_\beta$  of the  $\beta$ -shift, on  $\Sigma$ , is unique (Rényi [164]), ergodic (Parry [151]), maximal (Hofbauer [105]) and absolutely continuous with respect to the Lebesgue measure  $dt$ , with Radon-Nikodym derivative (Lasota and Yorke [120], Parry [151], Takahashi [199]) :

$$h_\beta = C \sum_{n: x < T_\beta^n(1)} \frac{1}{\beta^{n+1}}, \quad \text{so that} \quad d\mu_\beta = h_\beta dt,$$

for some constant  $C > 0$ . These results were independently discovered by A.O. Gelfond [79]. We denote by  $P_{T_\beta}$  the Perron-Frobenius operator associated with  $T_\beta$ .

The  $\beta$ -transformation  $T_\beta$  is a piecewise monotone map of the interval  $[0, 1]$  with weight function  $g = 1$ . In the context of noncompact operators, the objective consists in giving a sense to

$$(3.1.4) \quad \text{‘det’}(Id - zL_t) = \exp\left(-\sum_{n \geq 1} \frac{\text{‘tr’} L_t^n}{n} z^n\right),$$

where  $L_t$  is a dynamically defined weighted transfer operator acting on a suitable Banach space [13] [107].

Let  $1 < \beta < (1 + \sqrt{5})/2$  be a real number and  $0 = a_0 < a_1 = \frac{1}{\beta} < a_2 = 1$  be the finite partition of  $[0, 1]$ . The map  $T_\beta$  is strictly monotone and continuous on  $[a_0, a_1)$  and  $[a_1, 1]$ . The  $\beta$ -transformation  $T_\beta$  is one of the simplest transformations among piecewise monotone intervals maps (Baladi and Ruelle [16], Milnor and Thurston [140], Pollicott [159]). For each function  $f : [0, 1] \rightarrow \mathbb{C}$ , let

$$\text{var}(f) := \sup \left\{ \sum_{i=1}^n |f(e_i) - f(e_{i-1})| \mid n \geq 1, 0 \leq e_1 \leq e_2 \leq \dots \leq e_n \leq 1 \right\},$$

$$\|f\|_{BV} := \text{var}(f) + \sup(|f|),$$

and denote by  $BV$  the Banach space of functions with bounded variation [112] [113]:

$$BV := \{f : [0, 1] \rightarrow \mathbb{C} \mid \|f\|_{BV} < \infty\}.$$

For  $g \in BV$ , one can define the following transfer operator

$$L_{t\beta, g} : BV \rightarrow BV, \quad L_{t\beta, g} f(x) := \sum_{y, T_\beta(y)=x} g(y) f(y).$$

We will only consider the case  $g \equiv 1$  in the sequel and put  $L_{t\beta} := L_{t\beta, 1}$ .

**Theorem 3.3.** *Let  $\beta \in (1, \theta_2^{-1})$ . Then,*

(i) *the Artin-Mazur dynamical zeta function  $\zeta_\beta(z)$  defined by*

$$(3.1.5) \quad \zeta_\beta(z) := \exp\left(\sum_{n=1}^{\infty} \frac{\#\{x \in [0, 1] \mid T_\beta^n(x) = x\}}{n} z^n\right),$$

*counting the number of periodic points of period dividing  $n$ , is nonzero and meromorphic in  $\{|z| < 1\}$ , and such that  $1/\zeta_\beta(z)$  is holomorphic in  $\{|z| < 1\}$ ,*

(ii) *suppose  $|z| < 1$ . Then  $z$  is a pole of  $\zeta_\beta(z)$  of multiplicity  $k$  if and only if  $z^{-1}$  is an eigenvalue of  $L_{t\beta}$  of multiplicity  $k$ .*

*Proof.* Theorem 2 in [15], assuming that the set of intervals  $([0, a_1), [a_1, 1])$  forming the partition of  $[0, 1]$  is generating; In [174] [177] Ruelle shows that this assumption is not necessary, showing how to remove this obstruction.  $\square$

Theorem 3.3 was stated in Baladi and Keller [15] under more general assumptions. Theorem 3.3 has been conjectured by Hofbauer and Keller [106] for piecewise monotone maps, for the case where the function  $g$  is piecewise constant. (cf also Mori [141] [142]). The case  $g = 1$  in the transfer operators was studied by Milnor and Thurston [140], Hofbauer [105], Preston [162]. The fact (Theorem 3.3 (ii)) that the poles of  $\zeta_\beta(z)$ , lying in the open

unit disc, are of the same multiplicity of the inverses of the eigenvalues of the transfer operator  $L_{\iota\beta}$  is an extension of Theorem 2, Theorem 3 and Theorem 4 in Grothendieck [94] in the context of the Fredholm theory with compact operators.

When  $\beta > 1$  is a reciprocal algebraic integer and tends to  $1^+$ , we will prove in Section § 5 that the multiplicity  $k$  is equal to 1 for the first pole  $1/\beta$  of  $\zeta_\beta(z)$  and for a subcollection of Galois conjugates of  $1/\beta$  in an angular sector.

The relations between the poles of the dynamical zeta function  $\zeta_\beta(z)$ , the zeroes of the Parry Upper function  $f_\beta(z)$  and the eigenvalues of the transfer operator  $L_{\iota\beta}$  come from Theorem 3.3 and from the following theorem.

**Theorem 3.4.** *Let  $\beta > 1$  be a real number. Then the Parry Upper function  $f_\beta(z)$  satisfies*

$$(3.1.6) \quad (i) \quad f_\beta(z) = -\frac{1}{\zeta_\beta(z)} \quad \text{if } \beta \text{ is not a simple Parry number,}$$

and

$$(3.1.7) \quad (ii) \quad f_\beta(z) = -\frac{1-z^N}{\zeta_\beta(z)} \quad \text{if } \beta \text{ is a simple Parry number}$$

where  $N$ , which depends upon  $\beta$ , is the minimal positive integer such that  $T_\beta^N(1) = 0$ . It is holomorphic in the open unit disk  $\{|z| < 1\}$ . It has no zero in  $|z| \leq 1/\beta$  except  $z = 1/\beta$  which is a simple zero. The Taylor series of  $f_\beta(z)$  at  $z = 1/\beta$  is  $f_\beta(z) = c_{\beta,1}(z - \frac{1}{\beta}) + c_{\beta,2}(z - \frac{1}{\beta})^2 + \dots$  with

$$(3.1.8) \quad c_{\beta,m} = \sum_{n=m}^{\infty} \frac{n!}{(n-m)! m!} [\beta T_\beta^{n-1}(1)] \left(\frac{1}{\beta}\right)^{n-m} > 0, \quad \text{for all } m \geq 1.$$

*Proof.* Theorem 2.3 and Appendix A in Flatto, Lagarias and Poonen [84]; Theorem 1.2 in Flatto and Lagarias [83], I; Theorem 3.2 in Lagarias [116]. From Takahashi [199], Ito and Takahashi [109], these authors deduce

$$(3.1.9) \quad \zeta_\beta(z) = \frac{1-z^N}{(1-\beta z) \left( \sum_{n=0}^{\infty} T_\beta^n(1) z^n \right)}$$

where “ $z^N$ ” has to be replaced by “0” if  $\beta$  is not a simple Parry number. Since  $\beta T_\beta^n(1) = [\beta T_\beta^n(1)] + \{\beta T_\beta^n(1)\} = t_{n+1} + T_\beta^{n+1}(1)$  by (2.2.4), for  $n \geq 1$ , expanding the power series of the denominator (3.1.9) readily gives:

$$(3.1.10) \quad -1 + t_1 z + t_2 z^2 + \dots = f_\beta(z) = -(1-\beta z) \left( \sum_{n=0}^{\infty} T_\beta^n(1) z^n \right).$$

The zeroes of smallest modulus are characterized in Lemma 5.2, Lemma 5.3 and Lemma 5.4 in [84]. The coefficients  $c_{\beta,m}$  readily come from the derivatives of  $f_\beta(z)$ .  $\square$

From Theorem 3.4, since the roots of cyclotomic polynomials are of modulus 1, the zeroes of  $f_\beta(z)$  within  $|z| < 1$  are always exactly the poles of  $\zeta_\beta(z)$  in this domain, whatever the Rényi-Parry dynamics of  $\beta > 1$  is.

**Definition 3.5.** If  $\beta$  is a simple Parry number, with  $d_\beta(1) = 0.t_1t_2\dots t_m$ ,  $t_m \neq 0$ , the polynomial

$$(3.1.11) \quad P_{\beta,P}(X) := X^m - t_1X^{m-1} - t_2X^{m-2} - \dots - t_m$$

is called the *Parry polynomial* of  $\beta$ . If  $\beta$  is a Parry number which is not simple, with  $d_\beta(1) = 0.t_1t_2\dots t_m(t_{m+1}t_{m+2}\dots t_{m+p+1})^\omega$  and not purely periodic ( $m \neq 0$ ), then

$$(3.1.12) \quad \begin{aligned} P_{\beta,P}(X) := & X^{m+p+1} - t_1X^{m+p} - t_2X^{m+p-1} - \dots - t_{m+p}X - t_{m+p+1} \\ & - X^m + t_1X^{m-1} + t_2X^{m-2} + \dots + t_{m-1}X + t_m \end{aligned}$$

is the Parry polynomial of  $\beta$ . If  $\beta$  is a nonsimple Parry number such that  $d_\beta(1) = 0.(t_1t_2\dots t_{p+1})^\omega$  is purely periodic (i.e.  $m = 0$ ), then

$$(3.1.13) \quad P_{\beta,P}(X) := X^{p+1} - t_1X^p - t_2X^{p-1} - \dots - t_pX - (1 + t_{p+1})$$

is the Parry polynomial of  $\beta$ . By definition the degree  $d_P$  of  $P_{\beta,P}(X)$  is respectively  $m, m+p+1, p+1$  in the three cases.

If  $\beta$  is a Parry number, the Parry polynomial  $P_{\beta,P}(X)$ , belonging to the ideal  $P_\beta(X)\mathbb{Z}[X]$ , admits  $\beta$  as simple root and is often not irreducible [210] [25]. The polynomial  $\frac{P_{\beta,P}(X)}{P_\beta(X)}$  has been called *complementary factor* by Boyd. For the two cases (3.1.11) and (3.1.13) the constant term is  $\neq 0$ ; hence  $\deg(P_{\beta,P}^*) = \deg(P_{\beta,P})$ . In the case of (3.1.12) denote by  $q_\beta := 0$  if  $t_m \neq t_{m+p+1}$  and, if  $t_m = t_{m+p+1}$ ,  $q_\beta := 1 + \max\{r \in \{0, 1, m-1\} \mid t_{m-l} = t_{m+p+1-l} \text{ for all } 0 \leq l \leq r\}$ . Then  $p+1 \leq \deg(P_{\beta,P}) - q_\beta = \deg(P_{\beta,P}^*) \leq \deg(P_{\beta,P})$ .

Applying the Carlson-Polya dichotomy (Bell and Chen [19], Bell, Miles and Ward [20], Carlson [52] [53], Dienes [60], Pólya [161], Robinson [169], Szegő [198]) to the power series  $f_\beta(z)$ , for which the coefficients belong to the finite set  $\mathcal{A}_\beta \cup \{-1\}$ , gives the following equivalence.

**Theorem 3.6.** *The real number  $\beta > 1$  is a Parry number if and only if the Parry Upper function  $f_\beta(z)$  is a rational function, equivalently if and only if  $\zeta_\beta(z)$  is a rational function.*

*The set of Parry numbers, resp. of nonParry numbers, in  $(1, \infty)$ , is not empty. If  $\beta$  is not a Parry number, then  $|z| = 1$  is the natural boundary of  $f_\beta(z)$ . If  $\beta$  is a Parry number, with Rényi  $\beta$ -expansion of 1 given by*

$$d_\beta(1) = 0.t_1t_2\dots t_m(t_{m+1}t_{m+2}\dots t_{m+p+1})^\omega, \quad t_1 = \lfloor \beta \rfloor, \quad t_i \in \mathcal{A}_\beta, \quad i \geq 2,$$

*the preperiod length being  $m \geq 0$  and the period length  $p+1 \geq 1$ ,  $f_\beta(z)$  admits an analytic meromorphic extension over  $\mathbb{C}$ , of the following form:*

$$\begin{aligned} f_\beta(z) &= -P_{\beta,P}^*(z) && \text{if } \beta \text{ is simple,} \\ f_\beta(z) &= \frac{-P_{\beta,P}^*(z)}{1-z^{p+1}} && \text{if } \beta \text{ is nonsimple,} \end{aligned}$$

*where the Parry polynomial is given by (3.1.11), (3.1.12) or (3.1.13).*

*If  $\beta \in (1, 2)$  is a Parry number, the (naïve) height  $H(P_{\beta,P})$  of  $P_{\beta,P}$  is equal to 1 except when:  $\beta$  is nonsimple and that  $t_{p+1} = \lfloor \beta T_\beta^p(1) \rfloor = 1$ , in which case the Parry polynomial of  $\beta$  has naïve height  $H(P_{\beta,P}) = 2$ .*

*Proof.* Verger-Gaugry [214]. The set of nonParry numbers  $\beta$  in  $(1, \infty)$  is not empty as a consequence of Fekete-Szegő's Theorem [82] since the radius of convergence of  $f_\beta(z)$  is equal to 1 in any case (whatever the Rényi-Parry dynamics of  $\beta$  is), and that its domain of definition always contains the open unit disk which has a transfinite diameter equal to 1. The set of Parry numbers  $\beta$  in  $(1, \infty)$  is also nonempty. Indeed Pisot numbers, of degree  $\geq 2$ , are Parry numbers (Schmidt [183], Bertrand-Mathis [24]). Therefore the dichotomy between Parry and nonParry numbers in  $(1, \infty)$  has a sense.  $\square$

**Definition 3.7.** Let  $\beta > 1$  be a Parry number. If the Parry polynomial  $P_{\beta,p}(z)$  of  $\beta$  is not irreducible, the roots of  $P_{\beta,p}(z)$  which are not Galois conjugates of  $\beta$  are called the *beta-conjugates* of  $\beta$ .

Beta-conjugates were studied in [212] [213] in terms of Puiseux theory and in association with germs of curves.

**3.2. Distribution of zeroes of Parry Upper functions  $f_\beta(z)$  in Solomyak's fractal.** Let  $\beta > 1$  be a real number (algebraic or transcendental). The Parry Upper function  $f_\beta(z)$  has its zeroes of modulus  $< 1$  in a region of the open unit disk, called Solomyak's fractal, whose construction is given in [193] §3. Let us recall it and summarize its arithmetic properties in Theorem 3.8. From Theorem 3.4 and (3.1.9), the zeroes of  $f_\beta(z)$  in  $|z| < 1$ , which are  $\neq 1/\beta$ , are the zeroes of modulus  $< 1$  of the power series  $1 + \sum_{j=1}^{\infty} T_\beta^j(1)z^j$  where the coefficients are real numbers in the interval  $[0, 1]$ . Then, in full generality, let

$$\mathcal{B} := \left\{ h(z) = 1 + \sum_{j=1}^{\infty} a_j z^j \mid a_j \in [0, 1] \right\}$$

be the class of power series defined on  $|z| < 1$  equipped with the topology of uniform convergence on compact sets of  $|z| < 1$ . The subclass  $\mathcal{B}_{0,1}$  of  $\mathcal{B}$  denotes functions whose coefficients are all zeros or ones. The space  $\mathcal{B}$  is compact and convex. Let

$$\mathcal{G} := \left\{ \lambda \mid |\lambda| < 1, \exists h(z) \in \mathcal{B} \text{ such that } h(\lambda) = 0 \right\} \subset D(0, 1)$$

be the set of zeroes of the power series belonging to  $\mathcal{B}$ . The zeroes gather within the unit circle and curves in  $|z| < 1$  given in polar coordinates, by [210]. The complement  $D(0, 1) \setminus (\mathcal{G} \cup \{\frac{1}{\beta}\})$  is a zero-free region for  $f_\beta(z)$ ; the domain  $D(0, 1) \setminus \mathcal{G}$  is star-convex due to the fact that:  $h(z) \in \mathcal{B} \implies h(z/r) \in \mathcal{B}$ , for any  $r > 1$  ([193], §3), and that  $1/\beta$  is the unique root of  $f_\beta(z)$  in  $(0, 1)$ .

For every  $\phi \in (0, 2\pi)$ , there exists  $\lambda = re^{i\phi} \in \mathcal{G}$ ; the point of minimal modulus with argument  $\phi$  is denoted  $\lambda_\phi = \rho_\phi e^{i\phi} \in \mathcal{G}$ ,  $\rho_\phi < 1$ . A function  $h \in \mathcal{B}$  is called  $\phi$ -optimal if  $h(\lambda_\phi) = 0$ . Denote by  $\mathcal{H}$  the subset of  $(0, \pi)$  for which there exists a  $\phi$ -optimal function belonging to  $\mathcal{B}_{0,1}$ . Denote by  $\partial\mathcal{G}_S$  the "spike":  $[-1, \frac{1}{2}(1 - \sqrt{5})]$  on the negative real axis.

**Theorem 3.8 (Solomyak).** (i) *The union  $\mathcal{G} \cup \mathbb{T} \cup \partial\mathcal{G}_S$  is closed, symmetrical with respect to the real axis, has a cusp at  $z = 1$  with logarithmic tangency (Figure 1 in [193]),*

(ii) *the boundary  $\partial\mathcal{G}$  is a continuous curve, given by  $\phi \rightarrow |\lambda_\phi|$  on  $[0, \pi)$ , taking its values in  $[\frac{\sqrt{5}-1}{2}, 1)$ , with  $|\lambda_\phi| = 1$  if and only if  $\phi = 0$ . It admits a left-limit at  $\pi^-$ ,  $1 > \lim_{\phi \rightarrow \pi^-} |\lambda_\phi| > |\lambda_\pi| = \frac{1}{2}(-1 + \sqrt{5})$ , the left-discontinuity at  $\pi$  corresponding to the extremity of  $\partial\mathcal{G}_S$ .*



(iii) at all points  $\rho_\phi e^{i\phi} \in \mathcal{G}$  such that  $\phi/\pi$  is rational in an open dense subset of  $(0, 2)$ ,  $\partial\mathcal{G}$  is non-smooth,

(iv) there exists a nonempty subset of transcendental numbers  $L_{tr}$ , of Hausdorff dimension zero, such that  $\phi \in (0, \pi)$  and  $\phi \notin \mathcal{K} \cup \pi\mathbb{Q} \cup \pi L_{tr}$  implies that the boundary curve  $\partial\mathcal{G}$  has a tangent at  $\rho_\phi e^{i\phi}$  (smooth point).

*Proof.* [193], § 3 and § 4. □

**Definition 3.9.** The set  $\mathcal{G} \cup \mathbb{T} \cup \partial\mathcal{G}_S$  is called Solomyak's fractal.

Solomyak's fractal contains the set  $\overline{W}$ , where  $W$  consists of the zeroes  $\lambda$ ,  $|\lambda| < 1$ , of the polynomials  $1 + \sum_{j=1}^q a_j z^j$  having all coefficients  $a_j$  zeroes and ones, studied by Odlyzko and Poonen [149].

Let  $1 < \beta < (1 + \sqrt{5})/2$  be a reciprocal algebraic integer and apply the Carlson-Polya dichotomy to  $f_\beta(z)$  :

- if  $\beta$  is a Parry number, then  $f_\beta(z)$  has a finite number of zeroes by Theorem 3.6,  $\mathcal{G}$  contains all the Galois-conjugates and the inverses of the beta-conjugates (if any) of  $\beta$  of modulus  $< 1$ , and the unit circle  $|z| = 1$  is not a natural boundary of  $f_\beta(z)$ . Note that the Galois conjugates of  $\beta$  are the Galois conjugates of  $1/\beta$ , but the  $\beta$ -conjugates of  $\beta$  could be roots of non-reciprocal factors.

- if  $\beta$  is not a Parry number the zeroes of  $f_\beta(z)$  in the subfractal  $\mathcal{G}$  is more difficult to describe. In this case, the unit circle  $|z| = 1$  is the natural boundary of  $f_\beta(z)$ . The study of the zeroes of power series which lie very close to natural boundaries, having moderate lacunarity, is a difficult problem. This problem is more difficult than for sparse power series having Hadamard lacunarity for instance (Fuchs [91], Levinson [124] Chap. VI, Robinson [169]). Let us view the problem from the side of dynamical zeta functions since  $\zeta_\beta(z) = -1/f_\beta(z)$ . It is classical to study the analytic behaviour of the dynamical zeta function on its natural disk of convergence which is centered at 0 and of radius of convergence  $\exp(-\mathcal{H})$ , where  $\mathcal{H}$  is the topological entropy of the dynamical system [154]. In the case of the  $\beta$ -shift the topological entropy is  $\mathcal{H} = \text{Log } \beta$  (Proposition 5.1 in [154]). Therefore the important subregion of  $\mathcal{G} \cup \mathbb{T} \cup \partial\mathcal{G}_S$ , in the open unit disk, to be investigated for the existence, the number (eventually infinite) and the geometry of zeroes of  $f_\beta(z)$  is the annular region  $\{z \mid \exp(-\mathcal{H}) = \beta^{-1} < |z| < 1\}$ , in particular when  $\beta$  tends to  $1^+$ . This problem is the general problem of the extension of the meromorphy of  $\zeta_\beta(z)$ . Then the main objective is the extended research of zeroes of  $f_\beta(z)$  in  $\{\beta^{-1} < |z| < 1\}$ , which is mostly concerned with (i) the meromorphic extension of the dynamical zeta function of a dynamical system outside the disk of convergence whose radius is  $\exp(-\mathcal{H})$  with  $\mathcal{H}$  the topological entropy, the pressure, etc, of the dynamical system, and their poles in this annular region (Haydn [99], Hilgert and Rilke [103], Parry and Pollicott [154], Pollicott [159], Ruelle [176]), (ii) the structure theorems of orthogonal decomposition of the transfer operators (eventually the Perron-Frobenius operators), with the geometry of their isolated eigenvalues (e.g. Theorem 1 in Baladi and Keller [15]).

In the sequel (in Section § 5.3 and Section § 6.1) we will not solve the problem of the exact determination of the zeroes of  $f_\beta(z)$  in  $\{\beta^{-1} < |z| < 1\}$ . We will overcome this difficulty for the problem of Lehmer. Instead, we will use approximate values of the zeroes, by using à la Poincaré divergent series. It will be sufficient to observe the separation of the collection of zeroes into two categories: - those which are very close to the unit circle,

called *nonlenticular zeroes*, - those which lie off the unit circle, called *lenticular zeroes*. The lenticular zeroes, spreading inside the cusp region of  $\mathcal{G} \cup \mathbb{T} \cup \partial\mathcal{G}_S$ , stemming from  $\beta^{-1}$ , in the neighbourhood of  $z = 1$  towards  $e^{\pm i\frac{\pi}{3}}$ , are identified as Galois conjugates of  $\beta^{-1}$  in Section §5.4. Lenticuli of zeroes are exemplified in [73]. Then we will obtain the asymptotic expansion of the minorant  $M_r(\beta)$  of the Mahler measure  $M(\beta)$ , for  $\beta > 1$  being a reciprocal algebraic integer, from the lenticular roots of  $f_\beta(z)$ .

**3.3. Carlson-Polya dichotomy of reciprocal algebraic integers  $\beta > 1$  close to one.** The set  $\mathbb{P}$  of Perron numbers is dense in  $(1, +\infty)$ . It contains the subset  $\mathbb{P}_P$  of Parry numbers by a result of Lind [128] (Blanchard [27], Boyle [41], Denker, Grillenberger and Sigmund [58], Frougny in [131] chap.7). The set  $\mathbb{P} \setminus \mathbb{P}_P$  is not empty (by Akiyama’s Theorem 3.10 recalled below, also as a consequence of Fekete-Szegő’s Theorem [82]) ; it would contain all Salem numbers of large degrees, by Thurston [207] p. 11. Parry ([151], Theorem 5) proved that the subcollection of simple Parry numbers is dense in  $[1, \infty)$ . Simple Parry numbers  $\beta$ , as nonreciprocal algebraic integers, satisfy the minoration  $M(\beta) \geq \Theta$ . In the opposite direction a Conjecture of K. Schmidt [183] asserts that Salem numbers are all Parry numbers. For Salem numbers  $\beta$  of degree  $\geq 6$ , Boyd [39] established a simple probabilistic model, based on the frequencies of digits occurring in the Rényi  $\beta$ -expansions of unity, to conjecture that, more realistically, Salem numbers are dispatched into the two sets of Parry numbers and nonParry numbers, each of them with densities  $> 0$ . This model, coherent with Thurston’s one ([207], p. 11), is in contradiction with the conjecture of K. Schmidt. This dichotomy of Salem numbers was verified by Hichri [100] [101] [102] for Salem numbers of degree 8. The coding of small neighbourhoods of Salem numbers by Stieltjes continued fractions has been investigated in [95], in view of characterizing this dichotomy locally. Salem numbers of degree 4 are Parry numbers [37]. Boyd’s model covers the set of Salem numbers smaller than Lehmer’s number, if any. Few examples of nonParry algebraic numbers  $> 1$  exist; Solomyak ([193] p. 483) gives  $\frac{1}{2}(1 + \sqrt{13})$ .

**Theorem 3.10** (Akiyama). *The dominant root  $\gamma_n > 1$  of  $-1 - z + z^n$ , for  $n \geq 2$ , is a Perron number which is a Parry number if and only if  $n = 2, 3$ . If  $n = 2, 3$ ,  $\gamma_2 = \theta_2^{-1}$  and  $\gamma_3 = \theta_5^{-1} = \Theta$  are Pisot numbers which are simple Parry numbers.*

*Proof.* Theorem 1.1 and Lemma 2.2 in [3] using Lagrange inversion formula. Let us recall the dynamics of the Perron numbers  $\theta_n^{-1}$  for  $n \geq 2$ : ... □

This dichotomy, due to the Rényi-Parry dynamics, would separate the set of real reciprocal algebraic integers  $\beta > 1$  into two disjoint nonempty subsets. Since it corresponds exactly to the dichotomy of Parry Upper functions  $f_\beta(z)$ , we speak of *Carlson-Polya dichotomy of real reciprocal algebraic integers  $\beta > 1$* . The small Salem numbers found by Lehmer in [123], reported in the Survey [215], either given by their minimal polynomial or equivalently by their  $\beta$ -expansion, are Parry numbers. The two smallest ones Lehmer [123] has found:

$\text{deg}(\beta)$	$\beta = M(\beta)$	minimal pol. of $\beta$	$d_\beta(1)$
8	1.2806...	$X^8 - X^5 - X^4 - X^3 + 1$	$0.1(0^5 10^5 10^7)^\omega$
10	1.17628... “Lehmer’s number”	$X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$	$0.1(0^{10} 10^{18} 10^{12} 10^{18} 10^{12})^\omega$

of respective dynamical degrees  $\text{dyg}(\beta)$  7 and 12, with Parry polynomials of respective degrees 20 and 75, given by (3.1.12), are such that their respective Parry Upper functions take the form given in Proposition 3.2, namely  $f_\beta(z) =$

$$(3.3.1) \quad -\frac{z^{20} - z^{19} - z^{13} - z^7 - z + 1}{1 - z^{19}} = -1 + z + z^7 + z^{13} + \dots = G_7(z) + z^{13} + \dots,$$

resp.

$$(3.3.2) \quad \begin{aligned} f_\beta(z) &= -\frac{z^{75} - z^{74} - z^{63} - z^{44} - z^{31} - z^{12} - z + 1}{1 - z^{74}} \\ &= -1 + z + z^{12} + z^{31} + \dots = G_{12}(z) + z^{31} + \dots \end{aligned}$$

The relations between the digits  $(t_i)$  in the Rényi  $\beta$ -expansion of unity of an algebraic integer  $\beta > 1$  and the coefficient vector of its minimal polynomial are still obscure in general, except in a few cases: e.g. for Salem numbers of degree 4 and 6 (Boyd [35] [36] [38]), for Salem numbers of degree 8 (Hichri [100] [101] [102]), for Pisot numbers (Boyd [39], Frougny and Solomyak [90], Bassino [18] in the cubic case, Hare [97] [98], Panju [150] for regular Pisot numbers). A more abstract ergodic viewpoint is developed in Schmidt [184], with potential applications to limit Mahler measures.

By (3.1.10), the topological properties of the set  $\{T_\beta^n(1)\}$  control the Carlson-Polya dichotomy of reciprocal algebraic integers  $> 1$ . On this basis Blanchard [27] proposed a classification of real numbers  $\beta > 1$  into five classes; Verger-Gaugry in [209] refined it in terms of asymptotic gappiness in the direction of more enlighting the algebraicity of  $\beta$ :

- Class C1:  $d_\beta(1)$  is finite,
- Class C2:  $d_\beta(1)$  is ultimately periodic but not finite,
- Class C3:  $d_\beta(1)$  contains bounded strings of zeroes, but is not ultimately periodic (0 is not an accumulation point of  $\{T_\beta^n(1)\}$ ),
- Class C4:  $\{T_\beta^n(1)\}$  is not dense in  $[0, 1]$ , but admits 0 as an accumulation point,
- Class C5:  $\{T_\beta^n(1)\}$  is dense in  $[0, 1]$ .

Apart from C1, resp. C2, which is exactly the set of simple, resp. nonsimple, Parry numbers, how the remaining algebraic numbers  $> 1$  are dispatched in the classes C3, C4 and C5 is obscure. The specification property, meaning that 0 is not an accumulation point for  $\{T_\beta^n(1)\}$ , was weakened by Pfister and Sullivan [156] and Thompson [206]. For unique  $q$ -expansions the specification and synchronization properties were studied by Alcaraz Barreira [4] [5] [6]. Schmeling [182] proved that the class C3 has full Hausdorff dimension and that the class C5, probably mostly occupied by transcendental numbers, is of full Lebesgue measure 1. Lacunarity and Diophantine approximation were investigated by Bugeaud and Liao [47], Hu, Tong and Yu [108], Li, Persson, Wang and Wu [126]. For any  $x_0 \in [0, 1]$  the asymptotic distance  $\liminf_{n \rightarrow \infty} |T_\beta^n(1) - x_0|$ , for almost all  $\beta > 1$  (for the Lebesgue measure), was studied by Persson and Schmeling [155] [182], Ban and Li [17], Cao [51], Fang, Wu and Li [81], Li and Chen [125], Lü and Wu [133], Tan and Wang [204]. Kwon [115] studies the subset of Parry numbers whose conjugates lie close to the unit circle, using technics of combinatorics of words. The separation between algebraic numbers and transcendental numbers was studied by Dubickas [70]. Bugeaud [46] investigates Diophantine approximation properties and  $\beta$ -representations in algebraic bases. Adamczewski

and Bugeaud ([1], Theorem 4) show that the class C4 contains self-lacunary numbers, all transcendental, from Schmidt's Subspace Theorem and results of Corvaja and Zannier.

**3.4. Cyclotomic jumps in families of Parry Upper functions, right-continuity.** Allowing the real base  $\beta$  to vary continuously in the neighbourhood  $[1, \theta_2^{-1})$  of 1, except 1, asks the question whether it has a sense to consider the continuity of the bivariate Parry Upper function  $(\beta, z) \rightarrow f_\beta(z)$ , and, if it is the case, on which subsets, in  $z$ , of the complex plane.

Theorem 3.13 and its Corollary show that the open unit disk is a domain where the continuity of the roots of  $f_\beta(z)$  in  $|z| < 1$  holds though the functions  $f_\beta(z)$  are only right continuous in  $\beta$ , with infinitely many cyclotomic jumps, while, in the complement  $|z| \geq 1$ , either the Parry Upper functions are not defined, or may exhibit drastic changes on the unit circle. In the present attack of the Conjecture of Lehmer only the open unit disk is of interest.

**Lemma 3.11.** *Let  $1 < \beta < \theta_2^{-1}$  and  $0 < x < 1$ . Then (i) the bivariate  $\beta$ -transformation map  $(\beta, x) \rightarrow T_\beta(x) = \{\beta x\} = \beta x - \lfloor \beta x \rfloor$  is continuous, in  $\beta$  and  $x$ , when  $\beta x$  is not a positive integer. If  $\beta x$  is a positive integer,  $x = 1/\beta$  and*

$$(3.4.1) \quad \lim_{y \rightarrow \frac{1}{\beta}^-, \gamma \rightarrow \beta^-} T_\gamma(y) = 1, \quad \lim_{y \rightarrow \frac{1}{\beta}^+, \gamma \rightarrow \beta^+} T_\gamma(y) = T_\beta\left(\frac{1}{\beta}\right) = 0;$$

(ii) for any  $(\beta, x)$ , there exists  $\varepsilon = \varepsilon_{\beta, x}$  such that  $T_\gamma(y)$  is increasing both in  $\gamma \in [\beta, \beta + \varepsilon)$  and in  $y \in [x, x + \varepsilon)$ .

*Proof.* Lemma 3.1 in [84]. (i) If  $\beta x$  is an integer, this integer is 1 necessarily. The value  $x = 1/\beta$  is a negative power of  $\beta$ . The Rényi  $\beta$ -expansion of  $1/\beta$  is deduced from  $d_\beta(1)$  by a shift, given in (2.2.3) and (2.2.6); the sequence  $(T_\beta^n(\frac{1}{\beta}))_{n \geq 1}$  is directly obtained from  $(T_\beta^n(1))_{n \geq 1}$ . The fractional part  $\gamma \rightarrow \{\gamma\} = T_\gamma(1)$  is right continuous, hence the result; (ii) obvious.  $\square$

**Lemma 3.12.** *Let  $\beta \in (1, \theta_2^{-1})$ . (i) If  $\beta$  is a simple Parry number, then, for all  $n \geq 1$ , the map  $\gamma \rightarrow T_\gamma^n(1)$  is right continuous at  $\beta$ :*

$$(3.4.2) \quad \lim_{\gamma \rightarrow \beta^+} T_\gamma^n(1) = T_\beta^n(1),$$

(ii) if  $\beta$  is a simple Parry number, such that  $T_\beta^N(1) = 0$  with  $T_\beta^k(1) \neq 0$ ,  $1 \leq k < N$ , then, for all  $n \geq 1$ ,

$$(3.4.3) \quad \lim_{\gamma \rightarrow \beta^-} T_\gamma^n(1) = \begin{cases} T_\beta^n(1), & n < N \quad (\text{left continuity}) \\ T_\beta^{n_N}(1), & n \geq N, \end{cases}$$

where  $n_N \in \{0, 1, \dots, N-1\}$  is the residue of  $n$  modulo  $N$ ,

(iii) if  $\beta$  is a nonsimple Parry number, then  $\gamma \rightarrow T_\gamma^n(1)$  is continuous at  $\beta$ :

$$(3.4.4) \quad \lim_{\gamma \rightarrow \beta^-} T_\gamma^n(1) = T_\beta^n(1) = \lim_{\gamma \rightarrow \beta^+} T_\gamma^n(1), \quad \text{for all } n \geq 1.$$

*Proof.* Lemma 3.2 in [84].  $\square$

Denote by

$$\mathcal{F} := \{f_{\beta}|_{|z|<1}(z) \mid 1 < \beta < \theta_2^{-1}\}$$

the set of the restrictions of the Parry Upper functions  $f_{\beta}(z)$ ,  $1 < \beta < \theta_2^{-1}$ , to the open unit disk. The set  $\mathcal{F}$  is equipped with the topology of the uniform convergence on compact subsets of  $|z| < 1$ .

**Theorem 3.13.** *In  $\mathcal{F}$  the following right and left limits hold: (i) if  $\beta$  be a nonsimple Parry number, then continuity occurs as:*

$$(3.4.5) \quad \lim_{\gamma \rightarrow \beta^-} f_{\gamma}(z) = f_{\beta}(z) = \lim_{\gamma \rightarrow \beta^+} f_{\gamma}(z),$$

(ii) if  $\beta$  is a simple Parry number, and  $N$  the minimal value for which  $T_{\beta}^N(1) = 0$ , then

$$(3.4.6) \quad \lim_{\gamma \rightarrow \beta^+} f_{\gamma}(z) = f_{\beta}(z),$$

$$(3.4.7) \quad \lim_{\gamma \rightarrow \beta^-} f_{\gamma}(z) = \frac{f_{\beta}(z)}{(1-z^N)}.$$

*Proof.* Let  $\gamma, \beta \in (1, \theta_2^{-1})$  with  $|\gamma - \beta| \leq \varepsilon$ ,  $\varepsilon > 0$ ,  $d_{\gamma}(1) = 0.t'_1 t'_2 \dots$  and  $d_{\beta}(1) = 0.t_1 t_2 \dots$ . Any compact subset of  $|z| < 1$  is included in a closed disk centered at 0 of radius  $r$  for some  $0 < r < 1$ . Assume  $|z| \leq r$ . (i) Assume  $\beta$  nonsimple. Since  $|T_{\gamma}^m(1) - T_{\beta}^m(1)| \leq 2$  for  $m \geq 1$ , then

$$(3.4.8) \quad \begin{aligned} |f_{\gamma}(z) - f_{\beta}(z)| &= \left| \sum_{n \geq 1} (t'_n - t_n) z^n \right| = \left| \sum_{n \geq 1} [(\gamma T_{\gamma}^{n-1}(1) - \beta T_{\beta}^{n-1}(1)) - (T_{\gamma}^n(1) - T_{\beta}^n(1))] z^n \right| \\ &\leq \sum_{n \geq 1} \left| (\gamma T_{\gamma}^{n-1}(1) - \beta T_{\beta}^{n-1}(1)) - (T_{\gamma}^n(1) - T_{\beta}^n(1)) \right| r^n \leq 2(\varepsilon + \beta + 1) \sum_{n \geq 1} r^n, \end{aligned}$$

which is convergent. By (3.4.4) and the Lebesgue dominated convergence theorem, taking the limit termwise in the summation,

$$\lim_{\gamma \rightarrow \beta} |f_{\gamma}(z) - f_{\beta}(z)| = 0, \quad \text{uniformly for } |z| \leq r.$$

(ii) By (3.4.2) and (3.4.3), the iterates of 1 under the  $\gamma$ -transformation  $T_{\gamma}^n(1)$  behave differently at  $\beta$  if  $\gamma < \beta$  or resp.  $\gamma > \beta$  when  $\gamma$  tends to  $\beta$ : if  $\gamma > \beta$ , we apply the Lebesgue dominated convergence theorem in (3.4.8) to obtain the right continuity at  $\beta$ , i.e. (3.4.6); if  $\gamma \rightarrow \beta^-$ , (3.4.7) comes from the dominated convergence theorem applied to

$$f_{\gamma}(z) - \frac{1}{1-z^N} f_{\beta}(z) = (\beta z - 1) \left[ \left( \sum_{n=0}^{\infty} T_{\gamma}^n(1) \frac{\gamma z - 1}{\beta z - 1} z^n \right) - \left( \sum_{q=0}^{\infty} \sum_{m=0}^{N-1} T_{\beta}^m(1) z^{m+qN} \right) \right].$$

□

Theorem B in Mori [141], on the continuity properties of spectra of Fredholm matrices, admits the following counterpart in terms of the Parry Upper functions:

**Corollary 3.14.** *The root functions of  $f_{\beta}(z)$  valued in  $|z| < 1$  are all continuous, as functions of  $\beta \in (1, \theta_2^{-1}) \setminus \bigcup_{n \geq 3} \{\theta_n^{-1}\}$ .*

*Proof.* Let  $(\gamma_i)_{i \geq 1}$  be a sequence of real numbers tending to  $\beta$ . The (restrictions, to the open unit disk, of the) functions  $f_{\gamma_i}(z)$  constitute a convergent sequence in  $\mathcal{F}$ , tending either to  $f_\beta(z)$  or  $f_\beta(z)/(1-z^N)$  for some integer  $N \geq 1$ . By Hurwitz's Theorem ([179] (11.1)) any disk in  $|z| < 1$ , whose closure does not intersect the unit circle, which contains a zero  $w(\beta)$  of  $f_\beta(z)$  also contains a zero of  $f_{\gamma_i}(z)$  for all  $i \geq i_0$ , for some  $i_0$ . The multiplicity of  $w(\beta)$  is equal to the number of zeroes  $w(\gamma_i)$ , counted with multiplicities, in this disk.  $\square$

Another consequence, in  $\mathcal{F}$ , is the disappearance of the cyclotomic jumps of the left-discontinuities at the reciprocal algebraic integers  $\beta > 1$  close to  $1^+$ .

**Corollary 3.15.** *If  $\beta \in (1, \theta_2^{-1})$  is a reciprocal algebraic integer, then the left and right continuity of the Parry Upper function occurs in  $\mathcal{F}$  as:*

$$(3.4.9) \quad \lim_{\gamma \rightarrow \beta^-} f_\gamma(z) = f_\beta(z) = \lim_{\gamma \rightarrow \beta^+} f_\gamma(z).$$

*Proof.* From Theorem 3.13 the case (3.4.7) cannot occur since a reciprocal algebraic integer  $> 1$  cannot be a simple Parry number.  $\square$

#### 4. ASYMPTOTIC EXPANSIONS OF THE MAHLER MEASURES $M(-1 + X + X^n)$ , A DOBROWOLSKI TYPE MINORATION

**4.1. Factorization of the trinomials  $-1 + X + X^n$ , lenticuli of roots.** The notations used throughout this note come from the factorization of  $G_n(X) := -1 + X + X^n$  (Selmer [185], Verger-Gaugry [214] Section 2). Summing in pairs over complex conjugated imaginary roots, the indexation of the roots and the factorization of  $G_n(X)$  are taken as follows:

$$(4.1.1) \quad G_n(X) = (X - \theta_n) \left( \prod_{j=1}^{\lfloor \frac{n}{6} \rfloor} (X - z_{j,n})(X - \bar{z}_{j,n}) \right) \times q_n(X),$$

where  $\theta_n$  is the only (real) root of  $G_n(X)$  in the interval  $(0, 1)$ , where

$$q_n(X) = \begin{cases} \left( \prod_{j=1+\lfloor \frac{n}{6} \rfloor}^{\frac{n-2}{2}} (X - z_{j,n})(X - \bar{z}_{j,n}) \right) \times (X - z_{\frac{n}{2},n}) & \text{if } n \text{ is even, with} \\ & z_{\frac{n}{2},n} \text{ real} < -1, \\ \prod_{j=1+\lfloor \frac{n}{6} \rfloor}^{\frac{n-1}{2}} (X - z_{j,n})(X - \bar{z}_{j,n}) & \text{if } n \text{ is odd,} \end{cases}$$

where the index  $j = 1, 2, \dots$  is such that  $z_{j,n}$  is a (nonreal) complex zero of  $G_n(X)$ , except if  $n$  is even and  $j = n/2$ , such that the argument  $\arg(z_{j,n})$  of  $z_{j,n}$  is roughly equal to  $2\pi j/n$  (Proposition 4.7) and that the family of arguments  $(\arg(z_{j,n}))_{1 \leq j < \lfloor n/2 \rfloor}$  forms a strictly increasing sequence with  $j$ :

$$0 < \arg(z_{1,n}) < \arg(z_{2,n}) < \dots < \arg(z_{\lfloor \frac{n}{2} \rfloor, n}) \leq \pi.$$

For  $n \geq 2$  all the roots of  $G_n(X)$  are simple, and the roots of  $G_n^*(X) = 1 + X^{n-1} - X^n$ , as inverses of the roots of  $G_n(X)$ , are classified in the reversed order (Figure 1).

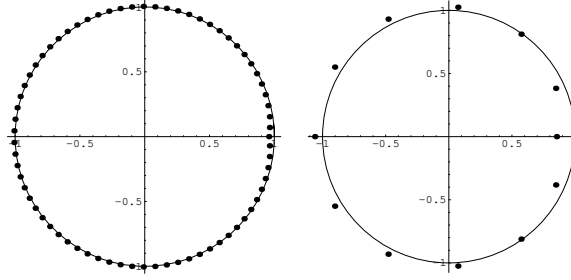


FIGURE 1. The roots (black bullets) of  $G_n(z)$  (represented here with  $n = 71$  and  $n = 12$ ) are uniformly distributed near  $|z| = 1$  according to the theory of Erdős-Turán-Amoroso-Mignotte. A slight bump appears in the half-plane  $\Re(z) > 1/2$  in the neighbourhood of 1, at the origin of the different regimes of asymptotic expansions. The dominant root of  $G_n^*(z)$  is the Perron number  $\theta_n^{-1} > 1$ , with  $\theta_n$  the unique root of  $G_n$  in the interval  $(0, 1)$ .

**Proposition 4.1.** *Let  $n \geq 2$ . If  $n \not\equiv 5 \pmod{6}$ , then  $G_n(X)$  is irreducible over  $\mathbb{Q}$ . If  $n \equiv 5 \pmod{6}$ , then the polynomial  $G_n(X)$  admits  $X^2 - X + 1$  as irreducible factor in its factorization and  $G_n(X)/(X^2 - X + 1)$  is irreducible.*

*Proof.* Selmer [185]. □

**Proposition 4.2.** *For all  $n \geq 2$ , all zeros  $z_{j,n}$  and  $\theta_n$  of the polynomials  $G_n(X)$  have a modulus in the interval*

$$(4.1.2) \quad \left[ 1 - \frac{2\text{Log } n}{n}, 1 + \frac{2\text{Log } 2}{n} \right],$$

(ii) *the trinomial  $G_n(X)$  admits a unique real root  $\theta_n$  in the interval  $(0, 1)$ . The sequence  $(\theta_n)_{n \geq 2}$  is strictly increasing,  $\lim_{n \rightarrow +\infty} \theta_n = 1$ , with  $\theta_2 = \frac{2}{1+\sqrt{5}} = 0.618\dots$ ,*

(iii) *the root  $\theta_n$  is the unique root of smallest modulus among all the roots of  $G_n(X)$ ; if  $n \geq 6$ , the roots of modulus  $< 1$  of  $G_n(z)$  in the closed upper half-plane have the following properties:*

(iii-1)  $\theta_n < |z_{1,n}|$ ,

(iii-2) *for any pair of successive indices  $j, j + 1$  in  $\{1, 2, \dots, \lfloor n/6 \rfloor\}$ ,*

$$|z_{j,n}| < |z_{j+1,n}|.$$

*Proof.* (i)(ii) Selmer [185], pp 291–292; (iii-1) Flatto, Lagarias and Poonen [84], (iii-2) Verger-Gaugry [214]. □

The Pisot number (golden mean)  $\theta_2^{-1} = \frac{1+\sqrt{5}}{2} = 1.618\dots$  is the largest Perron number in the family  $(\theta_n^{-1})_{n \geq 2}$ . The interval  $(1, \frac{1+\sqrt{5}}{2}]$  is partitioned by the strictly decreasing sequence of Perron numbers  $(\theta_n^{-1})$  as

$$(4.1.3) \quad \left(1, \frac{1+\sqrt{5}}{2}\right] = \left(\bigcup_{n=2}^{\infty} [\theta_{n+1}^{-1}, \theta_n^{-1})\right) \cup \{\theta_2^{-1}\}.$$

By the direct method of asymptotic expansions of the roots, as in [214], or by Smyth's Theorem [189] (Dubickas [67]), since the trinomials  $G_n(X)$  are not reciprocal, the Mahler measure of  $G_n$  satisfies

$$(4.1.4) \quad M(\theta_n) = M(G_n) \geq \Theta = 1.3247\dots, \quad n \geq 2,$$

where  $\Theta = \theta_5^{-1}$  is the smallest Pisot number, dominant root of the Pisot polynomial  $X^3 - X - 1 = -G_5^*(X)/(X^2 - X + 1)$ .

**Proposition 4.3.** *Let  $n \geq 2$ . Then (i) the number  $p_n$  of roots of  $G_n(X)$  which lie inside the open sector  $\mathcal{S} = \{z \mid |\arg(z)| < \pi/3\}$  is equal to*

$$(4.1.5) \quad 1 + 2 \lfloor \frac{n}{6} \rfloor,$$

*(ii) the correlation between the geometry of the roots of  $G_n(X)$  which lie inside the unit disk and the upper half-plane and their indexation is given by:*

$$(4.1.6) \quad j \in \{1, 2, \dots, \lfloor \frac{n}{6} \rfloor\} \iff \Re(z_{j,n}) > \frac{1}{2} \iff |z_{j,n}| < 1,$$

*and the Mahler measure  $M(G_n)$  of the trinomial  $G_n(X)$  is*

$$(4.1.7) \quad M(G_n) = M(G_n^*) = \theta_n^{-1} \prod_{j=1}^{\lfloor n/6 \rfloor} |z_{j,n}|^{-2}.$$

*Proof.* Verger-Gaugry [214], Proposition 3.7. □

**4.2. Asymptotic expansions: roots of  $G_n$  and relations.** The (Poincaré) asymptotic expansions of the roots of  $G_n$  (and  $G_n^*$ ) are generically written:  $\Re(z_{j,n}) = D(\Re(z_{j,n})) + \text{tl}(\Re(z_{j,n}))$ ,  $\text{Im}(z_{j,n}) = D(\text{Im}(z_{j,n})) + \text{tl}(\text{Im}(z_{j,n}))$ ,  $\theta_n = D(\theta_n) + \text{tl}(\theta_n)$ , where "D" and "tl" stands for "development" (or "limited expansion", or "lowest order terms") and "tl" for "tail" (or "remainder", or "terminant" in [61]). They are given at a sufficiently high order allowing to deduce the asymptotic expansions of the Mahler measures  $M(G_n)$ . The terminology *order* comes from the general theory (Borel [29], Copson [55], Dingle [61], Erdélyi [77]); the approximant solutions of a polynomial equation say  $G(z) = 0$  which arise naturally correspond to *order 1*. The solutions corresponding to *order 2* are obtained by inserting the *order 1 approximant solutions* into the equation  $G(z) = 0$ , for getting *order 2 approximant solutions*. And so on, as a function of  $\deg G$ . The order is the number of steps in this iterative process. Poincaré [158] introduced this method of divergent series for the  $N$  - body problem in celestial mechanics; this method does not appear in number theory in the book "Divergent series" of Hardy. The equivalent of the variable time  $t$  (in celestial mechanics) will be the dynamical degree  $\text{dyg}(\overline{\alpha})$  of the house of the algebraic integer  $\alpha$  in number theory (with  $|\alpha| > 1$ ), a new "variable concept" introduced in the present study; for the trinomials  $G_n$  it will be  $n$ .

The asymptotic expansions of  $\theta_n$  and those roots  $z_{j,n}$  of  $G_n(z)$  which lie in the first quadrant are (divergent) sums of functions of only *one* variable, which is  $n$ , while those of the other roots  $z_{j,n}$  are functions of a couple of *two* variables which is:

- $(n, j/n)$  in the angular sector  $\pi/4 > \arg z > 2\pi \text{Log } n/n$ , and
- $(n, j/\text{Log } n)$  in the angular sector  $2\pi \text{Log } n/n > \arg z > 0$ .



The first sector is the main angular sector. The second sector is the bump angular sector. A unique regime of asymptotic expansion exists in the main angular sector, whereas two regimes of asymptotic expansions do exist in the bump sector (Appendix, and [214]). These regimes are separated by two sequences  $(u_n)$  and  $(v_n)$ , to which the second variable  $j/n$ , resp.  $j/\text{Log } n$ , is compared. Details can be found in the Appendix.

**Proposition 4.4.** *Let  $n \geq 2$ . The root  $\theta_n$  can be expressed as:  $\theta_n = D(\theta_n) + \text{tl}(\theta_n)$  with  $D(\theta_n) = 1 -$*

$$(4.2.1) \quad \frac{\text{Log } n}{n} \left( 1 - \left( \frac{n - \text{Log } n}{n \text{Log } n + n - \text{Log } n} \right) \left( \text{Log } \text{Log } n - n \text{Log} \left( 1 - \frac{\text{Log } n}{n} \right) - \text{Log } n \right) \right)$$

and

$$(4.2.2) \quad \text{tl}(\theta_n) = \frac{1}{n} O \left( \left( \frac{\text{Log } \text{Log } n}{\text{Log } n} \right)^2 \right),$$

with the constant  $1/2$  involved in  $O(\cdot)$ .

*Proof.* [214] Proposition 3.1. □

**Lemma 4.5.** *Given the limited expansion  $D(\theta_n)$  of  $\theta_n$  as in (4.2.1), denote*

$$\lambda_n := 1 - (1 - D(\theta_n)) \frac{n}{\text{Log } n}.$$

Then  $\lambda_n = D(\lambda_n) + \text{tl}(\lambda_n)$ , with

$$(4.2.3) \quad D(\lambda_n) = \frac{\text{Log } \text{Log } n}{\text{Log } n} \left( \frac{1}{1 + \frac{1}{\text{Log } n}} \right), \quad \text{tl}(\lambda_n) = O \left( \frac{\text{Log } \text{Log } n}{n} \right)$$

with the constant  $1$  in the Big  $O$ .

*Proof.* [214] Lemma 3.2. □

In the sequel, for short, we write  $\lambda_n$  instead of  $D(\lambda_n)$ .

**Proposition 4.6.** *Let  $n \geq n_0 = 18$  and  $1 \leq j \leq \lfloor \frac{n-1}{4} \rfloor$ . The roots  $z_{j,n}$  of  $G_n(X)$  have the following asymptotic expansions:  $z_{j,n} = D(z_{j,n}) + \text{tl}(z_{j,n})$  in the following angular sectors:*

(i) Sector  $\frac{\pi}{2} > \arg z > 2\pi \frac{\text{Log } n}{n}$  (main sector):

$$D(\Re(z_{j,n})) = \cos(2\pi \frac{j}{n}) + \frac{\text{Log}(2 \sin(\pi \frac{j}{n}))}{n},$$

$$D(\Im(z_{j,n})) = \sin(2\pi \frac{j}{n}) + \tan(\pi \frac{j}{n}) \frac{\text{Log}(2 \sin(\pi \frac{j}{n}))}{n},$$

with

$$\text{tl}(\Re(z_{j,n})) = \text{tl}(\Im(z_{j,n})) = \frac{1}{n} O \left( \left( \frac{\text{Log } \text{Log } n}{\text{Log } n} \right)^2 \right)$$

and the constant  $1$  in the Big  $O$ ,

(ii) “Bump” sector  $2\pi \frac{\text{Log } n}{n} > \arg z > 0$  :

• Subsector  $2\pi \frac{\sqrt{(\text{Log } n)(\text{Log Log } n)}}{n} > \arg z > 0$ :

$$D(\Re(z_{j,n})) = \theta_n + \frac{2\pi^2}{n} \left( \frac{j}{\text{Log } n} \right)^2 (1 + 2\lambda_n),$$

$$D(\Im(z_{j,n})) = \frac{2\pi \text{Log } n}{n} \left( \frac{j}{\text{Log } n} \right) \left[ 1 - \frac{1}{\text{Log } n} (1 + \lambda_n) \right],$$

with

$$\text{tl}(\Re(z_{j,n})) = \frac{1}{n \text{Log } n} \left( \frac{j}{\text{Log } n} \right)^2 O\left( \left( \frac{\text{Log Log } n}{\text{Log } n} \right)^2 \right),$$

$$\text{tl}(\Im(z_{j,n})) = \frac{1}{n \text{Log } n} \left( \frac{j}{\text{Log } n} \right) O\left( \left( \frac{\text{Log Log } n}{\text{Log } n} \right)^2 \right),$$

• Subsector  $2\pi \frac{\text{Log } n}{n} > \arg z > 2\pi \frac{\sqrt{(\text{Log } n)(\text{Log Log } n)}}{n}$ :

$$D(\Re(z_{j,n})) = \theta_n + \frac{2\pi^2}{n} \left( \frac{j}{\text{Log } n} \right)^2 \left( 1 + \frac{2\pi^2}{3} \left( \frac{j}{\text{Log } n} \right)^2 (1 + \lambda_n) \right)$$

$$D(\Im(z_{j,n})) =$$

$$\frac{2\pi \text{Log } n}{n} \left( \frac{j}{\text{Log } n} \right) \left[ 1 - \frac{1}{\text{Log } n} \left( 1 - \frac{4\pi^2}{3} \left( \frac{j}{\text{Log } n} \right)^2 \left( 1 - \frac{1}{\text{Log } n} (1 - \lambda_n) \right) \right) \right],$$

with

$$\text{tl}(\Re(z_{j,n})) = \frac{1}{n} O\left( \left( \frac{j}{\text{Log } n} \right)^6 \right), \text{tl}(\Im(z_{j,n})) = \frac{1}{n} O\left( \left( \frac{j}{\text{Log } n} \right)^5 \right).$$

*Proof.* [214] Proposition 3.4. □

Outside the “bump sector” the moduli of the roots  $z_{j,n}$  are readily obtained as (Proposition 3.5 in [214]):

$$(4.2.4) \quad |z_{j,n}| = 1 + \frac{1}{n} \text{Log} \left( 2 \sin\left(\frac{\pi j}{n}\right) \right) + \frac{1}{n} O\left( \frac{(\text{Log Log } n)^2}{(\text{Log } n)^2} \right),$$

with the constant 1 in the Big  $O$  (independent of  $j$ ). The following expansions of the  $|z_{j,n}|$ s at the order 3 will be needed in the method of Rouché.

**Proposition 4.7.**

$$\arg(z_{j,n}) = 2\pi \left( \frac{j}{n} + A_{j,n} \right) \quad \text{with} \quad A_{j,n} = -\frac{1}{2\pi n} \left[ \frac{1 - \cos\left(\frac{2\pi j}{n}\right)}{\sin\left(\frac{2\pi j}{n}\right)} \text{Log} \left( 2 \sin\left(\frac{\pi j}{n}\right) \right) \right]$$

$$\text{and} \quad \text{tl}(\arg(z_{j,n})) = \frac{1}{n} O\left( \left( \frac{\text{Log Log } n}{\text{Log } n} \right)^2 \right).$$

*Proof.* §6 in [214]. □

**Proposition 4.8.** For all  $j$  such that  $\pi/3 \geq \arg z_{j,n} > 2\pi \frac{[v_n]}{n}$ , the asymptotic expansions of the moduli of the roots  $z_{j,n}$  are

$$|z_{j,n}| = D(|z_{j,n}|) + \text{tl}(|z_{j,n}|)$$

with

$$(4.2.5) \quad D(|z_{j,n}|) = 1 + \frac{1}{n} \text{Log} \left( 2 \sin \left( \frac{\pi j}{n} \right) \right) + \frac{1}{2n} \left( \frac{\text{Log Log } n}{\text{Log } n} \right)^2$$

and

$$(4.2.6) \quad \text{tl}(|z_{j,n}|) = \frac{1}{n} O \left( \frac{(\text{Log Log } n)^2}{(\text{Log } n)^3} \right)$$

where the constant involved in  $O(\cdot)$  is 1 (does not depend upon  $j$ ).

*Proof.* [214] Section 5.1. □

The following asymptotic expansions in Proposition 4.9, Proposition 4.10 and Proposition 4.11 will be used in the method of Rouché in §5.

**Proposition 4.9.** For  $n \geq 18$ , the modulus of the first root  $z_{1,n}$  of  $G_n(z) = -1 + z + z^n$  is

$$(4.2.7) \quad |z_{1,n}| = 1 - \frac{\text{Log } n - \text{Log Log } n}{n} + \frac{1}{n} O \left( \frac{\text{Log Log } n}{\text{Log } n} \right)$$

and

$$(4.2.8) \quad |-1 + z_{1,n}| = \frac{\text{Log } n - \text{Log Log } n}{n} + \frac{1}{n} O \left( \frac{\text{Log Log } n}{\text{Log } n} \right)$$

with the constant 1 in the two Big Os.

*Proof.* The root  $z_{1,n}$  belongs to the subsector  $2\pi \frac{\sqrt{(\text{Log } n)(\text{Log Log } n)}}{n} > \arg z > 0$ : first, from Lemma 4.5, the asymptotic expansion of  $\lambda_n$  is

$$\lambda_n = \frac{\text{Log Log } n}{\text{Log } n} + O \left( \frac{\text{Log Log } n}{(\text{Log } n)^2} \right)$$

with the constant 1 in the Big  $O$ . Since  $D(|z_{1,n}|) = D(\Re(z_{1,n})) \left( 1 + \left( \frac{D(\Im(z_{1,n}))}{D(\Re(z_{1,n}))} \right)^2 \right)^{1/2}$ , that

$$D(\Re(z_{1,n})) = \theta_n + \frac{2\pi^2}{n} \left( \frac{1}{\text{Log } n} \right)^2 (1 + 2\lambda_n), \quad D(\Im(z_{1,n})) = \frac{2\pi}{n} \left[ 1 - \frac{1}{\text{Log } n} (1 + \lambda_n) \right]$$

(Proposition 4.6) and

$$\theta_n = 1 - \frac{\text{Log } n}{n} (1 - \lambda_n) + \frac{1}{n} O \left( \left( \frac{\text{Log Log } n}{\text{Log } n} \right)^2 \right)$$

(Proposition 4.4) we deduce (4.2.7) and the expansion (4.2.8) from the expansion of  $\lambda_n$ . □

**Proposition 4.10.** For  $n \geq 18$ , the modulus of  $-1 + z_{j,n}$ , where  $z_{j,n}$  is the  $j$ -th root of  $G_n(z) = -1 + z + z^n$ ,  $[v_n] \leq j \leq [n/6]$ , is

$$(4.2.9) \quad |-1 + z_{j,n}| = 2 \sin \left( \frac{\pi j}{n} \right) + \frac{1}{n} O \left( \left( \frac{\text{Log Log } n}{\text{Log } n} \right)^2 \right)$$

with the constant 1 in the Big  $O$ .

*Proof.* From (4.2.4), Proposition 4.6 and Proposition 4.8, the identity

$$|-1 + z_{j,n}|^2 = (-1 + \Re(z_{j,n}))^2 + (\Im(z_{j,n}))^2 = 1 + |z_{j,n}|^2 - 2\Re(z_{j,n})$$

implies:  $|-1 + z_{j,n}|^2 =$

$$2 - 2\cos\left(2\pi\frac{j}{n}\right) + \frac{1}{n}O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right) = 4\sin^2\left(\frac{\pi j}{n}\right) + \frac{1}{n}O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right)$$

with the constant 4 in the Big  $O$ . We deduce (4.2.9).  $\square$

**Proposition 4.11.** *For  $n \geq 18$ , the modulus of  $(-1 + z_{j,n})/z_{j,n}$ , where  $z_{j,n}$  is the  $j$ -th root of  $G_n(z) = -1 + z + z^n$ ,  $\lceil v_n \rceil \leq j \leq \lfloor n/6 \rfloor$ , is*

$$(4.2.10) \quad \frac{|-1 + z_{j,n}|}{|z_{j,n}|} = 2\sin\left(\frac{\pi j}{n}\right)\left(1 - \frac{1}{n}\text{Log}\left(2\sin\left(\frac{\pi j}{n}\right)\right)\right) + \frac{1}{n}O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right)$$

with the constant 2 in the Big  $O$ .

*Proof.* The expansion (4.2.10) readily comes (4.2.9) and  $|z_{j,n}|$  given by Proposition 4.8.  $\square$

**4.3. Minoration of the Mahler measure.** In [214] “à la Poincaré” asymptotic expansions are shown to give “controlled” approximants of the set of the values of the Mahler measures  $M(G_n)$  and an exact value of its limit point. Compared to several methods (Amoroso [8] [9], Boyd and Mossinghoff [40], Dixon and Dubickas [62], Langevin [117], Smyth [191]), the present approach is new in the sense that the use of auxiliary functions by Dobrowolski [63] is replaced by the Rényi-Parry dynamics of the Perron numbers  $(\theta_n^{-1})_{n \geq 2}$ . Let us briefly mention the results. The product

$$(4.3.1) \quad \Pi_{G_n} := D(M(G_n)) = D(\theta_n)^{-1} \times \prod_{\substack{z_{j,n} \text{ in } |z| < 1 \\ \text{outside bump}}} D(|z_{j,n}|)^{-2}$$

is considered, instead of

$$(4.3.2) \quad M(G_n) = \theta_n^{-1} \prod_{j=1}^{\lfloor n/6 \rfloor} |z_{j,n}|^{-2} = \prod_{\mathcal{L}_{\theta_n^{-1}}} |z|^{-1}$$

as approximant value of  $M(G_n)$ . In (4.3.1) the zeroes  $z_{j,n}$  present in the bump sector are discarded since they do not contribute to the limited asymptotic expansions, as shown in [214] Section 4.2. In [214] Section 4, the two limits  $\lim_{n \rightarrow +\infty} \Pi_{G_n}$  and  $\lim_{n \rightarrow +\infty} M(G_n)$  are shown to exist, to be equal (and greater than  $\Theta$ ).

**Theorem 4.12.** *Let  $\chi_3$  be the uniquely specified odd character of conductor 3 ( $\chi_3(m) = 0, 1$  or  $-1$  according to whether  $m \equiv 0, 1$  or  $2 \pmod{3}$ ), equivalently  $\chi_3(m) = \left(\frac{m}{3}\right)$  the Jacobi symbol), and denote  $L(s, \chi_3) = \sum_{m \geq 1} \frac{\chi_3(m)}{m^s}$  the Dirichlet  $L$ -series for the character  $\chi_3$ . Then, with  $\Lambda$  given by (1.0.17),  $\lim_{n \rightarrow +\infty} M(G_n) = M(-1 + z + y) = \Lambda = 1.38135\dots$*

*Proof.* [214] Theorem 1.1; Smyth [190], using Boyd-Smyth’s method of bivariate Mahler measures ([214] Section 4.1).  $\square$

Introduced in the product (4.3.2), the terminants of the asymptotic expansions of the moduli of the roots  $z_{j,n}$  and of  $\theta_n$  provide the higher-order terms of the asymptotic expansion of  $M(G_n)$ .

**Theorem 4.13.** *Let  $n_0$  be an integer such that  $\frac{\pi}{3} > 2\pi \frac{\text{Log } n_0}{n_0}$ , and let  $n \geq n_0$ . Then,*

$$(4.3.3) \quad M(G_n) = \Lambda \left( 1 + r(n) \frac{1}{\text{Log } n} + O \left( \frac{(\text{Log } \text{Log } n)^2}{\text{Log } n} \right) \right)$$

with the constant  $1/6$  involved in the Big  $O$ , and with  $r(n)$  real,  $|r(n)| \leq 1/6$ .

*Proof.* [214] Theorem 1.2. □

In Theorem 4.13 we take  $n_0 = 18$ . For the small values of  $n$ , we have:

$$M(G_2) = \theta_2^{-1} = \frac{1 + \sqrt{5}}{2} = 1.618\dots$$

and the following lower bound.

**Proposition 4.14.**  $M(G_n) \geq M(G_5) = \theta_5^{-1} = \Theta = 1.3247\dots$  for all  $n \geq 3$ , with equality if and only if  $n = 5$ .

*Proof.* [214] Corollary 1.4. □

The minoration of the residual distance between the two algebraic integers 1 and  $\theta_n^{-1}$  is deduced from the Zhang-Zagier height and Doche's improvement.

**Proposition 4.15.** *Let  $u = 0$  except if  $n \equiv 5 \pmod{6}$  in which case  $u = -2$ . Then,*

$$(4.3.4) \quad M(\theta_n^{-1} - 1) \geq \frac{\eta^{n+u}}{\Lambda} \left( 1 - \frac{1}{6 \text{Log } n} \right), \quad n \geq 2,$$

with  $\eta = 1.2817770214$ .

*Proof.* Except for a finite subset of algebraic numbers, the minoration  $M(\alpha)M(1 - \alpha) \geq (\theta_2^{-1/2})^{\deg(\alpha)}$  was established by Zagier [220] and improved by Doche [64], with the lower bound  $\eta > \theta_2^{-1/2}$  instead of  $\theta_2^{-1/2}$  itself. The minorant (4.3.4) follows from (4.3.3). □

Recall Dobrowolski's minoration [63]:

$$(4.3.5) \quad M(\alpha) > 1 + (1 - \varepsilon) \left( \frac{\text{Log } \text{Log } d}{\text{Log } d} \right)^3, \quad d > d_1(\varepsilon),$$

for any nonzero algebraic number  $\alpha$  of degree  $d$ . Voutier in [216] obtained other effective minorations, improving (4.3.5) for  $d \geq 2$ . A survey on effective minorations is given in [215]. Theorem 4.13 implies the following minoration of  $M(\theta_n^{-1})$  which is better than (4.3.5) in the sense that the constant term of the minorant is  $> 1$ .

**Theorem 4.16.**

$$(4.3.6) \quad M(\theta_n^{-1}) > \Lambda - \frac{\Lambda}{6} \left( \frac{1}{\text{Log } n} \right), \quad n \geq n_1 = 2.$$

*Proof.* [214] Corollary 1.6. □

The problem of extremality of Perron numbers is still open (cf Boyd [34], [215]). The extremality of the Perron numbers  $\theta_n^{-1}$  occurs only for  $n = 2, 3$ . In general, if extremality holds, by Lind-Boyd's Conjecture, it would be associated with a lenticular distribution of roots (of modulus  $> 1$ ) which admits a proportion asymptotically equal to  $\frac{2}{3}n$ . For the trinomials  $G_n^*$ ,  $n \geq 4$ , this proportion is only  $\frac{1}{3}n$ , for  $n$  large.

## 5. THE LENTICULAR MINORANT OF THE MAHLER MEASURE $M(\beta)$ FOR $\beta > 1$ A REAL RECIPROCAL ALGEBRAIC INTEGER CLOSE TO ONE

The starting point of the proof of Lehmer's Conjecture is the investigation of the zeroes of the Parry Upper functions  $f_\beta(z)$  in the annular region  $\{\beta^{-1} < |z| < 1\} \cup \{\beta^{-1}\}$  for  $\beta > 1$  any reciprocal algebraic integer tending to one. The annular domain  $\{\beta^{-1} < |z| < 1\}$  is the extended domain of meromorphy of the dynamical zeta function  $\zeta_\beta(z)$  in the open unit disk. The case of the zero  $\beta^{-1}$ , as  $= D(\beta^{-1}) + \text{tl}(\beta^{-1})$ , is readily deduced from Section §5.1 where the asymptotic expansion of a real number  $\beta > 1$  close to 1 is obtained as a function of its dynamical degree  $\text{dyg}(\beta)$ . The geometry of the other zeroes is studied in Section §5.2 and §5.3. The lenticular zeroes are identified as Galois conjugates of  $\beta^{-1}$  in Section §5.4.

### 5.1. Asymptotic expansions of a real number $\beta > 1$ close to one and of the dynamical degree $\text{dyg}(\beta)$ .

**Lemma 5.1.** *Let  $n \geq 6$ . The difference  $\theta_n - \theta_{n-1} > 0$  admits the following asymptotic expansion, reduced to its terminant:*

$$(5.1.1) \quad \theta_n - \theta_{n-1} = \frac{1}{n} O \left( \left( \frac{\text{Log Log } n}{\text{Log } n} \right)^2 \right),$$

with the constant 1 involved in  $O(\cdot)$ .

*Proof.* From (4.2.1) and Lemma 4.5, we have

$$\theta_n = 1 - \frac{\text{Log } n}{n} (1 - \lambda_n) + \frac{1}{n} O \left( \left( \frac{\text{Log Log } n}{\text{Log } n} \right)^2 \right)$$

with the constant 1/2 involved in  $O(\cdot)$ , and

$$\lambda_n = \frac{\text{Log Log } n}{\text{Log } n} \left( \frac{1}{1 + \frac{1}{\text{Log } n}} \right) + O \left( \frac{\text{Log Log } n}{n} \right)$$

with the constant 1 in the Big O. Then we deduce

$$D(\theta_n) - D(\theta_{n-1}) = \frac{\text{Log } n}{n^2} + O \left( \frac{\text{Log Log } n}{n^2} \right).$$

The real function  $x^{-2} \text{Log } x$  on  $(1, +\infty)$  is decreasing for  $x \geq \sqrt{e}$ . Hence the sequence  $(D(\theta_n) - D(\theta_{n-1}))$  is decreasing for  $n$  large enough. By Proposition 4.2  $(\theta_n - \theta_{n-1})_n$  is already known to tend to 0.

Since  $\text{tl}(\theta_n) = \frac{1}{n}O\left(\left(\frac{\text{LogLog}n}{\text{Log}n}\right)^2\right)$ , we have

$$\begin{aligned} \theta_n - \theta_{n-1} &= (\theta_n - D(\theta_n)) + [D(\theta_n) - D(\theta_{n-1})] - (\theta_{n-1} - D(\theta_{n-1})) \\ &= \text{tl}(\theta_n) + \left(\frac{\text{Log}n}{n^2} + O\left(\frac{\text{LogLog}n}{n^2}\right)\right) - \text{tl}(\theta_{n-1}) \\ (5.1.2) \quad &= \frac{1}{n}O\left(\left(\frac{\text{LogLog}n}{\text{Log}n}\right)^2\right) \end{aligned}$$

where the constant involved in  $O(\cdot)$  is now  $1 = 1/2 + 1/2$ . Hence the claim.  $\square$

**Theorem 5.2.** *Let  $n \geq 6$ . Let  $\beta > 1$  be a real number of dynamical degree  $\text{dyg}(\beta) = n$ . Then  $\beta^{-1}$  can be expressed as:  $\beta^{-1} = D(\beta^{-1}) + \text{tl}(\beta^{-1})$  with  $D(\beta^{-1}) = 1 -$*

$$(5.1.3) \quad \frac{\text{Log}n}{n} \left(1 - \left(\frac{n - \text{Log}n}{n\text{Log}n + n - \text{Log}n}\right) \left(\text{LogLog}n - n\text{Log}\left(1 - \frac{\text{Log}n}{n}\right) - \text{Log}n\right)\right)$$

and

$$(5.1.4) \quad \text{tl}(\beta^{-1}) = \frac{1}{n}O\left(\left(\frac{\text{LogLog}n}{\text{Log}n}\right)^2\right),$$

with the constant 1 involved in  $O(\cdot)$ .

*Proof.* By definition  $\theta_n \leq \beta^{-1} < \theta_{n-1}$ . The development term of  $\beta^{-1}$  is  $D(\beta^{-1}) = D(\beta^{-1} - \theta_n) + D(\theta_n)$ , with  $|\beta^{-1} - \theta_n| < \theta_n - \theta_{n-1}$ . By Lemma 5.1,  $D(\theta_n - \theta_{n-1}) = 0$ . Therefore  $\beta^{-1} = D(\beta^{-1}) + \text{tl}(\beta^{-1})$  is deduced from  $D(\theta_n)$  in (4.2.1).  $\square$

**Theorem 5.3.** *Let  $\beta \in (1, \theta_6^{-1})$  be a real number. The asymptotic expansion of the locally constant function  $n = \text{dyg}(\beta)$ , as a function of the variable  $\beta - 1$ , is*

$$(5.1.5) \quad n = -\frac{\text{Log}(\beta - 1)}{\beta - 1} \left[1 + O\left(\left(\frac{\text{Log}(-\text{Log}(\beta - 1))}{\text{Log}(\beta - 1)}\right)^2\right)\right]$$

with the constant 1 in  $O(\cdot)$ .

*Proof.* Inverting (5.1.3) gives the asymptotic expansion of  $n$  as a function of  $\beta$ : from (5.1.3) readily comes

$$(5.1.6) \quad n = \frac{\beta}{\beta - 1} \text{Log}\left(\frac{\beta}{\beta - 1}\right) \left[1 + O\left(\left(\frac{\text{LogLog}\left(\frac{\beta}{\beta - 1}\right)}{\text{Log}\left(\frac{\beta}{\beta - 1}\right)}\right)^2\right)\right]$$

then (5.1.5) as  $\beta \rightarrow 1$ .  $\square$

**Remark 5.4.** (*Simplified forms*): If  $\beta$  runs over the set of Perron numbers  $\theta_n^{-1}$ ,  $n = 5, 6, \dots, 12$ , and over the smallest Parry - Salem numbers  $\beta \leq 1.240726\dots$ , the dynamical degree of  $\beta$  (cf Table 1, Column 1, in [215]) is the integer part of  $D(n)$  in (5.1.6):

$$(5.1.7) \quad \text{dyg}(\beta) = \lfloor \frac{\beta}{\beta - 1} \text{Log}\left(\frac{\beta}{\beta - 1}\right) \rfloor.$$

From (5.1.3) a (approximate) simplified form of  $\beta^{-1}$  is deduced:

$$(5.1.8) \quad D(\beta^{-1}) = 1 - \frac{1}{n} \left( \text{Log } n - \text{Log } \text{Log } n + \frac{\text{Log } \text{Log } n}{\text{Log } n} \right).$$

**5.2. Fracturability of the minimal polynomial by the Parry Upper function.** The algebraic integer  $\beta > 1$  is assumed reciprocal and close to one. The relations between the Parry Upper function  $f_\beta(z)$  and the minimal polynomial  $P_\beta(z)$ , as analytical functions, are given in Theorem 5.5 and will be complemented by Proposition 5.36, after the characterization of the common lenticular zeroes, for  $n = \text{dyg}(\beta)$  large enough.

**Theorem 5.5.** *Let  $1 < \beta < (1 + \sqrt{5})/2$  be a reciprocal algebraic integer. The following formal decomposition of the minimal polynomial*

$$(5.2.1) \quad P_\beta(X) = P_\beta^*(X) = U_\beta(X) \times f_\beta(X),$$

*holds, as a product of the Parry Upper function*

$$(5.2.2) \quad f_\beta(X) = G_{\text{dyg}(\beta)}(X) + X^{m_1} + X^{m_2} + X^{m_3} + \dots$$

*with  $m_0 := \text{dyg}(\beta)$ ,  $m_{q+1} - m_q \geq \text{dyg}(\beta) - 1$  for  $q \geq 0$ , and the invertible formal series  $U_\beta(X) = -\zeta_\beta(X)P_\beta(X) \in \mathbb{Z}[[X]]$ . The specialization  $X \rightarrow z$  of the formal variable to the complex variable leads to the identity between analytic functions, obeying the Carlson-Polya dichotomy:*

$$(5.2.3) \quad P_\beta(z) = U_\beta(z) \times f_\beta(z) \quad \left\{ \begin{array}{l} \text{on } \mathbb{C} \quad \text{if } \beta \text{ is a Parry number, with} \\ \quad \quad \quad U_\beta \text{ and } f_\beta \text{ both meromorphic,} \\ \text{on } |z| < 1 \quad \text{if } \beta \text{ is a nonParry number, with } |z| = 1 \\ \quad \quad \quad \text{as natural boundary for both } U_\beta \text{ and } f_\beta. \end{array} \right.$$

*In both cases, the domain of holomorphy of the function  $U_\beta(z)$  contains the open disk  $D(0, \theta_{\text{dyg}(\beta)-1})$ .*

*Proof.* The reciprocal algebraic integer  $\beta$  lies between two successive Perron numbers of the family  $(\theta_n^{-1})_{n \geq 5}$ , as  $\theta_n^{-1} < \beta < \theta_{n-1}^{-1}$ ,  $\text{dyg}(\beta) = n \geq 6$ . By Proposition 3.2 the Parry Upper function  $f_\beta(z)$  at  $\beta$  has the form (from (5.2.2)):

$$(5.2.4) \quad f_\beta(z) = -1 + z + z^n + z^{m_1} + z^{m_2} + z^{m_3} + \dots$$

The algebraic integer  $\beta$  is a Parry number or a nonParry number. In both cases,  $f_\beta(\beta^{-1}) = 0$ . If  $f_\beta(z) = -1 + \sum_{j \geq 1} t_j z^j$ , the zero  $\beta^{-1}$  of  $f_\beta(z)$  is simple since the derivative of  $f_\beta(z)$  satisfies  $f'_\beta(\beta^{-1}) = \sum_{j \geq 1} j t_j \beta^{-j+1} > 0$ . The other zeroes of  $f_\beta(z)$  of modulus  $< 1$  lie in  $1/\beta \leq |z| < 1$ . Therefore the poles, if any, of  $U_\beta(z) = P_\beta(z)/f_\beta(z)$  of modulus  $< 1$  all lie in the annular region  $\theta_{\text{dyg}(\beta)-1} < |z| < 1$ .

The formal decomposition (5.2.1), in  $\mathbb{Z}[[X]]$ , is always possible. Indeed, if we put  $U_\beta(X) = -1 + \sum_{j \geq 1} b_j X^j$ , and  $P_\beta(X) = 1 + a_1 X + a_2 X^2 + \dots + a_{d-1} X^{d-1} + X^d$ , (with  $a_j = a_{d-j}$ ), the formal identity  $P_\beta(X) = U_\beta(X) \times f_\beta(X)$  leads to the existence of the coefficient vector  $(b_j)_{j \geq 1}$  of  $U_\beta(X)$ , as a function of  $(t_j)_{j \geq 1}$  and  $(a_i)_{i=1, \dots, d-1}$ , as:  $b_1 = -(a_1 + t_1)$ ,



and, for  $r = 2, \dots, d-1$ ,

$$(5.2.5) \quad b_r = -(t_r + a_r - \sum_{j=1}^{r-1} b_j t_{r-j}) \quad \text{with} \quad b_d = -(t_d + 1 - \sum_{j=1}^{d-1} b_j t_{r-j}),$$

$$(5.2.6) \quad b_r = -t_r + \sum_{j=1}^{r-1} b_j t_{r-j} \quad \text{for } r > d.$$

Then  $b_j \in \mathbb{Z}$ ,  $j \geq 1$ ; the integers  $b_r$ ,  $r > d$ , are determined recursively by (5.2.6) by the sequence  $(t_i)$  and from the finite subset  $\{b_0 = -1, b_1, b_2, \dots, b_d\}$ , itself determined from  $P_\beta(X)$  using (5.2.5). They inherit the asymptotic properties of the asymptotic lacunarity of  $(t_i)$  when  $r$  is very large [209]. If  $R_\beta$  denotes the radius of convergence of  $U_\beta(z)$  the inequality  $R_\beta \geq \theta_{\text{dyg}(\beta)-1}$  can be directly obtained using Hadamard's formula  $R_\beta^{-1} = \limsup_{r \rightarrow \infty} |b_r|^{1/r}$  and the following Lemma 5.6 (in which  $n = \text{dyg}(\beta)$ ) whose proof is immediate.

**Lemma 5.6.** *Let  $\varepsilon > 0$  such that  $\theta_{n-1}^{-1} < \exp(\varepsilon)$ . There exists a constant  $C = C(\varepsilon) \geq \max\{1, \exp(\varepsilon(n-1)) - 1\}$  such that:*

$$(5.2.7) \quad |b_r| \leq C \times \exp(\varepsilon r), \quad \text{for all } r \geq 0.$$

□

**5.3. A lenticulus of zeroes of  $f_\beta(z)$  in the cusp of Solomyak's fractal.** In this subsection  $\beta \in (1, \theta_6^{-1})$  is assumed to be a real number (algebraic or transcendental) such that  $\beta \notin \{\theta_n^{-1} \mid n \geq 7\}$ . In Theorem 5.14 it will be proved that, to such a  $\beta$ , is associated a lenticulus of zeroes of  $f_\beta(z)$  in the cusp of Solomyak's fractal  $\mathcal{G}$  (cf Figure 9 in [73]), located in the angular sector

$$|\arg(z)| < \pi/18.2880.$$

For  $\beta \in \{\theta_n^{-1} \mid n \geq 12\}$  the lenticuli of zeroes of  $f_\beta(z)$  have been investigated in the larger sector  $|\arg(z)| < \pi/3$  (cf Section § 4). Examples of lenticuli can be visualized in [73].

The method which will be used to detect the lenticuli of zeroes of  $f_\beta(z)$  is the method of Rouché. This method will be shown to be powerful enough to reach relevant minorants of the Mahler measure  $M(\beta)$  for  $\beta > 1$  any reciprocal algebraic integer (cf § 5.7).

Let  $n := \text{dyg}(\beta)$ . The algebraic integers  $z_{j,n}$ ,  $1 \leq j < \lfloor n/6 \rfloor$ , which constitute the lenticulus  $\mathcal{L}_{\theta_n^{-1}}$  in the (upper) Poincaré half-plane satisfy (cf §4):

$$f_{\theta_n^{-1}}(\theta_n) = f_{\theta_n^{-1}}(z_{1,n}) = f_{\theta_n^{-1}}(z_{2,n}) = f_{\theta_n^{-1}}(z_{3,n}) = \dots = f_{\theta_n^{-1}}(z_{\lfloor n/6 \rfloor, n}) = 0,$$

with  $f_{\theta_n^{-1}}(z) = -1 + z + z^n$ . The Parry Upper function at  $\beta$  is characterized by the sequence of exponents  $(m_q)_{q \geq 0}$ :

$$(5.3.1) \quad f_\beta(z) = -1 + z + z^n + z^{m_1} + z^{m_2} + z^{m_3} + \dots = G_n(z) + \sum_{q \geq 1} z^{m_q},$$

where  $m_0 := n$ , with the fundamental minimal gappiness condition:

$$(5.3.2) \quad m_{q+1} - m_q \geq n - 1 \quad \text{for all } q \geq 0.$$

The Rényi  $\beta$ -expansion  $d_\beta(1)$  of 1 is infinite or not, namely the sequence of exponents  $(m_q)_{q \geq 0}$  is either infinite or finite: if it is infinite the integers  $m_q$  never take the value  $+\infty$ ;

if not the power series  $f_\beta(z)$  is a polynomial of degree  $m_q$  for some integer  $m_q, q \geq 2$ . In both cases, the integer  $m_1 \geq 2n - 1$  is finite.

We will compute real numbers  $t_{j,n} \in (0, 1)$  such that the small circles  $C_{j,n} := \{z \mid |z - z_{j,n}| = \frac{t_{j,n}}{n}\}$  of respective centers  $z_{j,n}, |z_{j,n}| < 1$ , all satisfy the Rouché conditions:

$$(5.3.3) \quad |f_\beta(z) - G_n(z)| = \left| \sum_{q \geq 1} z^{mq} \right| < |G_n(z)| \quad \text{for } z \in C_{j,n}, \text{ for } j = 1, 2, \dots, J_n,$$

are pairwise disjoint, are small enough to avoid to intersect  $|z| = 1$ , with  $J_n \leq \lfloor \frac{n}{6} \rfloor$  the largest possible integer (in the sense of Definition 5.10 and Proposition 5.11). As a consequence, the number of zeroes of  $f_\beta(z)$  and  $G_n(z)$  in the open disk  $D_{j,n} := \{z \mid |z - z_{j,n}| < \frac{t_{j,n}}{n}\}$  will be equal, implying the existence of a simple zero of the Parry Upper function  $f_\beta(z)$  in each disk  $D_{j,n}$ . The maximality of  $J_n$  means that the conditions of Rouché cannot be satisfied as soon as  $J_n < j \leq \lfloor \frac{n}{6} \rfloor$  for the reason that the circles  $C_{j,n}$  are too close to  $|z| = 1$ .

The values  $t_{j,n}$  are necessarily smaller than  $\pi$  in order to avoid any overlap between two successive circles  $C_{j,n}$  and  $C_{j+1,n}$ . Indeed, since the argument  $\arg z_{j,n}$  of the  $j$ -th root  $z_{j,n}$  is roughly equal to  $2\pi j/n$  (Proposition 4.7), the distance  $|z_{j,n} - z_{j+1,n}|$  is approximately  $2\pi/n$ .

The problem of the choice of the radius  $t_{j,n}/n$  is a true problem. On one hand, a too small radius would lead to make impossible the application of the Rouché conditions, in particular for those disks  $C_{j,n}$  located very near the unit circle. Indeed, we do not know a priori whether the unit circle is a natural boundary or not for  $f_\beta(z)$ ; locating zeroes close to a natural boundary is a difficult problem in general. On the other hand, taking larger values of  $t_{j,n}$  readily leads to a bad localization of the zeroes of  $f_\beta(z)$ , and hence, for algebraic integers  $\beta > 1$ , to a trivial minoration of the Mahler measure  $M(\beta)$ . The sequel reports a compromise, after many trials of the author, which works (cf § 5.7).

For any real number  $\beta \in (1, \theta_6^{-1})$  such that  $\beta \notin \{\theta_n^{-1} \mid n \geq 7\}$  let us denote by  $\omega_{j,n} \in D_{j,n}$  the simple zero of  $f_\beta(z)$ ; then  $|\omega_{j,n}| < 1$  and

$$(5.3.4) \quad |\omega_{j,n}| \leq |z_{j,n}| + \frac{t_{j,n}}{n} \quad \text{with } z_{j,n} \neq \omega_{j,n}, \quad j = 1, 2, \dots, J_n;$$

if, in addition,  $\beta > 1$  is a reciprocal algebraic integer, the strategy for obtaining a minorant of  $M(\beta)$  will be the following: to identify the zeroes  $\omega_{j,n}$  as roots of the minimal polynomial  $P_\beta^*(z) = P_\beta(z)$ , then to obtain a lower bound of the Mahler measure  $M(\beta)$  (will be made explicit in §5.7) from these roots by

$$(5.3.5) \quad \beta \times \prod |\omega_{j,n}|^{-2} \geq \theta_n^{-1} \times \prod_j (|z_{j,n}| + \frac{t_{j,n}}{n})^{-2},$$

where  $j$  runs over  $\{1, 2, \dots, J_n\}$ .

In general, for any real number  $\beta \in (1, \theta_6^{-1})$  such that  $\beta \notin \{\theta_n^{-1} \mid n \geq 7\}$ , the quantities  $t_{j,n}$  will be estimated by the following inequalities:

$$(5.3.6) \quad \frac{|z|^{2(n-1)+1}}{1 - |z|^{n-1}} = \frac{|z|^{2n-1}}{1 - |z|^{n-1}} < |G_n(z)| \quad \text{for } z \in C_{j,n}, \quad j = 1, 2, \dots, J_n$$

instead of (5.3.3), too complicated to handle. In (5.3.6) the exponent “ $n - 1$ ” comes from the minimal gappiness condition (5.3.2), that is from the dynamical degree  $n$  of  $\beta$  itself, as

unique variable. Indeed, due to the great variety of possible infinite admissible sequences  $(m_q)_{q \geq 1}$  in the power series  $f_\beta(z)$  in (5.3.1), for which  $m_q \geq q(n-1) + n$  for all  $q \geq 1$ , we will proceed by taking the upper bound condition (5.3.6) which comes from the general inequality:

$$|f_\beta(z) - G_n(z)| = \left| \sum_{q \geq 1} z^{m_q} \right| \leq \sum_{q \geq 1} |z^{m_q}| \leq \frac{|z|^{2n-1}}{1 - |z|^{n-1}}, \quad |z| < 1.$$

The radius  $t_{0,n}/n$  of the first circle  $C_{0,n} := \{z \mid |z - \theta_n| = \frac{t_{0,n}}{n}\}$ , which contains  $\beta^{-1}$ , is readily obtained without the method of Rouché.

**Lemma 5.7.** *Let  $n \geq 7$ .*

$$t_{0,n} := \left( \frac{\text{Log Log } n}{\text{Log } n} \right)^2.$$

*Proof.* Since  $\beta^{-1}$  runs over the open interval  $(\theta_{n-1}, \theta_n)$ , this interval  $(\theta_{n-1}, \theta_n)$  is necessarily completely included in  $D_{0,n}$ , and the radius of  $C_{0,n}$  is  $\theta_n - \theta_{n-1}$ . We deduce the result from Lemma 5.1. From Proposition 4.6 the root  $z_{1,n}$  admits  $\Im(z_{1,n}) = \frac{2\pi}{n} (1 - \frac{1}{\text{Log } n} + \dots)$  as imaginary part. Then, for any  $t_{1,n} \in (0, 1)$ , the circle  $C_{0,n}$ , of radius  $t_{0,n}/n$ , and  $C_{1,n}$  are disjoint and do not intersect  $|z| = 1$ .  $\square$

By Proposition 4.3 the only angular sector to be considered for the roots  $z_{j,n}$  of  $G_n$  and the Rouché circles  $C_{j,n}$ , up to complex-conjugation, is  $0 \leq \arg(z) \leq +\frac{\pi}{3}$ . In this sector the “bump” angular sector,  $\arg z \in (0, 2\pi(\text{Log } n)/n)$  (cf Appendix; Remark 3.3 in [214]), will be shown to contribute negligibly.

The existence of the roots  $\omega_{j,n}$  in the main subsector is proved in Theorem 5.8, then in Theorem 5.14 in a refined version. Proposition 5.13 completes the proof of their existence in the bump angular sector. In the complement of the family of the adjustable Rouché disks Theorem 5.20 asserts the existence of a zerofree region depending upon the dynamical degree of  $\beta$ .

In the following we consider the problem of the parametrization of the radii  $t_{j,n}/n$  by a unique real number  $a \geq 1$ , allowing to adjust continuously and uniformly the size of each circle  $C_{j,n}$ . We solve it by finding an optimal value.

**Theorem 5.8.** *Let  $n \geq n_1 = 195$ ,  $a \geq 1$ , and  $j \in \{\lceil v_n \rceil, \lceil v_n \rceil + 1, \dots, \lfloor n/6 \rfloor\}$ . Denote by  $C_{j,n} := \{z \mid |z - z_{j,n}| = \frac{t_{j,n}}{n}\}$  the circle centered at the  $j$ -th root  $z_{j,n}$  of  $-1 + X + X^n$ , with  $t_{j,n} = \frac{\pi |z_{j,n}|}{a}$ . Then the condition of Rouché*

$$(5.3.7) \quad \frac{|z|^{2n-1}}{1 - |z|^{n-1}} < |-1 + z + z^n|, \quad \text{for all } z \in C_{j,n},$$

*holds true on the circle  $C_{j,n}$  for which the center  $z_{j,n}$  satisfies*

$$(5.3.8) \quad \frac{|-1 + z_{j,n}|}{|z_{j,n}|} < \frac{1 - \exp\left(\frac{-\pi}{a}\right)}{2 \exp\left(\frac{\pi}{a}\right) - 1}.$$

*The condition  $n \geq 195$  ensures the existence of such roots  $z_{j,n}$ . Taking the value  $a = a_{\max} = 5.87433\dots$  for which the upper bound of (5.3.8) is maximal, equal to  $0.171573\dots$ , the*

roots  $z_{j,n}$  which satisfy (5.3.8) all belong to the angular sector, independent of  $n$ :

$$(5.3.9) \quad \arg(z) \in \left[0, +\frac{\pi}{18.2880}\right].$$

For any real number  $\beta > 1$  having  $\text{dyg}(\beta) = n$ ,  $f_\beta(z)$  admits a simple zero  $\omega_{j,n}$  in  $D_{j,n}$  for which the center  $z_{j,n}$  satisfies (5.3.8) with  $a = a_{\max}$ , and  $j$  in the range  $\{\lceil n/6 \rceil, \lceil n/6 \rceil + 1, \dots, \lfloor n/6 \rfloor\}$ .

*Proof.* Denote by  $\varphi := \arg(z_{j,n})$  the argument of the  $j$ -th root  $z_{j,n}$ . Since  $-1 + z_{j,n} + z_{j,n}^n = 0$ , we have  $|z_{j,n}|^n = |-1 + z_{j,n}|$ . Let us write  $z = z_{j,n} + \frac{1_{j,n}}{n} e^{i\psi} = z_{j,n} \left(1 + \frac{\pi}{an} e^{i(\psi - \varphi)}\right)$  the generic element belonging to  $C_{j,n}$ , with  $\psi \in [0, 2\pi]$ . Let  $X := \cos(\psi - \varphi)$ . Let us show that if the inequality (5.3.7) of Rouché holds true for  $X = +1$ , for a certain circle  $C_{j,n}$ , then it holds true for all  $X \in [-1, +1]$ , that is for every argument  $\psi \in [0, 2\pi]$ , i.e. for every  $z \in C_{j,n}$ . Let us show

$$\left|1 + \frac{\pi}{an} e^{i(\psi - \varphi)}\right|^n = \exp\left(\frac{\pi X}{a}\right) \times \left(1 - \frac{\pi^2}{2a^2 n} (2X^2 - 1) + O\left(\frac{1}{n^2}\right)\right)$$

and

$$\arg\left(\left(1 + \frac{\pi}{an} e^{i(\psi - \varphi)}\right)^n\right) = \text{sgn}(\sin(\psi - \varphi)) \times \left(\frac{\pi \sqrt{1 - X^2}}{a} \left[1 - \frac{\pi X}{an}\right] + O\left(\frac{1}{n^2}\right)\right).$$

Indeed, since  $\sin(\psi - \varphi) = \pm \sqrt{1 - X^2}$ , then

$$\begin{aligned} \left(1 + \frac{\pi}{an} e^{i(\psi - \varphi)}\right)^n &= \exp\left(n \text{Log}\left(1 + \frac{\pi}{an} e^{i(\psi - \varphi)}\right)\right) \\ &= \exp\left(\frac{\pi}{a} (X \pm i\sqrt{1 - X^2}) + \left[-\frac{n}{2} \left(\frac{\pi}{an} (X \pm i\sqrt{1 - X^2})\right)^2 + O\left(\frac{1}{n^2}\right)\right]\right) \\ &= \exp\left(\frac{\pi X}{a} - \frac{\pi^2}{2a^2 n} (2X^2 - 1) + O\left(\frac{1}{n^2}\right)\right) \times \exp\left(\pm i \frac{\pi \sqrt{1 - X^2}}{a} \left[1 - \frac{\pi X}{an}\right] + O\left(\frac{1}{n^2}\right)\right). \end{aligned}$$

Moreover,

$$\left|1 + \frac{\pi}{an} e^{i(\psi - \varphi)}\right| = \left|1 + \frac{\pi}{an} (X \pm i\sqrt{1 - X^2})\right| = 1 + \frac{\pi X}{an} + O\left(\frac{1}{n^2}\right).$$

with

$$\arg\left(1 + \frac{\pi}{an} e^{i(\psi - \varphi)}\right) = \text{sgn}(\sin(\psi - \varphi)) \times \frac{\pi \sqrt{1 - X^2}}{an} + O\left(\frac{1}{n^2}\right).$$

For all  $n \geq 18$  (Proposition 3.5 in [214]), let us recall that

$$(5.3.10) \quad |z_{j,n}| = 1 + \frac{1}{n} \text{Log}\left(2 \sin \frac{\pi j}{n}\right) + \frac{1}{n} O\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2.$$

Then the left-hand side term of (5.3.7) is

$$\frac{|z|^{2n-1}}{1 - |z|^{n-1}} = \frac{|-1 + z_{j,n}|^2 \left|1 + \frac{\pi}{an} e^{i(\psi - \varphi)}\right|^{2n}}{|z_{j,n}| \left|1 + \frac{\pi}{an} e^{i(\psi - \varphi)}\right| - |-1 + z_{j,n}| \left|1 + \frac{\pi}{an} e^{i(\psi - \varphi)}\right|^n}$$

$$(5.3.11) \quad = \frac{|-1 + z_{j,n}|^2 \left(1 - \frac{\pi^2}{an}(2X^2 - 1)\right) \exp\left(\frac{2\pi X}{a}\right)}{\left(1 + \frac{1}{n} \operatorname{Log}\left(2 \sin \frac{\pi j}{n}\right) + \frac{\pi X}{an}\right) - |-1 + z_{j,n}| \left(1 - \frac{\pi^2}{2an}(2X^2 - 1)\right) \exp\left(\frac{\pi X}{a}\right)}$$

up to  $\frac{1}{n} O\left(\frac{\operatorname{Log} \operatorname{Log} n}{\operatorname{Log} n}\right)^2$ -terms (in the terminant). The right-hand side term of (5.3.7) is

$$(5.3.12) \quad \begin{aligned} & \left| -1 + z + z^n \right| = \left| -1 + z_{j,n} \left(1 + \frac{\pi}{na} e^{i(\psi - \varphi)}\right) + z_{j,n}^n \left(1 + \frac{\pi}{na} e^{i(\psi - \varphi)}\right)^n \right| \\ & = \left| -1 + z_{j,n} \left(1 \pm i \frac{\pi \sqrt{1 - X^2}}{an}\right) \left(1 + \frac{\pi X}{an}\right) + \right. \\ & \quad \left. (1 - z_{j,n}) \left(1 - \frac{\pi^2}{2a^2 n} (2X^2 - 1)\right) \exp\left(\frac{\pi X}{a}\right) \exp\left(\pm i \left(\frac{\pi \sqrt{1 - X^2}}{a} \left[1 - \frac{\pi X}{an}\right]\right)\right) + O\left(\frac{1}{n^2}\right) \right| \end{aligned}$$

Let us consider (5.3.11) and (5.3.12) at the first order for the asymptotic expansions, i.e. up to  $O(1/n)$ -terms instead of up to  $O\left(\frac{1}{n}(\operatorname{Log} \operatorname{Log} n / \operatorname{Log} n)^2\right)$ -terms or  $O(1/n^2)$ -terms. (5.3.11) becomes:

$$\frac{|-1 + z_{j,n}|^2 \exp\left(\frac{2\pi X}{a}\right)}{|z_{j,n}| - |-1 + z_{j,n}| \exp\left(\frac{\pi X}{a}\right)}$$

and (5.3.12) is equal to:

$$\left| -1 + z_{j,n} \right| \left| 1 - \exp\left(\frac{\pi X}{a}\right) \exp\left(\pm i \frac{\pi \sqrt{1 - X^2}}{a}\right) \right|$$

and is independent of the sign of  $\sin(\psi - \varphi)$ . Then the inequality (5.3.7) is equivalent to (5.3.13)

$$\frac{|-1 + z_{j,n}|^2 \exp\left(\frac{2\pi X}{a}\right)}{|z_{j,n}| - |-1 + z_{j,n}| \exp\left(\frac{\pi X}{a}\right)} < \left| -1 + z_{j,n} \right| \left| 1 - \exp\left(\frac{\pi X}{a}\right) \exp\left(\pm i \frac{\pi \sqrt{1 - X^2}}{a}\right) \right|,$$

and (5.3.13) to

$$(5.3.14) \quad \frac{|-1 + z_{j,n}|}{|z_{j,n}|} < \frac{\left| 1 - \exp\left(\frac{\pi X}{a}\right) \exp\left(i \frac{\pi \sqrt{1 - X^2}}{a}\right) \right| \exp\left(\frac{-\pi X}{a}\right)}{\exp\left(\frac{\pi X}{a}\right) + \left| 1 - \exp\left(\frac{\pi X}{a}\right) \exp\left(i \frac{\pi \sqrt{1 - X^2}}{a}\right) \right|} =: \kappa(X, a).$$

Denote by  $\kappa(X, a)$  the right-hand side term, as a function of  $(X, a)$ , on  $[-1, +1] \times [1, +\infty)$ . It is routine to show that, for any fixed  $a$ , the partial derivative  $\partial \kappa_X$  of  $\kappa(X, a)$  with respect to  $X$  is strictly negative on the interior of the domain. The function  $x \rightarrow \kappa(x, a)$  takes its minimum at  $X = 1$ , and this minimum is always strictly positive. Hence the inequality of Rouché (5.3.7) will be satisfied on  $C_{j,n}$  once it is satisfied at  $X = 1$ .

For which range of values of  $j/n$ ? Up to  $O(1/n)$ -terms in (5.3.14), it is given by the set of integers  $j$  for which  $z_{j,n}$  satisfies:

$$(5.3.15) \quad \frac{|-1 + z_{j,n}|}{|z_{j,n}|} < \kappa(1, a) = \frac{\left| 1 - \exp\left(\frac{\pi}{a}\right) \right| \exp\left(\frac{-\pi}{a}\right)}{\exp\left(\frac{\pi}{a}\right) + \left| 1 - \exp\left(\frac{\pi}{a}\right) \right|}.$$

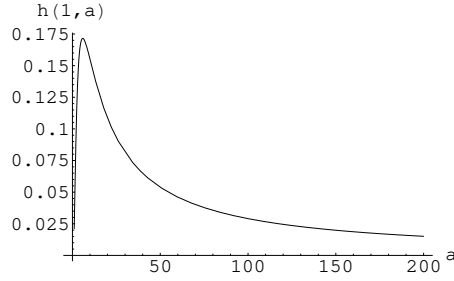


FIGURE 2. Curve of the Rouché condition  $a \rightarrow \kappa(1, a)$  (upper bound in (5.3.8)), for the circles  $C_{j,n} = \{z \mid |z - z_{j,n}| = \pi|z_{j,n}|/(an)\}$  centered at the zeroes  $z_{j,n}$  of the trinomial  $-1 + X + X^n$ , as a function of the size of the circles  $C_{j,n}$  parametrized by the adjustable real number  $a \geq 1$ .

In order to take into account a collection of roots of  $z_{j,n}$  as large as possible, i.e. in order to have a minorant of the Mahler measure  $M(\beta)$  the largest possible, the value of  $a \geq 1$  has to be chosen such that  $a \rightarrow \kappa(1, a)$  is maximal in (5.3.15).

The function  $a \rightarrow \kappa(1, a)$  reaches its maximum  $\kappa(1, a_{\max}) := 0.171573\dots$  at  $a_{\max} = 5.8743\dots$  (Figure 2). Denote by  $J_n$  the maximal integer  $j$  for which (5.3.15) is satisfied and in which  $a$  is taken equal to  $a_{\max}$  (Definition 5.10 and Proposition 5.11). From Proposition 5.11, in which are reported the asymptotic expansions of  $J_n$  and  $\arg(z_{J_n,n})$ , we deduce

$$(5.3.16) \quad \arg(z_{j,n}) < \frac{\pi}{18.2880\dots} = 0.171784\dots \quad \text{for } j = \lceil v_n \rceil, \lceil v_n \rceil + 1, \dots, J_n.$$

**Remark 5.9.** The minimal value  $n_1 = 195$  is calculated by the condition  $2\pi \frac{v_n}{n} < \frac{\pi}{18.2880\dots} = 0.171784\dots$ , for all  $n \geq n_1$ , for having a strict inclusion, of the “bump sector” inside the angular sector defined by the maximal opening angle  $0.171784\dots$  (cf Appendix for the sequence  $(v_n)$ )

This finishes the proof. □

Let us calculate the argument of the last root  $z_{j,n}$  for which (5.3.14) is an equality with  $X = 1$ .

**Definition 5.10.** Let  $n \geq 195$ . Denote by  $J_n$  the largest integer  $j \geq 1$  such that the root  $z_{j,n}$  of  $G_n$  satisfies

$$(5.3.17) \quad \frac{|-1 + z_{j,n}|}{|z_{j,n}|} \leq \kappa(1, a_{\max}) = \frac{1 - \exp\left(\frac{-\pi}{a_{\max}}\right)}{2\exp\left(\frac{\pi}{a_{\max}}\right) - 1} = 0.171573\dots$$

Let us observe that the upper bound  $\kappa(1, a_{\max})$  is independent of  $n$ . From this independence we deduce the following “limit” angular sector in which the Rouché conditions can be applied.

**Proposition 5.11.** Let  $n \geq 195$ . Let us put  $\kappa := \kappa(1, a_{\max})$  for short. Then

$$(5.3.18) \quad \arg(z_{J_n,n}) = 2 \arcsin\left(\frac{\kappa}{2}\right) + \frac{\kappa \text{Log } \kappa}{n\sqrt{4 - \kappa^2}} + \frac{1}{n} O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right),$$

$$(5.3.19) \quad J_n = \frac{n}{\pi} \left( \arcsin\left(\frac{\kappa}{2}\right) \right) + \frac{\kappa \operatorname{Log} \kappa}{\pi \sqrt{4 - \kappa^2}} + O\left(\left(\frac{\operatorname{Log} \operatorname{Log} n}{\operatorname{Log} n}\right)^2\right)$$

with, at the limit,

$$(5.3.20) \quad \lim_{n \rightarrow +\infty} \arg(z_{J_n, n}) = \lim_{n \rightarrow +\infty} 2\pi \frac{J_n}{n} = 2 \arcsin\left(\frac{\kappa}{2}\right) = 0.171784\dots$$

*Proof.* Since  $\lim_{n \rightarrow +\infty} |z_{J_n, n}| = 1$ , we deduce from (5.3.17) that the limit argument  $\varphi_{lim}$  of  $z_{J_n, n}$  satisfies  $|-1 + \cos(\varphi_{lim}) + i \sin(\varphi_{lim})| = 2 \sin(\varphi_{lim}/2) = \kappa(1, a_{\max})$ . We deduce (5.3.20), and the equality between the two limits from (5.3.21).

From (5.3.17), the inequality  $|-1 + z_{j, n}| \leq |z_{j, n}| \kappa(1, a_{\max})$  already implies that  $\arg(z_{J_n, n}) < \varphi_{lim}$ . In the sequel, we will use the asymptotic expansions of the roots  $z_{J_n, n}$ . From Section 6 in [214] the argument of  $z_{J_n, n}$  takes the following form

$$(5.3.21) \quad \arg(z_{J_n, n}) = 2\pi \left( \frac{J_n}{n} + \mathfrak{R} \right) \quad \text{with} \quad \mathfrak{R} = -\frac{1}{2\pi n} \left[ \frac{1 - \cos\left(\frac{2\pi J_n}{n}\right)}{\sin\left(\frac{2\pi J_n}{n}\right)} \operatorname{Log} \left( 2 \sin\left(\frac{\pi J_n}{n}\right) \right) \right]$$

with

$$\operatorname{tl}(\arg(z_{J_n, n})) = +\frac{1}{n} O\left(\left(\frac{\operatorname{Log} \operatorname{Log} n}{\operatorname{Log} n}\right)^2\right).$$

Its modulus is

$$(5.3.22) \quad |z_{J_n, n}| = 1 + \frac{1}{n} \operatorname{Log} \left( 2 \sin \frac{\pi J_n}{n} \right) + \frac{1}{n} O\left(\frac{\operatorname{Log} \operatorname{Log} n}{\operatorname{Log} n}\right)^2.$$

Denote  $\varphi := \arg(z_{J_n, n})$ . Up to  $\frac{1}{n} O\left(\left(\frac{\operatorname{Log} \operatorname{Log} n}{\operatorname{Log} n}\right)^2\right)$ -terms, we have

$$\begin{aligned} | -1 + z_{J_n, n} |^2 &= \left| -1 + \left[ 1 + \frac{1}{n} \operatorname{Log} \left( 2 \sin \frac{\pi J_n}{n} \right) \right] (\cos(\varphi) + i \sin(\varphi)) \right|^2 \\ &= \left[ -1 + \left[ 1 + \frac{1}{n} \operatorname{Log} \left( 2 \sin \frac{\pi J_n}{n} \right) \right] \cos(\varphi) \right]^2 + \left[ 1 + \frac{1}{n} \operatorname{Log} \left( 2 \sin \frac{\pi J_n}{n} \right) \right]^2 \sin^2(\varphi) \\ &= 1 + \left[ 1 + \frac{1}{n} \operatorname{Log} \left( 2 \sin \frac{\pi J_n}{n} \right) \right]^2 - 2 \left[ 1 + \frac{1}{n} \operatorname{Log} \left( 2 \sin \frac{\pi J_n}{n} \right) \right] \cos(\varphi) \\ (5.3.23) \quad &= 4 \left( \sin\left(\frac{\varphi}{2}\right) \right)^2 + \frac{4}{n} \left( \sin\left(\frac{\varphi}{2}\right) \right)^2 \operatorname{Log} \left( 2 \sin \frac{\pi J_n}{n} \right) = 4 \left( \sin\left(\frac{\varphi}{2}\right) \right)^2 \left[ 1 + \frac{1}{n} \operatorname{Log} \left( 2 \sin \frac{\pi J_n}{n} \right) \right]. \end{aligned}$$

Up to  $\frac{1}{n} O\left(\left(\frac{\operatorname{Log} \operatorname{Log} n}{\operatorname{Log} n}\right)^2\right)$ -terms, due to the definition of  $J_n$ , let us consider (5.3.17) as an equality; hence, from (5.3.23) and (5.3.22), the following identity should be satisfied

$$(5.3.24) \quad 2 \sin\left(\frac{\varphi}{2}\right) = \kappa \left[ 1 + \frac{1}{2n} \operatorname{Log} \left( 2 \sin \frac{\pi J_n}{n} \right) \right]$$

We now use (5.3.24) to obtain an asymptotic expansion of  $\psi_n := 2\pi \frac{J_n}{n} - \varphi_{lim}$  as a function of  $n$  and  $\varphi_{lim}$  up to  $\frac{1}{n} O\left(\left(\frac{\operatorname{Log} \operatorname{Log} n}{\operatorname{Log} n}\right)^2\right)$ -terms. First, at the first order in  $\psi_n$ ,

$$\begin{aligned} \sin\left(\frac{\pi J_n}{n}\right) &= \frac{\psi_n}{2} \cos\left(\frac{\varphi_{lim}}{2}\right) + \sin\left(\frac{\varphi_{lim}}{2}\right), \quad \cos\left(\frac{\pi J_n}{n}\right) = -\frac{\psi_n}{2} \sin\left(\frac{\varphi_{lim}}{2}\right) + \cos\left(\frac{\varphi_{lim}}{2}\right), \\ \operatorname{Log} \left( 2 \sin\left(\frac{\pi J_n}{n}\right) \right) &= \operatorname{Log} \left( 2 \sin\left(\frac{\varphi_{lim}}{2}\right) \right) + \psi_n \frac{\cos\left(\frac{\varphi_{lim}}{2}\right)}{2 \sin\left(\frac{\varphi_{lim}}{2}\right)} = \operatorname{Log} \kappa + \psi_n \frac{\cos\left(\frac{\varphi_{lim}}{2}\right)}{h}. \end{aligned}$$

Moreover,

$$(5.3.25) \quad \left[ \frac{1 - \cos(\frac{2\pi J_n}{n})}{\sin(\frac{2\pi J_n}{n})} \text{Log} (2 \sin(\frac{\pi J_n}{n})) \right] \\ = \tan(\frac{\varphi_{lim}}{2}) (\text{Log } \kappa) \left[ 1 + \psi_n \left( \frac{1}{\sin(\varphi_{lim})} + \frac{\cos(\frac{\varphi_{lim}}{2})}{\kappa \text{Log } \kappa} \right) \right].$$

Hence, with  $2 \sin(\varphi/2) = 2 \sin(\pi J_n/n) \cos(\pi \Re) + 2 \cos(\pi J_n/n) \sin(\pi \Re)$ , and from (5.3.21), up to  $\frac{1}{n} O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right)$ -terms, the identity (5.3.24) becomes

$$\left[ \psi_n \cos\left(\frac{\varphi_{lim}}{2}\right) + 2 \sin\left(\frac{\varphi_{lim}}{2}\right) \right] + \left( \frac{-2 \cos\left(\frac{\varphi_{lim}}{2}\right) \tan\left(\frac{\varphi_{lim}}{2}\right) \text{Log } \kappa}{2n} \right) = \kappa \left[ 1 + \frac{\text{Log } \kappa}{2n} \right].$$

We deduce

$$(5.3.26) \quad \psi_n = \frac{\kappa \text{Log } \kappa}{n \cos\left(\frac{\varphi_{lim}}{2}\right)} + \frac{1}{n} O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right),$$

then  $2\pi J_n/n = \psi_n + \varphi_{lim}$  and (5.3.19). With  $2\pi J_n/n$ , and from (5.3.21) and (5.3.25) we deduce (5.3.18). This finishes the proof.  $\square$

**Remark 5.12.** (i) The maximal half-opening angle of the sector in which one can detect zeroes of  $f_\beta(z)$ , for any  $\beta$  such that  $\theta_{n-1} < \beta^{-1} < \theta_n$ , by the method of Rouché, is  $0.17178\dots = 2 \arcsin\left(\frac{\kappa(1, a_{max})}{2}\right)$ . Remarkably this upper bound  $2 \arcsin\left(\frac{\kappa(1, a_{max})}{2}\right)$  is independent of  $n$ . By comparison it is fairly small with respect to  $\pi/3$  for the Perron numbers  $\theta_n^{-1}$ .

(ii) The curve  $a \rightarrow \kappa(1, a)$ , given by Figure 2, is such that any value in the interval  $(0, \kappa(1, a_{max}))$  is reached by the function  $\kappa(1, a)$  from two values say  $a_1$  and  $a_2$ , of  $a$ , satisfying  $a_1 < a_{max} < a_2$ . On the contrary, the correspondence  $a_{max} \leftrightarrow \kappa(1, a_{max})$  is unique, corresponding to a double root. Denote  $D := \exp(\pi/a_{max})$  and  $\kappa := \kappa(1, a_{max})$ . It means that the quadratic algebraic equation  $2\kappa D^2 - (\kappa + 1)D + 1 = 0$  deduced from the upper bound in (5.3.17) has necessarily a discriminant equal to zero. The discriminant is  $\kappa^2 - 6\kappa + 1$ . Therefore  $D = (\kappa + 1)/(4\kappa)$  and the limit value  $x = 2 \arcsin(\kappa/2)$  in (5.3.20) satisfies the quadratic algebraic equation

$$4(\sin(x/2))^2 - 12 \sin(x/2) + 1 = 0.$$

**Proposition 5.13.** *Let  $n \geq n_1 = 195$ . The circles  $C_{j,n} := \{z \mid |z - z_{j,n}| = \frac{\pi |z_{j,n}|}{n a_{max}}\}$  centered at the roots  $z_{j,n}$  of the trinomial  $-1 + z + z^n$  which belong to the “bump sector”, namely for  $j \in \{1, 2, \dots, \lfloor v_n \rfloor\}$ , are such that the conditions of Rouché*

$$(5.3.27) \quad \frac{|z|^{2n-1}}{1 - |z|^{n-1}} < |-1 + z + z^n|, \quad \text{for all } z \in C_{j,n}, \quad 1 \leq j \leq \lfloor v_n \rfloor,$$

*hold true. For any real number  $\beta > 1$  having  $\text{dyg}(\beta) = n$ ,  $f_\beta(z)$  admits a simple zero  $\omega_{j,n}$  in  $D_{j,n}$  (with  $a = a_{max}$ ), for  $j$  in the range  $\{1, 2, \dots, \lfloor v_n \rfloor\}$ .*

*Proof.* The development terms “ $D$ ” of the asymptotic expansions of  $|z_{j,n}|$  change from the main angular sector  $\arg z \in (2\pi(\text{Log } n)/n, \pi/3)$  to the first transition region  $\arg z \asymp$



$2\pi(\text{Log } n)/n$ , the “bump sector”, further to the second transition region  $\arg z \asymp 2\pi\sqrt{(\text{Log } n)(\text{Log } \text{Log } n)}/n$ , and to a small neighbourhood of  $\theta_n$  (Section 4.2).

Then the proof of (5.3.27) is the same as that of Theorem 5.8 once (5.3.10) is substituted by the suitable asymptotic expansions which correspond to the angular sector of the “bump”. The terminants of the respective asymptotic expansions of  $|z_{j,n}|$  also change: this change imposes to reconsider (5.3.11) and (5.3.12) up to  $\text{Log } n/n$  - terms, and not up to  $1/n$  - terms, as in the proof of Theorem 5.8. It is remarkable that the inequality (5.3.14) remains the same, with the same upper bound function  $\kappa(X, a)$ . Then the equation of the curve of the Rouché condition  $a \rightarrow \kappa(1, a)$ , on  $[1, +\infty)$ , is the same as in Theorem 5.8 for controlling the conditions of Rouché. The optimal value  $a_{\max}$  of  $a$  also remains the same, and (5.3.7) also holds true for those  $z_{j,n}$  in the bump sector.  $\square$

From the inequalities (5.3.8) in Theorem 5.8, also used in the proof of Proposition 5.13, we now obtain a finer localization of a subcollection of the roots  $\omega_{j,n}$  of the Parry Upper function  $f_\beta(z)$ , and a definition of the lenticulus  $\mathcal{L}_\beta$  of  $\beta$ , as follows.

**Theorem 5.14.** *Let  $n \geq n_1 = 195$ . Let  $\beta > 1$  be any real number having  $\text{dyg}(\beta) = n$ . The Parry Upper function  $f_\beta(z)$  has an unique simple zero  $\omega_{j,n}$  in each disk  $D_{j,n} := \{z \mid |z - z_{j,n}| < \frac{\pi|z_{j,n}|}{na_{\max}}\}$ ,  $j = 1, 2, \dots, J_n$ , which satisfies the additional inequality:*

$$(5.3.28) \quad |\omega_{j,n} - z_{j,n}| < \frac{\pi|z_{j,n}|}{na_{j,n}} \quad \text{for } j = \lceil v_n \rceil, \lceil v_n \rceil + 1, \dots, J_n,$$

where  $a_{J_n, n} = a_{\max}$  and, for  $j = \lceil v_n \rceil, \dots, J_n - 1$ , the value  $a_{j,n} > a_{\max}$ , is defined by

$$(5.3.29) \quad D\left(\frac{\pi}{a_{j,n}}\right) = \text{Log} \left[ \frac{1 + B_{j,n} - \sqrt{1 - 6B_{j,n} + B_{j,n}^2}}{4B_{j,n}} \right]$$

with  $B_{j,n} := 2 \sin\left(\frac{\pi j}{n}\right) \left(1 - \frac{1}{n} \text{Log} \left(2 \sin\left(\frac{\pi j}{n}\right)\right)\right),$

and, putting  $D := D\left(\frac{\pi}{a_{j,n}}\right)$  for short,

$$(5.3.30) \quad \text{tl}\left(\frac{\pi}{a_{j,n}}\right) = \frac{2}{n} \times B_{j,n}^{-1} \left( \frac{-3 + \exp(-D) + 2 \exp(D)}{4 - \exp(-D) - 2 \exp(D)} \right) \times \left( \frac{\text{Log } \text{Log } n}{\text{Log } n} \right)^2.$$

An upper bound of the tails, independent of  $j$ , is given by

$$(5.3.31) \quad O\left(\frac{(\text{Log } \text{Log } n)^2}{(\text{Log } n)^3}\right)$$

with the constant  $\frac{1}{7\pi}$  in the Big  $O$ . The lenticulus  $\mathcal{L}_\beta$  associated with  $\beta$  is constituted by the following subset of roots of  $f_\beta(z)$ :

$$(5.3.32) \quad \mathcal{L}_\beta := \{1/\beta\} \cup \bigcup_{j=1}^{J_n} (\{\omega_{j,n}\} \cup \{\overline{\omega_{j,n}}\}).$$

*Proof.* The existence of the zeroes comes from Proposition 5.13 and Theorem 5.8, with the maximal value  $J_n$  of the index  $j$  given by Proposition 5.10. To refine the localization of  $\omega_{j,n}$  in the neighbourhood of  $z_{j,n}$ , in the main angular sector, i.e. for  $j \in \{\lceil v_n \rceil, \lceil v_n \rceil + 1, \dots, J_n\}$ , the conditions of Rouché (5.3.7) are now used to define the new radii.

The value  $a_{j,n}$  is defined by the development term  $D(\frac{\pi}{a_{j,n}})$ , itself defined as follows:

$$(5.3.33) \quad D\left(\frac{|-1+z_{j,n}|}{|z_{j,n}|}\right) =: \frac{1 - \exp(-D(\frac{\pi}{a_{j,n}}))}{2 \exp(D(\frac{\pi}{a_{j,n}})) - 1}$$

and the tail  $tl(\frac{\pi}{a_{j,n}})$  calculated from  $tl\left(\frac{|-1+z_{j,n}|}{|z_{j,n}|}\right)$  so that the Rouché condition

$$(5.3.34) \quad \frac{|-1+z_{j,n}|}{|z_{j,n}|} = D\left(\frac{|-1+z_{j,n}|}{|z_{j,n}|}\right) + tl\left(\frac{|-1+z_{j,n}|}{|z_{j,n}|}\right) < \frac{1 - \exp(-\frac{\pi}{a_{j,n}})}{2 \exp(\frac{\pi}{a_{j,n}}) - 1}$$

holds true. From Proposition 4.11, denote

$$B_{j,n} := D\left(\frac{|-1+z_{j,n}|}{|z_{j,n}|}\right) = 2 \sin\left(\frac{\pi j}{n}\right) \left(1 - \frac{1}{n} \text{Log}\left(2 \sin\left(\frac{\pi j}{n}\right)\right)\right).$$

Let  $W := \exp(D(\frac{\pi}{a_{j,n}}))$ . The identity (5.3.33) transforms into the equation of degree 2:

$$(5.3.35) \quad 2B_{j,n}W^2 - (B_{j,n} + 1)W + 1 = 0$$

from which (5.3.36) is deduced. For the calculation of  $tl(\frac{\pi}{a_{j,n}})$ , denote  $D := D(\frac{\pi}{a_{j,n}})$  and

$$tl_{j,n} := tl\left(\frac{\pi}{a_{j,n}}\right). \text{ Then, at the first order, } \frac{1 - \exp(-\frac{\pi}{a_{j,n}})}{2 \exp(\frac{\pi}{a_{j,n}}) - 1}$$

$$= \frac{1 - \exp(-D - tl_{j,n})}{2 \exp(D + tl_{j,n}) - 1} = B_{j,n} \left[1 + tl_{j,n} \times \left(\frac{4 - \exp(-D) - 2 \exp(D)}{-3 + \exp(-D) + 2 \exp(D)}\right)\right].$$

From (5.3.34) and (4.2.10) the following inequality should be satisfied, with the constant 2 in the Big O,

$$\frac{1}{n} O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right) = tl\left(\frac{|-1+z_{j,n}|}{|z_{j,n}|}\right) < tl_{j,n} \times B_{j,n} \left(\frac{4 - \exp(-D) - 2 \exp(D)}{-3 + \exp(-D) + 2 \exp(D)}\right).$$

The expression of  $tl_{j,n}$  in (5.3.30) follows, to obtain a strict inequality in (5.3.34). By (5.3.29) the quantity  $\exp(D)$  is a function of  $B_{j,n}$ , which tends to  $\frac{3}{4}$  when  $B_{j,n}$  tends to 0; hence, at the first order, a lower bound of the function  $B_{j,n} \rightarrow |B_{j,n} \left(\frac{4 - \exp(-D) - 2 \exp(D)}{-3 + \exp(-D) + 2 \exp(D)}\right)|$  is obtained for  $j = \lceil v_n \rceil$ , and given by  $2\pi \frac{\text{Log } n}{n} \times 7$ . Then it suffices to take

$$tl_{j,n} = cste \left(\frac{(\text{Log Log } n)^2}{(\text{Log } n)^3}\right)$$

with  $cste = 1/(7\pi)$ , to obtain a tail independent of  $j$ , and therefore the conditions of Rouché (5.3.34) satisfied with these new smaller radii and tails in the main angular sector.  $\square$

**Remark 5.15.** For  $n$  very large, up to second-order terms, (5.3.35) reduces to

$$4 \sin\left(\frac{\pi j}{n}\right) W^2 - \left(2 \sin\left(\frac{\pi j}{n}\right) + 1\right) W + 1 = 0$$

and (5.3.29) to

$$(5.3.36) \quad D\left(\frac{\pi}{a_{j,n}}\right) = \text{Log} \left[ \frac{1 + 2 \sin\left(\frac{\pi j}{n}\right) - \sqrt{1 - 12 \sin\left(\frac{\pi j}{n}\right) + 4\left(\sin\left(\frac{\pi j}{n}\right)\right)^2}}{8 \sin\left(\frac{\pi j}{n}\right)} \right].$$

**Lemma 5.16.** *Let  $n \geq 195$  and  $c_n$  defined by  $|z_{J_n,n}| = 1 - \frac{c_n}{n}$ . Let us put  $\kappa := \kappa(1, a_{\max})$  for short. Then*

$$(5.3.37) \quad c_n = -(\text{Log } \kappa) \left(1 + \frac{1}{n}\right) + \frac{1}{n} O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right),$$

with  $c = \lim_{n \rightarrow +\infty} c_n = -\text{Log } \kappa = 1.76274\dots$ , and, up to  $O\left(\frac{1}{n}\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right)\right)$ -terms,

$$(5.3.38) \quad \frac{\left(1 - \frac{c_n}{n}\right)^{2n}}{\left(1 - \frac{c_n}{n}\right) - \left(1 - \frac{c_n}{n}\right)^n} = \frac{e^{-2c}}{1 - e^{-c}} \left(1 + \frac{c}{2n(1 - e^{-c})} [2 - ce^{-c} - 2c]\right)$$

with  $e^{-2c}/(1 - e^{-c}) = 0.0355344\dots$

*Proof.* The asymptotic expansion (5.3.37) of  $c_n$  is deduced from the asymptotic expansions of  $\psi_n$  and  $z_{J_n,n}$  given by (5.3.26) and (5.3.22) (Proposition 3.5 in [214]). We deduce the limit  $c := -\text{Log}(\kappa(1, a_{\max})) = 1.76274\dots$  and then (5.3.38) follows.  $\square$

**Definition 5.17.** Let  $n \geq n_2 := 260$ . We denote by  $H_n$  the largest integer  $j \geq \lceil v_n \rceil$  such that

$$(5.3.39) \quad \arg(z_{J_n,n}) - \arg(z_{j,n}) \geq \frac{\left(1 - \frac{c_n}{n}\right)^{2n}}{\left(1 - \frac{c_n}{n}\right) - \left(1 - \frac{c_n}{n}\right)^n}.$$

**Proposition 5.18.** *Let  $n \geq 260$ . Let denote  $\kappa := \kappa(1, a_{\max})$  for short. Then*

$$(5.3.40) \quad \arg(z_{H_n,n}) = 2 \arcsin\left(\frac{\kappa}{2}\right) - \frac{\kappa^2}{1 - \kappa} + \frac{\text{Log } \kappa}{n} \left[ \frac{\kappa}{\sqrt{4 - \kappa^2}} + \frac{2 + \kappa \text{Log}(\kappa) + 2 \text{Log}(\kappa)}{2(1 - \kappa)} \right] + \frac{1}{n} O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right),$$

with, at the limit,

$$\lim_{n \rightarrow +\infty} \arg(z_{H_n,n}) = 2 \arcsin\left(\frac{\kappa}{2}\right) - \frac{\kappa^2}{1 - \kappa} = 0.13625.$$

*Proof.* The asymptotic expansion of the right-hand side term of (5.3.39) is

$$(5.3.41) \quad \frac{\left(1 - \frac{c_n}{n}\right)^{2n}}{\left(1 - \frac{c_n}{n}\right) - \left(1 - \frac{c_n}{n}\right)^n} = \frac{e^{-2c}}{1 - e^{-c}} \left(1 + \frac{c(2 - ce^{-c} - 2c)}{2n(1 - e^{-c})}\right) + \dots$$

Then the asymptotic expansion of  $\arg(z_{H_n,n})$  comes from (5.3.39) in which the inequality is replaced by an equality, and from the asymptotic expansion (5.3.18) of  $\arg(z_{J_n,n})$  (Proposition 5.11).  $\square$

For  $n$  large enough,  $\arg(z_{H_n,n})$  is equal to  $2\pi \frac{H_n}{n}$ , up to higher order - terms, and a definition of  $H_n$  in terms of asymptotic expansions could be:

$$(5.3.42) \quad H_n = \lfloor \frac{n}{2\pi} \left(2 \arcsin\left(\frac{\kappa}{2}\right) - \frac{\kappa^2}{1 - \kappa}\right) - \text{Log}(\kappa) \left[ \frac{\kappa}{\sqrt{4 - \kappa^2}} + \frac{2 + \kappa \text{Log}(\kappa) + 2 \text{Log}(\kappa)}{2(1 - \kappa)} \right] \rfloor,$$

For simplicity's sake, we will take the following definition of  $H_n$

$$(5.3.43) \quad H_n := \lfloor \frac{n}{2\pi} \left( 2 \arcsin\left(\frac{\kappa}{2}\right) - \frac{\kappa^2}{1-\kappa} \right) - 1 \rfloor.$$

**Remark 5.19.** The value  $n_2 = 260$  is calculated by the inequality  $\frac{2\pi v_n}{n} < \arg(z_{H_n, n})$  which should be valid for all  $n \geq 260$ , where  $H_n$  is given by (5.3.43),  $\arg(z_{H_n, n})$  by (5.3.40), where  $(v_n)$  is the delimiting sequence (cf Appendix) of the transition region of the boundary of the bump sector. A first minimal value of  $n$  is first estimated by  $2\pi \frac{\text{Log} n}{n} < D(\arg(z_{H_n, n}))$  using (5.3.40). Then it is corrected so that the numerical value of the tail of the asymptotic expansion in (5.3.40) be taken into account in this inequality.

**Theorem 5.20.** *Let  $n \geq n_2 := 260$ . Denote by  $\mathcal{D}_n$  the subdomain of the open unit disk, symmetrical with respect to the real axis, defined by the conditions:*

$$(5.3.44) \quad |z| < 1 - \frac{c_n}{n}, \quad \frac{1}{n} \left( \frac{\text{Log} \text{Log} n}{\text{Log} n} \right)^2 < |z - \theta_n|,$$

$$(5.3.45) \quad \frac{\pi |z_{j,n}|}{n a_{\max}} < |z - z_{j,n}|, \quad \text{for } j = 1, 2, \dots, J_n,$$

and, for  $j = J_n + 1, \dots, 2J_n - H_n + 1$ ,

$$(5.3.46) \quad \frac{\pi |z_{j,n}|}{n s_{j,n}} < |z - z_{j,n}|, \quad \text{with } s_{j,n} = a_{\max} \left[ 1 + \frac{a_{\max}^2 (j - J_n)^2}{\pi^2 J_n^2} \right]^{-1/2}.$$

Then, for any real number  $\beta > 1$  having  $\text{dyg}(\beta) = n$ , the Parry Upper function  $f_\beta(z)$  does not vanish at any point  $z$  in  $\mathcal{D}_n$ .

*Proof.* Assume  $\beta > 1$  such that  $\theta_{n-1} < \beta^{-1} < \theta_n$ . We will apply the general form of the Theorem of Rouché to the compact  $\mathcal{K}_n$  which is the adherence of the domain  $\mathcal{D}_n$ , i.e. we will show that the inequality (and symmetrically with respect to the real axis)

$$(5.3.47) \quad |f_\beta(z) - G_n(z)| < |G_n(z)|, \quad z \in \partial \mathcal{K}_n^{\text{ext}} \cup C_{1,n} \cup C_{2,n} \cup \dots \cup C_{J_n,n}$$

holds, with  $z \in \text{Im}(z) \geq 0$ , where  $\partial \mathcal{K}_n$  is the union of: (i) the arcs of the circles defined by the equalities in (5.3.45) and (5.3.46), arcs which lie in  $|z| \leq 1 - c_n/n$ , and circles for which the intersection with  $|z| = 1 - c_n/n$  is not empty, (ii) the arcs of  $C(0, 1 - c_n/n)$  which have empty intersections with the interiors of the disks defined by the inequalities “>”, instead of “<”, in (5.3.45) and (5.3.46), which join two successive circles. The two functions  $f_\beta(z)$  and  $G_n(z)$  are continuous on the compact  $\mathcal{K}_n$ , holomorphic in its interior  $\mathcal{D}_n$ , and  $G_n$  has no zero in  $\mathcal{K}_n$ . As a consequence the function  $f_\beta(z)$  will have no zero in the interior  $\mathcal{D}_n$  of  $\mathcal{K}_n$ .

Instead of using  $f_\beta(z)$  itself in (5.3.47), we will show that the following inequality holds true

$$(5.3.48) \quad \frac{|z|^{2n-1}}{1 - |z|^{n-1}} < |-1 + z + z^n|, \quad \text{for all } z \in \partial \mathcal{K}_n^{\text{ext}}$$

what will imply the claim.

The Rouché inequalities (5.3.47) (5.3.48) hold true on the (complete) circles  $C_{j,n}$ ,  $1 \leq j \leq J_n$  by Theorem 5.8 and Proposition 5.13; these conditions become out of reach for  $j$

taking higher values (i.e. in  $\{J_n + 1, \dots, \lfloor n/6 \rfloor\}$ ), but we will show that they remain true on the arcs defined by the equalities in (5.3.46). The domain  $\mathcal{D}_n$  only depends upon the dynamical degree  $n$  of  $\beta$ , not of  $\beta$  itself.

Let us prove that the external Rouché circle  $|z| = 1 - c_n/n$  intersects all the circles  $C_{J_n-k,n}, k = 0, 1, \dots, k_{\max}$ , with  $k_{\max} := \lfloor J_n(\frac{\pi}{a_{\max}}) \rfloor$ . Indeed, up to  $\frac{1}{n}O((\frac{\text{Log Log } n}{\text{Log } n})^2)$ - terms, from Proposition 5.11,

$$\begin{aligned} \text{Log}(2 \sin(\pi \frac{J_n}{n})) &= \text{Log}(2 \sin(\pi \frac{(J_n - k) + k}{n})) = \text{Log}(2\pi \frac{J_n - k}{n} (1 + \frac{k}{J_n - k})) \\ (5.3.49) \qquad \qquad \qquad &= \text{Log}(2 \sin(\pi \frac{J_n - k}{n})) + \frac{k}{J_n}. \end{aligned}$$

Since  $|z_{J_n,n}| = 1 - c_n/n = 1 + \frac{1}{n} \text{Log}(2 \sin(\pi \frac{J_n}{n})) + \frac{1}{n} O((\frac{\text{Log Log } n}{\text{Log } n})^2)$ , we deduce from (5.3.49), with  $k \leq k_{\max}$ , that the point  $z \in C(0, 1 - c_n/n)$  for which  $\arg(z) = \arg(z_{J_n-k,n})$  is such that

$$|z_{J_n-k,n} - z| = \frac{k}{nJ_n} \leq \frac{\lfloor J_n(\frac{\pi}{a_{\max}}) \rfloor}{nJ_n} \leq \frac{\pi}{na_{\max}}$$

up to  $\frac{1}{n}O((\frac{\text{Log Log } n}{\text{Log } n})^2)$ - terms. As soon as  $n$  is large enough, we deduce that  $z$  lies in the interior of  $D_{J_n-k,n}$ . Since the function  $x \rightarrow \text{Log}(2 \sin(\pi x))$  is negative and strictly increasing on  $(0, 1/6)$ , the sequence  $(|z_{j,n}|)_{j=H_n, \dots, J_n}$  is strictly increasing, by (5.3.22). Hence we deduce that the circle  $|z| = 1 - c_n/n$  intersects all the circles  $C_{j,n}$  for  $j = J_n - k_{\max}, \dots, J_n$ .

The same arguments show that the external Rouché circle  $|z| = 1 - c_n/n$  intersects all the circles  $C(z_{j,n}, \frac{\pi|z_{j,n}|}{ns_{j,n}})$  for  $j = J_n + 1, J_n + 2, \dots, 2J_n - H_n + 1$ .

The quantities  $s_{j,n}$ , for  $j = J_n + 1, \dots, 2J_n - H_n + 1$ , are easily calculated (left to the reader) so that the distance (length of the  $j$ -th circle segment)

$$\left| \frac{z_{j,n}}{|z_{j,n}|} (1 - \frac{c_n}{n}) - y_j \right| = \left| \frac{z_{j,n}}{|z_{j,n}|} (1 - \frac{c_n}{n}) - y'_j \right|$$

for  $y_j, y'_j \in C(z_{j,n}, \frac{\pi|z_{j,n}|}{ns_{j,n}}) \cap C(0, 1 - \frac{c_n}{n}), y_j \neq y'_j$ , be independent of  $j$  in the interval  $\{J_n + 1, \dots, 2J_n - H_n + 1\}$  and equal to

$$(5.3.50) \qquad \qquad \qquad \frac{\pi|z_{J_n,n}|}{na_{\max}}.$$

Then the two sequences of moduli of centers  $(|z_{j,n}|)_{j=J_n+1, \dots, 2J_n-H_n+1}$  and of radii  $(\frac{\pi|z_{j,n}|}{ns_{j,n}})_{j=J_n+1, \dots, 2J_n-H_n+1}$  are both increasing, with the fact that the corresponding disks  $D(z_{j,n}, \frac{\pi|z_{j,n}|}{ns_{j,n}})$  keep constant the intersection chord  $\arg(y_j) - \arg(y'_j) = \frac{\pi|z_{J_n,n}|}{na_{\max}}$  with the external Rouché circle  $|z| = 1 - c_n/n$ .

Let  $z \in C(0, 1 - \frac{c_n}{n}), \varphi := \arg(z) \in [0, \pi]$ . Denote by  $Z(\varphi) := |G_n((1 - \frac{c_n}{n})e^{i\varphi})|^2 = |-1 + (1 - \frac{c_n}{n})e^{i\varphi} + (1 - \frac{c_n}{n})^n e^{in\varphi}|^2$ . The expansion of the function  $Z(\varphi)$  as a function of  $\varphi$ , up to  $O(1/n)$ - terms, is the following:  $Z(\varphi) =$

$$\begin{aligned} &(-1 + (1 - \frac{c_n}{n}) \cos(\varphi) + (1 - \frac{c_n}{n})^n \cos(n\varphi))^2 + ((1 - \frac{c_n}{n}) \sin(\varphi) + (1 - \frac{c_n}{n})^n \sin(n\varphi))^2 \\ &= 2 + e^{-2c} - 2 \cos(\varphi) - 2e^{-c} \cos(n\varphi) + 2e^{-c} \cos(\varphi) \cos(n\varphi) + 2e^{-c} \sin(\varphi) \sin(n\varphi) \end{aligned}$$

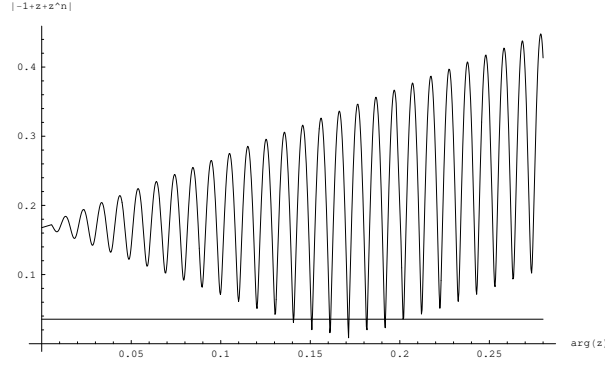


FIGURE 3. Oscillations of the upper bound  $| -1 + z + z^n |$  of the Rouché inequality (5.3.7), for  $z$  running over the curve  $|z| = 1 - c_n/n$  (here represented with  $n = 615$ ) as a function of  $\arg(z)$  in  $[0, 0.28]$ . The minima correspond to the angular positions of the zeroes  $z_{j,n}$  of the trinomial  $-1 + X + X^n$ , for  $j = 1, 2, \dots, H_n, \dots, J_n, \dots, 2J_n - H_n + 1, \dots$  ( $J_{615} = 17, H_{615} = 12$ ). The angular separation between two successive minima is  $\approx 2\pi/n$ . The difference between two successive minima is  $\approx 2\pi/n$ . For  $n = 615$ , the arguments  $2\pi(\text{Log } n)/n$  (limiting the bump sector),  $\arg(z_{H_n,n})$  and  $\arg(z_{J_n,n})$  are respectively equal to  $0.0656\dots, 0.12189\dots, 0.17129\dots$ . The horizontal line at the  $y$ -coordinate  $0.0354\dots$  is the value of the left-hand side term of the Rouché inequality (5.3.7) (Proposition 5.3.47); it is always strictly smaller than the minimal value of the oscillating function  $| -1 + z + z^n |$  on the external boundary  $\partial \mathcal{K}_n^{\text{ext}}$ , whose geometry surrounds the roots  $z_{j,n}$  for  $j$  between  $H_n + 1$  and  $2J_n - H_n + 1$ .

$$\begin{aligned}
 &= 2 + e^{-2c} - 2 \cos(\varphi) - 4e^{-c} \sin\left(\frac{\varphi}{2}\right) \left( \cos(n\varphi) \sin\left(\frac{\varphi}{2}\right) - \sin(n\varphi) \cos\left(\frac{\varphi}{2}\right) \right) \\
 (5.3.51) \quad &= 2 + e^{-2c} - 2 \cos(\varphi) + 4e^{-c} \sin\left(\frac{\varphi}{2}\right) \sin\left(n\varphi - \frac{\varphi}{2}\right).
 \end{aligned}$$

The function  $Z(\varphi)$ , defined on  $[0, \pi/3]$ , is almost-periodic (in the sense of Besicovitch and Bohr), takes the value 0 at  $\varphi = \arg(z_{J_n,n})$ , and therefore, up to  $O(1/n)$ -terms, has its minima at the successive arguments  $\arg(z_{J_n,n}) + \frac{2k\pi}{n}$  for  $|k| = 0, 1, 2, \dots, J_n - H_n + 1, \dots$  (Figure 3). For such integers  $k$ , from (5.3.51), we deduce the successive minima

$$(5.3.52) \quad | -1 + z_{J_n,n} e^{-2ik\pi/n} + (z_{J_n,n} e^{-2ik\pi/n})^n | = |G_n(z_{J_n,n})| + \frac{2|k|\pi}{n} = \frac{2|k|\pi}{n}$$

up to  $\frac{1}{n}O\left(\left(\frac{\text{Log } \text{Log } n}{\text{Log } n}\right)^2\right)$ - terms, with  $\arg(z_{J_n,n} e^{-2ik\pi/n}) = \arg(z_{J_n-k,n})$  up to  $O(1/n)$ -terms.

With the above notations, denote by  $y_j, y'_j$  the two points of  $C(0, 1 - \frac{c_n}{n})$  which belong to  $C_{j,n}$  for  $2H_n - J_n \leq j \leq J_n$ , to  $C(z_{j,n}, \frac{\pi|z_{j,n}|}{ns_{j,n}})$  for  $J_n + 1 \leq j \leq 2J_n - H_n + 1$ . Writing them by increasing argument, we have:

$$(5.3.53) \quad y_{2H_n - J_n}, y'_{2H_n - J_n}, \dots, y_{H_n}, y'_{H_n}, \dots, y_{J_n}, y'_{J_n}, y_{J_n + 1}, y'_{J_n + 1}, \dots, y_{2J_n - H_n + 1}, y'_{2J_n - H_n + 1}.$$

The Rouché inequality (5.3.47) is obviously satisfied at each  $y_j$  and  $y'_j$  for  $j = 2H_n - J_n, \dots, J_n$ . Let us show that this inequality holds at each point  $y_j$  and  $y'_j$  for  $j = J_n + 1, \dots, 2J_n - H_n + 1$ . Indeed, for such a point, say  $y_j$ , there exists

$$\xi_j = w_j z_{J_n, n} e^{2i(j-J_n)\pi/n} + (1 - w_j) y_j, \quad \text{for some } w_j \in [0, 1],$$

lying in the segment  $\left[ z_{J_n, n} e^{2i(j-J_n)\pi/n}, y_j \right]$  such that

$$G_n(y_j) = G_n(z_{J_n, n} e^{2i(j-J_n)\pi/n}) + (y_j - z_{J_n, n} e^{2i(j-J_n)\pi/n}) G'_n(\xi_j)$$

with, using (5.3.50),

$$|G_n(y_j) - G_n(z_{J_n, n} e^{2i(j-J_n)\pi/n})| = |y_j - z_{J_n, n} e^{2i(j-J_n)\pi/n}| |G'_n(\xi_j)| = \frac{\pi |z_{J_n, n}|}{n a_{\max}} |G'_n(\xi_j)|.$$

The derivative of  $G_n(z)$  is  $G'_n(z) = 1 + nz^{n-1}$ . Up to  $O(1/n)$ -terms, the line generated by the segment  $\left[ z_{J_n, n} e^{2i(j-J_n)\pi/n}, y_j \right]$  is tangent to the circle  $C(0, 1 - c/n)$ , and the modulus  $\frac{1}{n} |G'_n(\xi_j)|$  satisfies

$$\frac{1}{n} |G'_n(\xi_j)| = \frac{1}{n} |G'_n(z_{J_n, n} e^{2i(j-J_n)\pi/n})| = \frac{1}{n} |G'_n(z_{J_n, n})| = \lim_{n \rightarrow +\infty} \frac{1}{n} |G'_n(z_{J_n, n})| = e^{-c}.$$

From  $|G_n(y_j)| \geq |G_n(y_j) - G_n(z_{J_n, n} e^{2i(j-J_n)\pi/n})| - |G_n(z_{J_n, n} e^{2i(j-J_n)\pi/n})|$  and (5.3.50) we deduce

$$(5.3.54) \quad |G_n(y_j)| \geq \frac{\pi |z_{J_n, n}|}{a_{\max}} e^{-c} - \frac{2\pi |j - J_n|}{n}.$$

But, by definition of  $H_n$ , still up to  $O(1/n)$ -terms, for  $|j - J_n| \leq J_n - H_n - 1$ ,

$$(5.3.55) \quad \frac{2\pi |j - J_n|}{n} \leq \frac{2\pi (J_n - H_n - 1)}{n} = \arg(z_{J_n, n}) - \arg(z_{H_n+1, n}) \leq \frac{e^{-2c}}{1 - e^{-c}}.$$

This inequality is in particular satisfied for the last two values of  $|j - J_n|$  which are  $J_n - H_n$  and  $J_n - H_n + 1$  up to  $O(1/n)$ -terms. Since the inequality

$$(5.3.56) \quad 0.0710\dots = 2 \frac{e^{-2c}}{1 - e^{-c}} < \frac{\pi |z_{J_n, n}|}{a_{\max}} e^{-c} = 0.0914\dots$$

holds, from (5.3.54), (5.3.55) and (5.3.56), as soon as  $n$  is large enough, we deduce the Rouché inequality

$$|G_n(y_j)| \geq \frac{\pi |z_{J_n, n}|}{a_{\max}} - \frac{e^{-2c}}{1 - e^{-c}} \geq \frac{e^{-2c}}{1 - e^{-c}}.$$

Therefore the conditions of Rouché (5.3.48) hold at all the points  $y_j$  and  $y'_j$  of (5.3.53).

Let us prove that the conditions of Rouché (5.3.48) hold on each arc  $y'_j y_{j+1}$  of the circle  $|z| = 1 - c/n$ , for  $j = 2H_n - J_n, 2H_n - J_n + 1, \dots, 2J_n - H_n$ . Indeed, from (5.3.51), the derivative  $Z'(\varphi)$  takes a positive value at the extremity  $y'_j$  while it takes a negative value at the other extremity  $y_{j+1}$ .  $Z(\varphi)$  is almost-periodic of almost-period  $2\pi/n$ . The function  $\sqrt{Z(\varphi)}$  is increasing on  $(\arg(z_{j, n}), \arg(z_{j, n}) + \frac{\pi}{n})$  and decreasing on  $(\arg(z_{j, n}) + \frac{\pi}{n}, \arg(z_{j, n}) + 2\frac{\pi}{n})$ ; on the arc  $y'_j y_{j+1}$  it takes the value  $|G_n(y'_j)| \geq \frac{e^{-2c}}{1 - e^{-c}}$ , admits a maximum,

and decreases to  $|G_n(y_{j+1})| \geq \frac{e^{-2c}}{1-e^{-c}}$ . Hence, (5.3.7) holds true for all  $z \in C(0, 1 - c_n/n)$  with  $\arg(y'_j) \leq \arg(z) \leq \arg(y_{j+1})$ .

Let us now prove that the condition of Rouché (5.3.7) is satisfied in the angular sector  $0 \leq \arg(z) \leq \arg(z_{H_n, n})$ . Indeed, in this angular sector, the successive minima of  $\sqrt{Z(\varphi)}$  are all above  $\frac{e^{-2c}}{1-e^{-c}}$  by the definition of  $H_n$  and (5.3.52). Hence the claim.

Let us prove that the condition of Rouché (5.3.7) is satisfied in the angular sector  $\arg(z_{2J_n - H_n + 1, n}) \leq \arg(z) \leq \frac{\pi}{2}$ . In this angular sector, the oscillations of  $\sqrt{Z(\varphi)}$  still occur by the form of (5.3.51) and the successive minima of  $\sqrt{Z(\varphi)}$  are all above  $\frac{e^{-2c}}{1-e^{-c}}$  for  $\frac{2J_n - H_n + 2}{J_n} \leq \arg(z) \leq \pi/2$ , by (5.3.52) for  $k \geq J_n - H_n + 1$ . We deduce the claim.

The condition of Rouché (5.3.7) is also satisfied in the angular sector  $\pi \leq \arg(z) \leq \pi/2$ , since then  $\cos(\varphi) \leq 0$  and therefore  $\sqrt{Z(\varphi)} \geq \sqrt{2 + e^{-2c} - 4e^{-c}} = 1.15\dots$ . Since this lower bound is greater than the value  $\frac{e^{-2c}}{1-e^{-c}} = 0.0354\dots$  we deduce the claim.

Let us show that the conditions of Rouché (5.3.7) are also satisfied on the arcs  $C(z_{j, n}, \frac{\pi|z_{j, n}|}{ns_{j, n}}) \cap \overline{D}(0, 1 - \frac{c_n}{n})$  for  $j = J_n + 1, \dots, 2J_n - H_n + 1$ . For such an integer  $j$ , let us denote such an arc by  $y_j y'_j$ . The two extremities  $y_j$  and  $y'_j$  of the arc  $y_j y'_j$  of the circle  $C(z_{j, n}, \frac{\pi|z_{j, n}|}{ns_{j, n}})$  define the same value of the difference cosine, say  $X_j := \cos(\arg(y_j - z_{j, n}) - \arg(z_{j, n})) = \cos(\arg(y'_j - z_{j, n}) - \arg(z_{j, n}))$ , by (5.3.50). The conditions of Rouché are already satisfied at the points  $y_j$  and  $y'_j$  by the above. Recall that, for any fixed  $a \geq 1$ , the function  $\kappa(X, a)$ , defined in (5.3.14), is such that the partial derivative  $\partial \kappa_X$  of  $\kappa(X, a)$  is strictly negative on the interior of  $[-1, +1] \times [1, +\infty)$ . In particular the function  $\kappa(X, s_{j, n})$  is decreasing. For any point  $\Omega$  of the arc  $y_j y'_j$ , we denote by  $X = \cos(\arg(\Omega - z_{j, n}) - \arg(z_{j, n}))$ . We deduce, up to  $O(1/n)$ -terms,

$$\frac{e^{-2c}}{1-e^{-c}} \leq \kappa(X_j, s_{j, n}) \leq \kappa(X, s_{j, n}), \quad \text{for all } X \in [-1, X_j],$$

hence the result. □

**Remark 5.21.** In the case where  $\beta \in (1, \theta_6^{-1})$  is an algebraic integer such that  $\beta \notin \{\theta_n^{-1} \mid n \geq 6\}$ , the lenticulus  $\mathcal{L}_\beta$  of Galois conjugates of  $1/\beta$  in the angular sector  $\arg z \in \{-\frac{\pi}{3}, +\frac{\pi}{3}\}$  is obtained by truncation and a slight deformation of  $\mathcal{L}_{\theta_6^{-1}}$ . The asymptotic expansion of the minorant of the Mahler measure  $M(\beta)$  will be obtained from this lenticulus as a function of the dynamical degree  $\text{dyg}(\beta)$ .

#### 5.4. Identification of the lenticuli of roots as conjugates.

In this paragraph,  $\beta \in (1, (1 + \sqrt{5})/2)$  is assumed to be a reciprocal algebraic integer which is fixed, with  $\text{dyg}(\beta) = n \geq 260$ ,  $M(\beta) \leq 1.176280\dots$  (we continue to be under the assumptions of the Important Nota (N) in the Introduction - cf Section 5.9 also to discard some non-relevant families of reciprocal algebraic integers  $\beta > 1$  tending to 1). Since  $\beta$  is reciprocal the Parry Upper function  $f_\beta(x)$  is a power series which is never a polynomial. The Mahler measure  $M(\beta)$  is a function of the roots of  $P_\beta^*(z) = P_\beta(z)$  as

$$(5.4.1) \quad M(\beta) = \prod_{\gamma \text{ conjugate of } \beta^{-1}, |\gamma| < 1} |\gamma|^{-1}.$$



This product is over the set of conjugates  $\gamma$  of  $\beta^{-1}$ ,  $|\gamma| < 1$ . Up till now, the only element of this set, coming from the zeroes of  $f_\beta(z)$ , is  $\beta^{-1}$ . It is a common (simple) zero of  $P_\beta^*(z)$  and  $f_\beta(z)$ . We will show that this set contains other zeroes of  $f_\beta(z)$ . In §5.4.1 we prove that the separation of the roots of  $f_\beta(z)$  of modulus  $< 1$  into two distinct subcollections occurs inside the angular sector, roughly (“the cusp region”)

$$\arg(z) \in \left[-\frac{\pi}{18}, +\frac{\pi}{18}\right]$$

given by (5.3.9): those which are off the unit circle form a lenticular shape, those which are very close to the unit circle lie in a thin annular neighbourhood of  $|z| = 1$ . In §5.4.2 we use the Rényi-Parry numeration system to construct sequences of rewriting polynomials “between” the minimal polynomial  $P_\beta(z) = P_\beta^*(z)$  and  $f_\beta(z)$ . Two sequences of polynomial functions are used to identify the conjugates of  $\beta^{-1}$  with the zeroes of  $f_\beta(z)$ : the sequence of polynomial sections of  $f_\beta(z)$ , the sequence of rewriting polynomials. As a consequence, the product (5.4.1) will be over a set containing the collection of lenticular zeroes of  $f_\beta(z)$ .

5.4.1. *Lenticular roots of  $f_\beta(x)$  and its polynomial sections.* For  $n \geq 3$ , the polynomial sections of  $f_\beta(x)$ ,  $\beta \in (1, (1 + \sqrt{5})/2)$ , are of the type

$$(5.4.2) \quad -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s}$$

where  $s \geq 0$ ,  $m_1 - n \geq n - 1$ ,  $m_{q+1} - m_q \geq n - 1$  for  $1 \leq q < s$ . Denote by  $\mathcal{B}$  the class of the polynomials defined by (5.4.2), and by  $\mathcal{B}_n$  those whose third monomial is exactly  $x^n$ , so that

$$\mathcal{B} = \cup_{n \geq 3} \mathcal{B}_n.$$

The case “ $s = 0$ ” corresponds to the trinomials  $G_n(z) := -1 + z + z^n$  (Selmer [185]).

**Theorem 5.22.** *Let  $c_{lent} := \min_{n \geq 260} (c_n - \frac{\pi}{a_{max}})$ . Let  $n \geq 260$  and  $\beta > 1$  be a reciprocal algebraic integer such that  $\text{dyg}(\beta) = n$ ,  $M(\beta) \leq 1.176280\dots$ . Denote by*

$$f_\beta(z) = -1 + z + z^n + z^{m_1} + z^{m_2} + \dots + z^{m_j} + z^{m_{j+1}} + \dots,$$

where  $m_1 - n \geq n - 1$ ,  $m_{j+1} - m_j \geq n - 1$  for  $j \geq 1$ , the Parry Upper function at  $\beta$ . Then the zeroes of  $f_\beta(z)$  of modulus  $< 1$  which lie in  $-\arg(z_{J_n, n}) - \frac{\pi}{na_{max}} < \arg z < +\arg(z_{J_n, n}) + \frac{\pi}{na_{max}}$  either belong to

$$\left\{z \mid \left||z| - 1\right| < \frac{1}{3} \frac{c_{lent}}{n}\right\} \quad \text{or to} \quad \left\{z \mid \left||z| - 1\right| > \frac{c_{lent}}{n}\right\}.$$

In the second class of zeroes, all the zeroes are simple, and lie in the union

$$D_{0,n} \cup \bigcup_{j=1}^{J_n} (\overline{D_{j,n}} \cup D_{j,n});$$

there is one zero per disk  $D_{j,n}$ ,  $\overline{D_{j,n}}$ , the disk  $D_{0,n}$  containing the element  $\beta^{-1}$ .

*Proof.* Denote by

$$\mathcal{S}_n := \left\{z \mid \theta_{n-1} \leq |z| < 1, -\arg(z_{J_n, n}) - \frac{\pi}{na_{max}} \leq \arg z \leq +\arg(z_{J_n, n}) + \frac{\pi}{na_{max}}\right\}$$

the truncated angular sector and let

$$\widehat{\mathcal{S}}_n := \overset{o}{\mathcal{S}}_n \setminus \left( \bigcup_{j=1}^{J_n} (D_{j,n} \cup \overline{D_{j,n}}) \cup D(\theta_n, \theta_n - \theta_{n-1}) \right)^{cl}$$

the open truncated angular sector obtained from  $\mathcal{S}_n$  by removing the closure of the Rouché disks  $\overline{D_{j,n}}, D_{j,n}$  centered at the zeroes  $z_{j,n}$  of  $G_n(z)$  in  $\mathcal{S}_n$  of respective radius  $\frac{\pi|z_{j,n}|}{na_{max}}$ , and of  $D(\theta_n, \theta_n - \theta_{n-1})$ . The argument  $\arg(z_{j,n})$  is defined in (5.3.18). The analytic function  $G_n(z)$  has no zero in the adherence  $\widehat{\mathcal{S}}_n$  of  $\widehat{\mathcal{S}}_n$  and reaches its infimum  $\inf_{z \in \widehat{\mathcal{S}}_n} |-1 + z + z^n| > 0$  on the boundary  $\partial \widehat{\mathcal{S}}_n$  of  $\widehat{\mathcal{S}}_n$ . On the Rouché circles  $C_{j,n}, \overline{C_{j,n}}, j = 1, \dots, J_n$ , using (5.3.7) and (5.3.27), this infimum is bounded from below by

$$\frac{|Z|^{2n-1}}{1 - |Z|^{n-1}}$$

where  $Z$  is the point of  $C_{1,n}$  of smallest modulus, which is such that  $|Z| = |\theta_n - \frac{\pi}{na_{max}}\theta_n|$  at the first order. Putting aside the Rouché circles, using the inequality  $|-1 + z + z^n| \geq ||-1 + z| - |z^n||$ , the minimum of  $|-1 + z + z^n|$  on the arcs  $|z| = 1, |z| = \theta_{n-1}$ , the segments  $\arg x = \pm(\arg(z_{J_n,n}) + \frac{\pi}{na_{max}})$  and the circle  $C(\theta_n, \theta_n - \theta_{n-1})$  on  $\partial \widehat{\mathcal{S}}_n$  is bounded from below by

$$|-1 + \theta_{n-1}| - |\theta_{n-1}^n| = (1 - \theta_{n-1})^2.$$

Denote

$$\delta_n := \min \left\{ (1 - \theta_{n-1})^2, \frac{|Z|^{2n-1}}{1 - |Z|^{n-1}} \right\}.$$

We have:  $0 < \delta_n \leq \inf_{z \in \widehat{\mathcal{S}}_n} |-1 + z + z^n|$  and  $\lim_{n \rightarrow \infty} \delta_n = 0$ . It is easy to show that

$$\lim_{n \rightarrow \infty} \frac{\text{Log } \delta_n}{n} = 0.$$

Using §5.3 in [73] this limit condition allows to calculate a first-order estimate of the thickness of the annular neighbourhood of the unit circle, in  $\widehat{\mathcal{S}}_n$ , which contains the roots of a polynomial section  $-1 + z + z^n + z^{m_1} + z^{m_2} + \dots + z^{m_s}$  of  $f_\beta(z)$ ; this estimate is

$$(5.4.3) \quad e(s) = 1 - \left( 1 - 2 \frac{(n-1)(s - \delta_n)}{(n-1)(s^2 + s) + 2(m_s - n)} \right)^{1/(n-1)}.$$

In the expression (5.4.3)  $n$  is fixed, as well as the sequence  $(m_j)_{j \geq 1}$  since  $\beta$  is fixed, therefore  $f_\beta(z)$  also; the integer  $m_s$  tends to infinity, if  $s$  tends to infinity, since  $m_s - n \geq (m_1 - n) + \sum_{j=2}^s (m_j - m_{j-1}) \geq s(n-1)$ ; the integer  $s$  is large enough (at least to have  $s - \delta_n > 0$ ) and  $\lim_{s \rightarrow \infty} e(s) = 0$ .

Among all the Rouché disks  $D_{j,n}, 1 \leq j \leq J_n$ , the  $J_n$ th Rouché disk  $D_{J_n,n}$  is the closest to the unit circle (by (iii-2) in Proposition 4.2). By Lemma 5.16 its center is  $z_{J_n,n}$ , of modulus  $|z_{J_n,n}| = 1 - \frac{c_n}{n}$ , and its radius is  $\frac{\pi|z_{J_n,n}|}{na_{max}} < \frac{\pi}{na_{max}}$ . By Lemma 5.16 the limit  $c = \lim_{n \rightarrow \infty} c_n$  exists, is positive, and, from a numerical viewpoint,  $c - \frac{\pi}{a_{max}} = 1.76274 - 0.53479 = 1.22794 \dots$ . By the asymptotic expansion of  $c_n$  in Lemma 5.16, the constant

$c_{lent} := \min_{n \geq 260} (c_n - \frac{\pi}{a_{max}})$  is positive. The disk  $\{z \mid |z| < 1 - \frac{c_{lent}}{n}\}$  contains all the Rouché disks  $D_{j,n}, \overline{D_{j,n}}, 1 \leq j \leq J_n$ , and  $D_{0,n}$ .

Let assume that  $f_\beta(z)$  has a zero in

$$\widehat{\mathcal{S}}_n \cap \left\{ z \mid |z| < 1 - \frac{1}{3} \frac{c_{lent}}{n} \right\}.$$

Denote it by  $z$ , counted with multiplicity. There exists  $r > 0$  small enough such that the open disk  $D(z, r)$  be included in  $\widehat{\mathcal{S}}_n \cap \{z \mid |z| < 1 - \frac{1}{3} \frac{c_{lent}}{n}\}$  and only contains the zero  $z$  of  $f_\beta(z)$ . By Hurwitz Theorem (for instance cf §11 in Chap. 2 in [179]) the number of zeroes of any polynomial section  $-1 + z + z^n + z^{m_1} + z^{m_2} + \dots + z^{m_s}$  of  $f_\beta(z)$  in  $D(z, r)$  should be equal to the multiplicity  $\geq 1$  of  $z$ , as soon as  $s$  is large enough, say  $s \geq s_0$  for some  $s_0$ .

Since  $\lim_{s \rightarrow 0} e(s) = 0$ , we obtain a contradiction by taking  $s_0$  such that  $e(s) \leq \frac{c_{lent}}{10n}$  for all  $s \geq s_0$ . The constant 10, at the denominator, is arbitrary and may be taken eventually larger. This means that all the zeroes of all the polynomial sections of  $f_\beta(z)$ , in  $\widehat{\mathcal{S}}_n$ , for all  $s \geq s_0$ , are contained in

$$1 - \frac{c_{lent}}{3n} < |z| < 1.$$

But  $\{z \mid 1 - \frac{c_{lent}}{3n} < |z| < 1\} \cap D(z, r) = \emptyset$ . Contradiction.

Therefore the zeroes of  $f_\beta(z)$  which lie in the open angular sector

$$\left\{ z \in \mathbb{C} : |z| < 1, -\arg(z_{J_n, n}) - \frac{\pi}{na_{max}} < \arg z < \arg(z_{J_n, n}) + \frac{\pi}{na_{max}} \right\}$$

are located either in the Rouché disks by Theorem 5.8 and Theorem 5.14, or in a small neighbourhood of the unit circle included in  $\{z \mid 1 - \frac{c_{lent}}{3n} < |z| < 1\}$ . This dichotomy naturally extends to the zeroes of any polynomial section of  $f_\beta(z)$  (cf the proofs of Theorem 5.8, Theorem 5.14 and Theorem 5.22).  $\square$

**Definition 5.23.** Let  $n \geq 260$ . Let  $\beta > 1$  be a reciprocal algebraic integer such that  $\text{dyg}(\beta) = n$ , with  $M(\alpha) < 1.176280\dots$ . The zeroes of  $f_\beta(z) := -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s} + \dots$ , resp. of any of its polynomial section  $f(x) := -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s} \in \mathcal{B}_n$ , which belong to the angular sector

$$(5.4.4) \quad \left\{ z \in \mathbb{C} : |z| < 1 - \frac{c_{lent}}{n}, |\arg z| \leq \arg(z_{J_n, n}) + \frac{\pi}{na_{max}} \right\}$$

are called the *lenticular zeroes* of  $f_\beta(z)$ , resp. of  $f$ .

**Theorem 5.24.** For any  $f \in \mathcal{B}_n, n \geq 3$ , denote by

$$f(x) = -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s} = A(x)B(x)C(x),$$

where  $s \geq 1, m_1 - n \geq n - 1, m_{j+1} - m_j \geq n - 1$  for  $1 \leq j < s$ , the factorization of  $f$  where  $A$  is the cyclotomic part,  $B$  the reciprocal noncyclotomic part,  $C$  the nonreciprocal part. Then

- (i) the nonreciprocal part  $C$  is nontrivial, irreducible, and never vanishes on the unit circle,
- (ii) if  $\gamma_s > 1$  denotes the real algebraic integer uniquely determined by the sequence  $(n, m_1, m_2, \dots, m_s)$  such that  $1/\gamma_s$  is the unique real root of  $f$  in  $(0, 1)$ , the polynomial  $-C^*(X)$ , opposite of the reciprocal polynomial of  $C(X)$ , is the minimal polynomial of  $\gamma_s$ , and  $\gamma_s$  is a nonreciprocal algebraic integer.

*Proof.* Theorem 3 in [73]. □

Let us precise the behaviour of the reciprocal noncyclotomic parts  $B$  of  $f(x)$  in Theorem 5.24 on the lenticular roots of  $f$ .

**Proposition 5.25.** *Let  $\beta > 1$  be a reciprocal algebraic integer having  $\text{dyg}(\beta) \geq 260$ . Let  $f_\beta(x) = -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s} + \dots$  be the Parry Upper function at  $\beta$  and, for  $s \geq 0$ , denote its  $s$ -th polynomial section by*

$$f(x) = -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s} \quad \text{factorized as } = A(x)B(x)C(x),$$

where  $s \geq 1$ ,  $m_1 - n \geq n - 1$ ,  $m_{j+1} - m_j \geq n - 1$  for  $1 \leq j < s$ , where  $A$  is the cyclotomic part,  $B$  the reciprocal noncyclotomic part,  $C$  the nonreciprocal part of  $f$ .

There exists  $s_0$  (depending upon  $n$ ) such that the reciprocal noncyclotomic part  $B$  of  $f(x)$ , if any, does not vanish on the lenticular roots of  $f$ , as soon as  $s \geq s_0$ .

*Proof.* The result is true in the case “ $s = 0$ ” since a trinomial  $G_n(X)$  is either nonreciprocal and irreducible or the product of a nonreciprocal irreducible polynomial by a cyclotomic polynomial (by Proposition 4.1) [185]. Let us assume  $n = \text{dyg}(\beta) \geq 260$  ( $n$  is fixed) and make it explicit. Let  $z_{j,n}$  be a zero of  $G_n(x) = -1 + x + x^n$  which belongs to the angular sector (5.4.4). Then the reciprocal trinomial  $G_n^*(z_{j,n}) = -z_{j,n}^n + z_{j,n}^{n-1} + 1$  at  $z_{j,n}$  is such that

$$\begin{aligned} |G_n^*(z_{j,n})| &= |z_{j,n}(1 + z_{j,n}^{n-2})| = |z_{j,n}(1 + (1 - z_{j,n})/z_{j,n}^2)| \\ &\geq \theta_{n-1}(1 - |\frac{1 - z_{j,n}}{z_{j,n}z}|) \geq \theta_{n-1}(1 - \kappa(1, a_{\max})) \end{aligned}$$

by Proposition 4.2 and Definition 5.10. Obviously  $\min_{n \geq 260} \theta_{n-1}(1 - \kappa(1, a_{\max}))$  is  $> 0$  since  $\lim_{n \rightarrow \infty} \theta_n(1 - \kappa(1, a_{\max})) = 1 - \kappa(1, a_{\max}) = 1 - 0.171573\dots = 0.828427\dots$ . In other terms  $G_n^*(x)$  does not vanish on the lenticular zeroes  $z$  of  $G_n(x)$  which lie in (5.4.4), with the lower bound  $\min_{n \geq 260} \theta_{n-1}(1 - \kappa(1, a_{\max}))$ , independent of  $n$  and uniform on all the lenticular roots of  $f$  in (5.4.4).

We now consider the general case “ $s \geq 1$ ” and extend the previous case. Let  $D_{j,n}$  (defined in Theorem 5.14, with  $D_{-j,n} = \overline{D_{j,n}}$ ) be a Rouché disk in the open angular sector (5.4.4), that is for  $j \in \{0, 1, \dots, J_n\}$ . Denote by  $r_s$  the unique zero in  $D_{j,n}$  of the  $s$ th polynomial section  $f(x) = -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s} \in \mathcal{B}_n$ . Recall that in  $D_{j,n}$  the unique zero of  $f_\beta(z)$  is  $\omega_{j,n}$  (with  $\omega_{0,n} := \beta^{-1}$ ) so that  $\lim_{s \rightarrow \infty} r_s = \omega_{j,n}$ . We will show that the polynomial  $f^*(x) = A^*(x)B^*(x)C^*(x) = x^{m_s}f(1/x)$ , reciprocal of  $f(x)$ , does not vanish on the lenticular zero  $r_s$  of  $f(x)$ .

Assume the contrary. Then

$$(5.4.5) \quad 0 = f^*(r_s) = G_n^*(r_s)r_s^{m_s-n} + (1 + r_s^{m_s-m_{s-1}} + r_s^{m_s-m_{s-2}} + \dots + r_s^{m_s-m_1}).$$

From the polynomial  $f^*$ , let us construct the associated Parry Upper function

$$\begin{aligned} f_\beta^{[s]}(z) &:= G_n(z) + z^{m_1}(f^*(z) - G_n^*(z)z^{m_s-n}) + (f_\beta(z) - f(z)) \\ &= -1 + z + z^n + z^{m_1} + z^{m_1+m_s-m_{s-1}} + z^{m_1+m_s-m_{s-2}} + \dots + z^{m_s} + \text{tail of } f_\beta(z). \end{aligned}$$

For  $1 \leq q \leq s - 1$  the conditions  $(m_s - m_{s-(q+1)}) - (m_s - m_{s-q}) = m_{s-q} - m_{s-(q+1)} \geq n - 1$  imply that Theorem 5.14 applies. For every  $s \geq 1$  the Parry Upper function  $f_\beta^{[s]}(z)$  has a

zero in  $D_{j,n}$ . But, from (5.4.5), we would have

$$(5.4.6) \quad |G_n^*(r_s) r_s^{m_s-n}| = \left| \frac{f_\beta^{[s]}(r_s) - G_n(r_s) - \sum_{q=s+1}^{\infty} r_s^{m_q}}{r_s^{m_1}} \right|.$$

The limit  $\lim_{s \rightarrow \infty} \sum_{q=s+1}^{\infty} r_s^{m_q} = 0$  holds and

$$\liminf_{s \rightarrow \infty} \left| \frac{f_\beta^{[s]}(r_s) - G_n(r_s) - \sum_{q=s+1}^{\infty} r_s^{m_q}}{r_s^{m_1}} \right| = \liminf_{s \rightarrow \infty} \left| \frac{f_\beta^{[s]}(\omega_{j,n}) - G_n(\omega_{j,n})}{\omega_{j,n}^{m_1}} \right|$$

admits a lower bound which is strictly positive. Indeed, this lower bound can be computed from any series  $\sum_{q=0}^{\infty} z^{d_q}$ ,  $d_0 = 0, d_{q+1} - d_q \geq n - 1, q \geq 0$ , by

$$|1 + \omega_{j,n}^{d_1} + \omega_{j,n}^{d_2} + \dots + \omega_{j,n}^{d_q} + \dots| \geq 1 - |\omega_{j,n}|^{n-1} \frac{1}{1 - |\omega_{j,n}|^{n-1}}$$

and approximated by

$$(5.4.7) \quad 1 - |z_{j,n}|^{n-1} \frac{1}{1 - |z_{j,n}|^{n-1}} = 1 - \frac{|1 - z_{j,n}|}{|z_{j,n}| - |1 - z_{j,n}|} \geq 1 - \frac{|1 - z_{j,n}|}{|z_{j,n}| - |1 - z_{j,n}|} > 0.$$

The contradiction comes from the fact that the lhs of (5.4.6) tends to 0 when  $s$  tends to infinity, whereas the rhs of (5.4.6) has a positive liminf by (5.4.7). We deduce that  $B(r_s) = B^*(r_s) = 0$  cannot hold as soon as  $s$  is large enough.  $\square$

**Corollary 5.26.** *Let  $\beta > 1$  be a reciprocal algebraic integer having  $n = \text{dyg}(\beta) \geq 260$ , with  $M(\alpha) < 1.176280\dots$ . For  $s$  large enough, if  $\gamma_s > 1$  denotes the real root of the polynomial  $C^*(z)$ , all the lenticular zeroes of the  $s$ -th polynomial section  $f(z) = A(z)B(z)C(z)$  of  $f_\beta(z)$  are conjugates of  $\gamma_s^{-1}$  where conjugation is relative to the irreducible nonreciprocal polynomial part  $C(z)$ . The degree of the irreducible nonreciprocal factor  $C(X)$  of  $f$  satisfies*

$$(5.4.8) \quad \deg(C) \geq 1 + 2J_n = 1 + \frac{n}{\pi} \left( 2 \arcsin\left(\frac{\kappa}{2}\right) \right) + \frac{2\kappa \text{Log } \kappa}{\pi \sqrt{4 - \kappa^2}} + O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right).$$

*Proof.* The irreducible nonreciprocal part  $C$  of  $f$  has at least  $1 + 2J_n$  zeroes, where  $J_n$  is given by (5.3.19). The constant  $2 \arcsin\left(\frac{\kappa}{2}\right) = 0.171784\dots$  is given by (5.3.20).  $\square$

5.4.2. *Rewriting trail and rewriting polynomials from “ $P_\beta$ ” to “ $f_\beta$ ”.* (cf [74] for more details) As mentioned above  $\beta$  is a reciprocal algebraic integer which is fixed such that  $\text{dyg}(\beta) = n \geq 260$ ,  $M(\beta) \leq 1.176280\dots$ , as well as its minimal polynomial  $P_\beta$  and its Parry Upper function  $f_\beta$ . Let  $P_\beta(X) = P_\beta^*(X) = 1 + a_1X + a_2X^2 + a_3X^3 + \dots + a_{d-1}X^{d-1} + X^d$ ,  $a_i \in \mathbb{Z}$ ,  $a_{d-j} = a_j, d \geq 1$ , be the minimal polynomial of  $\beta$ . Let  $f_\beta(z) = -1 + t_1z + t_2z^2 + t_3z^3 + \dots$  be the Parry Upper function at  $\beta$ , written in the generic form (with  $t_1 = 1, t_2 = t_3 = \dots = t_{n-1} = 0, t_n = 1$ , etc). Recall that the sequence  $(t_i)_{i \geq 1}$  is unique, entirely characterizes  $\beta$  and is Lyndon (self-admissible). Polynomial sections of  $f_\beta(X)$  are denoted by  $S_q$ : for  $q \geq 1$ ,  $S_q(z) = -1 + t_1z + t_2z^2 + \dots + t_qz^q, |z| < 1$ . Since  $f_\beta(\beta^{-1}) = 0$  and that, for any

$q \geq 1$ ,  $f_\beta(\beta^{-1}) - S_q(\beta^{-1})$  is a sum of positive terms, we can permute them and group them in order to obtain  $d$  components in the  $\mathbb{Q}$ -basis  $\{1, \beta^{-1}, \beta^{-2}, \dots, \beta^{-d+1}\}$ . With

$$g_{q,j}(z) := z^{-j} \times \sum_{\substack{i=q+1 \\ i \equiv q+1+j \pmod{d}}}^{\infty} t_i z^{i-(q+1)} \quad q \geq 1, j = 0, 1, \dots, d-1,$$

we obtain the existence of  $d$  power series  $g_{q,0}(z), g_{q,1}(z), \dots, g_{q,d-1}(z)$ , all defined on the open unit disk, such that

$$(5.4.9) \quad f_\beta(\beta^{-1}) - S_q(\beta^{-1}) = \frac{1}{\beta^{q+1}} \left( \sum_{j=0}^{d-1} g_{q,j}(\beta^{-1}) \beta^{-j} \right), \quad q \geq 1.$$

Let us restrict  $z$  to the open angular sector (5.4.4):

$$z \in \left\{ z \mid |z| < 1 - \frac{c_{lent}}{n}, |\arg z| \leq \arg(z_{J_n, n}) + \frac{\pi}{na_{max}} \right\}.$$

Then all the power series  $g_{q,j}(z)$  are absolutely convergent in this sector, having the same uniform upper bound in modulus

$$(5.4.10) \quad |g_{q,j}(z)| \leq \frac{1}{1 - (1 - c_{lent}/n)^d}, \quad \text{for all } q \geq 1, j = 0, 1, \dots, d-1,$$

which is independent of  $q$  and  $j$ .

Now, let us use the numeration system in base  $\beta$  in order to obtain alternate expressions of the  $d$  components  $g_{q,j}(\beta^{-1})$  in (5.4.9). It will allow to “restore” the digits  $t_i$  of  $f_\beta$  one after the other. The identities  $P_\beta(\beta^{-1}) = 0$  and  $f_\beta(\beta^{-1}) = 0$  give two  $\beta$ -representations of 1, the second one being the Rényi  $\beta$ -expansion of 1:

$$(5.4.11) \quad 1 = -a_1\beta^{-1} - a_2\beta^{-2} - a_3\beta^{-3} + \dots - a_{d-1}\beta^{-(d-1)} - \beta^{-d} = 1 - P_\beta(\beta^{-1}),$$

$$(5.4.12) \quad 1 = t_1\beta^{-1} + t_2\beta^{-2} + t_3\beta^{-3} + \dots = 1 + f_\beta(\beta^{-1}).$$

Let us construct an infinite chain of intermediate  $\beta$ -representations of 1 between them. Let us show that, for every  $q \geq 1$ , there exists a polynomial  $A_q \in \mathbb{Z}[X]$ , with  $\deg(A_q) \leq q$  and  $A_q(0) = -1$ , and a  $d$ -tuple  $(h_{q,0}, h_{q,1}, \dots, h_{q,d-1})$  of integers such that

$$(5.4.13) \quad A_q(z)P_\beta(z) = S_q(z) + z^{q+1} \left( \sum_{j=0}^{d-1} h_{q,j} z^j \right),$$

satisfying

$$(5.4.14) \quad h_{q,j} = g_{q,j}(\beta^{-1}), \quad \text{for } j = 0, 1, 2, \dots, d-1.$$

For obtaining  $A_1$  the quantity  $0 = \beta^{-1}(a_1 + 1)P_\beta(\beta^{-1})$  is added to (5.4.11). Then

$$1 = 1 + (-1 + (1 + a_1)\beta^{-1})P_\beta(\beta^{-1}) = 1 + S_1(\beta^{-1}) + \beta^{-2} \left( \sum_{j=0}^{d-1} h_{1,j} z^j \right)$$

with  $h_{1,j} = -a_{j+2} + a_{j+1}(a_1 + 1)$  for  $j = 0, 1, \dots, d-3$ ,  $h_{1,d-2} = -1 + a_{d-1}(a_1 + 1)$  and  $h_{1,d-1} = (a_1 + 1)$ . We deduce  $A_1(z) = -1 + (a_1 + 1)z$ . From (5.4.9) and the fact that the lhs of (5.4.13) is equal to 0 for  $z = \beta^{-1}$  we also deduce

$$h_{1,j} = g_{1,j}(\beta^{-1}), \quad \text{for } j = 0, 1, 2, \dots, d-1.$$

Now let us proceed recursively. Let us assume that  $A_1, A_2, \dots, A_q$  are already constructed, with (5.4.13) and (5.4.14) satisfied. For obtaining  $A_{q+1}$  let us first observe, from (5.4.13), that  $h_{q,0} \in \mathbb{Z}$  is the coefficient of  $\beta^{-(q+1)}$  in the  $\beta$ -representation of 1 which is

$$(5.4.15) \quad 1 = 1 + A_q(\beta^{-1})P_\beta(\beta^{-1}).$$

Let us add the quantity  $0 = \beta^{-(q+1)}(t_{q+1} - h_{q,0})P_\beta(\beta^{-1})$  to (5.4.15) and consider the polynomial  $A_{q+1}(z) = A_q(z) + (t_{q+1} - h_{q,0})z^{q+1}$ . Then we obtain the  $\beta$ -representation of 1:

$$1 = 1 + A_{q+1}(\beta^{-1})P_\beta(\beta^{-1}) = 1 + S_{q+1}(\beta^{-1}) + \beta^{-(q+1)} \left( \sum_{j=0}^{d-1} h_{q+1,j} \beta^{-j} \right)$$

where the  $d$ -tuple  $(h_{q+1,0}, h_{q+1,1}, \dots, h_{q+1,d-1})$  of integers can be readily computed from  $(h_{q,0}, h_{q,1}, \dots, h_{q,d-1})$  and  $t_{q+1}$ . From (5.4.9) and the fact that the lhs of (5.4.13) is equal to 0 for  $z = \beta^{-1}$  we deduce

$$h_{q+1,j} = g_{q+1,j}(\beta^{-1}), \quad \text{for } j = 0, 1, 2, \dots, d-1.$$

**Definition 5.27.** The sequence of  $\beta$ -representations of 1

$$(1 = 1 + A_q(\beta^{-1})P_\beta(\beta^{-1}))_{q \geq 1}$$

is called the *rewriting trail* from “ $P_\beta$ ” to “ $f_\beta$ ”. The polynomial

$$A_q(X) = -1 + (a_1 + 1)X + \sum_{j=1}^{q-1} (t_{j+1} - h_{j,0})X^{j+1} \in \mathbb{Z}[X]$$

is called the  $q$ th *rewriting polynomial* of the rewriting trail from “ $P_\beta$ ” to “ $f_\beta$ ”.

**Proposition 5.28.** *Let  $\beta > 1$  be a reciprocal algebraic integer,  $\text{dyg}(\beta) = n \geq 260$ , and denote by  $P_\beta$  its minimal polynomial and by  $f_\beta$  its Parry Upper function at  $\beta$ . If  $x \neq \beta^{-1}$  is a zero of  $P_\beta(z)$  in the open angular sector*

$$(5.4.16) \quad z \in \left\{ z \mid |z| < 1 - \frac{c_{lent}}{n}, |\arg z| \leq \arg(z_{J_n, n}) + \frac{\pi}{na_{max}} \right\},$$

*then  $x$  is a lenticular zero of  $f_\beta(z)$ .*

*Proof.* Let us use the rewriting trail and the rewriting polynomials from “ $P_\beta$ ” to “ $f_\beta$ ”. Let us assume that  $x \neq \beta^{-1}$  is a zero of  $P_\beta$  in the open angular sector (5.4.16), and denote by  $\sigma : \beta^{-1} \rightarrow x$  the conjugation. The image by  $\sigma$  of the  $\mathbb{Q}$ -basis  $\{1, \beta^{-1}, \dots, \beta^{-d+1}\}$  is the  $\mathbb{Q}$ -basis  $\{1, x, \dots, x^{d-1}\}$ . From (5.4.13) and (5.4.14) we have

$$(5.4.17) \quad 0 = A_q(x)P_\beta(x) = S_q(x) + x^{q+1} \left( \sum_{j=0}^{d-1} h_{q,j} x^j \right),$$

with

$$\sigma(h_{q,j}) = h_{q,j} = g_{q,j}(\sigma(\beta^{-1})), \quad \text{for } q \geq 1, j = 0, 1, 2, \dots, d-1.$$

Since  $|h_{q,j}| \leq (1 - (1 - c_{lent}/n)^d)^{-1}$  for all  $q \geq 1, j = 0, 1, \dots, d-1$  by (5.4.10), and that  $|x| < 1$ , we have

$$\lim_{q \rightarrow \infty} x^{q+1} \left( \sum_{j=0}^{d-1} h_{q,j} x^j \right) = 0.$$

Therefore, from (5.4.17), we deduce  $\lim_{q \rightarrow \infty} S_q(x) = 0$ . As a consequence the zero  $x$  necessarily belongs to the set of limit points of the zeroes of the polynomial sections  $S_q(z)$ , zeroes which lie in the angular sector (5.4.16). This set of limit points is the set of lenticular zeroes of  $f_\beta$ , by Hurwitz's Theorem and the uniform convergence of  $(S_q)$  to  $f_\beta$  on every compact of  $D(0, 1)$ . Hence the result.  $\square$

In the following Proposition we continue the investigation of the relative positioning of the zeroes of  $f_\beta(z)$  with respect to those of the minimal polynomial  $P_\beta(z)$  in the open angular sector

$$(5.4.18) \quad \left\{ z \mid |z| < 1 - \frac{c_{lent}}{n}, \mid \arg z \mid \leq \arg(z_{J_n, n}) + \frac{\pi}{na_{max}} \right\}.$$

Indeed, since  $\beta$  is fixed, both collections of zeroes are fixed. The purpose of Proposition 5.29 is only to discriminate the possible locations from the impossible ones. More explicitly, from Proposition 5.28 the zeroes of  $P_\beta(z)$  which lie in the angular sector (5.4.18) form a subcollection of the lenticular zeroes of  $f_\beta(z)$ . A natural question is whether the remaining zeroes of  $f_\beta(z)$  in this sector are also zeroes of  $P_\beta(z)$ , or not? The following Proposition answers this question, using the tails of the rewriting polynomials constructed from the polynomial sections “ $S_s$ ” of  $f_\beta$  to “ $P_\beta$ ” (with  $s$  large enough), and the structure of the type of factorization of the polynomial sections of  $f_\beta(z)$ .

**Proposition 5.29.** *Let  $\beta > 1$  be a reciprocal algebraic integer,  $\text{deg}(\beta) = n \geq 260$ ,  $M(\beta) \leq 1.176280\dots$ , and denote by  $P_\beta$  its minimal (reciprocal) polynomial and by  $f_\beta$  its Parry Upper function. The minimal polynomial  $P_\beta(X)$  of  $\beta > 1$  is such that there does not exist an irreducible integer polynomial  $\widetilde{P}_\beta(X)$  and an integer  $r \geq 2$  ( $r$  being the largest integer having this property) such that it is written under the form*

$$(5.4.19) \quad P_\beta(X) = \widetilde{P}_\beta(X^r).$$

*If  $x \neq \beta^{-1}$  is a lenticular zero of  $f_\beta(z)$ , then  $x$  is a zero of  $P_\beta$ .*

*Proof.* Let us assume that there exists a lenticular zero  $x$  of  $f_\beta(z)$ , then such that  $f_\beta(x) = 0$ , having the property  $P_\beta(x) \neq 0$ , and show the contradiction. We can add the assumption that its imaginary part  $\Im(x)$  is  $> 0$ . Denote  $\nu := |P_\beta(x)| > 0$ . Let us consider the  $s$ -th polynomial section  $S_s(z) = -1 + \sum_{j=1}^s t_j z^j$  of  $f_\beta(z)$ , where the integer  $s$ , taken large enough, will be fixed below.

The  $s$ -th polynomial section  $S_s(z)$  admits a unique real zero in  $(0, 1)$ . Indeed  $S_s(0) = -1$ ,  $S_s(1) > 1$ , and the derivative of the restriction of  $S_s(z)$  to  $[0, 1]$  is positive on  $[0, 1]$ . The polynomial  $S_s^*(z)$ , reciprocal polynomial of  $S_s(z)$ , admits a unique real zero, say  $\gamma_s > 1$ . We have:  $\text{deg}(S_s) \leq s$  and  $\lim_{s \rightarrow \infty} \gamma_s^{-1} = \beta^{-1}$ . The real number  $\gamma_s$  is a nonreciprocal algebraic integer which is such that  $1 < \gamma_s < \beta$ : indeed,  $y \rightarrow S_s(y)$  is strictly increasing on  $(0, 1)$  and  $S_s(\beta^{-1}) = -1 + \sum_{j=1}^s t_j \beta^{-j} = f_\beta(\beta^{-1}) - \sum_{j=s+1}^\infty t_j \beta^{-j} = -\sum_{j=s+1}^\infty t_j \beta^{-j} < 0$ , so that  $\beta^{-1} < \gamma_s^{-1}$ . There exists an integer, say  $W_\nu$ , such that: (i)  $\gamma_s^{-1}$  belongs to the open angular sector (5.4.16) for all  $s \geq W_\nu$  and (ii)  $s \geq W_\nu \implies |P_\beta(\gamma_s^{-1})| < \min\{1, \nu/2\}$ . In the following we take  $s \geq W_\nu$ .

There exists  $j \in \{1, 2, \dots, J_n\}$  such that  $x = \omega_{j,n}$  in the  $j$ -th Rouché disk  $D_{j,n}$ . In this disk  $D_{j,n}$  the polynomial section  $S_s(z)$  has a unique (lenticular) zero; let us denote it by  $r_s$ . We have:  $\lim_{s \rightarrow \infty} r_s = \omega_{j,n}$  and  $r_s$  is equal to the conjugate  $\sigma_s(\gamma_s^{-1})$  of  $\gamma_s^{-1}$  for some



$\sigma_s$  which is the conjugation relative to the irreducible nonreciprocal (never trivial) part  $C$  of  $S_s$  by Proposition 5.25 and Corollary 5.26. We have:  $C(\gamma_s^{-1}) = C(r_s) = 0$ . Denote by  $s_c = \deg(C)$  the degree of the component  $C$ . The irreducible polynomials  $P_\beta(X)$  and  $C(X)$  are coprime: indeed the first one is reciprocal while the second one is nonreciprocal. The integer  $s_c$  is a function of  $s$ . Denote

$$d := \deg(P_\beta) \quad \text{and} \quad H := \max_{j=1, \dots, d-1} \{|a_j|\} \geq 1$$

the (naïve) height of  $P_\beta(X) = 1 + \sum_{j=1}^{d-1} a_j X^j + X^d$ .

(Rewriting trail) **5.30.** [74] *In the same way as above let us construct the rewriting trail from “ $S_s$ ” to “ $P_\beta$ ”, at  $\gamma_s^{-1}$ . The starting point is the identity  $1 = 1$ , to which we add  $0 = S_{\gamma_s}(\gamma_s^{-1})$  in the (rhs) right hand side. Then we define the rewriting trail from the Rényi  $\gamma_s^{-1}$ -expansion of 1*

$$(5.4.20) \quad 1 = 1 + S_{\gamma_s}(\gamma_s^{-1}) = t_1 \gamma_s^{-1} + t_2 \gamma_s^{-2} + \dots + t_{s-1} \gamma_s^{-(s-1)} + t_s \gamma_s^{-s}$$

(with  $t_1 = 1, t_2 = t_3 = \dots = t_{n-1} = 0, t_n = 1$ , etc) to

$$(5.4.21) \quad -a_1 \gamma_s^{-1} - a_2 \gamma_s^{-2} + \dots - a_{d-1} \gamma_s^{-(d-1)} - \gamma_s^{-d} = 1 - P_\beta(\gamma_s^{-1}),$$

by “restoring” the digits of  $1 - P_\beta(X)$  one after the other, from the left. We obtain a sequence  $(A'_q(X))_{q \geq 1}$  of rewriting polynomials involved in this rewriting trail; for  $q \geq 1$ ,  $A'_q \in \mathbb{Z}[X]$ ,  $\deg(A'_q) \leq q$  and  $A'_q(0) = 1$ . At the first step we add  $0 = -(-a_1 - t_1) \gamma_s^{-1} S_{\gamma_s}^*(\gamma_s^{-1})$ ; and we obtain

$$1 = -a_1 \gamma_s^{-1} + (-(-a_1 - t_1)t_1 + t_2) \gamma_s^{-2} + (-(-a_1 - t_1)t_2 + t_3) \gamma_s^{-3} + \dots$$

so that the height of the polynomial

$$(-(-a_1 - t_1)t_1 + t_2)X^2 + (-(-a_1 - t_1)t_2 + t_3)X^3 + \dots$$

is  $\leq H + 2$ . At the second step we add  $0 = -(-a_2 - (-(-a_1 - t_1)t_1 + t_2)) \gamma_s^{-2} S_{\gamma_s}^*(\gamma_s^{-1})$ . Then we obtain

$$1 = -a_1 \gamma_s^{-1} - a_2 \gamma_s^{-2} - [(-a_2 - (-(-a_1 - t_1)t_1 + t_2))t_1 + (-(-a_1 - t_1)t_2 + t_3)] \gamma_s^{-3} + \dots$$

where the height of the polynomial

$$- [(-a_2 - (-(-a_1 - t_1)t_1 + t_2))t_1 + (-(-a_1 - t_1)t_2 + t_3)] X^3 + \dots$$

is  $\leq H + (H + 2) + (H + 2) = 3H + 4$ . Iterating this process  $d$  times we obtain

$$1 = -a_1 \gamma_s^{-1} - a_2 \gamma_s^{-2} - \dots - a_d \gamma_s^{-d} + \text{polynomial remainder in } \gamma_s^{-1}.$$

Denote by  $V(\gamma_s^{-1})$  this polynomial remainder in  $\gamma_s^{-1}$ , for some  $V(X) \in \mathbb{Z}[X]$ , and  $X$  specializing in  $\gamma_s^{-1}$ . If we denote the upper bound of the height of the polynomial remainder  $V(X)$ , at step  $q$ , by  $\lambda_q H + v_q$ , we readily deduce:  $v_q = 2^q$ , and  $\lambda_{q+1} = 2\lambda_q + 1$ ,  $q \geq 1$ , with  $\lambda_1 = 1$ ; then  $\lambda_q = 2^q - 1$ .

To summarize, the first rewriting polynomials of the sequence  $(A'_q(X))_{q \geq 1}$  involved in this rewriting trail are

$$A'_1(X) = -1 - (-a_1 - t_1)X,$$

$$A'_2(X) = -1 - (-a_1 - t_1)X - (-a_2 - (-(-a_1 - t_1)t_1 + t_2))X^2, \quad \text{etc.}$$

For  $q \geq \deg(P_\beta)$ , all the coefficients of  $P_\beta$  are “restored”; denote by  $(h_{q,j})_{j=0,1,\dots,s-1}$  the  $s$ -tuple of integers produced by this rewriting trail, at step  $q$ . It is such that

$$(5.4.22) \quad A'_q(\gamma_s^{-1})S_{\gamma_s}^*(\gamma_s^{-1}) = -P(\gamma_s^{-1}) + \gamma_s^{-q-1} \left( \sum_{j=0}^{s-1} h_{q,j} \gamma_s^{-j} \right).$$

Then take  $q = d$ . The (lhs) left-and side of (5.4.22) is equal to 0. Thus

$$P(\gamma_s^{-1}) = \gamma_s^{-d-1} \left( \sum_{j=0}^{s-1} h_{d,j} \gamma_s^{-j} \right) \quad \implies \quad P(\gamma_s) = \sum_{j=0}^{s-1} h_{d,j} \gamma_s^{-j-1}.$$

The height of the polynomial

$$(5.4.23) \quad W(X) := \sum_{j=0}^{s-1} h_{d,j} X^{j+1} \quad \text{is} \quad \leq (2^d - 1)H + 2^d,$$

and is independent of  $s \geq W_v$ .

For any  $s \geq W_v$ , let us observe that  $-P_\beta(\gamma_s^{-1})$  is  $> 0$ , and that the sequence  $(\gamma_s^{-1})_s$  is decreasing. Indeed, by Proposition 5.28, the polynomial function  $x \rightarrow P_\beta(x)$  is positive on  $(0, \beta^{-1})$ , vanishes at  $\beta^{-1}$ , and changes its sign for  $x > \beta^{-1}$ , so that  $P_\beta(\gamma_s^{-1}) < 0$ . We have:  $\lim_{s \rightarrow \infty} P_\beta(\gamma_s^{-1}) = P_\beta(\beta^{-1}) = 0$ .

Let us use the  $\gamma_s$ -shift and the greedy (Rényi)  $\gamma_s$ -expansion of  $-P_\beta(\gamma_s^{-1})$ : there exists a unique sequence of integers  $(\widehat{t}_i)_{i \geq 1} \not\equiv (0)$  in the alphabet  $\{0, 1\}$  of the  $\gamma_s$ -shift, such that

$$(5.4.24) \quad -P_\beta(\gamma_s^{-1}) = \frac{\widehat{t}_1}{\gamma_s} + \frac{\widehat{t}_2}{\gamma_s^2} + \frac{\widehat{t}_3}{\gamma_s^3} + \dots$$

The integers  $\widehat{t}_i$  are given by the  $\gamma_s$ -transformation  $T_{\gamma_s} : [0, 1] \rightarrow [0, 1], x \rightarrow \{\gamma_s x\}$ , as in (2.2.1). Explicitly, the digits, all in the alphabet  $\{0, 1\}$ , are

$$\begin{aligned} \widehat{t}_1 &= \lfloor \gamma_s(-P_\beta(\gamma_s^{-1})) \rfloor, \\ \widehat{t}_2 &= \lfloor \gamma_s \{ \gamma_s(-P_\beta(\gamma_s^{-1})) \} \rfloor, \\ \widehat{t}_3 &= \lfloor \gamma_s \{ \gamma_s \{ \gamma_s(-P_\beta(\gamma_s^{-1})) \} \} \rfloor, \dots, \end{aligned}$$

and depend upon  $\gamma_s$ .

Since  $\lim_{s \rightarrow \infty} P_\beta(\gamma_s^{-1}) = P_\beta(\beta^{-1}) = 0$  there exists an increasing sequence  $(u_s)_{s \geq W_v}$  of positive integers, satisfying  $\widehat{t}_1 = \widehat{t}_2 = \dots = \widehat{t}_{u_s-1} = 0, \widehat{t}_{u_s} = 1$ , such that the identity between  $-P_\beta(\gamma_s^{-1})$  and its greedy expansion holds, as:

$$(5.4.25) \quad -P_\beta(\gamma_s^{-1}) = \frac{\widehat{t}_{u_s}}{\gamma_s^{u_s}} + \frac{\widehat{t}_{u_s+1}}{\gamma_s^{u_s+1}} + \frac{\widehat{t}_{u_s+2}}{\gamma_s^{u_s+2}} + \dots$$

The sequence  $(u_s)$  is defined by the bounds

$$(5.4.26) \quad |\beta^{u_s}(P_\beta(\gamma_s^{-1}))| \geq |\gamma_s^{u_s}(P_\beta(\gamma_s^{-1}))| \geq 1$$

and

$$|\gamma_s^{u_s}(P_\beta(\gamma_s^{-1}))| \leq \frac{1}{1 - \gamma_s^{-1}}.$$

Therefore, in (5.4.25),

$$(5.4.27) \quad \widehat{t}_i \in \{0, 1\}, \quad i \geq 1, \quad \text{and} \quad \lim_{s \rightarrow +\infty} u_s = +\infty.$$

Now the lhs of (5.4.25) belongs to  $\mathbb{Q}(\gamma_s)$ . For conjugating (5.4.25) by  $\sigma_s$ , if the image by  $\sigma_s$  of the lhs of (5.4.25) belongs to  $\mathbb{Q}(r_s)$ , there are three cases for the conjugation of the rhs of (5.4.25):

- (i) the rhs of (5.4.25) is finite (ends in infinitely many zeroes),
- (ii-1) the rhs of (5.4.25) is eventually periodic (infinite and ultimately periodic),
- (ii-2) the rhs of (5.4.25) is infinite and not eventually periodic.

**Case (i):** say that  $\frac{\widehat{t}_{u_s}}{\gamma_s^{\mu_s}} + \frac{\widehat{t}_{u_s+1}}{\gamma_s^{\mu_s+1}} + \dots + \frac{\widehat{t}_{u_s+N}}{\gamma_s^{\mu_s+N}}$  is the rhs of (5.4.25). Then its image by  $\sigma_s$  is  $\widehat{t}_{u_s} r_s^{\mu_s} + \widehat{t}_{u_s+1} r_s^{\mu_s+1} + \dots + \widehat{t}_{u_s+N} r_s^{\mu_s+N}$  and we have the equality

$$\begin{aligned} P_\beta(r_s) &= \sigma_s \left( - \sum_{j=0}^{s_c-1} h_j'' \gamma_s^{-j-d-2} \right) = - \sum_{j=0}^{s_c-1} h_j'' r_s^{j+d+2} \\ &= \sigma_s \left( \frac{\widehat{t}_{u_s}}{\gamma_s^{\mu_s}} + \frac{\widehat{t}_{u_s+1}}{\gamma_s^{\mu_s+1}} + \dots + \frac{\widehat{t}_{u_s+N}}{\gamma_s^{\mu_s+N}} \right) = \widehat{t}_{u_s} r_s^{\mu_s} + \widehat{t}_{u_s+1} r_s^{\mu_s+1} + \dots + \widehat{t}_{u_s+N} r_s^{\mu_s+N}. \end{aligned}$$

Conjugation by  $\sigma_s$  is done term by term.

**Case (ii-1):** the rhs of (5.4.25) is eventually periodic. Let us write it

$$= \frac{\widehat{t}_{u_s}}{\gamma_s^{\mu_s}} + \frac{\widehat{t}_{u_s+1}}{\gamma_s^{\mu_s+1}} + \dots + \frac{\widehat{t}_{u_s+N}}{\gamma_s^{\mu_s+N}} + \sum_{i=0}^{\infty} \left( \frac{\widehat{t}_{u_s+N+1}}{\gamma_s^{\mu_s+N+iq+1}} + \frac{\widehat{t}_{u_s+N+2}}{\gamma_s^{\mu_s+N+iq+2}} + \dots + \frac{\widehat{t}_{u_s+N+q}}{\gamma_s^{\mu_s+N+iq+q}} \right).$$

The period is not equal to zero. The period length is  $q$ . We have:  $|r_s| = |\sigma_s(\gamma_s^{-1})| < 1$ . Then it is equal to

$$\begin{aligned} &= \frac{\widehat{t}_{u_s}}{\gamma_s^{\mu_s}} + \dots + \frac{\widehat{t}_{u_s+N}}{\gamma_s^{\mu_s+N}} + \sum_{i=0}^{\infty} \gamma_s^{-iq} \left( \frac{\widehat{t}_{u_s+N+1}}{\gamma_s^{\mu_s+N+1}} + \dots + \frac{\widehat{t}_{u_s+N+q}}{\gamma_s^{\mu_s+N+q}} \right) \\ &= \frac{\widehat{t}_{u_s}}{\gamma_s^{\mu_s}} + \dots + \frac{\widehat{t}_{u_s+N}}{\gamma_s^{\mu_s+N}} + \frac{1}{1 - \gamma_s^{-q}} \left( \frac{\widehat{t}_{u_s+N+1}}{\gamma_s^{\mu_s+N+1}} + \dots + \frac{\widehat{t}_{u_s+N+q}}{\gamma_s^{\mu_s+N+q}} \right) \end{aligned}$$

and its image by  $\sigma_s$  is

$$\begin{aligned} &= \widehat{t}_{u_s} r_s^{\mu_s} + \dots + \widehat{t}_{u_s+N} r_s^{\mu_s+N} + \frac{1}{1 - r_s^q} \left( \widehat{t}_{u_s+N+1} r_s^{\mu_s+N+1} + \dots + \widehat{t}_{u_s+N+q} r_s^{\mu_s+N+q} \right) \\ &= \widehat{t}_{u_s} r_s^{\mu_s} + \dots + \widehat{t}_{u_s+N} r_s^{\mu_s+N} + \sum_{i=0}^{\infty} \left( \widehat{t}_{u_s+N+1} r_s^{\mu_s+N+iq+1} + \dots + \widehat{t}_{u_s+N+q} r_s^{\mu_s+N+iq+q} \right). \end{aligned}$$

The series can be conjugated term by term by  $\sigma_s$ ; in this case we have the identity

$$P_\beta(r_s) = \sigma_s \left( \frac{\widehat{t}_{u_s}}{\gamma_s^{\mu_s}} + \frac{\widehat{t}_{u_s+1}}{\gamma_s^{\mu_s+1}} + \dots \right) = \widehat{t}_{u_s} r_s^{\mu_s} + \widehat{t}_{u_s+1} r_s^{\mu_s+1} + \dots$$

**Case (ii-2):** (the reader may have interest in [74] for more details) if the rhs of (5.4.25) is not eventually periodic its conjugation by  $\sigma_s$  cannot be done term by term. This difficulty is overcome by enlarging the alphabet  $\{0, 1\}$  to a bigger alphabet  $\mathcal{A}$  and by replacing the Rényi expansion by a  $(\gamma_s, \mathcal{A})$ -eventually periodic representation of  $-P_\beta(\gamma_s^{-1})$ .

Let us recall the definitions. The  $(\delta, \mathcal{A})$ -representations for a given  $\delta \in \mathbb{C}$ ,  $|\delta| > 1$  and a given alphabet  $\mathcal{A} \subset \mathbb{C}$  finite, are expressions of the form  $\sum_{k \geq -L} a_k \delta^{-k}$ ,  $a_k \in \mathcal{A}$ , for some integer  $L$ . We denote

$$\text{Per}_{\mathcal{A}}(\delta) := \{x \in \mathbb{C} : x \text{ has an eventually periodic } (\delta, \mathcal{A})\text{-representation}\}.$$

**Theorem 5.31** (Kala - Vavra [110]). *Let  $\delta \in \mathbb{C}$  be an algebraic number of degree  $d$ ,  $|\delta| > 1$ , and  $a_d x^d - a_{d-1} x^{d-1} - \dots - a_1 x - a_0 \in \mathbb{Z}[x]$ ,  $a_0 a_d \neq 0$ , be its minimal polynomial. Suppose that  $|\delta'| \neq 1$  for any conjugate  $\delta'$  of  $\delta$ . Then there exists a finite alphabet  $\mathcal{A} \subset \mathbb{Z}$  such that  $\mathbb{Q}(\delta) = \text{Per}_{\mathcal{A}}(\delta)$ .*

Let us apply Theorem 5.31 to  $\delta = \gamma_s$ . By Proposition 5 in [73]  $\gamma_s$  has no conjugate of modulus 1. Therefore there exists a finite alphabet  $\mathcal{A} \subset \mathbb{Z}$  such that the lhs of (5.4.25) be identified with a  $(\gamma_s, \mathcal{A})$ -representation which is eventually periodic, for some integer  $u_s \in \mathbb{Z}$ :

$$(5.4.28) \quad -P_\beta(\gamma_s^{-1}) = \frac{\widehat{t_{u_s}}}{\gamma_s^{u_s}} + \frac{\widehat{t_{u_s+1}}}{\gamma_s^{u_s+1}} + \frac{\widehat{t_{u_s+2}}}{\gamma_s^{u_s+2}} + \dots$$

Being eventually periodic, the representation (5.4.28) can now be conjugated term by term by  $\sigma_s$ , since  $|\sigma_s(\gamma_s^{-1})| < 1$ , as in case (ii-1). In (5.4.28) the digits  $\widehat{t_i}$  belong to a symmetrical alphabet  $\mathcal{A} = \{-m, \dots, 0, \dots, m\}$ ; the integer  $m$  is provided by the rewriting trail, given in “(Rewriting trail). 5.30”, and (5.4.23): we have  $m = \lceil 2((2^d - 1)H + 2^d)/3 \rceil$ .

Indeed, by Theorem 5.31 there exist a preperiod  $R(X) \in \mathcal{A}[X]$ , a period  $T(X) \in \mathcal{A}[X]$  such that

$$\widehat{W}(\gamma_s^{-1}) := -P_\beta(\gamma_s) = R(\gamma_s^{-1}) + \gamma_s^{-\deg R - 1} \sum_{j=0}^{\infty} \frac{1}{\gamma_s^{j(\deg T + 1)}} T(\gamma_s^{-1}),$$

both polynomials  $R$  and  $T$  depending upon  $s$ . Since the relation  $S_{\gamma_s}(\gamma_s^{-1}) = -1 + t_1 \gamma_s^{-1} + t_2 \gamma_s^{-2} + \dots + t_{s-1} \gamma_s^{-s+1} + t_s \gamma_s^{-s} = 0$  holds, we may assume  $\deg R \leq s - 1$ ,  $\deg T \leq s - 1$ . Then, for  $X$  specialized at  $\gamma_s^{-1}$ , we have the identity

$$(5.4.29) \quad \widehat{W}(X) = R(X) + X^L \frac{T(X)}{1 - X^r}$$

for some positive integers  $L, r$ . The height of  $(1 - X^r) \widehat{W}(X)$  is  $\leq 2((2^d - 1)H + 2^d)$  and, with  $\mathcal{A}$  assumed  $= \{-m, \dots, 0, \dots, m\}$ , the height of  $(1 - X^r)R(X) + X^L T(X)$  is less than  $3m$ . Therefore  $m$  is  $\leq 2((2^d - 1)H + 2^d)/3$ . We can take  $m = \lceil 2((2^d - 1)H + 2^d)/3 \rceil$ . The alphabet  $\mathcal{A} = \{-m, \dots, m\}$  only depends upon the degree  $d$  and the height  $H$  of the polynomial  $P_\beta$ , and does not depend upon  $s$ .

We now assume  $0 \neq |P(\gamma_s)| \ll 1$ . The  $(\gamma_s, \mathcal{A})$ -eventually periodic representation of  $-P(\gamma_s)$  starts as

$$-P(\gamma_s) = \widehat{W}(\gamma_s^{-1}) = \frac{\widehat{t}_{u_s}}{\gamma_s^{\mu_s}} + \frac{\widehat{t}_{u_s+1}}{\gamma_s^{\mu_s+1}} + \frac{\widehat{t}_{u_s+2}}{\gamma_s^{\mu_s+2}} + \dots, \quad \text{with } |\widehat{t}_j| \leq m, j = u_s, u_s + 1, \dots$$

with  $\widehat{t}_{u_s} \neq 0$ . The exponent  $u_s$  appearing in the first term is defined in [88]; by Theorem 4, Remarks 5 to 7, in [88], there exists a positive real number  $\kappa_{\gamma_s, \mathcal{A}} > 0$  such that  $u_s$  is the minimal integer such that

$$\gamma_s^{\mu_s-1} \geq \frac{\kappa_{\gamma_s, \mathcal{A}}}{|P(\gamma_s)|}.$$

Since  $\lim_{s \rightarrow \infty} \gamma_s = \beta > 1$  and that the alphabet  $\mathcal{A}$  does not depend upon  $s$ , from Theorem 4, Remarks 5 to 7, in [88], we can replace  $\kappa_{\gamma_s, \mathcal{A}}$  by a constant  $\kappa > 0$ , independent of  $s$  (cf also [74]). Thus

$$\lim_{s \rightarrow \infty} u_s = +\infty.$$

The sequence  $(u_s)$  is here defined by the bounds

$$(5.4.30) \quad |\beta^{u_s-1}(P_\beta(\gamma_s^{-1}))| \geq |\gamma_s^{\mu_s-1}(P_\beta(\gamma_s^{-1}))| \geq \kappa$$

and

$$|\gamma_s^{\mu_s}(P_\beta(\gamma_s^{-1}))| \leq \frac{m}{1 - \gamma_s^{-1}}.$$

To the collection  $(\gamma_s)_{s \geq W_v}$  is associated the collection  $(\sigma_s : \gamma_s \rightarrow r_s)_{s \geq W_v}$  of  $\mathbb{Q}$ -automorphisms of  $\mathbb{C}$ . Now, for any  $s \geq W_v$ , let us conjugate the eventually periodic representation of  $-P_\beta(\gamma_s^{-1})$  by  $\sigma_s$ , term by term. For the cases (i) and (ii-1) we consider (5.4.25), and in the case (ii-2) we consider (5.4.28). Using (5.4.16) we deduce:

**Case (i) and (ii-1):** with the minimal alphabet  $\{-1, 0, +1\}$ ,

$$(5.4.31) \quad |P_\beta(r_s)| \leq |r_s|^{u_s} \frac{1}{1 - |r_s|} \leq \frac{n}{c_{lent}} \left(1 - \frac{c_{lent}}{n}\right)^{u_s},$$

**Case (ii-2):** with the alphabet  $\mathcal{A} = \{-m, \dots, +m\}$ ,

$$(5.4.32) \quad |P_\beta(r_s)| \leq |r_s|^{u_s} \frac{m}{1 - |r_s|} \leq \frac{nm}{c_{lent}} \left(1 - \frac{c_{lent}}{n}\right)^{u_s}.$$

In both cases,  $\lim_{s \rightarrow \infty} u_s = +\infty$ . The rhs of (5.4.31), resp. of (5.4.32), tends to 0 if  $s$  tends to infinity. We have:  $\lim_{s \rightarrow \infty} P_\beta(r_s) = 0$ . But  $x = \lim_{s \rightarrow \infty} r_s$  and  $z \rightarrow P_\beta(z)$  is continuous. Therefore there exists  $s_0 \geq W_v$  such that  $s \geq s_0 \implies |P_\beta(r_s)| < v/2$ . Contradiction.

The only limit possibility is  $P_\beta(x) = 0$ , for all the lenticular zeroes  $x$  of  $f_\beta$ . This provides a collection of lenticular zeroes of  $P_\beta(z)$  in the cusp, and implies a complete identification of these zeroes with the lenticular poles of  $\zeta_\beta(z)$ .  $\square$

To summarize,

**Proposition 5.32.** *Under the assumptions of Definition 5.23, if  $\Omega$  is a lenticular zero of  $f_\beta(z)$ , then*

$$f_\beta(\Omega) = 0 \quad \implies \quad P_\beta(\Omega) = 0,$$

where  $P_\beta$  is the minimal polynomial of  $\beta$ .

*Nota:* the reader will notice that there is no assumption of continuity of the conjugation  $\sigma : \beta \rightarrow x$  in the proof of Proposition 5.29.

A direct proof of Proposition 5.29 is given in [74].

**5.5. Continuity of the lenticular Galois conjugates in the cusp.** Let  $n \geq 260$ . To  $n$  is associated the set of the lenticular zeroes of the trinomial  $-1 + x + x^n$  of imaginary part  $\geq 0$ , as:

$$\{\theta_n, z_{1,n}, z_{2,n}, \dots, z_{J_n,n}\},$$

and the set of Rouché disks

$$\{D_{1,n}D_{2,n}, \dots, D_{J_n,n}\},$$

the  $j$ th-disk  $D_{j,n}$  being centered at  $z_{j,n}$ . For  $j = 1, \dots, J_n$ , the disks  $D_{j,n}$  satisfy:  $\overline{D_{j,n}} \subset D(0, 1)$ ,  $\overline{D_{j,n}} \cap C(0, 1) = \emptyset$ ,  $z_{j,n-1} \in D_{j,n}$ , and, for any real number  $\beta \in [\theta_n^{-1}, \theta_{n-1}^{-1}]$ ,  $f_\beta(z)$  admits an unique zero  $\omega_{j,n}$  in  $D_{j,n}$ , which is simple (Theorem 5.14). Since  $\beta \rightarrow f_\beta(z)$  is injective on  $[\theta_n^{-1}, \theta_{n-1}^{-1}]$ , the map  $\beta \rightarrow \omega_{j,n}$  is well-defined. Let us denote this map by  $\omega_{j,n}$  and by  $\omega_{j,n}(\beta)$  the image of  $\beta$  (instead of  $\omega_{j,n}$ ). Let  $\widetilde{\mathcal{S}}_{j,n} := \overline{\omega_{j,n}([\theta_n^{-1}, \theta_{n-1}^{-1}])}$  be the adherence of the image of the closed interval  $[\theta_n^{-1}, \theta_{n-1}^{-1}]$ ; it is a compact subset of  $D_{j,n}$  such that  $\widetilde{\mathcal{S}}_{j,n} \cap \partial D_{j,n} = \emptyset$ , for which the image of the left extremity of the interval is the center of the disk  $D_{j,n}$ :  $\omega_{j,n}(\theta_n^{-1}) = z_{j,n}$ .

**Lemma 5.33.** *Let  $n \geq 3$ . The two analytic functions  $-1 + z + z^{n-1}$  and  $-1 + z + z^n + \sum_{q=1}^{\infty} z^{q(n-1)+n}$  have the same zeroes (with the same multiplicities = 1) inside the open unit disk.*

*Proof.* Let  $x_0, |x_0| < 1$ , be a zero of the trinomial  $-1 + z + z^{n-1}$ . Then

$$0 = -1 + x_0 + x_0^{n-1} = -1 + x_0 + x_0^n \frac{1}{x_0} = -1 + x_0 + x_0^n (1 + x_0^{n-2}) = -1 + x_0 + x_0^n + x_0^{n+n-2+1} \frac{1}{x_0}.$$

Let us replace the last  $\frac{1}{x_0}$  by  $1 + x_0^{n-2}$ . And so on, iteratively. Doing this operation infinitely many times provides the identity

$$0 = -1 + x_0 + x_0^{n-1} = -1 + x_0 + x_0^n + \sum_{q=1}^{\infty} x_0^{q(n-1)+n}.$$

The converse comes from

$$-1 + z + z^n + \sum_{q=1}^{\infty} z^{q(n-1)+n} = \frac{-1 + z + z^{n-1}}{1 - z^{n-1}}, \quad |z| < 1.$$

Multiplicities are equal to 1 by [185]. □

Denote  $\delta := \min\{1 - |z| \mid z \in \cup_{j=1}^{J_n} \widetilde{\mathcal{S}}_{j,n}\} > 0$ .

**Proposition 5.34.** *Let  $n \geq 260$ . For all  $1 \leq j \leq J_n$ , the map*

$$\omega_{j,n} : [\theta_n^{-1}, \theta_{n-1}^{-1}] \rightarrow D_{j,n}, \quad \beta \rightarrow \omega_{j,n}(\beta)$$

*is continuous.*

*Proof.* Let  $\beta_1, \beta_2$  be two real numbers in the open interval  $(\theta_n^{-1}, \theta_{n-1}^{-1})$ , and assume  $\beta_1 < \beta_2$ . To  $\beta_1$ , resp.  $\beta_2$ , is associated uniquely the sequence  $(t_i)_{i \geq 1} \in \{0, 1\}$ , resp.  $(t'_i)_{i \geq 1} \in \{0, 1\}$ , of the coefficients of the Rényi  $\beta_1$ -expansion of unity  $d_{\beta_1}(1) = 0.t_1 t_2 t_3 \dots$ , resp. of  $d_{\beta_2}(1) = 0.t'_1 t'_2 t'_3 \dots$ ; the two Parry Upper functions  $f_{\beta_1}(z)$  and  $f_{\beta_2}(z)$  are constructed from these sequences, by Definition 3.1. When  $\beta_2$  is close to  $\beta_1$ , the inequality ' $<$ ' is translated by the lexicographical inequality ' $<_{lex}$ ' on the sequences  $(t_i)_{i \geq 1}$ , resp.  $(t'_i)$ , by Proposition 2.4. The first digits are the same. We define the lexicographical metric  $\bar{d}$  by:

$$\bar{d}(\beta_1, \beta_2) := e^{-r} \quad \text{iff} \quad t_1 = t'_1, t_2 = t'_2, \dots, t_r = t'_r, t_{r+1} \neq t'_{r+1}.$$

To prove the continuity of  $\omega_{j,n}$  it suffices to show that, for all  $\varepsilon > 0$ , there exists  $\eta > 0$  such that

$$\bar{d}(\beta_1, \beta_2) < \eta \quad \implies \quad |\omega_{j,n}(\beta_1) - \omega_{j,n}(\beta_2)| < \varepsilon,$$

and to establish the continuity at the extremity  $\theta_{n-1}^{-1}$  of the closed interval  $[\theta_n^{-1}, \theta_{n-1}^{-1}]$ .

We have

$$f_{\beta_1}(\beta_1^{-1}) = f_{\beta_1}(\omega_{j,n}(\beta_1)) = f_{\beta_2}(\beta_2^{-1}) = f_{\beta_2}(\omega_{j,n}(\beta_2)) = 0.$$

Since  $\beta_1 \neq \beta_2$  and that the disks  $D_{j,n}$  have the property that  $f_{\beta_2}(z)$  contains an unique zero in it (cf Theorem 5.14),

$$\begin{aligned} 0 \neq f_{\beta_2}(\omega_{j,n}(\beta_1)) &= f_{\beta_2}(\omega_{j,n}(\beta_2) + (\omega_{j,n}(\beta_1) - \omega_{j,n}(\beta_2))) \\ &= (\omega_{j,n}(\beta_1) - \omega_{j,n}(\beta_2)) \left[ \sum_{q=1}^{\infty} \frac{f_{\beta_2}^{(q)}(\omega_{j,n}(\beta_2))}{q!} (\omega_{j,n}(\beta_1) - \omega_{j,n}(\beta_2))^{q-1} \right]. \end{aligned}$$

Now by Theorem 5.14 the unique zero  $\omega_{j,n}(\beta_2^{-1})$  in  $D_{j,n}$  is simple. Thus  $f'_{\beta_2}(\omega_{j,n}(\beta_2)) \neq 0$  and the function

$$z \rightarrow f_{\beta_2}(z)/(z - \omega_{j,n}(\beta_2)),$$

extended by continuity by  $f'_{\beta_2}(\omega_{j,n}(\beta_2))$  at  $\omega_{j,n}(\beta_2)$ , does not take the value 0 on the compact  $\widetilde{\mathcal{S}}_{j,n}$ . Hence, the function

$$z \rightarrow \left| \sum_{q=1}^{\infty} \frac{f_{\beta_2}^{(q)}(\omega_{j,n}(\beta_2))}{q!} (z - \omega_{j,n}(\beta_2))^{q-1} \right|$$

admits an infimum, say  $\tilde{\mu}_j$  on  $\widetilde{\mathcal{S}}_{j,n}$ . Denote  $\tilde{\mu} := \min\{\tilde{\mu}_j \mid j = 1, 2, \dots, J_n\} > 0$  the infimum, common to all the disks  $D_{j,n}$ .

Now, with  $\bar{d}(\beta_1, \beta_2) := e^{-r}$ ,

$$f_{\beta_2}(\omega_{j,n}(\beta_1)) = (f_{\beta_2} - f_{\beta_1})(\omega_{j,n}(\beta_1)) = \sum_{k=r+1}^{\infty} (t_k - t'_k)(\omega_{j,n}(\beta_1))^k.$$

Therefore

$$|f_{\beta_2}(\omega_{j,n}(\beta_1))| \leq \sum_{k=r+1}^{\infty} |\omega_{j,n}(\beta_1)|^k \leq \sum_{k=r+1}^{\infty} (1 - \delta)^k \leq (1 - \delta)^{r+1} \frac{1}{\delta}.$$

Hence

$$|\omega_{j,n}(\beta_1) - \omega_{j,n}(\beta_2)| \leq \frac{1}{\tilde{\mu}}(1 - \delta)^{r+1} \frac{1}{\delta}.$$

It suffices to take  $r$  large enough to have the property of continuity.

Let us consider the case of the extremity  $\beta_2 = \theta_{n-1}^{-1}$ . By Lemma 5.33, since  $\theta_{n-1}$  is root of  $-1 + x + x^{n-1}$  it satisfies:

$$0 = -1 + \theta_{n-1} + \theta_{n-1}^n + \sum_{q=1}^{\infty} \theta_{n-1}^{q(n-1)+n}.$$

The metric  $\bar{d}$  can be extended to  $(\theta_n^{-1}, \theta_{n-1}^{-1}] \times (\theta_n^{-1}, \theta_{n-1}^{-1}]$  since the sequence of coefficients of the power series  $-1 + z + z^n + \sum_{q \geq 1} z^{q(n-1)+n}$  is obviously in the adherence of the set

$$\{(t_i) \mid \text{the digits } t_i \text{ being those of } d_{\beta}(1) \text{ for all } \beta \in (\theta_n^{-1}, \theta_{n-1}^{-1})\},$$

and not an isolated point. The case is simpler for the other extremity  $\beta_1 = \theta_n^{-1}$ . □

**Remark 5.35.** For every  $j = 1, \dots, J_n$ , the image  $\omega_{j,n}([\theta_n^{-1}, \theta_{n-1}^{-1}] \cap \overline{\mathbb{Q}})$  is dense in  $\widetilde{\mathcal{S}}_{j,n}$ . The restriction of  $\omega_{j,n}$  to the collection of the algebraic integers in  $[\theta_n^{-1}, \theta_{n-1}^{-1}]$  is continuous, i.e. the lenticular Galois conjugates  $\omega_{j,n}(\beta)$  of an algebraic integer  $\beta$  are continuous functions of  $\beta$ . Comparatively the degree  $\deg \beta$  has good chances to vary very chaotically with  $\beta$ , if  $\beta$  varies continuously.

Now, seeking very small Mahler measures, below Lehmer's number, calls for the existence of reciprocal algebraic integers in the intervals  $(\theta_n^{-1}, \theta_{n-1}^{-1})$ .

Suppose the existence of a reciprocal algebraic integer  $\beta > 1$  in the interval  $(\theta_n^{-1}, \theta_{n-1}^{-1})$ . As a consequence of Proposition 5.28 and Proposition 5.29, the minimal polynomial  $P_{\beta}(X)$  of  $\beta$  would be the product of two components, the lenticular part, with roots inside and outside the unit circle, say

$$(X - \beta)(X - 1/\beta) \times \prod_{\substack{\text{lenticular roots} \\ \text{Im}\beta^{(j)} > 0, |\beta^{(j)}| < 1}} (X - \beta^{(j)})(X - \overline{\beta^{(j)}})(X - \beta^{(j)^{-1}})(X - \overline{(\beta^{(j)^{-1}})})$$

by the non-lenticular part

$$\prod_{\text{non-lenticular roots}} (X - \beta^{(j)}),$$

with the identifications between the Galois conjugates and the lenticular poles of  $\zeta_{\beta}(z)$ :

$$\beta^{(j)} = \omega_{j,n}(\beta), \quad j = 1, 2, \dots, J_n.$$

The problem of the localization of the lenticular Galois conjugates  $\beta^{(j)}$  of  $\beta$  then calls first for a geometrical characterization of the supports which contain these conjugates, that is of the compact sets  $\widetilde{\mathcal{S}}_{j,n}$  in the Rouché disks  $D_{j,n}$ .

In [84] Flatto, Lagarias and Poonen study the continuity of the modulus of the first root

$$(5.5.1) \quad \beta \rightarrow |\omega_{1,n}(\beta)|,$$

over the union of the intervals  $[\theta_n^{-1}, \theta_{n-1}^{-1}]$ . Their Figure 1 summarizes the fluctuations of the map (5.5.1), for  $1 < \beta < 2$ . The curve given by Figure 1 in [84] suggests that the map



$\omega_{1,n}$  is injective on  $(\theta_n^{-1}, \theta_{n-1}^{-1})$  but that the union  $\widetilde{\mathcal{S}}_{1,n-1} \cup \widetilde{\mathcal{S}}_{1,n} \cup \widetilde{\mathcal{S}}_{1,n+1}$ , for all  $n$  large enough, is a self-intersecting object [75].

**5.6. Minoration of the Mahler measure: a continuous lower bound.** Let  $n \geq 260$  and  $\beta > 1$  be an algebraic integer such that  $\theta_n^{-1} < \beta < \theta_{n-1}^{-1}$ , with  $M(\alpha) < 1.176280\dots$ . The factorization of the minimal polynomial of  $\beta$  is

$$(5.6.1) \quad P_\beta(z) = \prod_{\gamma \in \mathcal{L}_\beta} (z - \gamma) \times \prod_{\gamma \notin \mathcal{L}_\beta} (z - \gamma).$$

From Proposition 5.28 and Proposition 5.29 the lenticular zeroes  $\omega_{j,n}$  of the Parry Upper function  $f_\beta(z)$  are zeroes of the minimal polynomial of  $\beta$ . The lenticulus of conjugates is  $\mathcal{L}_\beta := \{\beta^{-1}\} \cup \bigcup_{j=1}^{J_n} (\{\omega_{j,n}\} \cup \{\overline{\omega_{j,n}}\})$ . In this Section we investigate the product, called *lenticular Mahler measure of  $\beta$* , defined by

$$(5.6.2) \quad M_r(\beta) := \prod_{\substack{\gamma \text{ conjugate of } \beta^{-1}, |\gamma| < 1 \\ \text{lenticular}}} |\gamma|^{-1}$$

and its asymptotic expansion. The subscript ‘‘r’’ added to the ‘‘M’’ of the Mahler measure stands for ‘‘reduced to the lenticulus’’. We have:  $M_r(\beta) \leq M(\beta)$ .

First let us complete Theorem 5.5.

**Proposition 5.36.** *Let  $n \geq 260$  and  $\beta > 1$  a reciprocal algebraic integer such that  $\text{dyg}(\beta) = n$ , with  $M(\alpha) < 1.176280\dots$ . Let  $\Omega_n$  be the subdomain of the open unit disk defined by the union of*

$$(5.6.3) \quad \left\{ z \mid |z| < 1 - \frac{c_n}{n}, |\arg z| > \arg z_{J_n, n} + \frac{\pi}{n a_{\max}} \right\},$$

$$(5.6.4) \quad \left\{ z \mid |z| < 1 - \frac{c_{\text{lent}}}{n}, |\arg z| < \arg(z_{J_n, n}) + \frac{\pi}{n a_{\max}} \right\},$$

and, for  $j = J_n + 1, \dots, 2J_n - H_n + 1$ ,

$$(5.6.5) \quad \frac{\pi |z_{j,n}|}{n s_{j,n}} < |z - z_{j,n}|, \quad \text{with } s_{j,n} = a_{\max} \left[ 1 + \frac{a_{\max}^2 (j - J_n)^2}{\pi^2 J_n^2} \right]^{-1/2}.$$

and symmetrically by complex conjugation with respect to the real axis.

Then the minimal polynomial  $P_\beta(X)$  of  $\beta$  is fracturable in the domain  $\Omega_n$  in the sense that the invertible power series  $U_\beta(z) = -\zeta_\beta(z) P_\beta(z) \in \mathbb{Z}[[z]]$ , satisfying  $P_\beta(z) = U_\beta(z) f_\beta(z)$ , is not constant, does not vanish and is holomorphic in  $\Omega_n$ . It satisfies

$$(5.6.6) \quad U_\beta(\omega_{j,n}) = \frac{P'_\beta(\omega_{j,n})}{f'_\beta(\omega_{j,n})} \neq 0, \quad j = 1, 2, \dots, J_n, \quad U_\beta(\beta^{-1}) = \frac{P'_\beta(\beta^{-1})}{f'_\beta(\beta^{-1})} \neq 0,$$

and obeys the Carlson-Polya dichotomy, simultaneously with  $f_\beta(z)$  as in Theorem 5.5.

*Proof.* The domain of holomorphy of  $U_\beta(z)$  contains  $D(0, \theta_{\text{dyg}(\beta)-1})$  by Theorem 5.5. The roots of the minimal polynomial  $P_\beta(z)$  are simple. The roots of  $f_\beta(z)$  in  $\Omega_n$  are also simple by Theorem 5.14 and Theorem 5.22. The lenticular roots of  $f_\beta(z)$  coincide with roots of  $P_\beta(z)$  by Proposition 5.28 and Proposition 5.29. We deduce the fracturability of  $P_\beta(z)$  on  $\Omega_n$ . The relations (5.6.6) follow from the derivatives of the identity  $P_\beta(z) = U_\beta(z) \times f_\beta(z)$ .

The unit circle is the natural boundary of  $U_\beta(z)$  if and only if  $\beta$  is not a Parry number, by Theorem 5.5.  $\square$

The modulus of the second smallest root of  $f_\beta(z)$  is a continuous function of  $\beta$  [84]. Theorem 5.37 extends this result.

**Theorem 5.37.** *Let  $n \geq 260$ . Let  $\beta > 1$  be a reciprocal algebraic integer such that  $\text{dyg}(\beta) = n$ , with  $M(\alpha) < 1.176280\dots$ . The product, called lenticular Mahler measure of  $\beta$ , defined by*

$$(5.6.7) \quad M_r(\beta) := \prod_{\omega \in \mathcal{L}_\beta} |\omega|^{-1}$$

is a continuous function of  $\beta$  on the open interval  $(\theta_n^{-1}, \theta_{n-1}^{-1})$ , which admits the following left and right limits

$$(5.6.8) \quad \lim_{\beta \rightarrow \theta_{n-1}^{-1-}} M_r(\beta) = \prod_{\omega \in \mathcal{L}_{\theta_{n-1}^{-1}}} |\omega|^{-1} = \theta_{n-1}^{-1} \times \prod_{\substack{1 \leq j \leq J_n \\ z_{j,n-1} \in \mathcal{L}_{\theta_{n-1}^{-1}}} |z_{j,n-1}|^{-2},$$

$$(5.6.9) \quad \lim_{\beta \rightarrow \theta_n^{-1+}} M_r(\beta) = \prod_{\omega \in \mathcal{L}_{\theta_n^{-1}}} |\omega|^{-1} = \theta_n^{-1} \times \prod_{\substack{1 \leq j \leq J_n \\ z_{j,n} \in \mathcal{L}_{\theta_n^{-1}}} |z_{j,n}|^{-2}.$$

The discontinuity (jump) of  $M_r(\beta)$  at the Perron number  $\theta_{n-1}^{-1}$ , given in the multiplicative form by

$$(5.6.10) \quad \frac{\lim_{\beta \rightarrow \theta_{n-1}^{-1-}} M_r(\beta)}{\lim_{\beta \rightarrow \theta_{n-1}^{-1+}} M_r(\beta)} = |z_{J_n, n-1}|^{-2},$$

tends to 1 (i.e. disappears at infinity) when  $n = \text{dyg}(\beta)$  tends to infinity.

*Proof.* From Corollary 3.14 in §3.4 all the maps  $\beta \rightarrow \omega(\beta) \in \mathcal{L}_\beta$  are continuous. Now the identification of the zeroes of the Parry Upper function  $f_\beta(z)$  as conjugates of  $\beta$ , from Theorem 5.36, allows to consider this continuity property as a continuity property over the conjugates of  $\beta$  which define the lenticulus  $\mathcal{L}_\beta$ . As a consequence all the maps  $\beta \rightarrow |\omega(\beta)| \in \mathcal{L}_\beta$ , are continuous, as well as their product (5.6.7). Let  $1 \leq j \leq J_n$ . Let us prove that  $z_{j,n-1} \in D_{j,n} = \{z \mid |z - z_{j,n}| < \frac{\pi |z_{j,n}|}{n a_{\max}}\}$ . Indeed,

$$|z_{j,n}| = 1 + \frac{1}{n} \text{Log} \left( 2 \sin \left( \frac{\pi j}{n} \right) \right) + \dots, \quad \arg(z_{j,n}) = \dots$$

and

$$|z_{j,n-1}| = 1 + \frac{1}{n-1} \text{Log} \left( 2 \sin \left( \frac{\pi j}{n-1} \right) \right) + \dots, \quad \arg(z_{j,n-1}) = \dots$$

so that, easily,

$$(5.6.11) \quad |z_{j,n} - z_{j,n-1}| < \frac{\pi |z_{j,n}|}{n a_{\max}}.$$

The image of the interval  $(\theta_n^{-1}, \theta_{n-1}^{-1}) \cap \mathcal{O}_{\mathbb{Q}}$  by a map  $\beta \rightarrow \omega_{j,n}(\beta) \in \mathcal{L}_\beta$  is a curve in  $D_{j,n}$  over  $\mathcal{O}_{\mathbb{Q}}$  with extremities  $z_{j,n}$  and  $z_{j,n-1}$ , both in  $D_{j,n}$  by (5.6.11). This curve does not intersect itself. Indeed, if it would be a self-intersecting curve we would have, for two

distinct algebraic integers  $\beta$  and  $\beta'$ , the same conjugate in  $D_{j,n}$ , what is impossible since  $P_\beta$  and  $P'_\beta$  are both irreducible, and therefore they cannot have a root in common. This curve does not ramify either by the uniqueness property imposed locally by the Theorem of Rouché. We deduce the left limit (5.6.8) and the right limit (5.6.9) by continuity.  $\square$

**Remark 5.38.** Decomposing the Mahler measure gives

$$M(\beta) = \prod_{\omega \in \mathcal{L}_\beta} |\omega|^{-1} \times \prod_{\substack{\omega \notin \mathcal{L}_\beta, |\omega| < 1 \\ P_\beta(\omega) = 0}} |\omega|^{-1}.$$

Theorem 5.37, for which the Rouché method has been applied, shows the continuity of the partial product  $\beta \rightarrow \prod_{\omega \in \mathcal{L}_\beta} |\omega|^{-1}$ , associated with the identified lenticulus of conjugates of  $\beta$ , with  $\beta$  running over each open interval of extremities two successive Perron numbers  $\theta_n^{-1}$ . It is very probable that a method finer than the method of Rouché would lead to a higher value of  $J_n$ , to more zeroes of  $f_\beta(z)$  identified as conjugates of  $\beta$ , and the disappearance of the discontinuities (jumps) in (5.6.10).

**Theorem 5.39.** *Let  $\beta > 1$  be a reciprocal algebraic integer such that  $\text{dyg}(\beta) \geq 260$ , with  $M(\alpha) < 1.176280\dots$ . Denote  $\kappa = \kappa(1, a_{\max})$ . The Mahler measure  $M(\beta)$  is bounded from below by the lenticular Mahler measure of  $\beta$  as*

$$M(\beta) = M_r(\beta) \times \prod_{\substack{\omega \notin \mathcal{L}_\beta, |\omega| < 1 \\ P_\beta(\omega) = 0}} |\omega|^{-1} \geq M_r(\beta).$$

Denoting

$$\Lambda_r := \exp\left(\frac{-1}{\pi} \int_0^{2\arcsin(\frac{\kappa}{2})} \text{Log}\left(2 \sin\left(\frac{x}{2}\right)\right) dx\right) = 1.16302\dots,$$

and

$$\begin{aligned} \mu_r &:= \exp\left(\frac{-1}{\pi} \int_0^{2\arcsin(\frac{\kappa}{2})} \text{Log}\left[\frac{1 + 2 \sin(\frac{x}{2}) - \sqrt{1 - 12 \sin(\frac{x}{2}) + 4(\sin(\frac{x}{2}))^2}}{8 \sin(\frac{x}{2})}\right] dx\right) \\ &= 0.992337\dots, \end{aligned}$$

the lenticular Mahler measure  $M_r(\beta)$  of  $\beta$  admits a *liminf* and a *limsup* when  $\beta$  tends to  $1^+$ , equivalently when  $\text{dyg}(\beta)$  tends to infinity, respectively bounded from below and above as

$$(5.6.12) \quad \liminf_{\text{dyg}(\beta) \rightarrow +\infty} M_r(\beta) \geq \Lambda_r \cdot \mu_r = 1.15411\dots,$$

$$(5.6.13) \quad \limsup_{\text{dyg}(\beta) \rightarrow +\infty} M_r(\beta) \leq \Lambda_r \cdot \mu_r^{-1} = 1.172\dots$$

Then the “limit minorant” of the Mahler measure  $M(\beta)$  of  $\beta$ ,  $\beta > 1$  running over  $\mathcal{O}_{\overline{\mathbb{Q}}}$ , when  $\text{dyg}(\beta)$  tends to infinity, is given by

$$(5.6.14) \quad \liminf_{\text{dyg}(\beta) \rightarrow \infty} M(\beta) \geq \Lambda_r \cdot \mu_r = 1.15411\dots$$

*Proof.* The value  $2 \arcsin(\kappa(1, a_{\max})/2) = 0.171784\dots$  is given by Proposition 5.11, and  $a_{\max} = 5.8743\dots$  by Theorem 5.8. The variations of the Mahler measure  $M(\beta)$  of  $\beta$  can be fairly large when  $\beta$  approaches  $1^+$ . On the contrary the lenticular Mahler measure  $M_r(\beta)$  is a continuous function of  $\beta$  on  $(1, \theta_{260}^{-1})$  except at the point discontinuities which are the Perron numbers  $\theta_n^{-1}$  by (5.6.8), (5.6.9) and (5.6.10), writing  $n = \text{dyg}(\beta)$  for short.

First, by Proposition 5.11 let us observe that the Riemann-Stieltjes sum

$$S(f, n) := -2 \sum_{j=1}^{J_n} \frac{1}{n} \text{Log} \left( 2 \sin \left( \frac{\pi j}{n} \right) \right) = \frac{-1}{\pi} \sum_{j=1}^{J_n} (x_j - x_{j-1}) f(x_j)$$

with  $x_j = \frac{2\pi j}{n}$  and  $f(x) := \text{Log} \left( 2 \sin \left( \frac{x}{2} \right) \right)$  converges to the limit

$$(5.6.15) \quad \lim_{n \rightarrow \infty} S(f, n) = \frac{-1}{\pi} \int_0^{0.171784\dots} f(x) dx = \text{Log} \Lambda_r = \text{Log}(1.16302\dots).$$

This limit is a log-sine integral [30] [31]. Let us now show how  $\Lambda_r$  is related to  $\liminf_{\text{dyg}(\beta) \rightarrow \infty} M_r(\beta)$  and  $\limsup_{\text{dyg}(\beta) \rightarrow \infty} M_r(\beta)$  to deduce (5.6.12) and (5.6.13).

Taking only into account the lenticular zeroes of  $P_\beta(z)$ , which constitute the lenticulus  $\mathcal{L}_\beta$ , from Theorem 5.8 and Proposition 5.13, we obtain

$$(5.6.16) \quad \begin{aligned} \text{Log} M_r(\beta) &= -\text{Log} \left( \frac{1}{\beta} \right) - 2 \sum_{j=1}^{J_n} \text{Log} |\omega_{j,n}| = \text{Log} \left( \frac{1}{\beta} \right) - 2 \sum_{j=1}^{J_n} \text{Log} |(\omega_{j,n} - z_{j,n}) + z_{j,n}| \\ &= \text{Log}(\beta) - 2 \sum_{j=1}^{J_n} \text{Log} |z_{j,n}| - 2 \sum_{j=1}^{J_n} \text{Log} \left| 1 + \frac{\omega_{j,n} - z_{j,n}}{z_{j,n}} \right|. \end{aligned}$$

Obviously the first term of (5.6.16) tends to 0 when  $\text{dyg}(\beta)$  tends to  $+\infty$  since  $\lim_{n \rightarrow \infty} \theta_n = 1$  (Proposition 4.4). Let us turn to the third summation in (5.6.16). The  $j$ -th root  $\omega_{j,n} \in \mathcal{L}_\beta$  of  $f_\beta(z)$  is the unique root of  $f_\beta(z)$  in the disk  $D_{j,n} = \{z \mid |\omega_{j,n} - z_{j,n}| < \frac{\pi |z_{j,n}|}{n a_{\max}}\}$ . From Theorem 5.14 we have the more precise localization in  $D_{j,n}$ :  $|\omega_{j,n} - z_{j,n}| < \frac{\pi |z_{j,n}|}{n a_{j,n}}$  for  $j = \lceil v_n \rceil, \dots, J_n$  (main angular sector), with

$$D\left(\frac{\pi}{a_{j,n}}\right) = \text{Log} \left[ \frac{1 + B_{j,n} - \sqrt{1 - 6B_{j,n} + B_{j,n}^2}}{4B_{j,n}} \right]$$

and  $B_{j,n} = 2 \sin\left(\frac{\pi j}{n}\right) \left(1 - \frac{1}{n} \text{Log} \left(2 \sin\left(\frac{\pi j}{n}\right)\right)\right)$  (from (5.3.29)).

For  $j = \lceil v_n \rceil, \dots, J_n$  the following inequalities hold:

$$1 - \frac{1}{n} D\left(\frac{\pi}{a_{j,n}}\right) \leq \left| 1 + \frac{\omega_{j,n} - z_{j,n}}{z_{j,n}} \right| \leq 1 + \frac{1}{n} D\left(\frac{\pi}{a_{j,n}}\right),$$

up to second order terms. Let us apply the remainder Theorem of alternating series: for  $x$  real,  $|x| < 1$ ,  $|\text{Log}(1+x) - x| \leq \frac{x^2}{2}$ . Then the third summation in (5.6.16) satisfies

$$-2 \lim_{n \rightarrow \infty} \sum_{j=1}^{J_n} \frac{1}{n} \text{Log} \left[ \frac{1 + 2 \sin\left(\frac{\pi j}{n}\right) - \sqrt{1 - 12 \sin\left(\frac{\pi j}{n}\right) + 4 \left(\sin\left(\frac{\pi j}{n}\right)\right)^2}}{8 \sin\left(\frac{\pi j}{n}\right)} \right]$$

$$(5.6.17) \quad \leq \liminf_{n \rightarrow \infty} \left( -2 \sum_{j=1}^{J_n} \text{Log} \left| 1 + \frac{\omega_{j,n} - z_{j,n}}{z_{j,n}} \right| \right)$$

and

$$(5.6.18) \quad \limsup_{n \rightarrow \infty} \left( -2 \sum_{j=1}^{J_n} \text{Log} \left| 1 + \frac{\omega_{j,n} - z_{j,n}}{z_{j,n}} \right| \right) \leq \\ +2 \lim_{n \rightarrow \infty} \sum_{j=1}^{J_n} \frac{1}{n} \text{Log} \left[ \frac{1 + 2 \sin(\frac{\pi j}{n}) - \sqrt{1 - 12 \sin(\frac{\pi j}{n}) + 4(\sin(\frac{\pi j}{n}))^2}}{8 \sin(\frac{\pi j}{n})} \right]$$

Let us convert the limits to integrals. The Riemann-Stieltjes sum

$$S(F, n) := -2 \sum_{j=1}^{J_n} \frac{1}{n} \text{Log} \left[ \frac{1 + 2 \sin(\frac{\pi j}{n}) - \sqrt{1 - 12 \sin(\frac{\pi j}{n}) + 4(\sin(\frac{\pi j}{n}))^2}}{8 \sin(\frac{\pi j}{n})} \right] \\ = \frac{-1}{\pi} \sum_{j=1}^{J_n} (x_j - x_{j-1}) F(x_j)$$

with  $x_j = \frac{2\pi j}{n}$  and  $F(x) := \text{Log} \left[ \frac{1 + 2 \sin(\frac{x}{2}) - \sqrt{1 - 12 \sin(\frac{x}{2}) + 4(\sin(\frac{x}{2}))^2}}{8 \sin(\frac{x}{2})} \right]$  converges to the limit

$$(5.6.19) \quad \lim_{n \rightarrow \infty} S(F, n) = \frac{-1}{\pi} \int_0^{0.171784\dots} F(x) dx = \text{Log } \mu_r \quad \text{with } \mu_r = 0.992337\dots$$

From the inequalities (5.6.17) and (5.6.18), with the limit (5.6.19) as an integral, and by taking the exponential of (5.6.16), we obtain the two multiplicative factors  $\mu_r$  and  $\mu_r^{-1}$  of  $\Lambda_r$  in (5.6.12), resp. in (5.6.13).

Let us show that the second summation in (5.6.16) gives the term  $\Lambda_r$  in the inequalities (5.6.12) and (5.6.13), when  $n$  tends to infinity. From (5.6.15) it will suffice to show that

$$(5.6.20) \quad \lim_{n \rightarrow \infty} S(f, n) = -2 \lim_{n \rightarrow \infty} \sum_{j=1}^{J_n} \text{Log} |z_{j,n}|$$

The identity (5.6.20) only concerns the roots of the trinomials  $G_n$ . It was already proved to be true, but with  $\lfloor n/6 \rfloor$  instead of  $J_n$  as maximal index  $j$ , in the summation, in [214] §4.2, pp 111–115. The arguments of the proof are the same, the domain of integration being now  $(0, \lim_{n \rightarrow \infty} 2\pi \frac{J_n}{n}]$  given by Proposition 5.11.  $\square$

**5.7. Poincaré asymptotic expansion of the lenticular Mahler measure.** The aim of this subsection is to prove Theorem 5.40, in the continuation of the last paragraph.

The logarithm of the lenticular Mahler measure  $M_r(\beta)$  of  $\beta > 1$ , with  $\text{dyc}(\beta) \geq 260$ , given by (5.6.16), admits the lower bound

(5.7.1)

$$L_r(\beta) = \text{Log}(\beta) - 2 \sum_{j=1}^{J_n} \text{Log} |z_{j,n}| - 2 \sum_{j=1}^{\lfloor v_n \rfloor} \text{Log} \left( 1 + \frac{\pi}{n a_{\max}} \right) - 2 \sum_{j=\lfloor v_n \rfloor}^{J_n} \text{Log} \left( 1 + \frac{\pi}{n a_{j,n}} \right)$$

which is only a function of  $n = \text{dyc}(\beta)$ , where  $(a_{j,n})$  is given by Theorem 5.14, the sequence  $(v_n)$  by the Appendix, and  $J_n$  by Definition 5.10 and Proposition 5.11. From

(5.6.17), (5.6.19) and (5.6.20), the limit is  $\lim_{\text{dyg}(\beta) \rightarrow \infty} L_r(\beta) = \text{Log } \Lambda_r + \text{Log } \mu_r$ . In Theorem 5.40, we will gather the asymptotic contributions of each term and obtain the asymptotic expansion of  $L_r(\beta)$  as a function of  $n$ .

(i) First term in (5.7.1): from Lemma 4.5 and Theorem 5.2,

$$(5.7.2) \quad \text{Log}(\beta) = \frac{\text{Log } n}{n}(1 - \lambda_n) + \frac{1}{n} O\left(\left(\frac{\text{Log } \text{Log } n}{\text{Log } n}\right)^2\right) = O\left(\frac{\text{Log } n}{n}\right);$$

(ii) second term in (5.7.1): from Proposition 4.8,  $\sum_{j=\lceil v_n \rceil}^{J_n} \text{Log} |z_{j,n}| =$

$$\sum_{j=\lceil v_n \rceil}^{J_n} \text{Log} \left(1 + \frac{1}{n} \text{Log} \left(2 \sin\left(\frac{\pi j}{n}\right)\right) + \frac{1}{2n} \left(\frac{\text{Log } \text{Log } n}{\text{Log } n}\right)^2 + \frac{1}{n} O\left(\frac{(\text{Log } \text{Log } n)^2}{(\text{Log } n)^3}\right)\right)$$

with the constant 1 involved in the Big O. Let us apply the remainder Theorem of alternating series: for  $x$  real,  $|x| < 1$ ,  $|\text{Log}(1+x) - x| \leq \frac{x^2}{2}$ . Then

$$(5.7.3) \quad \left| \sum_{j=\lceil v_n \rceil}^{J_n} \text{Log} |z_{j,n}| - \sum_{j=\lceil v_n \rceil}^{J_n} \frac{1}{n} \text{Log} \left(2 \sin\left(\frac{\pi j}{n}\right)\right) - \sum_{j=\lceil v_n \rceil}^{J_n} \frac{1}{2n} \left(\frac{\text{Log } \text{Log } n}{\text{Log } n}\right)^2 \right|$$

$$\leq \sum_{j=\lceil v_n \rceil}^{J_n} \frac{1}{n} \left| O\left(\frac{(\text{Log } \text{Log } n)^2}{(\text{Log } n)^3}\right) \right|$$

$$+ \frac{1}{2} \sum_{j=\lceil v_n \rceil}^{J_n} \frac{1}{n^2} \left[ \text{Log} \left(2 \sin\left(\frac{\pi j}{n}\right)\right) + \frac{1}{2} \left(\frac{\text{Log } \text{Log } n}{\text{Log } n}\right)^2 + O\left(\frac{(\text{Log } \text{Log } n)^2}{(\text{Log } n)^3}\right) \right]^2.$$

For  $1 \leq j \leq J_n$ , the inequalities  $0 < 2 \sin(\pi j/n) \leq 1$  and  $\text{Log}(2 \sin(\pi j/n)) < 0$  hold. Then  $|\text{Log}(2 \sin(\pi j/n))| \leq |\text{Log}(2 \sin(\pi/n))| = O(\text{Log } n)$ . On the other hand, the two  $O(\cdot)$  in the rhs of (5.7.3) involve a constant which does not depend upon  $j$ . Therefore, from Proposition 5.11, the rhs of (5.7.3) is

$$= O\left(\frac{(\text{Log } \text{Log } n)^2}{(\text{Log } n)^3}\right) + O\left(\frac{\text{Log}^2 n}{n}\right) = O\left(\frac{(\text{Log } \text{Log } n)^2}{(\text{Log } n)^3}\right).$$

On the other hand, the two regimes of asymptotic expansions in the Bump give (Appendix)

$$\sum_{j=\lceil u_n \rceil}^{\lfloor v_n \rfloor} \text{Log} |z_{j,n}| = O\left(\frac{(\text{Log } n)^{2+\varepsilon}}{n}\right), \quad \sum_{j=1}^{\lfloor u_n \rfloor} \text{Log} |z_{j,n}| = O\left(\frac{(\text{Log } n)^2}{n}\right)$$

and

$$\sum_{j=\lceil \text{Log } n \rceil}^{\lfloor v_n \rfloor} \frac{2}{n} \text{Log} \left(2 \sin\left(\frac{\pi j}{n}\right)\right) = O\left(\frac{(\text{Log } n)^{2+\varepsilon}}{n}\right).$$

Therefore

$$(5.7.4) \quad -2 \sum_{j=1}^{J_n} \text{Log} |z_{j,n}| = - \sum_{j=\lceil \text{Log } n \rceil}^{J_n} \frac{2}{n} \text{Log} \left(2 \sin\left(\frac{\pi j}{n}\right)\right) + O\left(\frac{(\text{Log } \text{Log } n)^2}{\text{Log } n}\right)$$

with the constant  $\frac{1}{2\pi} \arcsin(\frac{\kappa}{2})$  (from Proposition 5.11) involved in the Big O.

(iii) third term in (5.7.1): with the definition of  $\varepsilon$  and  $(v_n)$  (Appendix),

$$(5.7.5) \quad -2 \sum_{j=1}^{\lfloor v_n \rfloor} \text{Log} \left( 1 + \frac{\pi}{na_{j,n}} \right) = O \left( \frac{(\text{Log } n)^{1+\varepsilon}}{n} \right);$$

(iv) fourth term in (5.7.1): from the Theorem of alternating series,

$$(5.7.6) \quad \left| \sum_{j=\lfloor v_n \rfloor}^{J_n} \text{Log} \left( 1 + \frac{\pi}{na_{j,n}} \right) - \sum_{j=\lfloor v_n \rfloor}^{J_n} \frac{1}{n} D \left( \frac{\pi}{a_{j,n}} \right) - \sum_{j=\lfloor v_n \rfloor}^{J_n} \frac{1}{n} \text{tl} \left( \frac{\pi}{a_{j,n}} \right) \right| \leq \frac{1}{2} \sum_{j=\lfloor v_n \rfloor}^{J_n} \left( \frac{\pi}{na_{j,n}} \right)^2.$$

The terminant  $\text{tl} \left( \frac{\pi}{a_{j,n}} \right) = O \left( \frac{(\text{Log } \text{Log } n)^2}{(\text{Log } n)^3} \right)$  is given by (5.3.31). From Theorem 5.14, with  $B_{j,n} = 2 \sin \left( \frac{\pi j}{n} \right) \left( 1 - \frac{1}{n} \text{Log} \left( 2 \sin \left( \frac{\pi j}{n} \right) \right) \right)$ , it is easy to show

$$\begin{aligned} D \left( \frac{\pi}{a_{j,n}} \right) &= \text{Log} \left[ \frac{1 + B_{j,n} - \sqrt{1 - 6B_{j,n} + B_{j,n}^2}}{4B_{j,n}} \right] \\ &= \text{Log} \left[ \frac{1 + 2 \sin \left( \frac{\pi j}{n} \right) - \sqrt{1 - 12 \sin \left( \frac{\pi j}{n} \right) + 4 \sin \left( \frac{\pi j}{n} \right)^2}}{8 \sin \left( \frac{\pi j}{n} \right)} \right] + O \left( \frac{\text{Log } n}{n} \right). \end{aligned}$$

The rhs of (5.7.6) is  $= O \left( \frac{1}{n} \right)$ . Then  $-2 \sum_{j=\lfloor v_n \rfloor}^{J_n} \text{Log} \left( 1 + \frac{\pi}{na_{j,n}} \right) =$

$$(5.7.7) \quad \sum_{j=\lfloor v_n \rfloor}^{J_n} \frac{-2}{n} \text{Log} \left[ \frac{1 + 2 \sin \left( \frac{\pi j}{n} \right) - \sqrt{1 - 12 \sin \left( \frac{\pi j}{n} \right) + 4 \sin \left( \frac{\pi j}{n} \right)^2}}{8 \sin \left( \frac{\pi j}{n} \right)} \right] + O \left( \frac{(\text{Log } \text{Log } n)^2}{(\text{Log } n)^3} \right).$$

The summation  $\sum_{j=\lfloor v_n \rfloor}^{J_n}$  can be replaced by  $\sum_{j=\lfloor \text{Log } n \rfloor}^{J_n}$ . Indeed, from the definition of the sequence  $(v_n)$  (Appendix),

$$\sum_{j=\lfloor \text{Log } n \rfloor}^{\lfloor v_n \rfloor} \frac{2}{n} \text{Log} \left[ \frac{1 + 2 \sin \left( \frac{\pi j}{n} \right) - \sqrt{1 - 12 \sin \left( \frac{\pi j}{n} \right) + 4 \sin \left( \frac{\pi j}{n} \right)^2}}{8 \sin \left( \frac{\pi j}{n} \right)} \right] = O \left( \frac{(\text{Log } n)^{2+\varepsilon}}{n} \right).$$

Inserting the contributions (5.7.2) (5.7.4) (5.7.5) (5.7.7) in (5.7.1) leads to

$$\begin{aligned} L_r(\beta) &= \text{Log } \Lambda_r + \text{Log } \mu_r + \left( -\text{Log } \Lambda_r - \sum_{j=\lfloor \text{Log } n \rfloor}^{J_n} \frac{2}{n} \text{Log} \left( 2 \sin \left( \frac{\pi j}{n} \right) \right) \right) \\ &+ \left( -\text{Log } \mu_r - \sum_{j=\lfloor \text{Log } n \rfloor}^{J_n} \frac{2}{n} \text{Log} \left( \frac{1 + 2 \sin \left( \frac{\pi j}{n} \right) - \sqrt{1 - 12 \sin \left( \frac{\pi j}{n} \right) + 4 \sin \left( \frac{\pi j}{n} \right)^2}}{8 \sin \left( \frac{\pi j}{n} \right)} \right) \right) \\ (5.7.8) \quad &+ O \left( \left( \frac{\text{Log } \text{Log } n}{\text{Log } n} \right)^2 \right) \end{aligned}$$

with the constant  $\frac{1}{2\pi} \arcsin(\frac{\kappa}{2})$  involved in the Big O. Let us denote by  $\Delta_1$  the first term within brackets, resp.  $\Delta_2$  the second term within brackets, in (5.7.8) so that

$$(5.7.9) \quad D(L_r(\beta)) = \text{Log}(\Lambda_r \mu_r) + \Delta_1 + \Delta_2.$$

Calculation of  $|\Delta_1|$ : let us estimate and give an upper bound of  $|\Delta_1| =$

$$(5.7.10) \quad \left| \frac{-1}{\pi} \int_0^{2\arcsin(\kappa/2)} \text{Log} \left( 2 \sin(x/2) \right) dx - \sum_{j=\lceil \text{Log } n \rceil}^{J_n} \frac{-2}{n} \text{Log} \left( 2 \sin \left( \frac{\pi j}{n} \right) \right) \right|.$$

In (5.7.10) the sums are truncated Riemann-Stieltjes sums of  $\text{Log } \Lambda_r$ , the integral being  $\text{Log } \Lambda_r$ . Referring to Stoer and Bulirsch ([197], pp 126–128) we now replace  $\text{Log } \Lambda_r$  by an approximate value obtained by integration of an interpolation polynomial by the methods of Newton-Cotes; we just need to know this approximate value up to  $O\left(\left(\frac{\text{Log } \text{Log } n}{\text{Log } n}\right)^2\right)$ . Up to  $O\left(\left(\frac{\text{Log } \text{Log } n}{\text{Log } n}\right)^2\right)$ , we will show that:

(i–1) an upper bound of (5.7.10) is ( $\kappa$  stands for  $\kappa(1, a_{max})$  as in Proposition 5.11)

$$\frac{\arcsin(\kappa/2)}{\pi} \frac{1}{\text{Log } n},$$

(ii–1) the approximate value of  $\text{Log } \Lambda_r$  is independent of the integer  $m$  (i.e. step length) used in the Newton-Cotes formulas, assuming the weights  $(\alpha_q)_{q=0,1,\dots,m}$  associated with  $m$  all positive. Indeed, if  $m$  is arbitrarily large, the estimate of the integral should be very good by these methods, ideally exact at the limit ( $m \rightarrow +\infty$ ).

*Proof of (i–1):* we consider the decomposition of the interval of integration as  $(0, 2\arcsin(\kappa/2)] =$

$$(5.7.11) \quad \left( 0, \frac{2\pi \lceil \text{Log } n \rceil}{n} \right] \cup \left( \bigcup_{j=\lceil \text{Log } n \rceil}^{J_n-1} \left[ \frac{2\pi j}{n}, \frac{2\pi(j+1)}{n} \right] \right) \cup \left[ \frac{2\pi J_n}{n}, 2\arcsin(\kappa/2) \right]$$

and proceed by calculating the estimations of

$$(5.7.12) \quad \left| \frac{-1}{\pi} \int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} \text{Log} \left( 2 \sin(x/2) \right) dx - \frac{-2}{n} \text{Log} \left( 2 \sin \left( \frac{\pi j}{n} \right) \right) \right|$$

on the intervals  $\mathcal{I}_j := \left[ \frac{2\pi j}{n}, \frac{2\pi(j+1)}{n} \right]$ ,  $j = \lceil \text{Log } n \rceil, \lceil \text{Log } n \rceil + 1, \dots, J_n - 1$ . On each such  $\mathcal{I}_j$ , the function  $f(x)$  is approximated by its interpolation polynomial  $P_m(x)$ , where  $m \geq 1$  is the number of subintervals forming an uniform partition of  $\mathcal{I}_j$  given by

$$(5.7.13) \quad y_q = \frac{2\pi j}{n} + q \frac{2\pi}{n m}, \quad q = 0, 1, \dots, m,$$

of step length  $h_{NC} := \frac{2\pi}{nm}$ , and  $P_m$  the interpolating polynomial of degree  $m$  or less with

$$P_m(y_q) = f(y_q), \quad \text{for } q = 0, 1, \dots, m.$$

The Newton-Cotes formulas

$$\int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} P_m(x) dx = h_{NC} \sum_{q=0}^m \alpha_q f(y_q)$$

provide approximate values of  $\int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} f(x) dx$ , where the  $\alpha_q$  are the weights obtained by integrating the Lagrange's interpolation polynomials. Steffensen [194] ([197], p 127) showed



that the approximation error may be expressed as follows:

$$\int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} P_m(x) dx - \int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} f(x) dx = h_{NC}^{p+1} \cdot K \cdot f^{(p+1)}(\xi), \quad \xi \in \overset{o}{\mathcal{I}}_j,$$

where  $p \geq 2$  is an integer related to  $m$ , and  $K$  a constant.

Using [197], p. 128, and  $m = 1$ , the method being the ‘‘Trapezoidal rule’’, we have: ‘‘ $p = 2, K = 1/12, \alpha_0 = \alpha_1 = 1/2$ ’’. Then (5.7.12) is estimated by

$$(5.7.14) \quad \left| \frac{1}{2} \frac{2\pi}{n} \left[ \frac{-1}{\pi} \text{Log} \left( 2 \sin \left( \frac{\pi j}{n} \right) \right) + \frac{-1}{\pi} \text{Log} \left( 2 \sin \left( \frac{\pi(j+1)}{n} \right) \right) \right] - \frac{-2}{n} \text{Log} \left( 2 \sin \left( \frac{\pi j}{n} \right) \right) \right| \\ = \frac{1}{n} \left| \text{Log} \left( 2 \sin \left( \frac{\pi j}{n} \right) \right) - \text{Log} \left( 2 \sin \left( \frac{\pi(j+1)}{n} \right) \right) \right| = \frac{2\pi}{n^2} \left| \frac{\cos(\xi/2)}{2 \sin(\xi/2)} \right| \leq \frac{1}{n} \frac{1}{\text{Log} n}$$

for some  $\xi \in \overset{o}{\mathcal{I}}_j$ , for large  $n$ . The (Steffensen’s) approximation error ‘‘ $h_{NC}^3 \cdot (1/12) \cdot f^{(2)}(\xi)$ ’’ for the trapezoidal rule, relative to (5.7.12), is

$$(5.7.15) \quad \frac{1}{\pi} \left( \frac{2\pi}{n} \right)^3 \frac{1}{12} \left| \frac{-1}{4 \sin^2(\xi/2)} \right| \leq \frac{1}{6n} \frac{1}{(\text{Log} n)^2}.$$

By Proposition 5.11 the integral

$$\left| \frac{-1}{\pi} \int_{\frac{2\pi j_n}{n}}^{2 \arcsin \kappa/2} \text{Log} (2 \sin(x/2)) dx \right| \quad \text{is a} \quad O\left(\frac{1}{n}\right).$$

Then, summing up the contributions of all the intervals  $\mathcal{I}_j$ , we obtain the following upper bound of (5.7.10)

$$(5.7.16) \quad \left| \frac{-1}{\pi} \int_0^{(2\pi \text{Log} n)/n} \text{Log} (2 \sin(x/2)) dx \right| + \frac{\arcsin(\kappa/2)}{\pi} \frac{1}{\text{Log} n}.$$

with global (Steffensen’s) approximation error, from (5.7.15),

$$O\left(\frac{1}{(\text{Log} n)^2}\right)$$

By integrating by parts the integral in (5.7.16), for large  $n$ , it is easy to show that this integral is  $= O\left(\frac{(\text{Log} n)^2}{n}\right)$ . We deduce the following asymptotic expansion

$$(5.7.17) \quad \Delta_1 = \frac{\mathcal{R}}{\text{Log} n} + O\left(\frac{1}{(\text{Log} n)^2}\right) \quad \text{with} \quad |\mathcal{R}| < \frac{\arcsin(\kappa/2)}{\pi}.$$

*Proof of (ii-1):* Let us show that the upper bound  $\frac{\arcsin(\kappa/2)}{\pi} \frac{1}{\text{Log} n}$  is independent of the integer  $m$  used, once assumed the positivity of the weights  $(\alpha_q)_{q=0,1,\dots,m}$ . For  $m \geq 1$  fixed, this is merely a consequence of the relation between the weights in the Newton-Cotes formulas. Indeed, we have  $\sum_{q=0}^m \alpha_q = m$ , and therefore

$$\left| \int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} P_m(x) dx - h_{NC} m f(y_0) \right| = h_{NC} \left| \sum_{q=0}^m \alpha_q (f(y_q) - f(y_0)) \right|$$

$$\leq h_{NC} \left( \sum_{q=0}^m |\alpha_q| \right) \sup_{\xi \in \mathcal{I}_j} |f'(\xi)|.$$

Since  $h_{NC}m = \frac{2\pi}{n}$  and that the inequality  $\sup_{\xi \in \mathcal{I}_j} |f'(\xi)| \leq |f'((2\pi \text{Log } n)/n)|$  holds uniformly for all  $j$ , we deduce the same upper bound as in (5.7.14) for the Trapezoidal rule. Summing up the contributions over all the intervals  $\mathcal{I}_j$ , we obtain the same upper bound (5.7.16) of (5.7.10) as before.

As for the (Steffensen's) approximation errors, they make use of the successive derivatives of the function  $f(x) = \text{Log}(2 \sin(x/2))$ . We have:

$$f'(x) = \frac{\cos(x/2)}{2 \sin(x/2)}, \quad f''(x) = -\frac{1}{4 \sin^2(x/2)}, \quad f'''(x) = \frac{\cos(x/2)}{4 \sin^3(x/2)} \dots$$

Recursively, it is easy to show that the  $q$ -th derivative of  $f(x)$ ,  $q \geq 1$ , is a rational function of the two quantities  $\cos(x/2)$  and  $\sin(x/2)$  with bounded numerator on the interval  $(0, \pi/3]$ , and a denominator which is  $\sin^q(x/2)$ . For the needs of majoration in the Newton-Cotes formulas over each interval of the collection  $(\mathcal{I}_j)$ , this denominator takes its smallest value at  $\xi = (2\pi \lceil \text{Log } n \rceil)/n$ . Therefore, for large  $n$ , the (Steffensen's) approximation error " $h_{NC}^{p+1} \cdot K \cdot f^{(p)}(\xi)$ " on one interval  $\mathcal{I}_j$  is

$$O \left( \left( \frac{2\pi}{nm} \right)^{p+1} \cdot K \cdot \frac{n^p}{(\pi \text{Log } n)^p} \right) = O \left( \frac{1}{n(\text{Log } n)^p} \right).$$

By summing up over the intervals  $\mathcal{I}_j$ , we obtain the global (Steffensen's) approximation error ( $p \geq 2$ )

$$O \left( \frac{1}{(\text{Log } n)^p} \right) \quad \text{which is a} \quad O \left( \left( \frac{\text{Log } \text{Log } n}{\text{Log } n} \right)^2 \right).$$

*Calculation of  $|\Delta_2|$ :* we proceed as above for establishing an upper bound of

$$(5.7.18) \quad |\Delta_2| = \left| \frac{-1}{\pi} \int_0^{2 \arcsin(\frac{\kappa(1, a_{\max})}{2})} \text{Log} \left[ \frac{1 + 2 \sin(\frac{x}{2}) - \sqrt{1 - 12 \sin(\frac{x}{2}) + 4(\sin(\frac{x}{2}))^2}}{8 \sin(\frac{x}{2})} \right] dx \right. \\ \left. - \sum_{j=\lceil \text{Log } n \rceil}^{J_n} \frac{-2}{n} \text{Log} \left( \frac{1 + 2 \sin(\frac{\pi j}{n}) - \sqrt{1 - 12 \sin(\frac{\pi j}{n}) + 4 \sin(\frac{\pi j}{n})^2}}{8 \sin(\frac{\pi j}{n})} \right) \right|$$

In (5.7.18) the sums are truncated Riemann-Stieltjes sums of  $\text{Log } \mu_r$ , the integral being  $\text{Log } \mu_r$ . As above, the methods of Newton-Cotes (Stoer and Bulirsch ([197], pp 126–128)) will be applied to compute an approximate value of the integral up to  $O\left(\left(\frac{\text{Log } \text{Log } n}{\text{Log } n}\right)^2\right)$ . Up to  $O\left(\left(\frac{\text{Log } \text{Log } n}{\text{Log } n}\right)^2\right)$ , we will show that:

(i-2) *an upper bound of (5.7.18) is ( $\kappa$  stands for  $\kappa(1, a_{\max})$  as in Proposition 5.11)*

$$(5.7.19) \quad \frac{4 \arcsin(\kappa/2)}{\kappa \sqrt{2\kappa(3-\kappa)\text{Log}(1/\kappa)}} \frac{1}{\sqrt{n}} \quad \text{which is a} \quad O\left(\left(\frac{\text{Log } \text{Log } n}{\text{Log } n}\right)^2\right),$$

in other terms that (5.7.18) is equal to zero up to  $O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right)$ ,

(ii-2) the approximate value of  $\text{Log } \mu_r$  is independent of the step length  $m$  used in the Newton-Cotes formulas, assuming the weights  $(\alpha_q)_{q=0,1,\dots,m}$  associated with  $m$  all positive.

*Proof of (i-2):* The decomposition of the interval of integration  $(0, 2 \arcsin(\kappa/2)]$  remains the same as above, given by (5.7.11). Let us treat the complete interval of integration  $(0, 2 \arcsin(\kappa/2)]$  by subintervals. We first proceed by estimating an upper bound of

$$(5.7.20) \quad \left| \frac{-1}{\pi} \int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} \text{Log} \left[ \frac{1 + 2 \sin(\frac{x}{2}) - \sqrt{1 - 12 \sin(\frac{x}{2}) + 4(\sin(\frac{x}{2}))^2}}{8 \sin(\frac{x}{2})} \right] dx \right. \\ \left. - \frac{-2}{n} \text{Log} \left( \frac{1 + 2 \sin(\frac{\pi j}{n}) - \sqrt{1 - 12 \sin(\frac{\pi j}{n}) + 4 \sin(\frac{\pi j}{n})^2}}{8 \sin(\frac{\pi j}{n})} \right) \right|$$

on the intervals  $\mathcal{I}_j := [\frac{2\pi j}{n}, \frac{2\pi(j+1)}{n}]$ ,  $j = \lceil \text{Log } n \rceil, \lceil \text{Log } n \rceil + 1, \dots, J_n - 1$ . Let

$$F(x) := \text{Log} \left[ \frac{1 + 2 \sin(\frac{x}{2}) - \sqrt{1 - 12 \sin(\frac{x}{2}) + 4(\sin(\frac{x}{2}))^2}}{8 \sin(\frac{x}{2})} \right].$$

On each interval  $\mathcal{I}_j$  the function  $F(x)$  is approximated by its interpolation polynomial (say)  $P_{F,m}(x)$ , where  $m \geq 1$  is the number of subintervals of  $\mathcal{I}_j$  given by their extremities  $y_q$  by (5.7.13), of step length  $h_{NC} := \frac{2\pi}{nm}$ , and  $P_{F,m}$  the interpolating polynomial of degree  $m$  or less with

$$P_{F,m}(y_q) = F(y_q), \quad \text{for } q = 0, 1, \dots, m.$$

The Newton-Cotes formulas

$$(5.7.21) \quad \int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} P_{F,m}(x) dx = h_{NC} \sum_{q=0}^m \alpha_q F(y_q)$$

provide the approximate values  $\int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} F(x) dx$ , where the  $\alpha_q$ s are the weights obtained by integrating the Lagrange's interpolation polynomials. Using [197], p. 128, and  $m = 1$ , the method being the "Trapezoidal rule", we have:  $p = 2$ ,  $K = 1/12$ ,  $\alpha_0 = \alpha_1 = 1/2$ . Then (5.7.20) is estimated by

$$(5.7.22) \quad \left| \frac{1}{2} \frac{2\pi}{n} \left[ \frac{-1}{\pi} F\left(\frac{2\pi j}{n}\right) + \frac{-1}{\pi} F\left(\frac{2\pi(j+1)}{n}\right) \right] - \frac{-2}{n} F\left(\frac{2\pi j}{n}\right) \right| \\ = \frac{1}{n} \left| F\left(\frac{2\pi j}{n}\right) - F\left(\frac{2\pi(j+1)}{n}\right) \right| = \frac{2\pi}{n^2} |F'(\xi)|$$

for some  $\xi \in \overset{o}{\mathcal{I}}_j$ , for large  $n$ . As in Remark 5.12, let  $x = 2 \arcsin(\kappa/2)$ . The derivative

$$(5.7.23) \quad F'(y) = \frac{\cos(y/2)(-2 \sin(y/2) + 1 - \sqrt{4 \sin^2(y/2) - 12 \sin(y/2) + 1}}{4 \sin(y/2) \sqrt{4 \sin^2(y/2) - 12 \sin(y/2) + 1}} > 0$$

is increasing on the interval  $(0, x)$ . When  $y = \frac{2\pi J_n}{n} < x$  tends to  $x^-$ , by Proposition 5.11 and Remark 5.12, since  $0 < \sqrt{4 \sin^2(y/2) - 12 \sin(y/2) + 1} \leq 1$  is close to zero for  $y = 2\pi J_n/n$ , the following inequality holds

$$(5.7.24) \quad F' \left( \frac{2\pi J_n}{n} \right) \leq \frac{2/\kappa}{\sqrt{4 \sin^2 \left( \frac{\pi J_n}{n} \right) - 12 \sin \left( \frac{\pi J_n}{n} \right) + 1}}.$$

The upper bound is a function of  $n$  which comes from the asymptotic expansion of  $\frac{\pi J_n}{n} - \frac{x}{2}$ , as deduced from (5.3.19). Indeed, from (5.3.19) and using Remark 5.12 (ii),

$$(5.7.25) \quad \begin{aligned} 4 \sin^2 \left( \frac{\pi J_n}{n} \right) - 12 \sin \left( \frac{\pi J_n}{n} \right) + 1 &= \left( \frac{\pi J_n}{n} - \frac{x}{2} \right) [8 \sin(x/2) \cos(x/2) - 12 \cos(x/2)] + O \left( \frac{1}{n^2} \right) \\ &= \frac{2\kappa(3-\kappa) \text{Log}(1/\kappa)}{n} + \frac{1}{n} O \left( \left( \frac{\text{Log Log } n}{\text{Log } n} \right)^2 \right) \end{aligned}$$

From (5.7.24) and (5.7.25) we deduce  $|F'(\frac{2\pi J_n}{n})| < \frac{(2/\kappa)}{\sqrt{2\kappa(3-\kappa)\text{Log}(1/\kappa)}} \sqrt{n}$ . From (5.7.22), we deduce the following upper bound of (5.7.20) on each  $\mathcal{I}_j := [\frac{2\pi j}{n}, \frac{2\pi(j+1)}{n}]$ :

$$(5.7.26) \quad \frac{4\pi}{\kappa \sqrt{2\kappa(3-\kappa)\text{Log}(1/\kappa)}} \frac{1}{n^{3/2}}.$$

By summing up the contributions, for  $j = \lceil \text{Log } n \rceil, \dots, J_n - 1$ , from (5.7.26) and the asymptotics of  $J_n$  given by (5.3.19), we deduce the upper bound (5.7.19) of  $|\Delta_2|$ .

Let us prove that the method of numerical integration we use leads to a (Steffensen's) approximation error which is a  $O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right)$ . The (Steffensen's) approximation error “ $h_{NC}^3 \cdot (1/12) \cdot F^{(2)}(\xi)$ ” for the trapezoidal rule applied to (5.7.20) ([197], p. 127–128) is

$$(5.7.27) \quad \frac{1}{\pi} \left( \frac{2\pi}{n} \right)^3 \frac{1}{12} \left| F^{(2)}(\xi) \right| \quad \text{for some } \xi \in \overset{o}{\mathcal{I}_j}.$$

The second derivative  $F''(y)$  is positive and increasing on  $(0, \frac{2\pi J_n}{n})$ . It is easy to show that there exists a constant  $C > 0$  such that

$$F'' \left( \frac{2\pi J_n}{n} \right) \leq \frac{C}{(4 \sin^2 \left( \frac{\pi J_n}{n} \right) - 12 \sin \left( \frac{\pi J_n}{n} \right) + 1)^{3/2}}.$$

Using the asymptotic expansion of  $J_n$  ((5.3.19); Remark 5.12 (ii); (5.7.25)), there exist  $C_1 > 0$  such that

$$(5.7.28) \quad F'' \left( \frac{2\pi J_n}{n} \right) \leq C_1 n^{3/2}.$$

From (5.7.27) and (5.7.28), summing up the contributions for  $j = \lceil \text{Log } n \rceil, \dots, J_n - 1$ , the global (Steffensen's) approximation error of (5.7.18) for  $|\Delta_2|$  admits the following upper bound, for some constants  $C'_2 > 0, C_2 > 0$ ,

$$C'_2 \frac{J_n}{n^3} n^{3/2} = C_2 \frac{1}{\sqrt{n}} \quad \text{which is a } O \left( \left( \frac{\text{Log Log } n}{\text{Log } n} \right)^2 \right).$$

Now let us turn to the extremity intervals. Using the Appendix, and (5.3.19) in Proposition 5.11, it is easy to show that the two integrals

$$\frac{-1}{\pi} \int_0^{\frac{2\pi \lceil \text{Log } n \rceil}{n}} \quad \text{and} \quad \frac{-1}{\pi} \int_{\frac{2\pi J_n}{n}}^{2 \arcsin(\kappa/2)} \quad \text{are} \quad O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right).$$

*Proof of (ii-2):* On each interval  $\mathcal{I}_j := \left[\frac{2\pi j}{n}, \frac{2\pi(j+1)}{n}\right]$ ,  $j = \lceil \text{Log } n \rceil, \dots, J_n - 1$ , let us assume that the number  $m$  of subintervals of  $\mathcal{I}_j$  given by their extremities  $y_q$  by (5.7.13), is  $\geq 2$ . The weights  $\alpha_q$  in (5.7.21) are assumed to be positive.

The upper bound  $\frac{4 \arcsin(\kappa/2)}{\kappa \sqrt{2\kappa(3-\kappa)} \text{Log}(1/\kappa)} \frac{1}{\sqrt{n}}$  of (5.7.18) is independent of  $m \geq 2$ , once assumed the positivity of the weights  $(\alpha_q)_{q=0,1,\dots,m}$ , since, due to the relation between the weights in the Newton-Cotes formulas  $\sum_{q=0}^m \alpha_q = m$ ,

$$\begin{aligned} \left| \int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} P_m(x) dx - h_{NC} m F(y_0) \right| &= h_{NC} \left| \sum_{q=0}^m \alpha_q (F(y_q) - F(y_0)) \right| \\ &\leq h_{NC} \left( \sum_{q=0}^m |\alpha_q| \right) \sup_{\xi \in \mathcal{I}_j} |F'(\xi)|. \end{aligned}$$

Since  $h_{NC} m = \frac{2\pi}{n}$  and that  $\sup_{\xi \in \mathcal{I}_j} |F'(\xi)| \leq |F'((2\pi J_n)/n)|$  holds uniformly for all  $j = \lceil \text{Log } n \rceil, \dots, J_n - 1$ , we deduce the same upper bound (5.7.26) as for the Trapezoidal rule. Summing up the contributions over all the intervals  $\mathcal{I}_j$ , we obtain the same upper bound (5.7.19) of (5.7.18), as before.

As for the (Steffensen's) approximation errors involved in the numerical integration (5.7.21) there are " $h_{NC}^{p+1} \cdot K \cdot F^{(p)}(\xi)$ " on one interval  $\mathcal{I}_j$ , for some  $p \geq 2$ . They make use of the successive derivatives of the function  $F(x)$ . It can be shown that they contribute negligibly, after summing up over all the intervals  $\mathcal{I}_j$ , as  $O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right)$ .

Gathering the different terms from (i-1)(i-2), the Steffensen's error terms and the error terms due to the numerical integration by the Newton-Cotes method (ii-1)(ii-2), we have proved the following theorem.

**Theorem 5.40.** *Let  $\beta > 1$  be a reciprocal algebraic integer such that  $n = \text{dyg}(\beta) \geq 260$ , with  $M(\alpha) < 1.176280\dots$ . The asymptotic expansion of the minorant  $L_r(\beta)$  of  $\text{Log } M_r(\beta)$  is*

$$(5.7.29) \quad L_r(\beta) = \text{Log } \Lambda_r \mu_r + \frac{\mathcal{R}}{\text{Log } n} + O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right), \quad \text{with} \quad 0 < \mathcal{R} < \frac{\arcsin(\kappa/2)}{\pi}$$

and  $\mathcal{R}$  depending upon  $n$ .

**5.8. A Dobrowolski type minoration.** Denote by  $\mathcal{R}_n$  the positive real number  $\mathcal{R}$  in (5.7.29). Let us show that it is substantially smaller than the bound  $\frac{\arcsin(\kappa/2)}{\pi}$ .

**Lemma 5.41.** *With the same notations as in Theorem 5.40, there exists an integer  $\eta \geq 260$  such that*

$$(5.8.1) \quad \left| \mathcal{R}_n + O\left(\frac{(\text{Log Log } n)^2}{\text{Log } n}\right) \right| < \frac{\arcsin(\kappa/2)}{\pi}, \quad n \geq \eta.$$

*Proof.* Let  $X_n := c \lceil \text{Log } n \rceil$  with  $c$  a positive constant such that  $\lceil \text{Log } n \rceil < X_n < J_n$ . The limit  $\lim_{n \rightarrow \infty} X_n/n = 0$  holds. Recall that  $J_n$  is given by (5.3.19).

The quantity  $\mathcal{R}_n$  comes from the integration of the  $j$ th-subdivision step (5.7.12) by (5.7.14), in order to give an estimate of the development term of  $|\Delta_1|$  given by (5.7.10). This  $j$ th-subdivision step of integration provides the estimated term (cf (5.7.14))

$$(5.8.2) \quad \frac{2\pi}{n^2} \left| \frac{\cos(\xi/2)}{2\sin(\xi/2)} \right| \quad \text{for some } \xi \in \left( \frac{2\pi j}{n}, \frac{2\pi(j+1)}{n} \right).$$

Since the cotangent function is positive and strictly decreasing on  $(0, \pi/2)$ , the upper bound of (5.8.2) is naturally the one given by the first interval of the subdivision  $\left[ \frac{2\pi \lceil \text{Log } n \rceil}{n}, \frac{2\pi(\lceil \text{Log } n \rceil + 1)}{n} \right]$ , that is  $\frac{1}{n} \frac{1}{\text{Log } n}$ . Finding a smaller upper bound of every term (5.8.2) for the other values  $j \in \{\lceil \text{Log } n \rceil + 1, \dots, J_n\}$  is probably important. Our intention is not to do it. We will just cut the following summation into two parts.

$$(5.8.3) \quad \frac{-1}{\pi} \int_{\frac{2\pi \lceil \text{Log } n \rceil}{n}}^{\frac{2\pi J_n}{n}} \text{Log} \left( 2 \sin(x/2) \right) dx - \sum_{j=\lceil \text{Log } n \rceil}^{J_n} \frac{-2}{n} \text{Log} \left( 2 \sin\left(\frac{\pi j}{n}\right) \right)$$

$$(5.8.4) \quad = \sum_{j=\lceil \text{Log } n \rceil}^{X_n-1} \left( \frac{-1}{\pi} \int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} \text{Log} \left( 2 \sin(x/2) \right) dx - \frac{-2}{n} \text{Log} \left( 2 \sin\left(\frac{\pi j}{n}\right) \right) \right)$$

$$(5.8.5) \quad + \sum_{j=X_n}^{J_n-1} \left( \frac{-1}{\pi} \int_{\frac{2\pi j}{n}}^{\frac{2\pi(j+1)}{n}} \text{Log} \left( 2 \sin(x/2) \right) dx - \frac{-2}{n} \text{Log} \left( 2 \sin\left(\frac{\pi j}{n}\right) \right) \right).$$

Each term of (5.8.4) is bounded by  $\frac{1}{n} \frac{1}{\text{Log } n}$  from above, as previously. On the contrary, each term of (5.8.5) is such that

$$\frac{2\pi}{n^2} \left| \frac{\cos(\xi/2)}{2\sin(\xi/2)} \right| \leq \frac{1}{n^2} \left| \frac{1}{X_n/n} \right| = \frac{1}{n} \frac{1}{c \text{Log } n}.$$

Summing up the two contributions, we obtain the following upper bound of (5.8.3):

$$\begin{aligned} & (X_n - 1 - \lceil \text{Log } n \rceil) \frac{1}{n} \frac{1}{\text{Log } n} + (J_n - X_n - 1) \frac{1}{n} \frac{1}{c \text{Log } n} \\ & \leq (c-1) \frac{1}{n} + \frac{1}{c} \frac{\arcsin(\kappa/2)}{\pi \text{Log } n} \end{aligned}$$

The first term  $(c-1) \frac{1}{n}$  is a  $O\left(\frac{1}{n}\right)$  and, multiplied by  $\text{Log } n$ , is inserted in the Big O of (5.8.1). The second term  $\frac{1}{c} \frac{\arcsin(\kappa/2)}{\pi \text{Log } n}$  is an upper bound of  $\mathcal{R}_n/\text{Log } n$ . Let us fix the constant  $c$ . Take for instance  $c = 3$ . The function  $(\text{Log Log } x)^2/\text{Log } x$  tends to 0 when  $x$  goes to infinity.

Therefore there exists an integer  $\eta$  such that all the functions, depending upon  $n$ , “grouped in the Big O” of (5.8.1) satisfy (in short form):

$$\left| O\left(\frac{(\text{Log Log } n)^2}{\text{Log } n}\right) \right| < \frac{2 \arcsin(\kappa/2)}{3 \pi}, \quad n \geq \eta.$$

We deduce Lemma 5.41. □

The decomposition of  $L_r(\beta)$  in (5.7.29) provide the following Dobrowolski type minoration of the Mahler measure  $M(\beta) \geq M_r(\beta)$ .

**Theorem 5.42.** *Let  $\beta > 1$  be a reciprocal algebraic integer such that  $\text{dyg}(\beta) \geq \eta$ , with  $M(\alpha) < 1.176280\dots$ . Then*

$$(5.8.6) \quad M(\beta) \geq \Lambda_r \mu_r - \frac{\Lambda_r \mu_r \arcsin(\kappa/2)}{\pi} \frac{1}{\text{Log}(\text{dyg}(\beta))}$$

*Proof.* Taking the exponential of (5.7.29) gives

$$M_r(\beta) \geq \exp(L_r(\beta)) = \Lambda_r \mu_r \left( 1 + \frac{\mathcal{R}}{\text{Log } n} + O\left(\left(\frac{\text{Log Log } n}{\text{Log } n}\right)^2\right) \right)$$

and (5.8.6) follows from Lemma 5.41. □

**5.9. The impossible convergence of  $M(\beta)$  to 1 when  $\beta > 1$  tends to 1.** Let  $(\beta_q)_{q \geq 1}$  be an infinite sequence of real reciprocal algebraic integers  $> 1$  tending to 1. Without loss of generality, we assume that it is strictly decreasing, with  $\beta_1 \leq 1.176280\dots$ ,  $\lceil \beta_1 \rceil \leq 1.176280\dots$ ,  $M(\beta_1) \leq 1.176280\dots$ , and that there exists only one  $\beta_q$  in an interval  $(\theta_n^{-1}, \theta_{n-1}^{-1})$  for some  $n$ . We denote it by  $\beta_{q_n}$  and have:

$$\theta_n^{-1} < \beta_{q_n} < \theta_{n-1}^{-1}, \quad \beta_{q_n} \leq \lceil \beta_{q_n} \rceil \leq M(\beta_{q_n}) \quad (n \geq n_0).$$

We allow  $n$  to run over a strictly increasing sequence  $\mathcal{J}$  of integers  $n_0, n_1, n_2, \dots$ . We assume that the sequence of Mahler measures  $(M(\beta_{q_n}))_{n \in \mathcal{J}}$  and the sequence of houses  $(\lceil \beta_{q_n} \rceil)_{n \in \mathcal{J}}$  are decreasing:

$$\begin{aligned} 1 &< \dots < \beta_{q_{n+1}} < \beta_{q_n} < \beta_{q_{n-1}} < \dots < \beta_{q_{n_0}} \leq \dots \leq 1.176280\dots, \\ 1 &< \dots \leq \lceil \beta_{q_{n+1}} \rceil \leq \lceil \beta_{q_n} \rceil \leq \lceil \beta_{q_{n-1}} \rceil \leq \dots \leq \lceil \beta_{q_{n_0}} \rceil \leq \dots \leq 1.176280\dots, \\ 1 &\leq \dots \leq M(\beta_{q_{n+1}}) \leq M(\beta_{q_n}) \leq M(\beta_{q_{n-1}}) \leq \dots \leq 1.176280\dots \end{aligned}$$

Let us call “Main Case” when the sequence  $(M(\beta_{q_n}))_{n \in \mathcal{J}}$  is not stationary after a certain rank. Let us call “Second Case” when the sequence  $(M(\beta_{q_n}))_{n \in \mathcal{J}}$  is stationary after a certain rank. Whatever  $\mathcal{J}$ , by Theorem 5.42, we have the universal lower bound and the asymptotic limit satisfying:  $\lim_{n \in \mathcal{J}, n \rightarrow +\infty} M(\beta_{q_n}) \geq \Lambda_r \mu_r$ . The universal lower bound for  $M(\beta_{q_n})$ , for any  $n \in \mathcal{J}$ , is then this limit value  $\Lambda_r \mu_r$  diminished by a quantity calculated by means of the Dobrowolsky type inequality, as explained in Section 6.3.

Let us give two examples of sequences of reciprocal algebraic integers  $\beta_{q_n} > 1$  which converge to 1 which do not possess any lenticular conjugate (except  $\beta_{q_n}^{-1}$ ), to better understand the assumptions involved. Recall that the definition of a lenticular zero is given in Definition 5.23.

5.9.1. “Second Case”: Minimal polynomial  $P_\beta(X) = \widetilde{P}_\beta(X^r)$  for  $r \geq 2$  - Roots of unity.

Let  $\beta_1 = 2 + \sqrt{3} \in (1, +\infty)$ . The minimal polynomial of  $\beta_1$  is  $P_{\beta_1}(X) = X^2 - 4X + 1$ . It is reciprocal.

**Lemma 5.43.** *Let  $r \geq 1$  be an integer and denote by  $\beta_r$  the real number in  $(1, \infty)$  with  $\beta_r^r = \beta_1$ . Then  $\beta_r$  is a reciprocal algebraic integer,  $\lim_{r \rightarrow \infty} \beta_r = 1$ , with minimal polynomial  $X^{2r} - 4X^r + 1$ . The Mahler measure is constant on the family  $(\beta_r)_{r \geq 1}$ , i.e., for all  $r \geq 1$ ,  $M(\beta_r) = \beta_1$ .*

*Proof.* Say  $r \geq 2$  and let  $\gamma \in \mathbb{C}$  satisfy  $\gamma^r = \beta_1$ . Then  $\gamma$  is an algebraic integer. We claim  $\gamma \notin K = \mathbb{Q}(\sqrt{3})$ . Indeed, if  $\gamma \in K$ , then  $|\gamma| \leq \sqrt{2 + \sqrt{3}} < 2$  and  $|\gamma'| \leq \sqrt{2 - \sqrt{3}} < 0.6$  with  $\gamma'$  the conjugate of  $\gamma$ . We write  $\gamma = a + b\sqrt{3}$  with  $a, b \in \mathbb{Z}$ , so  $\gamma' = a - b\sqrt{3}$ . Then  $b = (\gamma - \gamma')/(2\sqrt{3})$  and so  $|b| \leq (2 + 0.6)/(2\sqrt{3}) < 1$  which implies  $b = 0$ . So  $|a| = |\gamma| < 2$  implies  $\gamma \in \{0, \pm 1\}$ , a contradiction.

So  $\beta_r$  is not the  $r$ -th power of any element of  $K$  if  $r \geq 2$ . Moreover  $-\beta_1/4 < 0$  is not a fourth power in the real quadratic field  $K$ . From this and from a classical result from the theory of fields, Theorem VI.9.1 in S. Lang, *Algebra*, Graduate Texts in Mathematics, **211**, (2002), we conclude that  $X^r - \beta_1$  is irreducible in  $K[X]$  for all  $r \geq 1$ .

So

$$X^{2r} - 4X^r + 1 = P_{\beta_1}(X^r)$$

is the  $\mathbb{Z}$ -minimal polynomial of  $\beta_r$ . In particular  $\beta_r$  is reciprocal.

Now, if  $\varepsilon$  runs over the set of  $r$ -th roots of unity,  $\varepsilon^r = 1$ , then the set of the conjugates of  $\beta_r$  is  $\{\beta_r \varepsilon \mid \text{all } \varepsilon, \varepsilon^r = 1\}$ , since  $(\beta_r \varepsilon)^r = \beta_1$  which implies  $M(\beta_r) = M(\beta_1)$  for all  $r \geq 2$ .

The action of the  $r$ -th roots of unity, for  $r$  tending to infinity, does not produce reciprocal algebraic integers of smaller Mahler measure than  $M(\beta_1)$ . We have

$$1 \dots < \beta_{r+1} < \beta_r < \dots < \beta_2 < \beta_1 = 2 + \sqrt{3}$$

with

$$\dots = M(\beta_r + 1) = M(\beta_r) = \dots = M(\beta_2) = M(\beta_1) = 2 + \sqrt{3}.$$

□

In this note we are concerned with the attack of the Conjecture of Lehmer. Therefore, for obtaining a minorant of  $M(\beta_r)$ , because of the Mahler measure remains constant on the sequence  $(\beta_r)_{r \geq 1}$ , we replace  $\text{dyg}(\beta_r)$  by  $\text{dyg}(\beta_r^r)$  and use the lenticular minorant relative to  $f_{\beta_r^r}(z) = f_{\beta_1}(z)$ .

The fact is that  $\beta_1 = 2 + \sqrt{3} \geq (1 + \sqrt{5})/2$  and that “dyg” has not yet be defined on  $[(1 + \sqrt{5})/2, \infty)$ . By convention, say  $\text{dyg}(\gamma) = 1$  when  $\gamma > (1 + \sqrt{5})/2$ .

So  $\text{dyg}(\beta_r)$  is replaced by  $\text{dyg}(\beta_r^r) = 1$ . The minimal polynomial of  $\beta_1$  is relative to the Main Case. Now the set of lenticular roots of  $f_{\beta_1}(z)$  is  $\{1/\beta_1\}$ , and  $M(1/\beta_1) = \beta_1$ . We have proved  $M(\beta_r) \geq \beta_1$ . The equality holds; in this case the lenticular minorant is exactly the Mahler measure  $M(\beta_1) = \beta_1$ .

The general strategy is the same. Let  $\beta > 1$  be a reciprocal algebraic integer of dynamical degree  $\text{dyg}(\beta) \geq 260$ , for which the minimal polynomial

$$P_\beta(X) = 1 + \sum_{j=1}^{d-1} a_j X^j + X^d = \prod_{k=1}^d (X - \beta^{(k)}), \quad \beta^{(1)} = \beta,$$



is relative to the Main Case. Then, for any  $q \geq 2$ , we define the reciprocal algebraic integer  $\beta_q > 1$ , root of the reciprocal polynomial

$$P_{\beta_q}(X) = 1 + \sum_{j=1}^{d-1} a_j X^{qj} + X^{qd} = P_{\beta}(X^q).$$

For all  $q \geq 2$ , we have:  $M(\beta_q) = M(\beta)$ , considering the equations  $X^q - \beta^{(k)}$ ,  $|\beta^{(k)}| \geq 1$ , and if  $\gamma^q = \beta^{(k)}$ ,  $(\gamma\varepsilon)^q = \beta^{(k)}$  for any  $\varepsilon$ ,  $\varepsilon^q = 1$ . For any  $q \geq 2$ , there exists  $n \geq 260$  such that

$$\theta_n^{-1} < \beta_q < \theta_{n-1}^{-1}.$$

Then to find a minorant of  $M(\beta_q)$  we

$$\text{replace } n = \text{dyg}(\beta_q) \quad \text{by} \quad \text{dyg}(\beta)$$

and consider the set of the lenticular zeroes of  $f_{\beta_q^q}(z) = f_{\beta}(z)$ . Then we obtain the lenticular minorant (of Definition 6.2)

$$M_r(\beta) \leq M(\beta_q).$$

This amounts to the case, called Main Case in (N), relative to  $\beta$  and  $P_{\beta}(X)$ .

5.9.2. *An example outside the Problem of Lehmer.* Let  $n \geq 2$  be an integer. Let  $\beta_n$  the unique root  $> 1$  of the reciprocal integer polynomial

$$P_n(X) = X^{2^n} - 2X^{2^{n-1}} - 8X^{2^{n-1}} - 2X + 1 \in \mathbb{Z}[X].$$

We show that the Conjecture of Lehmer is true for the family  $\{\beta_n : n \geq 2\}$  (in the sense that there is a common lower bound  $> 1$  for all the Mahler measures  $M(\beta_n)$ ). This is due to Lemma 5.45 and to the fact that the house  $[\beta_n] = \beta_n$  tends to 2 as  $n \rightarrow \infty$  ((iv) in Proposition 5.44). Since 2 does not belong to the interval  $(1, 1.32\dots)$ , a dynamical degree  $\text{dyg}$  of  $[\beta_n]$  cannot be defined. Therefore a lenticular minorant of the Mahler measure  $M(\beta_n)$  has no sense. We cannot expect any help from any lenticular root of  $f_{[\beta_n]}(z)$  in this case, since lenticular roots do not exist.

This example does not constitute an attack of the problem of Lehmer, but, interestingly, when we consider the integer polynomial

$$P_n(-X) = X^{2^n} + 2X^{2^{n-1}} - 8X^{2^{n-1}} + 2X + 1 \in \mathbb{Z}[X],$$

we find the sequence of reciprocal algebraic integers  $(-\gamma_n)_{n \geq 2}$  in  $(1, +\infty)$  which are roots of  $P_n(-z)$  and which converges to 1, by (iv) in Proposition 5.44. For this family it is tempting to try to use the lenticular roots of  $f_{-\gamma_n}(z)$  for  $n$  large enough, which exist, as  $n$  tends to infinity, to establish a minorant of  $M(-\gamma_n) = M(\beta_n)$ . It is hopeless. The conditions of identification of the lenticular roots of  $f_{-\gamma_n}(z)$  with some zeroes of  $P_n(-z)$  in  $\{z \in \mathbb{C} : -\pi/18 \leq \arg z \leq +\pi/18, |z| < 1\}$ , are not satisfied (cf [74]); in this angular sector, the only zero which is common to  $f_{-\gamma_n}(z)$  and  $P_n(-z)$  is  $(-\gamma_n)^{-1}$  (cf (v) in Proposition 5.44).

**Proposition 5.44.** *Let  $n \geq 2$ . Then*

- (i) *the polynomial  $P_n(X)$  is not of the form  $Q(X^r)$  for some integer  $r \geq 2$  and some integer polynomial  $Q$ ,*
- (ii)  *$P_n$  is irreducible in  $\mathbb{Q}[X]$ ,*

(iii)  $P_n$  has no root on the unit circle,

(iv) for  $n$  large enough,  $P_n$  admits four real roots  $\gamma_n, \gamma_n^{-1}, \beta_n, \beta_n^{-1}$  which have the following properties:

$$\gamma_n < -1 < \gamma_n^{-1} < 0 < \beta_n^{-1} < 1 < \beta_n,$$

$$\beta_n = \overline{|\gamma|} > 2 \text{ for any conjugate } \gamma \text{ of } \beta_n, \text{ in particular } \beta_n = \overline{|\gamma_n|} > 2,$$

$$\lim_{n \rightarrow +\infty} \gamma_n = -1, \quad \lim_{n \rightarrow +\infty} \beta_n = 2,$$

(v) for  $n$  large enough,  $P_n$  has no root in the annulus  $\{z \in \mathbb{C} : |\gamma_n^{-1}| < |z| < 1\}$ .

*Proof.* (i) This is readily due to the fact that  $2^n$  and  $2^n - 1$  are coprime.

(ii) The shifted polynomial  $P_n(X + 1)$  modulo 2 satisfies:

$$P_n(X + 1) \equiv (X + 1)^{2^n} + 1 \equiv X^{2^n} \pmod{2\mathbb{Z}[X]}.$$

Moreover  $P_n(-1) = -2$  is not divisible by  $2^2$ . By Eisenstein's criterium,  $P_n(X + 1)$  is irreducible in  $\mathbb{Z}[X]$ .

(iii) Assume  $z, |z| = 1$ , is a zero of  $P_n$ . We have:

$$|z^{2^n} + 2z^{2^n-1} + 2z + 1| \leq 1 + 2 + 2 + 1 = 6.$$

Then

$$|P_n(z)| \geq \left| \left| -8z^{2^{(n-1)}} \right| - \left| z^{2^n} + 2z^{2^n-1} + 2z + 1 \right| \right| \geq 2.$$

Contradiction.

(iv) For  $n$  large enough, let us prove

$$-1 - \frac{1}{2^{n-1}} < \gamma_n < -1.$$

Let us abbreviate  $m = 2^{n-1}$ . Then

$$\begin{aligned} P_n\left(-1 - \frac{1}{m}\right) &= \left(-1 - \frac{1}{m}\right)^{2m} - 2\left(-1 - \frac{1}{m}\right)^{2m-1} - 8\left(-1 - \frac{1}{m}\right)^m - 2\left(-1 - \frac{1}{m}\right) + 1 \\ &= \left(1 + \frac{1}{m}\right)^{2m} + 2\left(1 + \frac{1}{m}\right)^{2m-1} - 8\left(1 + \frac{1}{m}\right)^m + 2\left(1 + \frac{1}{m}\right) + 1, \end{aligned}$$

which converges to  $e^2 + 2e^2 - 8e + 3 > 0$  as  $m \rightarrow \infty$ . Since  $P_n(-1) < 0$ , we obtain the existence of a root in the interval  $(-1 - \frac{1}{2^{n-1}}, -1)$ . By Descartes's rule, the number of positive real roots of  $P_n(-X)$  is equal to the number of sign changes of the polynomial, which is equal to 2. Since  $P_n(-X)$  is reciprocal, there is only one root in  $(-1 - \frac{1}{m}, -1)$ ;  $\gamma_n$  is the only root of  $P_n(X)$  in this interval.

By Descartes's rule applied to  $P_n(X)$ , since  $P_n$  is reciprocal, there is only one root of  $P_n$  in  $(1, \infty)$ . For  $n \geq 2$ , observe that  $P_n(-2) < 0$ ,  $P_n(-3) > 0$ , and then  $-3 < \beta_n < -2$ . For  $n$  large enough, let us prove that

$$\beta_n - 2 = 2^{-2^{n-1}+4}(1 + o(1)), \quad \text{as } n \rightarrow \infty.$$

Indeed, if we set  $\beta_n = 2 + u$ ,

$$P_n(2 + u) = 0 = (2 + u)^{2m} - 2(2 + u)^{2m-1} - 8(2 + u)^m - 2(2 + u) + 1$$

gives, expanding at the first order,

$$u [2^{2m-1} - 8m2^{m-1} - 2 + \dots] = 3 + 8 \times 2^m,$$

hence the result.

Now, let  $\xi$  a primitive root of unity,  $\xi^m = 1$ . Let us prove that the other roots  $\gamma'$  of  $P_n$ ,  $\gamma' \neq \beta_n$ ,  $|\gamma'| > 1$ , are

$$(5.9.1) \quad \xi^j \gamma_n + v_j \quad \text{with } |v_j| \leq 2|\gamma_n|^{-2m+3}, \quad j = 1, 2, \dots, m-1.$$

Indeed, if we set  $\gamma' = \xi^j \gamma_n + v_j$ , we have

$$P_n(\gamma') = 0 = (\xi^j \gamma_n + v_j)^{2m} - 2(\xi^j \gamma_n + v_j)^{2m-1} - 8(\xi^j \gamma_n + v_j)^m - 2(\xi^j \gamma_n + v_j) + 1.$$

Consequently, for  $j \neq 0$ , expanding at the first order,

$$v_j [2 \times (\xi^j \gamma_n)^{2m-2} - 8m(\xi^j \gamma_n)^{m-1} + \dots] = 2(\xi^j - 1)\gamma_n,$$

hence the result. Since  $|\gamma_n| < 1 + 1/2^{n-1}$  we deduce from (5.9.1), for any  $\varepsilon > 0$ , that

$$\lim_{n \rightarrow \infty} |\gamma_n|^{-2 \times 2^{n-1} + 3} \leq e^{-2} + \varepsilon.$$

Thus, since  $e^{-2} < 0.5$ , for any root  $\gamma'$  of  $P_n$ , for  $n$  large enough, we have:  $|\gamma'| \leq |\gamma_n| + |v_j| \leq 1 + 1/m + e^{-2} + \varepsilon$ , hence  $|\gamma'| = \beta_n$ .

(v) We use Rouché's Theorem and assume that  $m$  is large enough. The root  $1/\gamma_n$  is the unique root in  $[-1, 0)$  of  $P_n(x)$ , from (iii). For all  $r \in [-1, 1/\gamma_n)$ , we find  $0 > P_n(r) = r^{2^n} - 2r^{2^n-1} - 8r^{2^n-1} - 2r + 1$ , hence

$$|r|^{2m} + 2|r|^{2m-1} + 2|r| + 1 < 8|r|^m.$$

Therefore, for all  $z \in \mathbb{C}$  with  $|z| = |r|$ , we find

$$|P_n(z) + 8z^m| = |z^{2m} - 2z^{2m-1} - 2z + 1| \leq |z|^{2m} + 2|z|^{2m-1} + 2|z| + 1 < |8z^m|.$$

By Rouché's Theorem  $z \mapsto P_n(z)$  and  $z \mapsto -8z^m$  have the same number of roots, counted with multiplicities, in  $\{z \in \mathbb{C} : |z| < |r|\}$ . This number equals  $m$  for all  $r \in [-1, 1/\gamma_n)$ . Therefore  $P_n(z)$  has no roots with absolute value in  $(|1/\gamma_n|, 1)$ .  $\square$

**Lemma 5.45.** *For  $n$  large enough,  $M(\beta_n) = M(\gamma_n) \geq 4.6$ .*

*Proof.* From Proposition 5.44, the number of conjugates of  $\beta_n$  of modulus  $> -\gamma_n$  is exactly  $2^{n-1}$ . Therefore, the Mahler measure of  $\beta_n$  satisfies

$$M(\beta_n) > \beta_n(-\gamma_n)^{2^{n-1}-1}.$$

A lower bound of  $-\gamma_n$ , as a function of  $n$ , is obtained as follows. We have:  $P_n(-1) = -2$ ,  $P_n(-1 - \frac{1}{m})$  which tends to  $e^2 + 2e^2 - 8e + 3 > 0$  as  $m \rightarrow \infty$ . Observe that  $P'_n(-1) = -2^n$  which tends to  $-\infty$  as  $n \rightarrow \infty$ , and

$$P'_n(-1 - \frac{1}{m}) = 2m(-1 - \frac{1}{m})^{2m-1} - 2(2m-1)(-1 - \frac{1}{m})^{2m-2} - 8m(-1 - \frac{1}{m})^{m-1} - 2$$

which tends to  $-\infty$  as  $m \rightarrow \infty$ , with  $\frac{1}{m}P'_n(-1 - \frac{1}{m})$  converging to  $-6e^2 + 8e < 0$ . Since  $P_n(-1 - 1/m) - P_n(\gamma_n) = P_n(-1 - 1/m) = (-1 - \frac{1}{m} - \gamma_n)P'_n(-1 - \frac{1}{m})$  at the first order, we

deduce that  $m(1 + \frac{1}{m} + \gamma_n)$  converges to  $\frac{3e^2 - 8e + 3}{6e^2 - 8e}$  as  $m$  tends to  $+\infty$ . Therefore, when  $m$  is large, we have approximately

$$-\gamma_n = +1 + \frac{1}{m} \frac{3(e^2 - 1)}{6e^2 - 8e}.$$

We deduce, for any  $\varepsilon > 0$ ,

$$M(P_n) = M(\beta_n) > 2 \times e^{\left(\frac{3(e^2-1)}{6e^2-8e}\right)} - \varepsilon, \approx 4.66\dots - \varepsilon.$$

as soon as  $n$  is large enough. □

**Remark 5.46.** Lemma 5.45 shows that the family  $(\beta_n)$  is not concerned with the Problem of Lehmer. It is concerned with the problem of the topology and the search for limit points of the set of Mahler measures of algebraic numbers in the half-line  $(1, \infty)$ . Indeed, the Mahler measure  $M(\beta_n)$  (calculated by Graeffe’s method) tends to a limit, as  $n \rightarrow \infty$ .

$n$	$M(\beta_n) = M(-\gamma_n)$
2	7.095126...
3	7.273581...
4	7.275408...
5	7.275409...
6	7.275409...
...	...

Whether this limit 7.275409... is algebraic or transcendental is not known. The characterization of limit points has been tackled by Boyd and Mossinghoff [40], Deninger [59] and many subsequent contributions [215], but never by means of dynamical zeta functions of numeration systems. It asks the question of how it could be investigated by such means.

### 6. MINORATION OF THE MAHLER MEASURE $M(\alpha)$ FOR $\alpha$ A RECIPROCAL COMPLEX ALGEBRAIC INTEGER OF HOUSE $|\overline{\alpha}| > 1$ CLOSE TO ONE. PROOFS OF THE CONJECTURES

Let  $\alpha$  be a nonreal complex reciprocal algebraic integer, for which  $|\overline{\alpha}| > 1$  (case **(ii)** in Section §1) is close to one. The minimal polynomial of  $\alpha$  is denoted by  $P_\alpha(X) \in \mathbb{Z}[X]$ . If  $\alpha^{(i)}$  is a conjugate of maximal modulus of  $\alpha$ ,  $\alpha^{(i)}$  is conjugated with  $\overline{\alpha^{(i)}}$ ,  $(\alpha^{(i)})^{-1}$ ,  $(\overline{\alpha^{(i)}})^{-1}$ ; the house  $|\overline{\alpha}|$  of  $\alpha$ , resp. its inverse  $|\overline{\alpha}|^{-1}$ , is root of the quadratic equation

$$X^2 - \alpha^{(i)} \overline{\alpha^{(i)}} = 0, \quad \text{resp. of} \quad X^2 - (\alpha^{(i)})^{-1} (\overline{\alpha^{(i)}})^{-1} = 0.$$

The house  $|\overline{\alpha}|$  and its inverse  $|\overline{\alpha}|^{-1}$  are real algebraic integers of degree  $\leq \deg(\alpha) + 2$  for which we assume  $1 < |\overline{\alpha}| < \Theta = \theta_5^{-1}$ . The mapping  $\alpha \rightarrow |\overline{\alpha}|, \mathcal{O}_{\overline{\mathbb{Q}}} \rightarrow \mathcal{O}_{\overline{\mathbb{Q}}} \cap (1, \infty)$  is not continuous.

Writing  $\beta = |\overline{\alpha}|$ , the preceding analytic functions  $\zeta_\beta(z), f_\beta(z)$  of the Rényi-Parry dynamical system of the  $\beta$ -shift, defined in Section §4, can be applied once  $|\overline{\alpha}| > 1$  is close enough to  $1^+$ ; the quantities  $\text{dyg}(\beta), \mathcal{L}_\beta, M(\beta), M_r(\beta)$  are also well-defined. The minoration of the Mahler measure  $M(|\overline{\alpha}|)$  is of Dobrowolski type as in Theorem 5.42, as a

function of the dynamical degree  $\text{dyg}(\beta)$ . The domain  $\Omega_n$  on which there is fracturability of the polynomial  $P_\beta(z)$  is defined in Proposition 5.36.

**6.1. Fracturability of the minimal polynomial of  $\alpha$  by the Parry Upper function at  $|\overline{\alpha}|$ .** The following Theorem is an extension of Theorem 5.5 and Proposition 5.36.

**Theorem 6.1.** *Let  $\alpha$  be a nonreal complex reciprocal algebraic integer, for which  $1 < |\overline{\alpha}| < \Theta = \theta_5^{-1}$ , with  $M(\alpha) < 1.176280\dots$ . Denote  $\beta = |\overline{\alpha}|$ . Then the following formal decomposition of the minimal polynomial of  $\alpha$*

$$(6.1.1) \quad P_\alpha(X) = P_\alpha^*(X) = U_\alpha(X) \times f_\beta(X)$$

holds, as the product of the Parry Upper function at  $\beta$

$$(6.1.2) \quad f_\beta(X) = G_{\text{dyg}(\beta)}(X) + X^{m_1} + X^{m_2} + X^{m_3} + \dots$$

with  $m_0 := \text{dyg}(\beta)$ ,  $m_{q+1} - m_q \geq \text{dyg}(\beta) - 1$  for  $q \geq 0$ , and the invertible formal series  $U_\alpha(X) \in \mathbb{Z}[[X]]$ , quotient of  $P_\alpha$  by  $f_\beta$ . The specialization  $X \rightarrow z$  of the formal variable to the complex variable leads to the identity between analytic functions, obeying the Carlson-Polya dichotomy as:

$$(6.1.3) \quad P_\alpha(z) = U_\alpha(z) \times f_\beta(z) \quad \left\{ \begin{array}{ll} \text{on } \mathbb{C} & \text{if } |\overline{\alpha}| \text{ is a Parry number,} \\ \text{on } |z| < 1 & \text{if } |\overline{\alpha}| \text{ is a nonParry number, with } |z| = 1 \\ & \text{as natural boundary for both } U_\alpha \text{ and } f_\beta. \end{array} \right.$$

Assume  $\text{dyg}(\beta) \geq 260$ . Then

- (i) the minimal polynomials of  $\alpha$  and  $\beta$  are equal:  $P_\alpha = P_\beta$  and  $\beta$  is reciprocal,
- (ii) the identity  $U_\alpha(z) = -\zeta_\beta(z) \times P_\beta(z)$  holds as meromorphic functions on the domain of definition of  $f_\beta$ ,
- (iii) the integer power series  $U_\alpha(z)$  is a nonconstant holomorphic function on the domain  $\Omega_n$ , and has no zero in  $\Omega_n$ ,
- (iv) the lenticulus of lenticular zeroes of  $\alpha$  is that of  $\beta$ , namely

$$\mathcal{L}_\alpha = \mathcal{L}_\beta = \{\beta^{-1}\} \cup \bigcup_{j=1}^{J_n} (\{\omega_{j,n}\} \cup \{\overline{\omega_{j,n}}\}) \subset \Omega_n,$$

where the  $\omega_{j,n}$  and  $\overline{\omega_{j,n}}$  are the conjugates of  $\beta^{-1}$ ,

- (v) for any zero  $\omega_{j,n} \in \mathcal{L}_\beta$ ,

$$(6.1.4) \quad U_\alpha(\omega_{j,n}) = \frac{P'_\alpha(\omega_{j,n})}{f'_\beta(\omega_{j,n})} \neq 0, \quad U_\alpha(\overline{\omega_{j,n}}) = \frac{P'_\alpha(\overline{\omega_{j,n}})}{f'_\beta(\overline{\omega_{j,n}})} \neq 0 \quad \text{and} \quad U_\alpha(\beta^{-1}) = \frac{P'_\alpha(\beta^{-1})}{f'_\beta(\beta^{-1})} \neq 0.$$

*Proof.* There exists an integer  $n \geq 6$  such that  $|\overline{\alpha}|$  lies between two successive Perron numbers of the family  $(\theta_n^{-1})_{n \geq 5}$ , as  $\theta_n^{-1} \leq |\overline{\alpha}| < \theta_{n-1}^{-1}$ . Then the Parry Upper function  $f_{|\overline{\alpha}|}(z)$  at  $|\overline{\alpha}|$  has the form:

$$(6.1.5) \quad f_{|\overline{\alpha}|}(z) = -1 + z + z^n + z^{m_1} + z^{m_2} + z^{m_3} + \dots$$

with  $m_0 = n$  and  $m_{q+1} - m_q \geq n - 1$  for  $q \geq 0$ . Whether  $|\overline{\alpha}|$  is a Parry number or a nonParry number is unknown. In any case,  $f_{|\overline{\alpha}|}(|\overline{\alpha}|^{-1}) = 0$  and the zero  $|\overline{\alpha}|^{-1}$  of  $f_{|\overline{\alpha}|}(z)$  is simple. Let us write the Parry Upper function in the generic form  $f_{|\overline{\alpha}|}(z) = -1 + \sum_{j \geq 1} t_j z^j$ .

Let us show that the formal decomposition (6.1.1) is always possible. We proceed as in the proof of Theorem 5.5. Indeed, if we put  $U_\alpha(X) = -1 + \sum_{j \geq 1} b_j X^j$ , and  $P_\alpha(X) = 1 + a_1 X + a_2 X^2 + \dots + a_{d-1} X^{d-1} + X^d$ , (with  $a_j = a_{d-j}$ ), the formal identity  $P_\alpha(X) = U_\alpha(X) \times f_{|\overline{\alpha}|}(X)$  leads to the existence of the coefficient vector  $(b_j)_{j \geq 1}$  of  $U_\alpha(X)$ , as a function of  $(t_j)_{j \geq 1}$  and  $(a_i)_{i=1, \dots, d-1}$ , as:  $b_1 = -(a_1 + t_1)$ , and, for  $r = 2, \dots, d - 1$ ,

$$(6.1.6) \quad b_r = -(t_r + a_r - \sum_{j=1}^{r-1} b_j t_{r-j}) \quad \text{with} \quad b_d = -(t_d + 1 - \sum_{j=1}^{d-1} b_j t_{r-j}),$$

$$(6.1.7) \quad b_r = -t_r + \sum_{j=1}^{r-1} b_j t_{r-j} \quad \text{for } r > d.$$

For  $j \geq 1$ ,  $b_j \in \mathbb{Z}$ , and the integers  $b_r, r > d$ , are determined recursively by (6.1.7) from  $\{b_0 = -1, b_1, b_2, \dots, b_d\}$ . Every  $b_j$  in  $\{b_1, b_2, \dots, b_d\}$  is computed from the coefficient vector of  $P_\alpha(X)$  using (6.1.6), starting by  $b_1 = -1 - a_1$ . The disk of convergence of  $U_\alpha(z)$  has a radius  $\geq \theta_{n-1}$  by Theorem 5.5.

Let us show (i). Assume the contrary, i.e.  $P_\alpha \neq P_\beta$ . We will proceed as in §5.4.2 by constructing another rewriting trail from "P $_\alpha$ " to "f $_\beta$ ", the one from "P $_\beta^*$ " to "f $_\beta$ " being already studied (a priori the two polynomials  $P_\beta$  and  $P_\beta^*$  may be different).

The starting point is the two identities  $P_\alpha(\alpha^{-1}) = 0$  and  $f_\beta(\beta^{-1}) = 0$ . They provide a  $\alpha$ -representation of 1 and a  $\beta$ -representation of 1, the second one being the Rényi  $\beta$ -expansion of 1:

$$(6.1.8) \quad 1 = -a_1 \alpha^{-1} - a_2 \alpha^{-2} - a_3 \alpha^{-3} + \dots - a_{d-1} \alpha^{-(d-1)} - \alpha^{-d} = 1 - P_\alpha(\alpha^{-1}),$$

$$(6.1.9) \quad 1 = t_1 \beta^{-1} + t_2 \beta^{-2} + t_3 \beta^{-3} + \dots = 1 + f_\beta(\beta^{-1}).$$

The goal consists in constructing an infinite chain of intermediate  $(\alpha, \beta)$ -representations of 1 between them, by "restoring" the digits  $t_i$  of  $f_\beta$  in (6.1.9) one after the other from (6.1.8). The first step is the addition of  $(\beta^{-1} + a_1 \alpha^{-1})P_\alpha(\alpha^{-1}) = 0$  to (6.1.8). Then, denoting by  $S_q(z) = -1 + \sum_{j=1}^q t_j z^j, q \geq 1$ , the  $q$ -th polynomial section of  $f_\beta$ , we obtain

$$1 = \beta^{-1} + R_1(\alpha^{-1}, \beta^{-1}) = (1 + S_1(\beta^{-1})) + R_1(\alpha^{-1}, \beta^{-1})$$

with  $R_1(X, Y) \in \mathbb{Z}[X, Y]$ ,

$$R_1(\alpha^{-1}, \beta^{-1}) = (a_1^2 - a_2) \alpha^{-2} + (a_1 a_2 - a_3) \alpha^{-3} + (a_1 a_3 - a_4) \alpha^{-4} + \dots \\ + a_1 \alpha^{-1} \beta^{-1} + a_2 \alpha^{-2} \beta^{-1} + a_3 \alpha^{-3} \beta^{-1} + \dots$$

Let  $A_1(\alpha^{-1}, \beta^{-1}) = -1 + (a_1 \alpha^{-1} + \beta^{-1})$ . The bivariate polynomial  $A_1(X, Y) = -1 + (a_1 X + Y)$  belongs to  $\mathbb{Z}[X, Y]$ . We deduce, at the first step,

$$0 = A_1(\alpha^{-1}, \beta^{-1})P_\alpha(\alpha^{-1}) = S_1(\beta^{-1}) + R_1(\alpha^{-1}, \beta^{-1})$$

Iterating this process we deduce, for every  $q \geq 1$ , the existence of two polynomials  $A_q, R_q \in \mathbb{Z}[X, Y]$ , with  $\deg_X(A_q) \leq q$ ,  $\deg_Y(A_q) \leq q$ ,  $A_q(0, 0) = -1$ ,  $R_q(0, 0) = 0$ , such that

$$(6.1.10) \quad 0 = A_q(\alpha^{-1}, \beta^{-1})P_\alpha(\alpha^{-1}) = S_q(\beta^{-1}) + R_q(\alpha^{-1}, \beta^{-1}).$$

But, for  $q \geq 1$ ,  $0 = S_q(\beta^{-1}) + (f_\beta(\beta^{-1}) - S_q(\beta^{-1}))$ . Hence, the quantities

$$R_q(\alpha^{-1}, \beta^{-1}) = f_\beta(\beta^{-1}) - S_q(\beta^{-1}), \quad q \geq 1,$$

do not depend upon  $\alpha^{-1}$ , but only upon  $\beta^{-1}$ . Let  $\sigma$  the  $\mathbb{Q}$ -endomorphism of the number field  $\mathbb{Q}(\alpha, \beta, \beta^{-1}) \neq \mathbb{Q}(\beta, \beta^{-1})$  defined by  $\sigma(\alpha^{-1}) = \beta^{-1}$  leaving invariant the subfield  $\mathbb{Q}(\beta, \beta^{-1})$ . Applying  $\sigma$  to (6.1.10) gives

$$(6.1.11) \quad 0 = A_q(\beta^{-1}, \beta^{-1})P_\alpha(\beta^{-1}), \quad q \geq 1.$$

If we assume that  $P_\alpha(\beta^{-1}) \neq 0$  then we should have all the (nonzero) polynomials  $A_q(X, X)$ ,  $q \geq 1$ , in the ideal generated by  $P_\beta^*(X)$  in  $\mathbb{Z}[X]$ . As multiples of  $P_\beta^*(X)$  we should have:  $\deg(A_q(X, X)) \geq \deg(P_\beta^*)$ . But  $\text{dyg}(\beta) \geq 260$  implies that  $\deg(\beta) = \deg(P_\beta^*)$  is large since the number of roots of  $P_\beta^*$  is at least the number  $1 + 2J_n$  of lenticular roots (Theorem 5.22). Therefore it suffices to take a value of  $q$  small enough to obtain a contradiction. We deduce  $P_\alpha(\beta^{-1}) = 0$ . Therefore  $P_\alpha = P_\beta^*$  and  $\beta = \overline{\alpha}$  is reciprocal since  $P_\alpha$  is reciprocal. We deduce  $P_\alpha = P_\beta$ .

Let us prove (ii). Since  $\beta > 1$  is a reciprocal algebraic integer,  $\beta$  is not a simple Parry number. By Theorem 3.4,  $\zeta_\beta(z) = -1/f_\beta(z)$ . The Parry Upper function  $f_\beta(z)$  has coefficients in the finite set  $\{-1, 0, 1\}$ , and therefore obeys the Carlson-Polya dichotomy. The domain of definition of  $\zeta_\beta(z)$ , as a meromorphic function, is that of  $f_\beta(z)$ , that is:  $\mathbb{C}$  if  $\beta$  is a Parry number, the open unit disk if  $\beta$  is not a Parry number. On this domain of definition the fracturability of the minimal polynomial  $P_\alpha$  comes from that of  $P_\beta$  by Proposition 5.36, as

$$P_\alpha(z) = P_\beta(z) = (-\zeta_\beta(z)P_\alpha(z)) \times f_\beta(X).$$

Let us prove (iii), (iv) and (v). The holomorphy of  $-\zeta_\beta(z)P_\alpha(z)$  on  $\Omega_n$  is a consequence of Proposition 5.36. The relations (6.1.4) come from the derivation of (6.1.3) at the conjugates  $\omega_{j,n}$  and  $\overline{\omega_{j,n}}$  of  $\beta^{-1}$  in the domain  $\Omega_n$ .

**Definition 6.2.** Let  $\alpha$  be a nonreal complex reciprocal algebraic integer such that  $1 < \overline{\alpha} < \Theta = \theta_5^{-1}$ . For  $n = \text{dyg}(\alpha) \geq 260$ , with  $M(\alpha) < 1.176280\dots$ , the *dynamical degree of  $\alpha$*  is defined by  $\text{dyg}(\alpha) := \text{dyg}(\overline{\alpha})$ ; the reduced Mahler measure of  $\alpha$  is

$$M_r(\alpha) := M_r(\overline{\alpha}) = \overline{\alpha} \prod_{j=1}^{J_n} |\omega_{j,n}|^{-2}.$$

The *domain of fracturability* of the minimal polynomial  $P_\alpha(X)$  is the largest domain in the open unit disk on which the analytic function  $-\zeta_\beta(z)P_\alpha(z)$  is a nonconstant holomorphic function which does not vanish in this domain. It contains  $\Omega_n$ . □

**6.2. A Dobrowolski type minoration with the dynamical degree of the house - Proof of Theorem 1.4.** Let  $\alpha$  be a reciprocal algebraic integer such that  $\beta = |\overline{\alpha}|$  has dynamical degree  $\text{dyg}(\beta) \geq 260$ , with  $M(\alpha) < 1.176280\dots$ . The first nonreal complex root  $\omega_{1,n}$  of the lenticulus  $\mathcal{L}_\alpha$  is a continuous function of  $\beta$  by [84]. By Corollary 3.14 and Theorem 6.1 the other lenticular roots of the Parry Upper function  $f_{|\overline{\alpha}|}(z)$  are continuous functions of  $\beta = |\overline{\alpha}|$ . These facts suggest the Conjecture that the (true) Mahler measure  $M(\alpha)$  is a continuous function of the house  $|\overline{\alpha}|$  of  $\alpha$ . On the contrary, the nonderivability of the function  $\beta = |\overline{\alpha}| \rightarrow \omega_{1,\text{dyg}(\beta)}$  conjectured in [84] suggests that the (true) Mahler measure  $M(\alpha)$  is nowhere derivable as a function of  $\beta = |\overline{\alpha}|$ .

By Theorem 6.1, since  $P_\alpha = P_{|\overline{\alpha}|}$ , the minoration of the Mahler measure  $M(\alpha)$  is deduced from the minoration of the Mahler measure  $M(\beta)$ . The following Theorems are readily deduced from Theorem 5.40, Lemma 5.41 and Theorem 5.42.

**Theorem 6.3.** *Let  $\alpha$ ,  $|\overline{\alpha}| > 1$ , be a reciprocal algebraic integer such that  $n = \text{dyg}(|\overline{\alpha}|) \geq 260$ , with  $M(\alpha) < 1.176280\dots$ . The asymptotic expansion of the minorant  $L_r(\alpha)$  of  $\text{Log } M_r(\alpha)$  is*

$$(6.2.1) \quad L_r(\alpha) := \text{Log } \Lambda_r \mu_r + \frac{\mathcal{R}}{\text{Log } n} + O\left(\left(\frac{\text{Log } \text{Log } n}{\text{Log } n}\right)^2\right), \quad \text{with } |\mathcal{R}| < \frac{\arcsin(\kappa/2)}{\pi}$$

and  $\mathcal{R}$  depending upon  $|\overline{\alpha}|$  and  $n$ .

**Theorem 6.4.** *Let  $\alpha$ ,  $|\overline{\alpha}| > 1$ , be a reciprocal algebraic integer such that  $\text{dyg}(\alpha) \geq \eta$ , with  $M(\alpha) < 1.176280\dots$ . Then*

$$(6.2.2) \quad M(\alpha) \geq \Lambda_r \mu_r - \frac{\Lambda_r \mu_r \arcsin(\kappa/2)}{\pi} \frac{1}{\text{Log}(\text{dyg}(\alpha))}$$

In the case where  $\alpha$  is the conjugate of a Perron number  $\theta_n^{-1}$ , for some  $n \geq 260$ , the minorant in (6.2.2) has to be replaced by that of Theorem 4.16 for the trinomials  $G_n$ , taking higher values. Comparatively, if  $\alpha$  is a nonzero nonreciprocal algebraic integer, which is not a root of unity, the Mahler measure  $M(\alpha)$  is uniformly bounded from below by Smyth's lower bound  $\Theta$  [189].

**6.3. Proof of the Conjecture of Lehmer (Theorem 1.2).** Let  $\alpha \neq 0$  be a reciprocal algebraic integer which is not a root of unity, such that  $\text{dyg}(\alpha) \geq \eta$  with  $\eta \geq 259$ . Since  $M(\alpha) = M(\alpha^{-1})$  there are three cases to be considered:

- (i) the house of  $\alpha$  satisfies  $|\overline{\alpha}| \geq \theta_5^{-1}$ ,
- (ii) the dynamical degree of  $\alpha$  satisfies:  $6 \leq \text{dyg}(\alpha) < \eta$ , with  $M(\alpha) < 1.176280\dots$ ,
- (iii) the dynamical degree of  $\alpha$  satisfies:  $\text{dyg}(\alpha) \geq \eta$ , with  $M(\alpha) < 1.176280\dots$

In the first case,  $M(\alpha) \geq \theta_5^{-1} \geq \theta_{259}^{-1} \geq \theta_\eta^{-1} > 1$ . In the second case,  $M(\alpha) \geq \theta_\eta^{-1}$ . In the third case, the Dobrowolski type inequality (1.0.12) gives the following lower bound of the Mahler measure

$$\begin{aligned} M(\alpha) &\geq \Lambda_r \mu_r - \frac{\Lambda_r \mu_r \arcsin(\kappa/2)}{\pi \text{Log}(\text{dyg}(\alpha))} \\ &\geq \Lambda_r \mu_r - \frac{\Lambda_r \mu_r \arcsin(\kappa/2)}{\pi \text{Log}(\eta)} \geq \Lambda_r \mu_r - \frac{\Lambda_r \mu_r \arcsin(\kappa/2)}{\pi \text{Log}(259)}, = 1.14843\dots \end{aligned}$$



by Theorem 5.39 and Theorem 6.4. This lower bound is numerically greater than  $\theta_{259}^{-1} = 1.016126\dots$ , itself greater than  $\theta_\eta^{-1}$ . Therefore, in any case, the lower bound  $\theta_\eta^{-1}$  of  $M(\alpha)$  holds true. We deduce the claim.

#### 6.4. Proof of the Conjecture of Schinzel-Zassenhaus (Theorem 1.3).

**Proposition 6.5.** *Let  $\alpha$ ,  $|\overline{\alpha}| > 1$ , be a reciprocal algebraic integer such that  $\text{dyg}(\alpha) \geq 260$ , with  $M(\alpha) < 1.176280\dots$ . The degree  $\text{deg}(\alpha)$  of  $\alpha$  is related to its dynamical degree  $\text{dyg}(\alpha)$  by*

$$(6.4.1) \quad \text{dyg}(\alpha) \left( \frac{2 \arcsin(\frac{\kappa}{2})}{\pi} \right) + \left( \frac{2\kappa \text{Log } \kappa}{\pi \sqrt{4 - \kappa^2}} \right) \leq \text{deg}(\alpha).$$

*Proof.* By Theorem 5.14 and Proposition 5.36 the number of zeroes in the lenticulus  $\mathcal{L}_\alpha$  is  $1 + 2J_n$ , with  $n := \text{dyg}(\alpha)$ ; these zeroes are all conjugates of  $\alpha$ . The total number of conjugates of  $\alpha$  is the degree  $\text{deg}(\alpha)$  of the minimal polynomial  $P_\alpha$ . By Proposition 5.11,

$$1 + 2J_n = \frac{2n}{\pi} \left( \arcsin\left(\frac{\kappa}{2}\right) \right) + \left( \frac{2\kappa \text{Log } \kappa}{\pi \sqrt{4 - \kappa^2}} \right) + \left( 1 + \frac{1}{n} O\left(\left(\frac{\text{Log } \text{Log } n}{\text{Log } n}\right)^2\right) \right).$$

The inequality (6.4.1) follows. □

**Theorem 6.6.** *Let  $\alpha$ ,  $|\overline{\alpha}| > 1$ , be a reciprocal algebraic integer which is not a root of unity, with  $M(\alpha) < 1.176280\dots$ . Then*

$$(6.4.2) \quad |\overline{\alpha}| \geq 1 + \frac{c}{\text{deg}(\alpha)}, \quad \text{with } c = \theta_\eta^{-1} - 1.$$

*Proof.* There are two cases: either (i)  $|\overline{\alpha}| \geq \theta_\eta^{-1}$ , or (ii)  $n \geq \eta + 1$ . (i) If  $|\overline{\alpha}| \geq \theta_\eta^{-1}$ , then, whatever the degree  $\text{deg}(\alpha) \geq 1$ ,

$$|\overline{\alpha}| \geq 1 + \frac{(\theta_\eta^{-1} - 1)}{\text{deg}(\alpha)}.$$

(ii) The minoration of the house  $\beta = |\overline{\alpha}|$  can easily be obtained as a function of the dynamical degree of  $\alpha$ . Let  $n = \text{dyg}(\beta)$  and assume  $n \geq \eta + 1$ . By definition  $\theta_n^{-1} \leq \beta < \theta_{n-1}^{-1}$ . Theorem 1.8 in [214] (cf also [214] §5.3) implies

$$(6.4.3) \quad \beta = |\overline{\alpha}| \geq \theta_n^{-1} \geq 1 + \frac{(\text{Log } n)(1 - \frac{\text{Log } \text{Log } n}{\text{Log } n})}{n}.$$

From Proposition 6.5,

$$(6.4.4) \quad \frac{1}{n} = \frac{1}{\text{dyg}(\beta)} \geq \frac{2 \arcsin(\kappa/2)}{\pi \text{deg}(\alpha)} \left( 1 + \frac{\kappa \text{Log } \kappa}{n \arcsin(\kappa/2) \sqrt{4 - \kappa^2}} \right).$$

The function  $\frac{\text{Log } x - \text{Log } \text{Log } x}{\text{Log } x} \left( 1 + \frac{\kappa \text{Log } \kappa}{x \arcsin(\kappa/2) \sqrt{4 - \kappa^2}} \right)$  is increasing for  $x \geq 260$ . From (6.4.3) and (6.4.4) we deduce

$$|\overline{\alpha}| \geq 1 + \frac{\tilde{c}}{\text{deg}(\alpha)}$$

with  $\tilde{c} = \frac{2}{\pi} \frac{\text{Log } 260 - \text{Log } \text{Log } 260}{\text{Log } 260} \left( \arcsin(\kappa/2) + \frac{\kappa \text{Log } \kappa}{260 \sqrt{4 - \kappa^2}} \right) = 0.0375522\dots$

From (i) and (ii), we deduce that (6.4.2) holds with  $c = \min\{\tilde{c}, (\theta_\eta^{-1} - 1)\} = (\theta_\eta^{-1} - 1)$  since  $\theta_{259}^{-1} - 1 = 0.016126\dots < \tilde{c}$ , for every nonzero reciprocal algebraic integer  $\alpha$  which is not a root of unity.  $\square$

## 7. PROOF OF THE CONJECTURE OF LEHMER FOR SALEM NUMBERS

The set of Pisot numbers admits the minorant  $\Theta$  by a result of Siegel [186]. Theorem 7.3 implies boundedness from below to the set of Salem numbers.

**7.1. Existence and localization of the first nonreal root of the Parry Upper function  $f_\beta(z)$  of modulus  $< 1$  in the cusp of the fractal of Solomyak.** The lenticular roots of the Parry Upper function  $f_\beta(z)$  were studied in Section §5 and identified as conjugates of the reciprocal algebraic integer  $\beta > 1$ , for  $\beta$  close enough to one,  $M(\beta) < 1.176280\dots$ . Their number is  $1 + 2J_n$ . This was done under the assumption that the quantity  $J_n$  has a sense, that is for a regime of asymptotic expansions of the roots  $z_{j,n}$  of  $G_n$  the closest to  $|z| = 1$  valid outside the "bump sector" (cf Appendix). This is the reason why  $n$  has been taken above 260.

In the present Section the "emergence" of such lenticuli of roots is used, at small values of  $n$ . By emergence is meant that the number of lenticular roots of  $f_\beta(z)$  takes the odd values  $1, 3, 5, \dots$  when  $n$  increases from 6 to higher values, the lenticuli being successively of the type

$$\{\beta^{-1}\}, \{\beta^{-1}, \omega_{1,n}, \overline{\omega_{1,n}}\}, \{\beta^{-1}, \omega_{1,n}, \overline{\omega_{1,n}}, \omega_{2,n}, \overline{\omega_{2,n}}\}, \dots$$

for  $\theta_n^{-1} \leq \beta < \theta_{n-1}^{-1}$ , the value 3 corresponding to the "emergence". This study of the emergence of 3-tuples of lenticular roots does not call for the regime of asymptotic expansions of the roots of  $G_n$  outside the "bump sector". On the contrary it calls for a regime relative to the lenticular roots which lie inside the "bump sector", in particular for the roots of  $G_n$  the closest to the real axis, to deal with the control of the existence of the lenticular root  $\omega_{1,n}$ . The different regimes of asymptotic expansions of the roots  $z_{j,n}$  of  $G_n$  are recalled in Section §4 and the limits of validity of the 3 regimes summarized in the Appendix.

The regime of asymptotic expansions dedicated to the first nonreal lenticular root  $\omega_{1,n}$  has Proposition 4.9 for consequence. Proposition 4.9 is used in the proof of Theorem 7.1. Corollary 7.2 asserts the existence of a lenticulus of at least 3 roots of  $f_\beta(z)$  as soon as  $n = \text{dyg}(\beta)$  is  $\geq 32$ . The identification of these lenticular roots as conjugates of  $\beta^{-1}$  is done in §7.2.

**Theorem 7.1.** *Let  $n \geq 32$ . Denote by  $C_{1,n} := \{z \mid |z - z_{1,n}| = \frac{\pi|z_{1,n}|}{na_{\max}}\}$  the circle centered at the first root  $z_{1,n}$  of  $G_n(z) = -1 + z + z^n$ . Then the condition of Rouché*

$$(7.1.1) \quad \frac{|z|^{2n-1}}{1 - |z|^{n-1}} < |-1 + z + z^n|, \quad \text{for all } z \in C_{1,n},$$

*holds true.*

*Proof.* Let  $a \geq 1$  and  $n \geq 18$ . Denote by  $\varphi := \arg(z_{1,n})$  the argument of the first root  $z_{1,n}$  (in  $\text{Im}(z) > 0$ ). Since  $-1 + z_{1,n} + z_{1,n}^n = 0$ , we have  $|z_{1,n}|^n = |-1 + z_{1,n}|$ . Let us write  $z = z_{1,n} + \frac{\pi|z_{1,n}|}{na} e^{i\psi} = z_{1,n}(1 + \frac{\pi}{an} e^{i(\psi - \varphi)})$  the generic element belonging to  $C_{1,n}$ , with  $\psi \in [0, 2\pi]$ . Let  $X := \cos(\psi - \varphi)$ . Let us show that if the inequality (7.1.1) of Rouché

holds true for  $X = +1$ , then it holds true for all  $X \in [-1, +1]$ , that is for every argument  $\psi \in [0, 2\pi]$ , i.e. for every  $z \in C_{1,n}$ . As in the proof of Theorem 5.8,

$$\left| 1 + \frac{\pi}{an} e^{i(\psi-\varphi)} \right|^n = \exp\left(\frac{\pi X}{a}\right) \times \left( 1 - \frac{\pi^2}{2a^2 n} (2X^2 - 1) + O\left(\frac{1}{n^2}\right) \right)$$

and

$$\arg\left(\left(1 + \frac{\pi}{an} e^{i(\psi-\varphi)}\right)^n\right) = \operatorname{sgn}(\sin(\psi - \varphi)) \times \left( \frac{\pi\sqrt{1-X^2}}{a} \left[1 - \frac{\pi X}{an}\right] + O\left(\frac{1}{n^2}\right) \right).$$

Moreover,

$$\left| 1 + \frac{\pi}{an} e^{i(\psi-\varphi)} \right| = \left| 1 + \frac{\pi}{an} (X \pm i\sqrt{1-X^2}) \right| = 1 + \frac{\pi X}{an} + O\left(\frac{1}{n^2}\right).$$

with

$$\arg\left(1 + \frac{\pi}{an} e^{i(\psi-\varphi)}\right) = \operatorname{sgn}(\sin(\psi - \varphi)) \times \frac{\pi\sqrt{1-X^2}}{an} + O\left(\frac{1}{n^2}\right).$$

For all  $n \geq 18$ , from Proposition 4.9, we have

$$(7.1.2) \quad |z_{1,n}| = 1 - \frac{\operatorname{Log} n - \operatorname{Log} \operatorname{Log} n}{n} + \frac{1}{n} O\left(\frac{\operatorname{Log} \operatorname{Log} n}{\operatorname{Log} n}\right).$$

from which we deduce the following equality, up to  $O\left(\frac{1}{n}\right)$  - terms,

$$|z_{1,n}| \left| 1 + \frac{\pi}{an} e^{i(\psi-\varphi)} \right| = |z_{1,n}|.$$

Then the left-hand side term of (7.1.1) is

$$(7.1.3) \quad \frac{|z|^{2n-1}}{1 - |z|^{n-1}} = \frac{|-1 + z_{1,n}|^2 \left| 1 + \frac{\pi}{an} e^{i(\psi-\varphi)} \right|^{2n}}{|z_{1,n}| \left| 1 + \frac{\pi}{an} e^{i(\psi-\varphi)} \right| - |-1 + z_{1,n}| \left| 1 + \frac{\pi}{an} e^{i(\psi-\varphi)} \right|^n}$$

$$= \frac{|-1 + z_{1,n}|^2 \left( 1 - \frac{\pi^2}{an} (2X^2 - 1) \right) \exp\left(\frac{2\pi X}{a}\right)}{|z_{1,n}| \left| 1 + \frac{\pi}{an} e^{i(\psi-\varphi)} \right| - |-1 + z_{1,n}| \left( 1 - \frac{\pi^2}{2an} (2X^2 - 1) \right) \exp\left(\frac{\pi X}{a}\right)}$$

up to  $\frac{1}{n} O\left(\frac{\operatorname{Log} \operatorname{Log} n}{\operatorname{Log} n}\right)$  -terms (in the terminant). The right-hand side term of (7.1.1) is

$$(7.1.4) \quad \left| -1 + z + z^n \right| = \left| -1 + z_{1,n} \left( 1 + \frac{\pi}{na} e^{i(\psi-\varphi)} \right) + z_{1,n}^n \left( 1 + \frac{\pi}{na} e^{i(\psi-\varphi)} \right)^n \right|$$

$$= \left| -1 + z_{1,n} \left( 1 \pm i \frac{\pi\sqrt{1-X^2}}{an} \right) \left( 1 + \frac{\pi X}{an} \right) + (1 - z_{1,n}) \left( 1 - \frac{\pi^2}{2a^2 n} (2X^2 - 1) \right) \right|$$

$$\times \exp\left(\frac{\pi X}{a}\right) \exp\left(\pm i \left( \frac{\pi\sqrt{1-X^2}}{a} \left[1 - \frac{\pi X}{an}\right] \right)\right) + O\left(\frac{1}{n^2}\right)$$

Let us consider (7.1.3) and (7.1.4) at the first order for the asymptotic expansions, i.e. up to  $O(1/n)$  - terms instead of up to  $O(\frac{1}{n}(\text{Log Log } n/\text{Log } n))$  - terms or  $O(1/n^2)$  - terms. (7.1.3) becomes:

$$\frac{|-1 + z_{1,n}|^2 \exp(\frac{2\pi X}{a})}{|z_{1,n}| - |-1 + z_{1,n}| \exp(\frac{\pi X}{a})}$$

and (7.1.4) is equal to:

$$|-1 + z_{1,n}| \left| 1 - \exp\left(\frac{\pi X}{a}\right) \exp\left(\pm i \frac{\pi \sqrt{1-X^2}}{a}\right) \right|$$

and is independent of the sign of  $\sin(\psi - \varphi)$ . Then the inequality (7.1.1) is equivalent to

$$(7.1.5) \quad \frac{|-1 + z_{1,n}|^2 \exp(\frac{2\pi X}{a})}{|z_{1,n}| - |-1 + z_{1,n}| \exp(\frac{\pi X}{a})} < |-1 + z_{1,n}| \left| 1 - \exp\left(\frac{\pi X}{a}\right) \exp\left(\pm i \frac{\pi \sqrt{1-X^2}}{a}\right) \right|,$$

and (7.1.5) to

$$(7.1.6) \quad \frac{|-1 + z_{1,n}|}{|z_{1,n}|} < \frac{\left| 1 - \exp\left(\frac{\pi X}{a}\right) \exp\left(i \frac{\pi \sqrt{1-X^2}}{a}\right) \right| \exp\left(\frac{-\pi X}{a}\right)}{\exp\left(\frac{\pi X}{a}\right) + \left| 1 - \exp\left(\frac{\pi X}{a}\right) \exp\left(i \frac{\pi \sqrt{1-X^2}}{a}\right) \right|} = \kappa(X, a).$$

The right-hand side function  $\kappa(X, a)$  is a function of  $(X, a)$ , on  $[-1, +1] \times [1, +\infty)$ . which is strictly decreasing for any fixed  $a$ , and reaches its minimum at  $X = 1$ ; this minimum is always strictly positive. Consequently the inequality of Rouché (7.1.1) will be satisfied on  $C_{1,n}$  once it is satisfied at  $X = 1$ , as claimed.

Hence, up to  $O(1/n)$ -terms, the Rouché condition (7.1.6), for any fixed  $a$ , will be satisfied (i.e. for any  $X \in [-1, +1]$ ) by the set of integers  $n = n(a)$  for which  $z_{1,n}$  satisfies:

$$(7.1.7) \quad \frac{|-1 + z_{1,n}|}{|z_{j,n}|} < \kappa(1, a) = \frac{\left| 1 - \exp\left(\frac{\pi}{a}\right) \right| \exp\left(\frac{-\pi}{a}\right)}{\exp\left(\frac{\pi}{a}\right) + \left| 1 - \exp\left(\frac{\pi}{a}\right) \right|},$$

equivalently, from Proposition 4.2.7,

$$(7.1.8) \quad \frac{\text{Log } n - \text{Log Log } n}{n} < \frac{\kappa(1, a)}{1 + \kappa(1, a)}.$$

In order to obtain the largest possible range of values of  $n$ , the value of  $a \geq 1$  has to be chosen such that  $a \rightarrow \kappa(1, a)$  is maximal in (7.1.8) (Figure 2). In the proof of Theorem 5.8 we have seen that the function  $a \rightarrow \kappa(1, a)$  reaches its maximum  $\kappa(1, a_{\max}) := 0.171573 \dots$  at  $a_{\max} = 5.8743 \dots$ . We take  $a = a_{\max}$ .

The slow decrease of the functions of the variable  $n$  involved in the terminants when  $n$  tends to infinity, as a factor of uncertainty on (7.1.8), has to be taken into account in (7.1.8). It amounts to check numerically whether (7.1.1) is satisfied for the small values  $18 \leq n \leq 100$  for  $a = a_{\max}$ , or not. Indeed, for the large enough values of  $n$ , the inequality (7.1.8) is satisfied since  $\lim_{n \rightarrow \infty} \frac{\text{Log } n - \text{Log Log } n}{n} = 0$ . On the computer, the critical threshold of  $n = 32$  is easily calculated, with  $(\text{Log } 32 - \text{Log Log } 32)/32 = 0.0694628 \dots$ . Then

$$\frac{\text{Log } n - \text{Log Log } n}{n} < \frac{\kappa(1, a_{\max})}{1 + \kappa(1, a_{\max})} = 0.146447 \dots \quad \text{for all } n \geq 32.$$

Let us note that the last inequality also holds for some values of  $n$  less than 32. □

**Corollary 7.2.** *Let  $\beta > 1$  be a reciprocal algebraic number such that  $\text{dyg}(\beta) \geq 32$ . Then the Parry Upper function  $f_\beta(z)$  admits a simple zero  $\omega_{1,n}$  (of modulus  $< 1$ ) in the open disk  $D(z_{1,n}, \frac{\pi|z_{1,n}|}{na_{\max}})$ .*

*Proof.* The polynomial  $G_n(z)$  has simple roots. Since (7.1.1) is satisfied, the Theorem of Rouché states that  $f_\beta(z)$  and  $G_n(z) = -1 + z + z^n$  have the same number of roots, counted with multiplicities, in the open disk  $D(z_{1,n}, \frac{\pi|z_{1,n}|}{na_{\max}})$ , giving the existence of an unique zero  $\omega_{1,n}$ .  $\square$

**7.2. Identification of the first lenticular root as a conjugate.** Let us prove that the first zero  $\omega_{1,n}$  of  $f_\beta(z)$  is a zero of the minimal polynomial of  $\beta$ , using rewriting polynomials as in §5.4.

**Theorem 7.3.** *Let  $\beta > 1$  be a reciprocal algebraic number such that  $\text{dyg}(\beta) \geq 32$  with  $M(\beta) < 1.176280\dots$ . Then (i) the first zero  $\omega_{1,n}$  of  $f_\beta(z)$  is a conjugate of  $\beta$ , (ii) the minimal polynomial  $P_\beta(X)$  of  $\beta$  is fracturable on the union of the disks  $D_{1,n} = \{z \mid |z - z_{1,n}| < \frac{\pi|z_{1,n}|}{na_{\max}}\}$  and  $\overline{D_{1,n}}$  in the sense that the analytic function  $U_\beta(z) = -\zeta_\beta(z)P_\beta(z)$  is a nonconstant holomorphic function which does not vanish on this domain, and that*

$$(7.2.1) \quad P_\beta(z) = (-\zeta_\beta(z)P_\beta(z)) \times f_\beta(z), \quad z \in D_{1,n} \cup \overline{D_{1,n}},$$

(iii) the function  $U_\beta(z) \in \mathbb{Z}[[z]]$  takes the following values at  $\omega_{1,n} \in D_{1,n}$ , resp.  $\overline{\omega_{1,n}} \in \overline{D_{1,n}}$ ,

$$(7.2.2) \quad U_\beta(\omega_{1,n}) = \frac{P'_\beta(\omega_{1,n})}{f'_\beta(\omega_{1,n})} \neq 0, \quad U_\beta(\overline{\omega_{1,n}}) = \frac{P'_\beta(\overline{\omega_{1,n}})}{f'_\beta(\overline{\omega_{1,n}})} \neq 0,$$

(iv) the first zero  $\omega_{1,n} = \omega_{1,n}(\beta)$  of  $f_\beta(z)$  is a nonreal complex zero of modulus  $< 1$  of the minimal polynomial  $P_\beta(z)$  which is a continuous function of  $\beta$ .

*Proof.* The domain of definition of the meromorphic function  $f_\beta(z)$  is  $\mathbb{C}$  or the open unit disk with  $|z| = 1$  as natural boundary, according to the Carlson-Polya dichotomy (Bell and Chen [19], Carlson [52] [53], Polya [161], Szegő [198]). In the first case  $\beta$  is a Parry number and it is a nonParry number in the second case. In both cases  $f_\beta(z)$  is holomorphic in  $D_{1,n} \cup \overline{D_{1,n}}$ , these disk being included in  $|z| < 1$ . The function  $f_\beta(z)$  admits only one simple zero in each disk by Corollary 7.2.

Let us prove (i) and (ii). Since  $\beta$  is assumed reciprocal, the power series  $f_\beta(z)$  is never a polynomial and  $\beta$  is not a simple Parry number. Therefore  $f_\beta(z) = -1/\zeta_\beta(z)$  so that the quotient  $P_\beta(z)/f_\beta(z) \in \mathbb{Z}[[z]]$  is equal to  $-\zeta_\beta(z) \times P_\beta(z)$  on the domain of definition of  $f_\beta(z)$ . Hence the identity  $P_\beta(z) = (-\zeta_\beta(z)P_\beta(z)) \times f_\beta(z)$  is satisfied for  $|z| < 1$ .

As in §5.4.2 we can construct the infinite sequence of  $\beta$ -representations of 1 from “ $P_\beta$ ” to “ $f_\beta$ ” by restoring the digits of  $f_\beta$  one after the other. The corresponding rewriting trail starts at (5.4.11) and ends at the Rényi  $\beta$ -expansion of 1 which is given by (5.4.12). Then we obtain the analogue statement of Proposition 5.28: *if  $x$  is a zero of  $P_\beta$  in  $D_{1,n} \cup \overline{D_{1,n}}$ , then  $x$  is a zero of  $f_\beta$ , that is  $x = \omega_{1,n}$  or  $x = \overline{\omega_{1,n}}$ .* Conversely, in order to show that  $x = \omega_{1,n}$  is a zero of  $P_\beta$ , we can construct the rewriting trail from the  $s$ -th polynomial section “ $S_s$ ” of  $f_\beta$  to “ $P_\beta$ ”, i.e. the sequence of  $\gamma_s$ -representations of 1 from the Rényi  $\gamma_s$ -expansion of 1 given by (5.4.20) to (5.4.21), at the unique zero  $\gamma_s^{-1} \in D_{0,n}$  of  $S_s$ , by restoring the digits of

$1 - P_\beta(X)$  one after the other. The important point is that  $s$  should be taken large enough by the analogue of Proposition 5.25: *let  $\beta > 1$  be a reciprocal algebraic integer having  $\text{dyg}(\beta) \geq 32$ . Let  $f_\beta(x) = -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s} + \dots$  be the Parry Upper function at  $\beta$  and, for  $s \geq 0$ , denote its  $s$ -th polynomial section by*

$$f(x) = -1 + x + x^n + x^{m_1} + x^{m_2} + \dots + x^{m_s} \quad \text{factorized as } = A(x)B(x)C(x),$$

where  $s \geq 1$ ,  $m_1 - n \geq n - 1$ ,  $m_{j+1} - m_j \geq n - 1$  for  $1 \leq j < s$ , where  $A$  is the cyclotomic part,  $B$  the reciprocal noncyclotomic part,  $C$  the nonreciprocal part of  $f$ .

There exists  $s_0$  (depending upon  $n$ ) such that the reciprocal noncyclotomic part  $B$  of  $f(x)$ , if any, does not vanish on the lenticular root  $\omega_{1,n}$  of  $f$ , as soon as  $s \geq s_0$ . Then, taking the limit  $s \rightarrow \infty$ , with  $\lim_{s \rightarrow \infty} \gamma_s^{-1} = \beta^{-1}$ , we obtain the analogue of Proposition 5.29: *the lenticular zero  $\omega_{1,n}$  of  $f_\beta$  in  $D_{1,n}$  is a zero of  $P_\beta$ .*

Let us prove (iii). It suffices to take the derivatives

$$P'_\beta(z) = U'_\beta(z) \times f_\beta(z) + U_\beta(z) \times f'_\beta(z)$$

of (7.2.1) at  $\omega_{1,n}$  and  $\overline{\omega_{1,n}}$ . Let us prove (iv). It is a consequence of Corollary 3.14. □

**7.3. A lower bound for the set of Salem numbers. Proof of Theorem 1.5.** Assume that  $\beta$  is a Salem number of dynamical degree  $n = \text{dyg}(\beta) \geq 32$ . Its minimal polynomial  $P_\beta(X)$  would admit  $\beta$ ,  $1/\beta$  as real roots, the remaining roots being on the unit circle, as (nonreal) complex-conjugated pairs. By Theorem 7.3 it would admit the pair of nonreal roots  $(\omega_{1,n}, \overline{\omega_{1,n}})$  as well, but  $|\omega_{1,n}| < 1$ . This fact is impossible. We deduce that  $\beta > \theta_{31}^{-1} = 1.08545\dots$

8. SEQUENCES OF RECIPROCAL ALGEBRAIC INTEGERS CONVERGING TO  $1^+$  IN HOUSE AND LIMIT EQUIDISTRIBUTION OF CONJUGATES ON THE UNIT CIRCLE.  
PROOF OF THEOREM 1.6

In this Section, given a sequence  $(\alpha_q)$  of reciprocal algebraic integers, such that  $|\alpha_q| > 1$ ,  $\text{dyg}(\alpha_q) \geq 260$  tending to infinity, we consider the limit geometry of all the conjugates of the  $\alpha_q$ s. The limit equidistribution, restricted to an arc of the unit circle, of the lenticular conjugates was used in the proof of Theorem 5.39. We generalize this fact to all the conjugates with limit arc the complete unit circle. We will make use of Belotserkovski's Theorem [21], recalled below as Theorem 8.1, which prefigures Bilu's theorem on the  $n$ -dimensional torus [26]; the discrepancy function of equidistribution given by this theorem is well adapted to become a function of only the dynamical degree.

**Theorem 8.1** (Belotserkovski). *Let  $F(x) = a \prod_{i=1}^m (x - \alpha^{(i)}) \in \mathbb{C}[x]$ ,  $m \geq 1$ , be a polynomial with roots  $\alpha^{(k)} = r_k e^{i\varphi_k}$ ,  $0 \leq \varphi_k \leq 2\pi$ . For  $0 \leq \varphi \leq \psi \leq 2\pi$ , denote  $N_F(\varphi, \psi) = \text{Card}\{k \mid \varphi \leq \varphi_k \leq \psi\}$ . Let  $0 \leq \varepsilon, \delta \leq 1/2$  and*

$$\sigma_{dis} = \max \left( m^{-1/2} \text{Log}(m+1), \sqrt{-\varepsilon \text{Log}(\varepsilon)}, \sqrt{-\delta \text{Log}(\delta)} \right).$$

*If  $|r_k - 1| \leq \varepsilon$  for  $1 \leq k \leq m$ , and  $|\text{Log } a| \leq \delta m$  are satisfied, then, for some (universal, in the sense that it does not depend upon  $F$ ) constant  $C > 0$ ,*

$$(8.0.1) \quad \left| \frac{1}{m} N_F(\varphi, \psi) - \frac{\psi - \varphi}{2\pi} \right| \leq C \sigma_{dis} \quad \text{for all } 0 \leq \varphi \leq \psi \leq 2\pi.$$

Let us recall some useful notations. The multiplicative group of nonzero elements of  $\mathbb{C}$ , resp.  $\mathbb{Q}$ , is denoted by  $\mathbb{C}^\times$ , resp.  $\mathbb{Q}^\times$ . The unit Dirac measure supported at  $\omega \in \mathbb{C}$  is denoted by  $\delta_\omega$ . We denote by  $\mu_{\mathbb{T}}$  the (normalized) Haar measure (unit Borel measure), invariant by rotation, that is supported on the unit circle  $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ , compact subgroup of  $\mathbb{C}^\times$ , i.e. with  $\mu_{\mathbb{T}}(\mathbb{T}) = 1$ . Given  $\alpha \in \overline{\mathbb{Q}}^\times$ , of degree  $m = \deg(\alpha)$ , we define the unit Borel measure (probability)

$$\mu_\alpha = \frac{1}{\deg(\alpha)} \sum_{j=1}^{\deg(\alpha)} \delta_{\sigma_j(\alpha)}$$

on  $\mathbb{C}^\times$ , the sum being taken over all  $m$  embeddings  $\sigma : \mathbb{Q}(\alpha) \rightarrow \mathbb{C}$ . A sequence  $\{\gamma_s\}$  of points of  $\overline{\mathbb{Q}}^\times$  is said to be *strict* if any proper subgroup of  $\overline{\mathbb{Q}}^\times$  contains  $\gamma_s$  for only finitely many values of  $s$ .

Theorem 6.2 in [214] shows that limit equidistribution of conjugates occurs on the unit circle for the sequence of Perron numbers  $\{\theta_n^{-1} \mid n = 2, 3, 4, \dots\}$ , as  $\mu_{\theta_n^{-1}} \rightarrow \mu_{\mathbb{T}}, n \rightarrow \infty$ . All these Perron numbers have a Mahler measure  $> \Theta$ . We now give a generalization of this limit result to convergent sequences of algebraic integers of small Mahler measure,  $< \Theta$ , where “convergence to 1” has to be taken in the sense of the “house”. The Theorem is Theorem 1.6.

*Proof of Theorem 1.6:* (i) Denote generically by  $\alpha \in \mathcal{O}_{\overline{\mathbb{Q}}}$  any element of  $(\alpha_q)_{q \geq 1}$ . Let  $m = \deg(\alpha)$  and  $\beta = |\overline{\alpha}| \in (\theta_n^{-1}, \theta_{n-1}^{-1}), n \geq 260$ . Using the inequality (6.4.1) between  $m$  and the dynamical degree  $n = \text{dyg}(\alpha) = \text{dyg}(\beta)$ , there exists a constant  $c_1 > 0$  such that

$$\frac{\text{Log}(m+1)}{\sqrt{m}} \leq c_1 \frac{\text{Log} n}{\sqrt{n}}.$$

On the other hand, the minimal polynomial  $P_\alpha = P_\beta$  is reciprocal and all its roots  $\alpha^{(k)}$ , including  $\beta$  and  $1/\beta$  by Theorem 6.1, lie in the annulus  $\{z \mid \frac{1}{\beta} \leq |z| \leq \beta\}$ . As a consequence, using Theorem 5.2, there exists a constant  $c_2 > 0$  such that

$$\left| |\alpha^{(k)}| - 1 \right| \leq \varepsilon, \quad 1 \leq k \leq m, \quad \text{with } \varepsilon = c_2 \frac{\text{Log} n}{n}.$$

We take  $\delta = 0$  in the definition of  $\sigma$  in Theorem 8.1 since  $P_\alpha$  is monic. We deduce that the discrepancy function, i.e. the upper bound in the rhs of (8.0.1), is equal to  $C\sigma_{dis} = c_3 \frac{\text{Log} n}{\sqrt{n}}$  for some constant  $c_3 > 0$ . Hence,

$$(8.0.2) \quad \left| \frac{1}{m} N_{P_\alpha}(\varphi, \psi) - \frac{\psi - \varphi}{2\pi} \right| \leq c_3 \frac{\text{Log} n}{\sqrt{n}} \quad \text{for all } 0 \leq \varphi \leq \psi \leq 2\pi.$$

The discrepancy function of (8.0.2) tends to 0 if  $n$  tends to infinity. By Theorem 5.2 and Theorem 5.3, for  $1 < \beta < \theta_{260}^{-1}$ ,

$$\beta \rightarrow 1^+ \iff n = \text{dyg}(\beta) \rightarrow \infty,$$

so that the sequence of Galois orbit measures in (1.0.21) converge for the weak topology as a function of the dynamical degree.

(ii) The sequence  $(\alpha_q)$  is strict since the sequence  $(|\overline{\alpha}_q|)$  only admits 1 as limit point:  $\limsup_{q \rightarrow \infty} |\overline{\alpha}_q| = \lim_{q \rightarrow \infty} |\overline{\alpha}_q| = 1$  and the number  $\text{Card}\{\alpha_q \in (\theta_n^{-1}, \theta_{n-1}^{-1})\}$  between two successive Perron numbers of  $(\theta_n^{-1})$ , for every  $n \geq 3$ , is finite. In the space of probability

measures equipped with the weak topology, the reformulation of (8.0.2) means (1.0.21), equivalently (1.0.22).

9. SOME CONSEQUENCES: SALEM NUMBERS, AND A CONJECTURE OF MARGULIS

A first consequence concerns the difference between two successive Salem numbers generating the same number field. We are in the context of root separation theorems [22] [48] [78] [96] [136] and the representability of real algebraic numbers as a difference of two Mahler measures [66].

**Theorem 9.1.** *Suppose that  $\tau$  and  $\tau'$  are Salem numbers with  $\tau' \in \mathbb{Q}(\tau)$ . Then*

$$\tau' > \tau \implies \tau' - \tau \geq \theta_{31}^{-1}(\theta_{31}^{-1} - 1) = 0.0927512\dots$$

*Proof.* Since  $\tau' - \tau = (\tau'/\tau - 1)\tau$  we deduce the lower bound from [180], p. 169, Proposition 3 in [192], since  $\tau'/\tau$  is a Salem number, and Theorem 1.5. □

**Remark 9.2.** For each Pisot number  $\theta$  the ‘‘Construction of Salem’’ ([95], Theorem IV in Salem [180]) gives two convergent sequences  $(\tau'_n)_n$  and  $(\tau_n)_n$  of Salem numbers, such that  $\tau_n < \theta < \tau'_n$  for  $n$  large enough, and

$$\lim_{n \rightarrow \infty} \tau_n = \theta = \lim_{n \rightarrow \infty} \tau'_n.$$

From Theorem 9.1, since  $\lim_{n \rightarrow \infty} (\tau'_n - \tau_n) = 0$ , we have:  $\mathbb{Q}(\tau_n) \neq \mathbb{Q}(\tau'_n)$  for  $n$  large enough.

Ghate and Hironaka (in [92] p. 304) mention that if Lehmer’s Conjecture is true, then the following Conjecture of Margulis is also true.

**Theorem 9.3 (ex-Margulis Conjecture).** *Let  $G$  be a connected semi-simple group over  $\mathbb{R}$ , with  $\text{rank}_{\mathbb{R}}(G) \geq 2$ . Then there is a neighbourhood  $U \subset G(\mathbb{R})$  of the identity such that for any irreducible cocompact lattice  $\Gamma \subset G(\mathbb{R})$ , the intersection  $\Gamma \cap U$  consists only of elements of finite order.*

Let us give a proof of Theorem 9.3 from the two arguments of Margulis ([137], Theorem (B) p. 322): (i) first, the arithmeticity Theorem 1.16 in [137], p. 299, and (ii) the following statement (Margulis [137], p. 322):

*Let  $P(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0$  be an irreducible monic polynomial with integral coefficients. Denote by  $\beta_1(P), \dots, \beta_n(P)$  the roots of  $P$  and by  $m(P)$  the number of those  $i$  with  $1 \leq i \leq n$  and  $|\beta_i(P)| \neq 1$ . Then*

$$(9.0.3) \quad M(P) = \prod_{1 \leq i \leq n} \max\{1, |\beta_i(P)|\} > d$$

where the constant  $d > 1$  depends only upon  $m(P)$  (and does not depend upon  $n$ ). The minorant  $d$  is universal and is given by Theorem 1.2 (ex-Lehmer’s Conjecture), hence the result.

But the dependency of the minorant  $d$  of  $M(P)$  in (9.0.3), expected by Margulis, with the number of roots  $m(P)/2$  lying outside the closed unit disk, or equivalently inside the open unit disk (the polynomial  $P$  can be assumed of small Mahler measure  $< \Theta$ , hence reciprocal by Smyth’s Theorem), and not with the degree  $n$  of  $P$ , is not clear in view of Theorem 5.14, Theorem 6.1 and Proposition 5.11. Indeed, the following minorant

$$m(P) \geq 2(1 + J_{\text{dyg}(P)})$$



can only be deduced from the present study, this minorant being two times the cardinal of the lenticulus of roots associated with the dynamical degree of the house of the polynomial  $P$ ; what can be said is that the degree  $n = \deg(P)$  of  $P$  is not involved in this minorant of  $m(P)$ , and therefore that the constant  $d$  in (9.0.3) is likely to depend upon the dynamical degree  $\text{dyg}(P)$ .

10. APPENDIX

**10.1. Notations.** Let  $P(X) \in \mathbb{Z}[X]$ ,  $m = \deg(P) \geq 1$ . The *reciprocal polynomial* of  $P(X)$  is  $P^*(X) = X^m P(\frac{1}{X})$ . The polynomial  $P$  is reciprocal if  $P^*(X) = P(X)$ . If  $P(X) = a_0 \prod_{j=1}^m (X - \alpha_j) = a_0 X^m + a_1 X^{m-1} + \dots + a_m$ , with  $a_i \in \mathbb{C}$ ,  $a_0 a_m \neq 0$ , and roots  $\alpha_j$ , the *Mahler measure* of  $P$  is

$$(10.1.1) \quad M(P) := |a_0| \prod_{j=1}^m \max\{1, |\alpha_j|\}.$$

The absolute Mahler measure of  $P$  is  $M(P)^{1/\deg(P)}$ , denoted by  $\mathcal{M}(P)$ . The Mahler measure of an algebraic number  $\alpha$  is the Mahler of its minimal polynomial  $P_\alpha$ :  $M(\alpha) := M(P_\alpha)$ . For any algebraic number  $\alpha$  the house  $|\overline{\alpha}|$  of  $\alpha$  is the maximum modulus of its conjugates, including  $\alpha$  itself; by Jensen's formula the Weil height  $h(\alpha)$  of  $\alpha$  is  $\text{Log} M(\alpha) / \deg(\alpha)$ . By its very definition,  $M(PQ) = M(P)M(Q)$  (multiplicativity). The Mahler measure of a nonzero polynomial  $P(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$  is defined by

$$(10.1.2) \quad M(P) := \exp \left( \frac{1}{(2i\pi)^n} \int_{\mathbb{T}^n} \text{Log} |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n} \right),$$

where  $\mathbb{T}^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_1| = \dots = |z_n| = 1\}$  is the unit torus in dimension  $n$ . If  $n = 1$ , by Jensen's formula, it is given by (10.1.1).

A *Perron number* is either 1 or a real algebraic integer  $\theta > 1$  such that the Galois conjugates  $\theta^{(i)}$ ,  $i \neq 0$ , of  $\theta^{(0)} := \theta$  satisfy:  $|\theta^{(i)}| < \theta$ . Denote by  $\mathbb{P}$  the set of Perron numbers. A *Pisot number* is a Perron number  $> 1$  for which  $|\theta^{(i)}| < 1$  for all  $i \neq 0$ . The smallest Pisot number is denoted by  $\Theta = 1.3247\dots$ , dominant root of  $X^3 - X - 1$ . A *Salem number* is an algebraic integer  $\beta > 1$  such that its Galois conjugates  $\beta^{(i)}$  satisfy:  $|\beta^{(i)}| \leq 1$  for all  $i = 1, 2, \dots, m - 1$ , with  $m = \deg(\beta) \geq 1$ ,  $\beta^{(0)} = \beta$  and at least one conjugate  $\beta^{(i)}$ ,  $i \neq 0$ , on the unit circle. All the Galois conjugates of a Salem number  $\beta$  lie on the unit circle, by pairs of complex conjugates, except  $1/\beta$  which lies in the open interval  $(0, 1)$ . Salem numbers are of even degree  $m \geq 4$ . The set of Pisot numbers, resp. Salem numbers, is denoted by  $S$ , resp. by  $T$ . If  $\tau \in S$  or  $T$ , then  $M(\tau) = \tau$ .

The set of algebraic numbers, resp. algebraic integers, in  $\mathbb{C}$ , is denoted by  $\overline{\mathbb{Q}}$ , resp.  $\mathcal{O}_{\overline{\mathbb{Q}}}$ . The  $n$ th cyclotomic polynomial is denoted by  $\Phi_n(z)$ . The (naïve) height of a polynomial  $P$  is the maximum of the absolute value of the coefficients of  $P$ .

For  $x > 0$ ,  $[x]$ ,  $\{x\}$  and  $\lceil x \rceil$  denotes respectively the integer part, resp. the fractional part, resp. the smallest integer greater than or equal to  $x$ . For  $\beta > 1$  any real number, the map  $T_\beta : [0, 1] \rightarrow [0, 1], x \rightarrow \{\beta x\}$  denotes the  $\beta$ -transformation. With  $T_\beta^0 := T_\beta$ , its iterates are denoted by  $T_\beta^{(j)} := T_\beta(T_\beta^{j-1})$  for  $j \geq 1$ . A real number  $\beta > 1$  is a *Parry number* if the sequence  $(T_\beta^{(j)}(1))_{j \geq 1}$  is eventually periodic; a Parry number is called *simple* if in

particular  $T_\beta^{(q)}(1) = 0$  for some integer  $q \geq 1$ . The set of Parry numbers is denoted by  $\mathbb{P}_P$ . The terminology chosen by Parry in [151] has changed:  $\beta$ -numbers are now called Parry numbers, in honor of W. Parry.

For  $x > 0$ ,  $\text{Log}^+x$  denotes  $\max\{0, \text{Log}x\}$ . Let  $\mathcal{F}$  be an infinite subset of the set of nonzero algebraic numbers which are not a root of unity; we say that the *Conjecture of Lehmer is true for  $\mathcal{F}$*  if there exists a constant  $c_{\mathcal{F}} > 0$  such that  $M(\alpha) \geq 1 + c_{\mathcal{F}}$  for all  $\alpha \in \mathcal{F}$ .

**10.2. Angular asymptotic sectorization of the roots  $z_{j,n}, \omega_{j,n}$ , of the Parry Upper functions, in lenticular sets of zeroes – notations for transition regions.** The Poincaré asymptotic expansions of the roots  $z_{j,n}$  of  $G_n(z) = -1 + z + z^n$ , lying in the first quadrant of  $\mathbb{C}$ , are divergent formal series of functions of the couple of *two variables* which is:

- $(n, \frac{j}{n})$ , in the angular sector:  $\frac{\pi}{2} > \arg z > 2\pi \frac{\text{Log} n}{n}$ ,
- $(n, \frac{j}{\text{Log} n})$ , in the angular sector (“bump” sector):  $2\pi \frac{\text{Log} n}{n} > \arg z \geq 0$ .

In the bump sector (cusp sector of Solomyak’s fractal  $\mathcal{G}$ , § 3.2), the roots  $z_{j,n}$  are dispatched into the two subsectors:

- $2\pi \frac{\sqrt{(\text{Log} n)(\text{Log} \text{Log} n)}}{n} > \arg z > 0$ ,
- $2\pi \frac{\text{Log} n}{n} > \arg z > 2\pi \frac{\sqrt{(\text{Log} n)(\text{Log} \text{Log} n)}}{n}$ .

The relative angular size of the bump sector, as  $(2\pi \frac{\text{Log} n}{n}) / (\frac{\pi}{2})$ , tends to zero, as soon as  $n$  is large enough. By transition region, we mean a small neighbourhood of the argument :

$$\arg z = 2\pi \frac{\text{Log} n}{n} \quad \text{or of} \quad 2\pi \frac{\sqrt{(\text{Log} n)(\text{Log} \text{Log} n)}}{n}.$$

Outside these two transition regions, a dominant asymptotic expansion of  $z_{j,n}$  exists. In a transition region an asymptotic expansion contains more  $n$ -th order terms of the same order of magnitude ( $n = 2, 3, 4$ ). These two neighbourhoods are defined as follows. Let  $\varepsilon \in (0, 1)$  small enough. Two strictly increasing sequences of real numbers  $(u_n), (v_n)$  are introduced, which satisfy:

$$[n/6] > v_n > \text{Log} n, \quad \text{Log} n > u_n > \sqrt{(\text{Log} n)(\text{Log} \text{Log} n)}, \quad \text{for } n \geq n_0 = 18,$$

such that

$$\lim_{n \rightarrow \infty} \frac{v_n}{n} = \lim_{n \rightarrow \infty} \frac{\sqrt{(\text{Log} n)(\text{Log} \text{Log} n)}}{u_n} = \lim_{n \rightarrow \infty} \frac{u_n}{\text{Log} n} = \lim_{n \rightarrow \infty} \frac{\text{Log} n}{v_n} = 0$$

and

$$(10.2.1) \quad v_n - u_n = O((\text{Log} n)^{1+\varepsilon})$$

with the constant 1 involved in the big O. The roots  $z_{j,n}$  lying in the first transition region about  $2\pi(\text{Log} n)/n$  are such that:

$$2\pi \frac{v_n}{n} > \arg z_{j,n} > 2\pi \frac{(2\text{Log} n - v_n)}{n},$$

and the roots  $z_{j,n}$  lying in the second transition region about  $\frac{2\pi\sqrt{(\text{Log } n)(\text{Log Log } n)}}{n}$  are such that:

$$2\pi \frac{u_n}{n} > \arg z_{j,n} > 2\pi \frac{2\sqrt{(\text{Log } n)(\text{Log Log } n)} - u_n}{n}.$$

In Proposition 4.6, for simplicity's sake, these two transition regions are schematically denoted by

$$\arg z \asymp 2\pi \frac{(\text{Log } n)}{n} \quad \text{resp.} \quad \arg z \asymp 2\pi \frac{\sqrt{(\text{Log } n)(\text{Log Log } n)}}{n}.$$

By complementarity, the other sectors are schematically written:

$$2\pi \frac{\sqrt{(\text{Log } n)(\text{Log Log } n)}}{n} > \arg z > 0$$

instead of

$$2\pi \frac{2\sqrt{(\text{Log } n)(\text{Log Log } n)} - u_n}{n} > \arg z > 0;$$

resp.

$$2\pi \frac{\text{Log } n}{n} > \arg z > 2\pi \frac{\sqrt{(\text{Log } n)(\text{Log Log } n)}}{n}$$

instead of

$$2\pi \frac{2\text{Log } n - v_n}{n} > \arg z > 2\pi \frac{u_n}{n};$$

resp.

$$\frac{\pi}{2} > \arg z > 2\pi \frac{\text{Log } n}{n} \quad \text{instead of} \quad \frac{\pi}{2} > \arg z > 2\pi \frac{v_n}{n}.$$

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LAMA, CNRS UMR 5127, UNIV. GRENOBLE ALPES, UNIV. SAVOIE MONT BLANC, F - 73000 CHAMBÉRY, FRANCE

*E-mail address:* Jean-Louis.Verger-Gaugry@univ-smb.fr