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Resolvable Decomposition of K_n^*

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In this paper, we prove that K_n^* admits a resolvable decomposition into TT_3 or C_3 if and only if $n \equiv 0 \pmod{3}$, $n \neq 6$.

I. INTRODUCTION

Let G and H be two given graphs (directed or not). We shall say that H can be decomposed into graphs G and we shall denote this by $H \rightarrow G$ if there exists a partition of the edges (or arcs) of H into partial subgraphs isomorphic to G . Furthermore, if we can arrange these subgraphs into classes, called parallel classes, where each class consists of a partition of the vertices of H , we shall say that we have a resolvable decomposition and we shall denote this by $H \xrightarrow{r} G$.

For previous results on the existence of such decompositions with different graphs G see [1].

Here we are interested to know for what values of n there exist resolvable decompositions of $H = K_n^*$ (the complete symmetric directed graph) into $G = TT_3$ or into C_3 , where TT_3 is the translative tournament on 3 vertices and C_3 the directed cycle of length 3.

Notations. In what follows we shall denote by (a, b, c) either the TT_3 with arcs (a, b) , (a, c) , and (b, c) or the C_3 with arcs (a, b) , (b, c) , and (c, a) .

1.1. EXAMPLE. $K_{12}^* \xrightarrow{r} TT_3$. Let the vertices of K_{12}^* be $Z_{11} \cup \{\infty\}$ (Z_n denotes the additive group of residues modulo n). The 11 classes of a resolvable decomposition are the following $\{(i, i+2, i+10)(i+7, i+1, i+5)(i+8, i+9, i+4)(i+3, \infty, i+6)\}$, $i \in Z_{11}$.

It is well known that $K_n \xrightarrow{r} C_3$ if and only if $n \equiv 3 \pmod{6}$ (solution of Kirkman's school girl problem by Ray-Chaudhuri and Wilson [5]). Thus, by associating two opposite TT_3 (or C_3) to each C_3 we have

1.2. THEOREM. If $n \equiv 3 \pmod{6}$ $K_n^* \xrightarrow{r} C_3$ and $K_n^* \xrightarrow{r} TT_3$.

It has also been proved for not necessarily resolvable decompositions:

1.3. THEOREM [4]. $K_n^* \rightarrow TT_3$ if and only if $n(n-1) \equiv 0 \pmod{3}$

1.4. THEOREM. $K_n^* \rightarrow C_3$ if and only if $n(n-1) \equiv 0 \pmod{3}$, $n \neq 6$ (see [1] for the references).

1.5. THEOREM (Hanani [3]). $2K_n \xrightarrow{r} C_3$ if and only if $n \equiv 0 \pmod{3}$ $n \neq 6$.

For the existence of resolvable decompositions of K_n^* into TT_3 or C_3 we have the following:

1.6. PROPOSITION. Necessary conditions for $K_n^* \xrightarrow{r} TT_3$ or $K_n^* \xrightarrow{r} C_3$ are $n \equiv 0 \pmod{3}$, $n \neq 6$.

Proof. The number of vertices of K_n^* must be a multiple of 3 (each class being a partition of the vertices) and then the number of arcs of K_n^* is a multiple of 3. Furthermore, for $n = 6$ if $K_n^* \xrightarrow{r} TT_3$ or $K_n^* \xrightarrow{r} C_3$ then $2K_n \rightarrow C_3$ (by deleting the orientation) and that is impossible by Theorem 1.5. ■

We shall prove that these conditions are sufficient:

1.7. THEOREM. $K_n^* \xrightarrow{r} TT_3$ and $K_n^* \xrightarrow{r} C_3$ if and only if $n \equiv 0 \pmod{3}$, $n \neq 6$.

As corollary we will obtain Theorem 1.5. of Hanani [3]. The proof given here will use some of the ideas of that paper.

II. PROOF OF THE THEOREM

Notations. $K_{[A]}^*$ shall denote the complete symmetric directed graph with vertex set A .

$K_{r \times n}$ shall denote the complete r -partite graph with vertex set the union of r disjoint sets of cardinality n .

2.1. LEMMA. $K_4^* \rightarrow TT_3$, $K_4^* \rightarrow C_3$.

Proof. Let the vertices of K_n^* be the four elements of $GF(4)$: $0, 1, x, x^2$ with $x^2 = x + 1$; the four TT_3 (or C_3) of a decomposition are the following $\{(\alpha, 1 + \alpha, x + \alpha), \alpha \in GF_4\}$ (that is, $\{(0, 1, x)(1, 0, x^2)(x, x^2, 0)(x^2, x, 1)\}$. ■

2.2. LEMMA. If $K_n^* \xrightarrow{r} TT_3$ (or C_3) then $K_{4n}^* \xrightarrow{r} TT_3$ (or C_3).

Proof. It is similar to that of [3, Lemma 7, p. 281] but to be complete we shall give it here in the case TT_3 (it is exactly the same for C_3).

Let the vertices of K_{4n}^* be partitioned into n sets $A_1 \cdots A_n$ with $|A_i| = 4$. The vertices of A_i will be denoted a_i^α , $\alpha \in GF(4)$.

Let $\mathcal{C}_1 \cdots \mathcal{C}_{n-1}$ be the parallel classes of a resolvable decomposition of K_n^* into TT_3 .

From \mathcal{C}_1 we construct seven parallel classes of a decomposition of K_{4n}^* by associating to each $TT_3(i, j, k)$ of \mathcal{C}_1 the seven following families:

$$\{(a_i^\alpha, a_j^\alpha, a_k^\alpha)(a_i^{\alpha+1}, a_i^{\alpha+\omega}, a_i^{\alpha+\omega^2})(a_j^{\alpha+1}, a_j^{\alpha+\omega}, a_j^{\alpha+\omega^2})(a_k^{\alpha+1}, a_k^{\alpha+\omega}, a_k^{\alpha+\omega^2})\},$$

where $\alpha \in GF(4)$ and

$$\{(a_i^{\alpha^p}, a_j^{\alpha^{p+1}}, a_k^{\alpha^{p+2}})(a_i^{\alpha^{p+1}}, a_j^{\alpha^{p+1+1}}, a_k^{\alpha^{p+2+1}})(a_i^{\alpha^{p+\omega}}, a_j^{\alpha^{p+1+\omega}}, a_k^{\alpha^{p+2+\omega}})(a_i^{\alpha^{p+\omega^2}}, a_j^{\alpha^{p+1+\omega^2}}, a_k^{\alpha^{p+2+\omega^2}})\},$$

where $p \in \{1, 2, 3\}$.

From each \mathcal{C}_l , $2 \leq l \leq n-1$, we construct four parallel classes of a decomposition of K_{4n}^* by associating to each $TT_3(i, j, k)$ of \mathcal{C}_l the four following families

$$\{(a_i^\alpha, a_j^\alpha, a_k^\alpha)(a_i^{1+\alpha}, a_j^{\alpha+\alpha}, a_k^{\alpha^2+\alpha})(a_i^{\alpha+\alpha}, a_j^{\alpha^2+\alpha}, a_k^{1+\alpha})(a_i^{\alpha^2+\alpha}, a_j^{1+\alpha}, a_k^{\alpha+\alpha})\},$$

where $\alpha \in GF(4)$.

As wanted we obtain $7 + 4(n-2) = 4n - 1$ parallel classes. ■

2.3. LEMMA.

$$\begin{aligned} K_{18}^* &\xrightarrow{r} TT_3, & K_{18}^* &\xrightarrow{r} C_3, \\ K_{24}^* &\xrightarrow{r} TT_3, & K_{24}^* &\xrightarrow{r} C_3, \\ K_{30}^* &\xrightarrow{r} TT_3, & K_{30}^* &\xrightarrow{r} C_3, \\ K_{42}^* &\xrightarrow{r} TT_3, & K_{42}^* &\xrightarrow{r} C_3. \end{aligned}$$

Proofs will be given in the Appendix.

2.4. LEMMA. $K_{[A \cup B]}^* - K_{[B]}^*$ with $|A| = 12$, $|B| = 6$, can be decomposed into 17 classes of TT_3 (or C_3) such that

- 12 classes D_j ($j = 1, \dots, 12$) partition the vertices of $A \cup B$.
- 5 classes E_j ($j = 13, \dots, 17$) partition only the vertices of A .

The *proof* will be given in the Appendix.

2.5. THEOREM (Brouwer, Hanani, and Schrijver [2]). For $r \geq 4$

$$K_{r \times 12} \rightarrow K_4.$$

(In fact in [2], they give all the values of r, n for which $K_{r \times n} \rightarrow K_4$.)

2.6. THEOREM. For any $u \geq 1$, $K_{12u+6}^* \rightarrow TT_3$ (or C_3).

Proof. For $u = 1, 2, 3$ see Lemma 2.3. Let $n = 12u + 6$, $u \geq 4$ and let us partition the set X of vertices of K_n^* as follows:

$$X = \bigcup_{i=1}^u A_i \cup B \quad \text{with} \quad |A_i| = 12, \quad 1 \leq i \leq u \quad \text{and} \quad |B| = 6.$$

By Lemma 2.3, $K_{[A_i \cup B]}^*$ isomorphic to K_{18}^* can be decomposed into 17 parallel classes C_j^1 , $1 \leq j \leq 17$.

By Lemma 2.4, for $i = 2, \dots, u$, $K_{[A_i \cup B]}^* - K_B^*$ can be decomposed into 17 classes of TT_3 (or C_3):

- 12 classes $D_1^i, D_2^i, \dots, D_{12}^i$ which partition the vertices of $A_i \cup B$.
- 5 classes $E_{13}^i, E_{14}^i, \dots, E_{17}^i$ which partition the vertices of A_i .

By Theorem 2.5 the graph $K_{u \times 12}^*$ constructed on $\bigcup_{i=1}^u A_i$ can be decomposed into K_4^* , and by Lemma 2.1 each of these K_4^* can be decomposed into TT_3 (or C_3). For any i , let a_i^j be a given vertex of the set A_i ; for each K_4^* of the decomposition of $K_{u \times 12}^*$ containing a_i^j we consider the TT_3 (or C_3) of the decomposition of this K_4^* not containing this vertex a_i^j (for example if $a = x^2$ in Lemma 2.1, we take $(0, 1, x)$). Then for any j , $1 \leq j \leq 12$, the set F_j^i of all these TT_3 (or C_3) associated to this vertex a_i^j form a partition of the vertices of the $K_{(u-1) \times 12}^*$ constructed on $\bigcup_{k=1}^u A_k - A_i$.

Thus we obtain a resolvable decomposition of K_{12u+6}^* into TT_3 (or C_3) with the $12u + 5$ following parallel classes:

- 12 classes $C_j^1 \cup F_j^1$ for $j = 1, \dots, 12$.
- 5 classes $C_j^1 \cup \bigcup_{i=2}^u E_j^i$ for $j = 13, \dots, 17$.
- 12 $(u - 1)$ classes $D_j^i \cup F_j^i$ for $j = 1, \dots, 12$, $2 \leq i \leq u$. ■

2.7. End of the Proof of Theorem 1.7

If $n \equiv 3 \pmod{6}$ the theorem follows from Theorem 1.2.

If $n \equiv 6 \pmod{12}$ $n \neq 6$ the theorem follows from Theorem 2.6.

If $n \equiv 0 \pmod{12}$; let $n = 4^\alpha q$, where $q \equiv 0 \pmod{3}$ but $q \not\equiv 0 \pmod{12}$.

In this case the theorem follows by repeated applications of Lemma 2.2 from the fact that the theorem is true for q , $q \neq 6$ and for $n = 24$ (Lemma 2.3). ■

III. APPENDIX—PROOFS OF LEMMAS 2.3 AND 2.4

In the following we shall denote by X the set of vertices of K_n^* and if \mathcal{C} is a class of TT_3 (or C_3), $\mathcal{C} + i$ will mean the class obtained from \mathcal{C} by adding i to each vertex of the TT_3 (or C_3) of \mathcal{C} , with the convention that $\infty + i = \infty$.

3.1. Proof of Lemma 2.3

$$— K_{18}^* \xrightarrow{r} TT_3.$$

Let $X = Z_{17} \cup \{\infty\}$. The 17 parallel classes of a resolvable decomposition of K_{18}^* into TT_3 are the $\mathcal{C} + i$, $0 \leq i \leq 16$ with $\mathcal{C} = \{(0, \infty, 11) (1, 9, 14) (7, 2, 5) (4, 3, 10) (6, 15, 8) (12, 16, 13)\}$.

$$— K_{24}^* \xrightarrow{r} TT_3.$$

Let $X = Z_{23} \cup \{\infty\}$. The 23 parallel classes are the $\mathcal{C} + i$, $0 \leq i \leq 22$ with $\mathcal{C} = \{(6, 18, 19) (13, 1, 22) (2, 5, 12) (20, 11, 16) (7, 9, 15) (4, 3, 21) (17, 10, 14) (8, \infty, 0)\}$.

$$— K_{30}^* \xrightarrow{r} TT_3.$$

Let $X = Z_{29} \cup \{\infty\}$. The 29 parallel classes are the $\mathcal{C} + i$, $0 \leq i \leq 28$ with: $\mathcal{C} = \{(0, 4, 15) (28, 1, 2) (22, 7, 14) (20, \infty, 11)\} \cup \{(2^{3\alpha}, 2^{3\alpha+1}, 2^{\alpha 3+2})/\alpha = 1, 2, 3, 5, 6, 7\}$.

$$— K_{42}^* \xrightarrow{r} TT_3.$$

Let $X = Z_{41} \cup \{\infty\}$. The 41 parallel classes are the $\mathcal{C} + i$, $0 \leq i \leq 40$ with: $\mathcal{C} = \{(15, 8, 0) (7, 4, 24) (21, 12, 31) (23, \infty, 22)\} \cup \{(6^{3\alpha}, 6^{3\alpha+1}, 6^{3\alpha+2})/\alpha = 0, 1, 2, 3, 5, 6, 7, 8, 10, 11\}$.

Remark. The four decompositions above are derived from the decompositions $2K_n \xrightarrow{r} C_3$ given by Hanani [3] by suitably assigning directions to the C_3 in these decompositions.

$$— K_{18}^* \xrightarrow{r} C_3.$$

Let $X = Z_{15} \cup \{\infty_1, \infty_2, \infty_3\}$. The 17 parallel classes are the $\mathcal{C}_1 + i$, $0 \leq i \leq 14$, \mathcal{C}_{16} and \mathcal{C}_{17} with

$$\mathcal{C}_1 = \{(0, 11, 12) (1, 3, 9) (2, 6, 5) (\infty_1, 10, 8) (\infty_2, 4, 13) (\infty_3, 14, 7)\},$$

$$\mathcal{C}_{16} = \{(\infty_1, \infty_2, \infty_3)\} \cup \{(i, i+5, i+10)/i = 0, 1, 2, 3, 4\},$$

$$\mathcal{C}_{17} = \{(\infty_1, \infty_3, \infty_2)\} \cup \{(i, i+10, i+5)/i = 0, 1, 2, 3, 4\}.$$

$$— K_{24}^* \xrightarrow{r} C_3.$$

Let $X = Z_{21} \cup \{\infty_1, \infty_2, \infty_3\}$. The 23 parallel classes are the $\mathcal{C}_1 + i$, $0 \leq i \leq 20$, \mathcal{C}_{22} , \mathcal{C}_{23} with

$$\mathcal{C}_1 = \{(0, 1, 4) (3, 5, 9) (12, 17, 2) (8, 7, 16) (11, 19, 14) (\infty_1, 15, 13) \\ (\infty_2, 6, 18) (\infty_3, 20, 10)\},$$

$$\mathcal{C}_{22} = \{(\infty_1, \infty_2, \infty_3)\} \cup \{(i, i+7, i+14)/i = 0, 1, 2, 3, 4, 5, 6\},$$

$$\mathcal{C}_{23} = \{(\infty_1, \infty_3, \infty_2)\} \cup \{(i, i+14, i+7)/i = 0, 1, 2, 3, 4, 5, 6\}.$$

$$— K_{30}^* \xrightarrow{r} C_3.$$

Let $X = Z_{27} \cup \{\infty_1, \infty_2, \infty_3\}$. The 29 parallel classes are the $\mathcal{C}_1 + i$, $0 \leq i \leq 26$, \mathcal{C}_{28} , \mathcal{C}_{29} with

$$\mathcal{C}_1 = \{(0, 1, 3) (4, 7, 11) (12, 17, 23) (6, 13, 21) (16, 14, 24) (10, 9, 22) \\ (15, 26, 20) (\infty_1, 2, 25) (\infty_2, 18, 8) (\infty_3, 5, 19)\},$$

$$\mathcal{C}_{28} = \{(\infty_1, \infty_2, \infty_3)\} \cup \{(i, i+9, i+18)/0 \leq i \leq 8\},$$

$$\mathcal{C}_{29} = \{(\infty_1, \infty_3, \infty_2)\} \cup \{(i, i+18, i+9)/0 \leq i \leq 8\}.$$

$$— K_{42}^* \xrightarrow{r} C_3 \quad (\text{the decomposition is due to A.E. Brouwer}).$$

Let $X = Z_{30} \cup \{\infty_i, 1 \leq i \leq 12\}$.

Let \mathcal{D}_j ($1 \leq j \leq 11$) be the 11 parallel classes of C_3 of the K_{12}^* constructed on the vertices $\{\infty_i, 1 \leq i \leq 12\}$.

The 41 parallel classes of the decomposition of K_{42}^* will be $\mathcal{C}_1 + i$, $0 \leq i \leq 29$, $\mathcal{D}_j \cup (\mathcal{C}_{30} + j)$, $1 \leq j \leq 10$, $\mathcal{C}_{41} \cup \mathcal{D}_{11}$ with

$$\mathcal{C}_1 = \{(0, 17, 22) (1, 7, 5) (8, 2, \infty_1) (3, 19, \infty_2) (4, 26, \infty_3) (6, 24, \infty_4) \\ (9, 13, \infty_5) (23, 12, \infty_6) (18, 27, \infty_7) (29, 20, \infty_8) (16, 11, \infty_9) \\ (28, 21, \infty_{10}) (15, 14, \infty_{11}) (10, 25, \infty_{12})\},$$

$$\mathcal{C}_{30} = \{(0, 10, 20)\} \cup \{(i+1, i+2, i+4) (i+3, i+16, i+19) \\ (i+5, i+17, i+28)/i = 0, 10, 20\},$$

$$\mathcal{C}_{41} = \{(i, i+20, i+10)/0 \leq i \leq 9\}.$$

3.2. Proof of Lemma 2.4

Let Z_{12} be the set of vertices of A and $B = \{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5, \infty_6\}$. To obtain the decomposition of $K_{[A \cup B]}^* - K_{[B]}^*$ into TT_3 with the wanted properties it suffices to take $D_j = \mathcal{C} + j$, $1 \leq j \leq 12$ with $\mathcal{C} = \{(0, \infty_1, 6) (3, \infty_2, 1) (2, \infty_3, 9) (4, \infty_4, 7) (5, \infty_5, 10) (11, \infty_6, 8)\}$ and

$$E_{13} = \{(1, 0, 2) (4, 3, 5) (7, 6, 8) (10, 9, 11)\},$$

$$E_{14} = E_{13} + 1,$$

$$E_{15} = E_{13} + 2,$$

$$E_{16} = \{(0, 4, 8) (1, 5, 9) (2, 6, 10) (3, 7, 11)\},$$

$$E_{17} = \{(8, 4, 0) (9, 5, 1) (10, 6, 2) (11, 7, 3)\}.$$

The decomposition of $K_{[A \cup B]}^* - K_{[B]}^*$ into C_3 is given by

$$D_j = \mathcal{C} + j, \quad 1 \leq j \leq 12,$$

with

$$\mathcal{C} = \{(11, \infty_1, 0) (10, \infty_2, 1) (2, \infty_3, 4) (3, \infty_4, 8) (9, \infty_5, 6) (7, \infty_6, 5)\}$$

and

$$E_{13} = \{(0, 1, 6) (4, 5, 10) (8, 9, 2) (3, 7, 11)\},$$

$$E_{14} = E_{13} + 1,$$

$$E_{15} = E_{13} + 2,$$

$$E_{16} = E_{13} + 3,$$

$$E_{17} = \{(10, 6, 2) (11, 7, 3) (0, 8, 4) (1, 9, 5)\}.$$

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