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Complex Portfolio Selection via Convex Mixed-Integer Quadratic Programming: A Survey

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1 Introduction

The first milestone in modern single-period portfolio selection theory is undoubtedly Harry Markowitz’s 1952 seminal paper [101] (for an historic perspective, see also [122] and surveys [142, 35, 53]), in which the mean-variance portfolio optimization model was proposed for the first time. Although several ideas and results are already introduced by de Finetti [42] (see [5] for the English translation of the first chapter “The problem in a single accounting period”), the of the Italian mathematician was discovered only recently by the financial international community (see [124, 123, 119]) and was acknowledged by Harry Markowitz himself [103]. The mean-variance approach is based on the fundamental observation that, according to what Markowitz states in [101], the investors should try to increase their portfolio return and contemporaneously to decrease, as much as possible, its volatility or its risk (see also [102, 106]). As pointed out by Markowitz in [105, 104], the previous assumptions are sufficient, but not necessary conditions. However, might be unrealistic: the probability distribution for expected returns is generally leptokurtic [109].

The resulting mathematical program , but it can be enriched with various constraints to model the different characteristics of the modern financial markets. Moreover, the mean-variance approach considers only the first- and second-order moments of probability distribution of returns: consequently, in specific situations, this approach might lead to counterintuitive or even paradoxical solutions [36].

Kallberg and Ziemba [84] compare the effects of different utility functions with respect to optimal portfolios , and show empirically that utility functions with similar absolute risk aversion indices – defined by Arrow [3] and Pratt [118], but originally introduced by de Finetti [43] (see also [27, 112]) – give rise to similar optimal portfolios.

The remainder of the paper is organized as follows. In the next Section we give the mathematical formulation of the portfolio problem according to Markowitz

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[101] and we discuss its main drawbacks. . Section 6 describes several equivalent mathematical reformulations for the . In Section 7 we survey exact methods, proposed in literature, to solve mean-variance portfolio problems. Conclusions are finally drawn in Section 8. A summary of the main notation used throughout the paper is reported in the appendix.

2 Portfolio optimization

We consider r possibly risky assets characterized by a mean return vector $\bar{\mu} \in \mathbb{R}^r$. Let $x \in \mathbb{R}_+^r$ be the vector whose generic entry represents the fraction of the portfolio value invested in asset j ($j = 1, \dots, r$). For the moment, following Markowitz [101], we assume that the entries of $\bar{\mu}$ and of the covariance return matrix $\bar{\Sigma} \in \mathbb{R}^{r \times r}$. In mean-variance approach, we aim to minimize the portfolio variance $x^T \bar{\Sigma} x$ under the constraint that the portfolio return is at least equal to a given level $R > 0$:

$$\min x^T \bar{\Sigma} x \tag{1a}$$

$$\text{s.t. } \bar{\mu}^T x \geq R \tag{1b}$$

$$\mathbf{e}^T x = 1 \tag{1c}$$

$$x \geq 0, \tag{1d}$$

where $\mathbf{e} \in \mathbb{R}^r$ is the all-one vector. Then, by repeatedly solving problem (1) for different values of return R , we can compute the efficient frontier, i.e., the set of the non-dominated portfolios in the sense of Pareto optimality. In several cases (see, e.g., [22]) an additional non-risky asset with mean $\bar{\mu}_0$ and zero variance is also considered, in order to algorithmically derive the efficient frontier (see Section 4.8). Several papers (see, e.g., [40]) consider an equality version for the return constraint (1b), namely $\bar{\mu}^T x = R$.

Constraint (1c) ensures that the whole is invested in the portfolio. Constraint (1d) short selling, i.e., the possibility for the investor to sell financial assets in his/her portfolio. This financial operation is generally performed with speculative intents when the investor expects a bearish trend in the financial . In case short selling is allowed, (1c) can be replaced by the constraint $\mathbf{e}^T x = 0$, which defines the so-called dollar neutral portfolio, by requiring the exposure on long part of the portfolio the one on the short part. Several authors consider the decision variables x represent the absolute amount invested per asset so that the inequality (1c) becomes $\mathbf{e}^T x \leq B$, where B is the investor's .

In [25], Buchheim et al. introduce the budget constraint:

$$v^T x \leq B, \tag{2}$$

where the decision variables x are the units of financial asset held in the investor's portfolio and v is the vector of the costs per unit of corresponding asset.

Problem (1) is a convex continuous linearly constrained quadratic program, because, by definition, matrix $\bar{\Sigma}$ is symmetric and positive semidefinite; hence, we have a computationally tractable problem. However, the main drawback of this model consists in the sensitivity of the optimal solutions with respect to (expected returns and covariance matrix), which are clearly unknown in real-world applications . Furthermore, when $\bar{\Sigma}$ is estimated starting from empirical

measurements, it might happen that semidefiniteness is not directly satisfied and some ad-hoc procedures are required (see [39, 75, 128]).

Chopra [33] empirically analyzes the effects of slight differences in the estimate. Best and Grauer [15] conduct a analysis with computational results about the sensitivity of mean-variance efficient portfolios changes in asset means. the instability and of problem (1): for instance, Kallberg and Ziemba [85] consider estimation errors in the investor's utility function and the mean vector and covariance matrix of the for normally distributed portfolio selection problems and observe that errors in mean vector give rise to significant problems. Chopra and Ziemba [34] show that the estimating errors the expected is generally one order of magnitude than the one corresponding to estimating errors in asset variances or covariances, assuming negative exponential utility function with joint normal distribution of returns.

3 Robust and Approaches

3.1 Robust approaches

The robust version of the mean-variance problem (1) has been considered in quite recent works (see the surveys [54, 66]). It consists in assuming that the expected returns are uncertain and their expected values and variances belong to a given set. By input parameters belong to a given uncertainty set (see [10, 11, 50, 52]), it is possible to show some theoretical results.

Goldfarb and Iyengar [68]

Under the assumption that the return mean belongs to a convex polytope, whose vertices are known, Costa and Paiva [37] prove that program (1) Linear Matrix Inequalities (LMI) problem (see [23, 115]). El Ghaoui et al. [51] show that, when the mean and the covariance are unknown, but bounded, a Semidefinite Program (SDP) (see [12, 115, 126, 140]).

Ye et al. [144] introduce uncertain sets both for the mean vector and the second moment matrix of the returns, showing the connection between the fully robust portfolio selection problem with box uncertain set for the mean and ellipsoid uncertain set for the second moment of returns and SOCP, SDP, and Programming (see [145]).

3.2

Bonami and Lejeune [22] take into account the uncertainty in the expected assets returns a probabilistic constraint, which imposes that the expected return of the optimal portfolio should be not less than a given return level R with a high probability $p > 0$.

Let ξ be the random vector representing the expected returns of the r risky assets. We assume that the random vector ξ admits a probability density function and the density function of $\xi^T x$ is strictly positive. Moreover, let $\mu \in \mathbb{R}^r$ with $\mu = \mathbb{E}[\xi]$ and $\Sigma = \mathbb{E}[(\xi - \mu)(\xi - \mu)^T]$ be the mean and the covariance matrix for the r -variate distribution of ξ , respectively. Let $\psi = (\xi^T x - \mu^T x) / \sqrt{x^T \Sigma x}$ be the standardized random variable representing the normalized port-

folio return. (??) can be equivalently rewritten, as follows:

$$\mathbb{P}(\xi^T x \geq R) = \mathbb{P}\left(\psi \geq \frac{R - \mu^T x}{\sqrt{x^T \Sigma x}}\right) = 1 - F_{(x)}\left(\frac{R - \mu^T x}{\sqrt{x^T \Sigma x}}\right), \quad (3)$$

where $F_{(x)}(\cdot)$ is the cumulative distribution of the standardized portfolio return. We assume that $F_{(x)}(\cdot)$ is a continue strictly increasing function. Moreover, we point out that the analytic form of the probability distribution F depends on the portfolio weights x . It follows that the probabilistic constraint (3) becomes

$$\begin{aligned} 1 - F_{(x)}\left(\frac{R - \mu^T x}{\sqrt{x^T \Sigma x}}\right) \geq p &\iff 1 - p \geq F_{(x)}\left(\frac{R - \mu^T x}{\sqrt{x^T \Sigma x}}\right) \\ &\iff \mu^T x + F_{(x)}^{-1}(1 - p) \sqrt{x^T \Sigma x} \geq R, \end{aligned} \quad (4)$$

where $F_{(x)}^{-1}(\cdot)$ is the inverse of the cumulative distribution $F_{(x)}(\cdot)$ and $F_{(x)}^{-1}(1 - p)$ is the $(1 - p)$ -quantile of $F_{(x)}(\cdot)$. Therefore, the deterministic equivalent of optimization problem (??) corresponds to the following NLP [87]:

$$\min x^T \Sigma x \quad (5a)$$

$$\text{s.t. } \mu^T x + F_{(x)}^{-1}(1 - p) \sqrt{x^T \Sigma x} \geq R \quad (5b)$$

$$\mathbf{e}^T x = 1 \quad (5c)$$

$$x \geq 0. \quad (5d)$$

In the following, we survey for which the problem SOCP (see, e.g., [1, 12, 24]). We thus recall the definition of centrally distributed random variable.

Definition 1. (Serfling [131]) Let $\xi \in \mathbb{R}^r$ be a random variable, whose probability density function is $f : \mathbb{R}^r \rightarrow \mathbb{R}$. If $f(\xi - \theta) = f(\theta - \xi)$, then ξ has a symmetric about $\theta \in \mathbb{R}^r$.

Theorem 1. (Bonami and Lejeune [22]) *the deterministic constraint (5b), equivalent to (3), is a Second-Order Cone Constraint (SOCC).*

Therefore, optimization problem (??) is a SOCP because its objective function is convex quadratic and its feasible region is described by the intersection of a second-order cone and several linear constraints. Constraint $\mu^T x \geq R - F_{(x)}^{-1}(1 - p) \sqrt{x^T \Sigma x}$ ensures that the expected portfolio return is greater than the given return plus a penalty term, which is function of the portfolio variance and is increasing the confidence level p [60].

We recall also the definition of the skewness of a multi-variate distribution of a real-valued random variable ξ with mean μ and standard deviation σ [4]:

$$\text{skew}(\xi) = \frac{\mathbb{E}[\xi - \mu]^3}{\sigma^3}, \quad (6)$$

The skewness is basically an asymmetry index of the distribution: perfectly symmetric distributions have zero skewness.

Theorem 2. (Bonami and Lejeune [22]) *the deterministic constraint (5b), equivalent to (3), is a SOCC.*

The exact value of the $(1 - p)$ -quantile, $F_{(x)}(1 - p)$, is known for probability distributions. If we assume, for example, the distribution of expected returns is Gaussian, which is a restrictive assumption (see, for example, [56, 57, 98, 120]), but rather common in several theoretical frameworks (see [72, 84, 112]), then the numerical values of quantiles $F_{(x)}^{-1}(1 - p)$ of the normalized portfolio return ψ are computationally known.

4

Beyond the of problem (1), the other serious drawback of Markowitz's original proposal is represented by (see, e.g., [39, 135]) to problems (1) or (??), which describe the most common restrictions observed in real-world financial markets (see [55, 89, 93, 96, 99]). However, this kind of constraints could make the efficient frontier discontinuous and more challenging to compute [81].

4.1 Buy-in thresholds

Generally, investors, because, on one side, they have a limited impact on the return value of the portfolio and, on the other side, they could be quite expensive finance fees and monitoring costs [129].

$$\underline{x}_j \leq x_j \leq \bar{x}_j, \quad j = 1, \dots, r. \quad (7)$$

Several authors (see, [31, 32, 63, 81]) require x to be a semi-continuous variable [137], i.e., they require $x_j \in [\underline{x}_j, \bar{x}_j] \cup \{0\}$ for all $j = 1, \dots, r$: they introduce extra binary variables $\delta \in \{0, 1\}^r$ such that, for all $j = 1, \dots, r$, $\delta_j = 1$ if the investor holds the asset j , i.e. if $x_j > 0$, and add the following constraints avoiding too small or huge holding positions:

$$\underline{x}_j \delta_j \leq x_j \leq \bar{x}_j \delta_j, \quad j = 1, \dots, r. \quad (8)$$

Note that constraints (8) directly $0 \leq x_j \leq \delta_j$

4.2 Round lot purchasing

Usually, investors manage only given lots of shares and other financial agreements, because of facility in monitoring and operations. Furthermore, for small, splitting a large lot, that has to be paid to the broker. Round lot purchasing constraints, for the risky asset j ($j = 1, \dots, r$), batches or lots of S_j stocks.

Let us define $\gamma \in \mathbb{Z}_+^r$ a vector of general integer variables. that the number of the shares of asset j ($j = 1, \dots, r$), namely $\eta_j \in \mathbb{Z}_+$, is an integer multiple of the lot-size S_j :

$$\eta_j = \gamma_j S_j, \quad j = 1, \dots, r. \quad (9)$$

Let q_j be the of asset j ($j = 1, \dots, r$) held in portfolio, then we have $\eta_j = x_j B / q_j$ and constraint (10) can be equivalently rewritten as follows,

$$x_j = \frac{q_j \gamma_j S_j}{B}, \quad j = 1, \dots, r. \quad (10)$$

The reader is referred to [22] for further discussion.

Mansini and Speranza [100] have shown that finding a feasible solution of problem (1) with round lot constraints (9), upper bound on γ_j , i.e., the number ($j = 1, \dots, r$) of minimum lots, and bound constraints with respect to the total portfolio expenditure is .

4.3 Sector diversification

Generally, either there exist law limitations about the risk exposure (this is the case, for instance, of) or investors try to hold a representative portion of their portfolio in a prescribed number of asset categories or industrial sectors. However, in general, optimal portfolios for problem (1) are not well-diversified [70]. Usually, given are lower bound on the fraction of portfolio value held in specific sets of shares. For classical empirical analysis about of a well-diversified portfolio, we refer the reader to [36, 58, 134].

Let us assume that every asset can be allocated to a specific financial category and let C_k ($k = 1, \dots, n$) be the index set of all risky assets connected with the category k . Moreover, we suppose that sets C_k define a partition of $\{1, \dots, r\}$. We introduce a binary variable $\zeta_k \in \{0, 1\}$ for each financial category, such that $\zeta_k = 1$ if and only if the investment in financial category k ($k = 1, \dots, n$) is above a prescribed minimum level \underline{s} :

$$\underline{s}\zeta_k \leq \sum_{j \in C_k} x_j \leq \underline{s} + (1 - \underline{s})\zeta_k. \quad (11)$$

Moreover, we have to consider an additional constraint in order to satisfy the diversification prescription [22], which requires to hold portions of assets in at least $\underline{n} > 0$ categories:

$$\sum_{k=1}^n \zeta_k \geq \underline{n}. \quad (12)$$

4.4 Cardinality constraints

Beyond diversification requirements, asset managers (for instance in index tracking) wish to replicate as accurately as possible a market index with a limited number of financial agreements, namely $\overline{K} > 0$. This can be modeled through the following cardinality constraint:

$$\|x\|_0 = \sum_{j=1}^r \text{sign}(|x_j|) \leq \overline{K}. \quad (13)$$

By introducing additional decision variables , already presented for constraints (8), we can straightforwardly reformulate the previous constraint in the following equivalent form [93]:

$$\sum_{j=1}^r \delta_j \leq \overline{K}. \quad (14)$$

Bienstock [16] (see also [132]) shows that problem (1) with cardinality constraint (14) is , even when $r = 3$. Several authors (see, e.g., [32, 44, 59, 133, 143])

consider an equality version for cardinality constraint (14) and propose mainly heuristic methods to solve the corresponding problem:

$$\sum_{j=1}^r \delta_j = \bar{K}. \quad (15)$$

Using the theoretical results in [130, 139] and extending [28, 29, 30], Cesarone et al. [31] have shown that problem (1) with cardinality constraints (14) has the same optimal solution of problem (1) with equality cardinality constraints (15) and reduce this kind of programs to Standard Quadratic Programming Problem (see [18, 19]), avoiding to explicitly introduce binary variables and considering an exact tailored solving procedure, called Increasing Set Algorithm. and is an problem when the Hessian matrix of the objective function is indefinite, i.e., if the Hessian matrix of the objective function is neither positive nor negative semidefinite [18].

Di Gaspero et al. [45] consider an “interval” version for the cardinality constraint (14): $\underline{K} \leq \sum_{j=1}^r \delta_j \leq \bar{K}$, where \underline{K} and \bar{K} are such that $1 \leq \underline{K} \leq \bar{K} \leq r$. Cardinality constraints are closely related to buy-in threshold constraints [81]. Finally, in several papers (see, e.g., [31, 32, 81]) it is observed that problem (1), with cardinality constraints (14) and with minimum and maximum buy-in thresholds (8) can be straightforwardly reformulated as a convex mixed-integer quadratic problem .

4.5 Sector capitalization

Sector capitalization constraints are introduced by Soleimani et al. [133], in order to mathematically formulate the behavior of investors generally inclined to hold assets in with higher capitalization value to reduce the total portfolio risk.

Let ℓ be the number of economic sectors and suppose, without loss of generality, that they are sorted in non-increasing way according to their capitalization value. Define L_l as the set of assets for economic sector l ($l \in \{1, \dots, \ell\}$). We introduce additional binary variables y_l such that

$$\frac{1}{M} \sum_{j \in L_l} \delta_j \leq y_l \leq M \sum_{j \in L_l} \delta_j \quad l \in \{1, \dots, \ell\} \quad (16a)$$

$$\sum_{j \in L_l} \bar{\mu}_j + (1 - y_l) \geq \sum_{j \in L_{l+1}} \bar{\mu}_j \quad , \quad (16b)$$

where $M \in \mathbb{R}_+$ is a sufficiently large positive number. The “big-M” constraints (16) ensure the assets belonging to the sectors with higher capitalization values have basically higher probability to be in the optimal portfolios than .

4.6 Turnover and trading

Frequently, investors already hold a portfolio $x^{(0)}$ and, because of mutations in the financial market or others, they want to change their portfolio, by considering the new financial environment and by limiting, however, the variations the portfolio already held [117].

Crama and Schyns [40] propose to introduce restrictions on purchasing and selling variations. In particular, let \overline{P}_j and \overline{S}_j be respectively the maximum purchasing and selling levels for asset j ($j = 1, \dots, r$), turnover constraints can be stated as follows,

$$\max \{x_j - x_j^{(0)}, 0\} \leq \overline{P}_j \quad j = 1, \dots, r \quad (17a)$$

$$\max \{x_j^{(0)} - x_j, 0\} \leq \overline{S}_j \quad j = 1, \dots, r. \quad (17b)$$

Because of fixed transaction costs (see Section 4.1), additional constraints are, generally, introduced in order to prevent small variations between portfolios. Let \underline{P}_j and \underline{S}_j be respectively the minimum purchasing and selling levels for asset j , trading disjunctive constraints can be stated as follows, $(x_j = x_j^{(0)}) \vee (x_j \leq x_j^{(0)} + \underline{P}_j) \vee (x_j \leq x_j^{(0)} - \underline{S}_j) j = 1, \dots, r$.

4.7 Benchmark constraints

Often, investors want to obtain a portfolio which is as close as possible to a benchmark (or target) portfolio x^B [13]. economic sector diversified investments, [14] introduce the following additional constraints in order to bound variances between the optimal and the target portfolios:

$$\left| \sum_{j \in S_l} (x_j - x_j^B) \right| \leq \varepsilon_l \quad l = 1, \dots, \ell. \quad (18)$$

4.8 Collateral constraints

Di Gaspero et al. [47] (see also [80]) discuss the following legal constraints for short selling portfolios imposed by US Regulation T, a set of US laws concerning the margin requirements for the collateral agreement. In particular, they introduce a free-risk asset with mean μ_0 and zero variance, the so-called collateral agreement, such that

$$x_0 \geq -a \sum_{j=1}^r \min\{0, x_j\} \quad (19a)$$

$$\sum_{j=0}^r |x_j| \leq 2 \quad (19b)$$

where $a \in \mathbb{N}_+$ is the security level for the collateral agreement.

In this case the decision variables x are not constrained to be positive, since short-selling is allowed, variables δ_j defined in (8) are replaced by ternary variables $z \in \{-1, 0, 1\}^r$, such that, for each j ($j = 1, \dots, r$), $z_j = 1$ if the investor bought the asset j , i.e., if $x_j > 0$, if the investor sold the asset j , i.e., if $x_j < 0$, and if the investor does not hold asset j . Therefore, cardinality constraint (14) becomes $\sum_{j=1}^r |z_j| \leq \overline{K}$.

5

Besides (1a), several different objective functions have been proposed in the literature in order to make problems (1) and (??) simpler with respect to com-

putational tractability or to better model real behaviors of investors and money savers. We consider only objective functions involving quadratic risk measure, namely portfolio variance .

5.1 Penalty functions

In order to define an unconstrained NLP, Bartholomew-Biggs and Kane [6] introduce the following penalty function for problem (1) with minimum buy-in threshold constraints (7) with $\underline{x}_j := \underline{x}$ and $\bar{x}_j := 1$ for all $j \in \{1, \dots, r\}$,

$$\phi(x_j) = \frac{4x_j(x_j - \underline{x})}{\underline{x}^2}, \quad j = 1, \dots, r \quad (20)$$

which is when $x_j \leq 0$ or $x_j \geq \underline{x}$. Moreover, when $x_j \in (0, \underline{x})$, so that (7) can be replaced by the following continuous one:

$$\phi(x_j) \geq 0, \quad j = 1, \dots, r. \quad (21)$$

Therefore, an unconstrained NLP can be easily defined, by introducing additional continuous variables $s \in \mathbb{R}^r$, such that $x_j := s_j^2$ for all $j = 1, \dots, r$ and considering the resulting objective function, adjoint with penalty terms, one replacing each set of constraints:

$$x^T \bar{\Sigma} x + \rho(1 - \mathbf{e}^T x)^2 + \rho \left(\frac{\bar{\mu}^T x}{R} - 1 \right)^2 + \tau \sum_{j=1}^r \kappa_j(x_j)^2, \quad (22)$$

where ρ and τ are suitable positive parameters and $\kappa_j(x_j) := \min\{0, \phi(x_j)\}$ for all $j = 1, \dots, r$.

A similar approach is stated also for the round lot purchasing constraints (9) or (10), that can be replaced by the following constraints:

$$\kappa'_j(x_j) = \left(\frac{Bx_j}{q_j} - \left\lfloor \frac{Bx_j}{q_j} \right\rfloor \right) \left(1 - \left(\frac{Bx_j}{q_j} - \left\lfloor \frac{Bx_j}{q_j} \right\rfloor \right) \right) = 0, \quad j = 1, \dots, r \quad (23)$$

where $\lfloor v \rfloor$ denotes the integer part of $v \in \mathbb{R}$.

However, round lot purchasing constraints (9) might make impossible satisfy at the same time request (1c): consequently, the following new quadratic risk measure [110] is considered:

$$\frac{x^T \bar{\Sigma} x}{(\mathbf{e}^T x)^2}, \quad (24)$$

leading to an alternative definition of (22):

$$\frac{x^T \bar{\Sigma} x}{(\mathbf{e}^T x)^2} + \rho(\min\{0, 1 - \mathbf{e}^T x\})^2 + \rho \left(\frac{\bar{\mu}^T x}{R} - 1 \right)^2 + \tau \sum_{j=1}^r \kappa'_j(x_j)^2. \quad (25)$$

Bartholomew-Biggs and Kane [6] apply a DIRECT (DIviding RECTangles) type global algorithm (see [61, 64, 65, 82, 83]) to previous unconstrained (22) and (25).

5.2 Balanced objective functions

Mean-variance portfolio selection problems (1) and (??) are naturally multi-objective optimization programs usually investors want to gain the maximum profit at the minimum risk: these are, of course, conflicting targets, that have to be considered at the same time.

Several authors (see, e.g., [32]) use standard (linear) scalarization techniques such as the Weighted Sum approach (see, for example, [49]). Namely, they consider the “balanced” objective function

$$\lambda(x^T \bar{\Sigma} x) - (1 - \lambda)(\bar{\mu}^T x), \quad (26)$$

where $\lambda \in [0, 1]$ is a investor’s risk . Let $\theta_1, \theta_2 \in \mathbb{R}_+$ be two parameters, a more general variant is proposed by Schaerf [127]:

$$\theta_1(x^T \bar{\Sigma} x) + \theta_2 \max\{0, \bar{\mu}^T x - R\}. \quad (27)$$

Bertsimas and Shioda [14] (see also [13]) introduce an extended “balanced” objective function, considering also trading and turnover requirements with respect to a given initial portfolio $x^{(0)}$:

$$\frac{1}{2}x^T \bar{\Sigma} x - \bar{\mu}^T x + \sum_{j=1}^r \iota_j (x_j - x_j^{(0)})^2, \quad (28)$$

where $\iota_j > 0$ is a coefficient for asset j symmetric purchasing/selling impact with respect to the stock price. Finally, Tadonki and Vial [138] and Shaw et al. [132] consider respectively constant and linear transaction costs embedded in a quadratic “balanced” objective function, namely respectively

$$\lambda_1(x^T \bar{\Sigma} x) - \lambda_2(\bar{\mu}^T x) + c^T \delta \quad (29)$$

$$\lambda_1(x^T \bar{\Sigma} x) - \lambda_2(\bar{\mu}^T x) + c^T x, \quad (30)$$

where $\lambda_1 \in \mathbb{R}_+$ and $\lambda_2 \in \mathbb{R}_+$ are two positive scalars, c is a vector, whose entries represent the transaction costs for the portfolio assets and $\delta \in \mathbb{R}^r$ is the binary vector defined in constraints (8).

6 Compact reformulations

In this section we present several different possible reformulations and approximations for the mean-variance portfolio optimization problem.

6.1 SOCC inner approximations

As observed in Section 3.2, given F , it is not always possible to write the problem (5) in a closed form: the exact value for the quantile $F_{(x)}^{-1}(1 - p)$ is known only for special distributions (e.g., normal distribution, Student distribution, uniform distribution on an ellipsoid). However, if the probability distribution of the expected returns is only partially known, the value of its quantiles can be approximately computed using several probability inequalities [93], e.g., Cantelli [22], Chebyshev [22], and Camp-Meidell [92] inequalities (see also [73, 79, 95, 107]).

Theorem 3. (Bonami and Lejeune [22]) *Assume the first and the second moments of the probability distribution of the portfolio return are finite. The SOCC*

$$\mu^T x - \sqrt{\frac{p}{1-p}} \sqrt{x^T \Sigma x} \geq R \quad (31)$$

is an approximation of chance constraint (??).

Theorem 4. (Bonami and Lejeune [22]) *Assume the first and the second moment of the probability distribution of the portfolio return are finite and the distribution is symmetric. The SOCC*

$$\mu^T x - \sqrt{\frac{1}{2(1-p)}} \sqrt{x^T \Sigma x} \geq R \quad (32)$$

is an approximation of chance constraint (??).

Theorem 5. (Lejeune [92]) *Assume the first and the second moment of the probability distribution of the portfolio return are finite and the distribution is symmetric and unimodal. The SOCC*

$$\mu^T x - \sqrt{\frac{1}{9(1-p)}} \sqrt{x^T \Sigma x} \geq R \quad (33)$$

is an approximation of chance constraint (??).

We remark that the approximation given by Theorem 4 for a symmetric probability distribution is tighter than the one given by Theorem 3 and that the approximation given by Theorem 5 for a symmetric unimodal probability distribution is tighter than the one given by Theorem 4 (see [22, 92]).

6.2 Variance reformulation

Given the symmetric positive definite matrix Σ , we consider its Cholesky decomposition $\Sigma = CC^T$, where $C \in \mathbb{R}^{r \times r}$ is a lower triangular matrix. From a computational viewpoint, the Cholesky decomposition is twice faster and more stable than LU factorization or Gauss elimination method (see [88, 108, 136]) and it is implemented in High Performance Computing numerical software libraries (see [2, 17, 48]).

Note that Cholesky decomposition exists and is unique if matrix Σ is positive definite (see [69, 78]) and this property is verified by matrix, if we exclude the case of exact collinearity of the random variables, i.e., we assume that none of the risky asset can be exactly replicated by a linear combination of the other ones. The hypotheses to apply Cholesky decomposition to positive semidefinite matrices are identified in [74, 78, 111] and error analysis is instead formally stated in [111] for idempotent matrices and in [74] for the general case.

By assuming positive definiteness for covariance matrix Σ and introducing non negative decision variable $h \in \mathbb{R}_+$, we obtain the following problem, equiv-

alent to (5):

$$\min_{x,h} \|C^T x\|_2^2 \quad (34a)$$

$$\text{s.t. } \mu^T x - R \geq h \quad (34b)$$

$$F_{(x)}^{-1}(1-p) \|C^T x\|_2 \geq -h \quad (34c)$$

$$\mathbf{e}^T x = 1 \quad (34d)$$

$$x \geq 0, \quad h \geq 0. \quad (34e)$$

Theorem 6. (Filomena and Lejeune [60]) *Program (34) is equivalent to the following NLP:*

$$\begin{aligned} \min_{x,h} \quad & h \\ \text{s.t.} \quad & (34b), (34c), (34d), (34e). \end{aligned} \quad (35)$$

6.3 Period-separable reformulation

As pointed out by Filomena and Lejeune [60], the variance of the portfolio can be reformulated as the Euclidean norm of a vector, whose number of components T corresponds to the number of data points, by using the following preliminary result:

Theorem 7. (Konno and Suzuki [90]) *Let ν_{jt} be the (observed) return of asset j at time t and introduce the extra variables $b_t = \sum_{j=1}^r (\nu_{jt} - \mu_j) x_j$ ($t = 1, \dots, T$). The variance of the portfolio return can be rewritten as*

$$x^T \Sigma x = \frac{1}{T} \|b\|_2^2.$$

The probabilistic Markowitz portfolio model (??) can be reformulated as the following convex NLP:

$$\min_{x,h,b} \frac{1}{T} \|b\|_2^2 \quad (36a)$$

$$\text{s.t. } \mu^T x - R \geq h \quad (36b)$$

$$\frac{F_{(x)}^{-1}(1-p)}{\sqrt{T}} \|b\|_2 \geq -h \quad (36c)$$

$$b_t - \sum_{j=1}^r (\nu_{jt} - \mu_j) x_j = 0, \quad t = 1, \dots, T \quad (36d)$$

$$\mathbf{e}^T x = 1 \quad (36e)$$

$$x \geq 0, \quad h \geq 0. \quad (36f)$$

Furthermore, Filomena and Lejeune [60] observe that mathematically compute the variance in problems (??) and (35) the estimate of only $r(r+1)/2$ covariance terms : this situation can to several coherence problems for the covariance matrix (see Section 1). Moreover, the approach described in this section does not require any assumption on matrix Σ . Finally, we can consider the corresponding equivalent epigraph formulation of problem (36):

$$\begin{aligned} \min_{x,h,b} \quad & h \\ \text{s.t.} \quad & (36b), (36c), (36d), (36e), (36f). \end{aligned}$$

7 Exact algorithms

Mean-variance portfolio selection problem with \bar{x} gives rise to a convex MIQP, which is at least as difficult as \bar{x} , because it includes Mixed-Integer Linear Problem (MILP) as special case [86, 67]. Nowadays, MIQPs can be solved via commercial and open-source solvers (see, e.g., [21, 41, 141]). In this section we overview specialized and more efficient computational procedures recently proposed in literature.

In [16] Bienstock proposes a tailored Branch-and-Cut (BC) procedure to solve \bar{x} with the “surrogate” constraint

$$\sum_{j=1}^r \frac{x_j}{\bar{x}_j} \leq \bar{K}. \quad (37)$$

Several types of cutting planes, namely mixed-integer rounding inequalities, knapsack cuts, intersection cuts, and disjunctive cuts are also considered in the same paper.

Bertsimas and Shioda [14] develop a BC algorithm [38]. The portfolio problem with objective function (29) and cardinality (14) and buy-in (8) constraints was solved by Tadonki and Vial [138] with BB techniques together with a Bender decomposition approach.

Lee and Mitchell [91] describe a parallel BB framework for \bar{x} cardinality constrained portfolio selection problem, in which each node is approximated by means of Sequential Quadratic Programming (SQP) and each quadratic subproblem is solved via interior-point method (see, e.g., [114, 115]).

Frangioni and Gentile [63] solve problem (1) with minimum and maximum buy-in thresholds additional constraints (8) with a BC method improved by using Perspective Cuts (see also [62]), a family of valid inequalities, related to the perspective function (see [76, 77]) and to the convex envelope of the objective function (see [71]).

Zheng et al. [147] propose a difference of convex functions approach to \bar{x} , by replacing \bar{x} cardinality constraint (13) with the following piecewise linear approximation:

$$\frac{1}{\omega} \left(\|x\|_1 - \sum_{j=1}^r \max\{x_j - \omega, 0\} + \max\{-x_j - \omega, 0\} \right) \leq 0, \quad (38)$$

where $\omega > 0$ is a given parameter. Non-smooth approximation (1) with constraint (38) is solved by means of Successive Convex Approximation method. This algorithm determines a Karush-Kuhn-Tucker (KKT) point or defines a sequence of points converging to a KKT point for the ω -parametrized approximation. Moreover, the authors show that, letting $\omega \rightarrow 0^+$, the optimal value of the approximate problem approaches the optimal value of the original problem.

Shaw et al. [132] solve cardinality constrained portfolio problem under the assumption that vector $\bar{\mu}$ of assets returns can be decomposed according to a multiple factor model [27], i.e., $\bar{\mu} = \Xi f + u$, where r' represents the number of different factors, $\Xi \in \mathbb{R}^{r \times r'}$ is the sensitivity-factor matrix, $f \in \mathbb{R}^{r'}$ is the factor-return vector, and $u \in \mathbb{R}^r$ is the asset-specific (non-factor) returns vector. A Lagrangian relaxation of the problem is then solved by means of sub-gradient procedure [121] and embedded in a BB framework.

In [22] Bonami and Lejeune deal with deterministic equivalent (5) of with buy-in threshold (8), round lot purchasing (10), and diversification (11)-(12) constraints, by proposing a Nonlinear BB algorithm [9] with tailored branching rules.

Buchheim et al. [25] consider portfolio selection problem with objective function (26), constraints (1d) and (2) and integrality requirement on the decision variables, i.e.,

$$x \in \mathbb{Z}^r, \quad (39)$$

which represents the units of assets held in the investor's portfolio. They introduce a new BB algorithm

We end this section with Table 1 that summarizes the main characteristics of the papers described above. In particular, the columns report the authors, the year of publication of the paper, the objective function and constraints of the tackled problem, the proposed algorithm, the competitors employed as benchmarks, and the instances that were used for the computational experiments.

8 Conclusions

in terms of mathematical modeling and robustness. In fact, in practical application the return and the variance of the assets are unknown and are estimated from observed data: the optimal solutions of the original Markowitz model is not robust variations in estimation of the input parameters. Moreover, the original Markowitz work

The first limitation is faced via robust and . In the first case, we assume the assets returns (and eventually their variances) are unknown, but belonging to a given uncertain set, while in the second one we substitute the return constraint with , the return of the portfolio above a given level with high probability. The second limitation , as, for example, buy-in threshold, round lot purchasing, diversification, and cardinality constraints.

Furthermore, several variants of the original objective function have been proposed in the literature for computational and algorithmic reasons (for instance, by introducing several penalty terms in the objective to obtain an unconstrained optimization problem) or for better modeling the real-world financial markets (for example, by considering constant or linear transaction costs). Then, we discussed possible exact and (inner) approximate reformulations for of the mean-variance portfolio problem.

Finally, in the last part of the paper, we have reviewed exact tailored algorithms designed to solve the deterministic and to optimality.

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Author(s)	Year	Objective	Constraints	Algorithm	Benchmark(s)	Instances (number and type)
[16]	1996	(1a)	(1c), (1b), (1d), (7), (14)	Branch-and-Cut	–	13 problems (real-life data)
[91]	2000	(1a)	(1c), (1b), (1d), (14)	Branch-and-Bound	–	5 portfolio instances and 16 instances from MIPLIB1
[138]	2003	(29)	(1c), (1b), (1d), (8), (14)	Branch-and-Bound	–	OR-Library ² [7, 8]
[63]	2006	(1a)	(1c), (1b), (1d), (8)	Branch-and-Cut	CPLEX 8.0	20 self-generated ³ (see [116])
[132]	2008	(30)	(7), (14)	Branch-and-Bound	CPLEX 8.1	8 real representative instances
[6]	2009	(22)	–	DIRECT	–	2 illustrative examples
[6]	2009	(25)	–	DIRECT	–	2 illustrative examples
[14]	2009	(28)	(1c), (1d), (7), (14), (18)	Branch-and-Bound	CPLEX 8.1	50 randomly generated
[22]	2009	(1a)	(1c), (1d), (5b), (8) (10), (11), (12), (39)	Branch-and-Bound	[20], CPLEX 10.1 MINLP-BB [94]	36 self-generated instances
[147]	2012	(1a)	(1c), (1b), (1d), (7), (13)	SQA Algorithm	[97]	[63]
[31]	2013	(1a)	(1c), (1b), (1d), (8), (14)	Increasing Set	[46], [113], [125], [127]	5 additional data sets ⁴
[146]	2013	(1a)	(1c), (1b), (1d), (8), (14)	Branch-and-Cut	–	OR-Library ² [7, 8]
[25]	2015	(26)	(1d), (2), (39)	Branch-and-Bound with Frank-Wolfe	CPLEX 12.6	[31]
[26]	2016	(1a)	(1c), (1d), (??), (7), (14)	Regularization Method	GUROBI 5.6.2	[63]

Table 1: Deterministic exact approaches to mean-variance portfolio selection problem (see also [99] and references within). References are sorted in chronological order; papers published in the same year are sorted according to alphabetic order of the last name of the corresponding first author.

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A Notation

$r \in \mathbb{N}_+$	number of possibly risky assets
$r' \in \mathbb{N}_+$	number of number of different factors
$n \in \mathbb{N}_+$	number of financial categories
$\ell \in \mathbb{N}_+$	number of economic sectors
$R \in \mathbb{N}_+$	minimum return level for the portfolio
$p \in \mathbb{N}_+$	confidence level for the probabilistic return constraint
$B \in \mathbb{N}_+$	total investor initial budget
$\bar{s} \in \mathbb{N}_+$	prescribed minimum level per financial category
$\underline{n} \in \mathbb{N}_+$	minimum number of categories with positive positions
$\underline{K}, \bar{K} \in \mathbb{N}_+$	minimum and maximum number of assets in the portfolio
$\bar{P} \in \mathbb{R}_+$	maximum purchasing level per asset
$\bar{S} \in \mathbb{R}_+$	maximum selling level per asset
$T \in \mathbb{N}_+$	number of time period for observing the returns of quoted assets
$m \in \mathbb{N}_+$	number of factors in the multiple factor model
$\bar{\mu} \in \mathbb{R}^r$	mean return vector of the assets
$\xi \in \mathbb{R}^r$	random vector of expected returns
$\mu \in \mathbb{R}^r$	mean of the r -variate distribution of ξ
$\psi \in \mathbb{R}^r$	normalized portfolio return
$\underline{x}, \bar{x} \in \mathbb{R}^r$	lower and upper bound for the fraction of the portfolio value
$S \in \mathbb{R}^r$	size of the batches of the assets
$q \in \mathbb{R}_+^r$	market value of the quoted assets
$x^{(0)} \in \mathbb{R}^r$	fraction of the portfolio value already invested
$x^B \in \mathbb{R}^r$	benchmark (or target) portfolio
$c \in \mathbb{R}^r$	transaction costs per asset
$v \in \mathbb{R}^r$	costs per unit of asset
$f \in \mathbb{R}^m$	factor-return vector
$u \in \mathbb{R}^r$	asset-specific (non-factor) returns vector
$\Xi \in \mathbb{R}^{r \times r'}$	sensitivity-factor matrix
$\Sigma \in \mathbb{R}^{r \times r}$	covariance matrix of the r -variate distribution of ξ
$\bar{\Sigma} \in \mathbb{R}^{r \times r}$	covariance return matrix of the assets
$L_l \subseteq \{1, \dots, r\}$	set of assets for economic sector l ($l = 1, \dots, \ell$)
$C_k \subseteq \{1, \dots, r\}$	set of indexes of all risky assets connected with the category k ($k = 1, \dots, n$)

Decision Variables:

$x \in \mathbb{R}^r$	(fraction of the) portfolio value invested per asset
$\eta \in \mathbb{Z}_+^r$	integer multiple of the lot-size S
$\delta \in \{0, 1\}^r$	additional binary variables such that $\delta_j = 1$ ($j \in \{1, \dots, r\}$) iff $x_j > 0$ ($j \in \{1, \dots, r\}$)
$\gamma \in \mathbb{Z}_+^r$	additional vector of general integer variables
$\zeta \in \{0, 1\}^n$	additional binary variables such that $\zeta_k = 1$ ($k \in \{1, \dots, n\}$) iff $\sum_{j \in C_k} x_j \leq \bar{s}$
$y \in \{0, 1\}^\ell$	additional binary variables such that $y_l = 1$ ($l \in \{1, \dots, \ell\}$) iff $\sum_{j \in L_l} \delta_j = 1$
$z \in \{-1, 0, 1\}^r$	additional ternary variables such that $z_j = 1$ ($j \in \{1, \dots, r\}$) iff $x_j > 0$ ($j \in \{1, \dots, r\}$) and $z_j = -1$ ($j \in \{1, \dots, r\}$) iff $x_j < 0$ ($j \in \{1, \dots, r\}$)
$s \in \mathbb{R}^r$	additional continuous variables such that $s_j = \sqrt{x_j}$ ($j \in \{1, \dots, r\}$)
$b \in \mathbb{R}^T$	additional continuous variables such that $b_t = \sum_{j=1}^r (\nu_{jt} - \bar{\mu}_j) x_j$ ($t \in \{1, \dots, T\}$)

Functions:

$F(x) : \mathbb{R} \rightarrow \mathbb{R}$

$\phi : \mathbb{R} \rightarrow \mathbb{R}$

$\theta : \mathbb{R} \rightarrow \mathbb{R}$

cumulative distribution of the normalized portfolio return

penalty function for the minimum buy-in threshold constraint (7)

penalty function for the round lot purchasing constraints (9)