



Hamiltonian Decompositions of Graphs, Directed Graphs and Hypergraphs

Jean-Claude Bermond

► To cite this version:

Jean-Claude Bermond. Hamiltonian Decompositions of Graphs, Directed Graphs and Hypergraphs. Annals of Discrete Mathematics, 1978, 3, pp.21-28. 10.1016/S0167-5060(08)70494-1 . hal-02321683

HAL Id: hal-02321683

<https://hal.science/hal-02321683>

Submitted on 21 Oct 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

HAMILTONIAN DECOMPOSITIONS OF GRAPHS, DIRECTED GRAPHS AND HYPERGRAPHS

J.-C. BERMOND

C.M.S., 54 Bd. Raspail, 75006 Paris, Cédex 06, France

0. Definitions

Definitions not given here can be found in [3].

We will say that a graph G (undirected or directed) can be *decomposed into Hamiltonian cycles or paths* if we can partition its edges (arcs in the directed case) into hamiltonian cycles or paths (directed cycles or directed paths in the directed case). Our notation is as follows:

K_n — the complete graph on n vertices;

K_n^* — the complete symmetric directed graph on n vertices;

$K_{r \times n}$ — the complete r -partite graph whose vertex set is the disjoint union of r sets of n elements, two vertices being joined iff they belong to two different sets;

C_r (resp. \vec{C}_r) — a cycle (resp. directed cycle) of length r ;

S_n — an independent set of n vertices;

$G_1 \times G_2$ — the cartesian sum (also called product) of two graphs $G_1 = (X_1, E_1)$ and $G_2 = (X_2, E_2)$ is the graph with vertex set $X_1 \times X_2$ in which (x_1, x_2) is joined to (y_1, y_2) whenever $x_1 = y_1$ and x_2 is joined to y_2 in G_2 , or $x_2 = y_2$ and x_1 is joined to y_1 in G_1 ;

$G_1 \otimes G_2$ — the lexicographic product (also called composition) of two graphs G_1 and G_2 is the graph with vertex set $X_1 \times X_2$ in which (x_1, x_2) is joined to (y_1, y_2) whenever x_1 is joined to y_1 in G_1 or $x_1 = y_1$ and x_2 is joined to y_2 in G_2 ;

$G_1 \cdot G_2$ — the cartesian product (also called conjunction) of two graphs G_1 and G_2 is the graph with vertex set $X_1 \times X_2$ in which (x_1, x_2) is joined to (y_1, y_2) whenever x_1 is joined to y_1 in G_1 and x_2 is joined to y_2 in G_2 .

1. Hamiltonian decompositions of graphs

The first two results are folklore.

1.1. Theorem. K_{2n} can be decomposed into n hamiltonian paths.

1.2. Theorem. K_{2n+1} can be decomposed into n hamiltonian cycles and K_{2n+2} can be decomposed into n hamiltonian cycles and a perfect matching (or 1-factor).

1.3. Theorem (Auerbach and Laskar [2]). $K_{r \times n}$ can be decomposed into hamiltonian cycles iff $n(r-1)$ is even. If $n(r-1)$ is odd $K_{r \times n}$ can be decomposed into hamiltonian cycles and a perfect matching.

We will now give a survey of hamiltonian decompositions of the three products defined in Section 0 and pose some problems concerning them.

1.4. Theorem (Kotzig [12]). $C_r \times C_n$ can be decomposed into 2 hamiltonian cycles.

1.5. Remark. The case $r = n$ is also proved in Myers [15].

1.6. Corollary. If each of G_1 and G_2 can be decomposed into p hamiltonian cycles, then $G_1 \times G_2$ can be decomposed into $2p$ hamiltonian cycles.

1.7. Theorem (Myers [15]). $K_n \times K_n$ can be decomposed into $(n-1)$ hamiltonian cycles.

1.8. Conjecture (Kotzig [12]). $C_r \times C_n \times C_m$ can be decomposed into hamiltonian cycles.

1.9. Remarks on Conjecture 1.8. G. Koester (personal communication, 1977) has proved that $C_4 \times C_4 \times C_4$ can be decomposed into hamiltonian cycles. In fact he informed me that the problem of the existence of a decomposition of $C_4 \times C_4 \times \cdots \times C_4$ (n times) was posed by Ringel [16, Problem 2] as the existence of a decomposition of the $2n$ -cube ($2n$ -dimensional Würfel) into hamiltonian cycles. Ringel [16] proved this conjecture for n a power of 2: this also follows from Theorem 1.4 with $r = n = 4$ and Corollary 1.6.

Very recently I learned that the existence of a decomposition of $C_3 \times C_3 \times C_3$ into hamiltonian cycles was proved by M. Foregger (personal communication of R. Brualdi, 1977).

1.10. Conjecture. $K_m \times K_n$ can be decomposed into $\frac{1}{2}(n+m-2)$ hamiltonian cycles iff $n+m$ is even and into $\frac{1}{2}(n+m-3)$ hamiltonian cycles and a perfect matching if $n+m$ is odd.

1.11. Conjecture. If G_1 can be decomposed into p_1 hamiltonian cycles and if G_2 can be decomposed into p_2 hamiltonian cycles, then $G_1 \times G_2$ can be decomposed into $p_1 + p_2$ hamiltonian cycles.

1.12. Theorem (Laskar [13]). $C_r \otimes S_n$ can be decomposed into n hamiltonian cycles.

1.13. Remark. Theorem 1.12 can be used to give a short proof of Theorem 1.3 since

$K_{r \times n} = K_r \otimes S_n$. If r is odd, then by Theorem 1.2 K_r can be decomposed into hamiltonian cycles, that is $K_r = \bigcup_i C_r^{(i)}$ (where \bigcup means the edge-disjoint union and $1 \leq i \leq r$) and $K_{r \times n} = \bigcup_i C_r^{(i)} \otimes S_n = \bigcup_{i,j} C_m^{(i,j)}$ by Theorem 1.12. If r is even, then $K_{\frac{1}{2}r \times 2}$ is the graph obtained from K_r by deleting a perfect matching and so can be decomposed into hamiltonian cycles by Theorem 1.2. Finally $K_{r \times 2p} = (K_{r \times 2}) \otimes S_p = \bigcup_i C_{2r}^{(i)} \otimes S_p = \bigcup_{i,j} C_{2rp}^{(i,j)}$. And thus if $n(r-1)$ is even $K_{r \times n}$ can be decomposed into hamiltonian cycles.

1.14. Theorem (Laskar [13]). $C_r \otimes C_n$ can be decomposed into $n+1$ hamiltonian cycles if n is odd or r is even.

1.15. Conjecture. $C_r \otimes C_n$ can always be decomposed into $n+1$ hamiltonian cycles.

1.16. Conjecture. If G_1 can be decomposed into p_1 hamiltonian cycles and if G_2 can be decomposed into p_2 hamiltonian cycles, then $G_1 \otimes G_2$ can be decomposed into $p_1 n_2 + p_2$ hamiltonian cycles (where n_2 is the number of vertices of G_2).

1.17. Remark. I can prove that Conjecture 1.15 implies the truth of Conjecture 1.16 for $p_1 \geq p_2$.

1.18. Theorem. $C_r \cdot C_n$ can be decomposed into 2 hamiltonian cycles.

Proof. We can suppose $r \geq n$; let the vertex set be $\mathbb{Z}_r \times \mathbb{Z}_n$ (where \mathbb{Z}_n denotes the additive group of residues mod n). Then two hamiltonian cycles are x_0, x_1, \dots, x_{m-1} and y_0, y_1, \dots, y_{m-1} where $x_{i+j} = (i+j, j)$ and $y_{i+j} = (-i-j, j)$, $0 \leq i \leq r-1, 0 \leq j \leq n-1$. \square

1.19. Corollary. If G_1 and G_2 can be decomposed into hamiltonian cycles, then $G_1 \cdot G_2$ also can be decomposed into hamiltonian cycles.

Proof. This follows from the distributivity of the product \cdot with respect to the edge disjoint union of graphs (a property not holding for the cartesian sum and the lexicographic product). \square

1.20. Many other problems similar to those above can be considered; in particular we can consider decompositions into cycles of given length (see [6]). We mention also that Huang and Rosa [9] have considered “orthogonal” hamiltonian decompositions, and finally we give the following conjecture of Kotzig [11].

1.21. Conjecture (Kotzig [11]). K_{2n} can be decomposed into perfect matchings, i.e. has a 1 factorisation, in such a manner that the union of any two perfect matchings is a hamiltonian cycle.

Partial results have been obtained on this problem, see for example Anderson [1].

2. Hamiltonian decompositions of directed graphs

The same problems can be asked for the directed graphs, but they are more difficult. Many of the known results are obtained in the following easy way; associate with a hamiltonian decomposition of G a hamiltonian decomposition of the directed graph G^* (obtained from G by associating to each edge of G two opposite arcs) by associating with each hamiltonian cycle two opposite directed hamiltonian cycles. For example Theorem 1.2 gives the following.

2.1. Theorem. K_{2n+1}^* can be decomposed into $2n$ directed hamiltonian cycles.

The problem of the existence of a hamiltonian decomposition of K_n^* has been solved only recently.

2.2. Theorem (Tillson [17]). *If $2n \geq 8$, then K_{2n}^* can be decomposed into $2n-1$ directed hamiltonian cycles.*

2.3. For $2n=4$ and $2n=6$, such a decomposition is impossible. The problem seems to have been asked first by Strauss for hamiltonian paths (see Mendelsohn [14]). In [14] Mendelsohn showed how the existence of sequenceable groups implies the existence of a hamiltonian decomposition of K_{2n}^* and that gives the result for $2n=22$ [14], 28 [10], 40, 56, 58 [18]. By computer the existence of a hamiltonian decomposition of K_{2n}^* for $8 \leq 2n \leq 18$ was obtained (see [5]). With Faber we proposed Theorem 2.2 as conjecture in [5] and [4]. After that A. Bouchet (personal communication, 1976) showed that if K_{2n}^* can be decomposed into directed hamiltonian cycles, then so can K_{4n-2}^* .

2.4. Conjecture. *One can easily ask many other problems, for example the directed versions of the results or problems of Section 1. But there are also problems peculiar to the directed case like Kelly's conjecture that every regular tournament can be decomposed into directed hamiltonian cycles.*

3. Hamiltonian decompositions of hypergraphs

3.1. For hypergraphs the number of problems grows quickly, because one can give different definitions of a hamiltonian cycle. I will restrict myself to a definition and a problem considered in [7]. If H is a hypergraph with n vertices then a hamiltonian cycle is a sequence $x_1 E_1 x_2 \cdots x_i E_i x_{i+1} \cdots x_n E_n x_1$ such that

(i) the n vertices x_i are all different (and thus are the n vertices of the hypergraph),

(ii) the n edges E_i are all different,

(iii) $\{x_i, x_{i+1}\} \subset E_i (1 \leq i \leq n-1)$ and $\{x_1, x_n\} \subset E_n$.

Let K_n^h denote the complete h -uniform hypergraph; its edges are all the h -subsets of a set X of cardinality n .

In [7] we conjectured that K_n^h can be decomposed into hamiltonian cycles if and only if $\binom{n}{h}/n$ is an integer and proved this conjecture for n a prime.

Here I want to prove two theorems concerning the case $h=3$.

3.2. Theorem. *If K_n^3 can be decomposed into hamiltonian cycles then K_{2n}^3 also can be decomposed into hamiltonian cycles.*

Proof. (This proof was obtained with D. Sotteau.) In order to shorten the writing, I will write a hamiltonian cycle as $E_1, \dots, E_i, \dots, E_n$ where $E_i = (x_i, y_i, x_{i+1})$. Let the vertex set of K_{2n}^3 be $X \cup X'$ with $|X| = |X'| = n$. With each hamiltonian cycle of the decomposition of K_n^3 we associate 4 hamiltonian cycles of K_{2n}^3 in the following manner:

(i) if n is even, we associate with $(x_1 y_1 x_2)(x_2 y_2 x_3), \dots, (x_n y_n x_1)$ the following

$$\begin{aligned} & (x_1 y_1 x_2)(x_2 y_2 x_3) \cdots (x_{n-1} y_{n-1} x_n)(x_n y_n x'_1) \\ & \quad (x'_1 y'_1 x'_2) \cdots (x'_{n-1} y_{n-1} x'_n)(x'_n y_n x_1), \\ & (x_1 y_1 x'_2)(x'_2 y'_2 x'_3)(x'_3 y'_3 x'_4) \cdots (x_{n-1} y_{n-1} x'_n)(x'_n y_n x'_1) \\ & \quad (x'_1 y'_1 x_2)(x_2 y_2 x'_3) \cdots (x'_{n-1} y_{n-1} x_n)(x_n y_n x_1), \end{aligned}$$

and the two cycles obtained by exchanging the vertices of X and those of X' :

(ii) if n is odd we associate with $(x_1 y_1 x_2)(x_2 y_2 x_3), \dots, (x_n y_n x_1)$ the following:

$$\begin{aligned} & (x_1 y_1 x_2)(x_2 y_2 x_3) \cdots (x_{n-2} y_{n-2} x_{n-1})(x_{n-1} y'_{n-1} x'_n)(x'_n y_n x'_1) \\ & \quad (x'_1 y'_1 x'_2)(x'_2 y'_2 x'_3) \cdots (x'_{n-2} y'_{n-2} x'_{n-1})(x'_{n-1} y'_{n-1} x_n)(x_n y_n x_1), \\ & (x'_1 y'_1 x'_2)(x'_2 y'_2 x'_3) \cdots (x'_{n-2} y'_{n-2} x'_{n-1})(x'_{n-1} y'_{n-1} x'_n)(x'_n y'_n x_1) \\ & \quad (x_1 y'_1 x_2)(x_2 y'_2 x_3) \cdots (x_{n-2} y'_{n-2} x_{n-1})(x_{n-1} y'_{n-1} x_n)(x_n y'_n x_1), \\ & (x_1 y_1 x'_2)(x'_2 y'_2 x'_3) \cdots (x_{n-2} y_{n-2} x'_{n-1})(x'_{n-1} y_{n-1} x_n)(x_n y_n x'_1) \\ & \quad (x'_1 y'_1 x_2)(x_2 y_2 x'_3) \cdots (x'_{n-2} y_{n-2} x_{n-1})(x_{n-1} y_{n-1} x'_n)(x'_n y_n x_1), \end{aligned}$$

and

$$\begin{aligned} & (x'_1 y'_1 x_2)(x_2 y'_2 x'_3) \cdots (x'_{n-2} y'_{n-2} x_{n-1})(x_{n-1} y_{n-1} x_n)(x_n y'_n x_1) \\ & \quad (x_1 y'_1 x'_2)(x'_2 y'_2 x'_3) \cdots (x_{n-2} y'_{n-2} x'_{n-1})(x'_{n-1} y_{n-1} x'_n)(x'_n y'_n x'_1). \end{aligned}$$

Thus we have obtained a decomposition of the edges of K_{2n}^3 not of the form (x, x', y) or (x, x', y') . We will use Theorems 2.1 and 2.2 to decompose these remaining triples. Indeed with the directed hamiltonian cycle x_1, \dots, x_n of a

decomposition of K_n^* we associate the following hamiltonian cycle of K_n^3 :

$$(x_1 x'_1 x_2)(x_2 x'_2 x_3) \cdots (x_{n-1} x'_{n-1} x_n)(x_n x'_n x'_1) \\ (x'_1 x_1 x'_2) \cdots (x'_{n-1} x_{n-1} x'_n)(x'_n x_n x_1).$$

Thus the proof is complete. One can check that we have found $[\frac{4}{6}(n-1)(n-2)] + n - 1 = \frac{1}{6}(2n-1)(2n-2)$ hamiltonian cycles in K_{2n}^3 . \square

3.3. Theorem. *If $n \equiv 2 \pmod{3}$ K_n^3 can be decomposed into hamiltonian cycles.*

We are grateful to A.E. Brouwer (personal communication, 1976) for the following idea on which the proof is based.

3.4. *A choice design of order n is a system of representatives of the triples of K_n^3 such that:*

- (i) each point is chosen equally often as a representative;
- (ii) among the $n-2$ triples containing a given pair $\{a, b\}$, a is chosen $\frac{1}{3}(n-2)$ times and b $\frac{1}{3}(n-2)$ times also.

For example, when $n = 5$, we have underlined the element chosen.

$$012; 01\bar{3}; 0\bar{1}4; 0\bar{2}3; 0\bar{3}4; 034; 12\bar{3}; 12\bar{4}; 1\bar{3}4; 2\bar{3}4.$$

3.5. Theorem. *A choice design of order n exists if and only if $n \equiv 2 \pmod{3}$.*

Proof. The necessary condition is obvious as $\frac{1}{3}(n-2)$ must be an integer. We will prove that the condition is sufficient by induction. Suppose there exists a choice design of order n . Let the elements of K_{n+3}^3 be $\{1, 2, \dots, n\} \cup \{\alpha, \beta, \gamma\}$. For a triple of elements of $\{1, 2, \dots, n\}$ we choose the element defined by the choice design of order n . For the triples we choose

$$\begin{array}{lll} (i, j, \alpha) & (i, j, \beta) & (i, j, \gamma) \quad \text{with } i < j, \\ i & j & \gamma \quad \text{if } i+j \equiv 0 \pmod{3}, \\ j & \beta & i \quad \text{if } i+j \equiv 1 \pmod{3}, \\ \alpha & i & j \quad \text{if } i+j \equiv 2 \pmod{3}. \end{array}$$

For the triples we choose

$$\begin{array}{lll} (i, \alpha, \beta) & (i, \alpha, \gamma) & (i, \beta, \gamma) \\ i & \gamma & \gamma \quad \text{if } i \equiv 0 \pmod{3}, \\ \beta & \alpha & i \quad \text{if } i \equiv 1 \pmod{3}, \\ \alpha & i & \beta \quad \text{if } i \equiv 2 \pmod{3}. \end{array}$$

For the triple (α, β, γ) we choose γ .

We leave to the reader the care of checking that we obtain a choice design of order $n+3$; the only non-immediate part is to check property (ii) for the triples containing a pair (i, α) or (i, β) or (i, γ) . \square

3.6. Proof of Theorem 3.3. If n is odd, there exists a decomposition of K_n into $\frac{1}{2}(n-1)$ hamiltonian cycles. To each of these cycles (x_1, x_2, \dots, x_n) we associate the following $\frac{1}{3}(n-2)$ hamiltonian cycles of K_n^3 : $(x_1, y_1^i, x_2)(x_2, y_2^i, x_3) \dots (x_{n-1}, y_{n-1}^i, x_n)(x_n, y_n^i, x_1)$, where $i = 1, 2, \dots, \frac{1}{3}(n-2)$, $y_j^i \neq y_j^k$ for $i \neq k$ and where the y_j^i are defined according to the existence of a choice design of order n (by Theorem 3.5). The set $\{y_j^i : i = 1, 2, \dots, \frac{1}{3}(n-2)\}$ consists of the $\frac{1}{3}(n-2)$ elements representatives of the $\frac{1}{3}(n-2)$ triples (x_j, x_{j+1}, y) containing the pair $\{x_j, x_{j+1}\}$ and where neither x_j nor x_{j+1} has been chosen. Thus we have constructed $\frac{1}{6}(n-1)(n-2)$ hamiltonian cycles of K_n^3 and it suffices to verify that no triple (edge) appears twice, but that follows from the definition of a choice design of order n .

3.7. Example. Let $(0, 1, 2, 3, 4)$ and $(0, 2, 4, 1, 3)$ be two hamiltonian cycles of K_5 ; by using the choice design of the example we obtain the two hamiltonian cycles of K_5^3 :

$$(0, 3, 1)(1, 4, 2)(2, 0, 3)(3, 1, 4)(4, 2, 0),$$

$$(0, 1, 2)(2, 3, 4)(4, 0, 1)(1, 2, 3)(3, 4, 0).$$

If n is even the proof is similar. We use a decomposition of K_n^* into $n-1$ directed hamiltonian cycles (Theorem 2.2). To each of these directed cycles (x_1, x_2, \dots, x_n) we associate $\frac{1}{6}(n-2)$ hamiltonian cycles of K_n^3 :

$$(x_1, y_1^i, x_2)(x_2, y_2^i, x_3) \dots (x_{n-1}, y_{n-1}^i, x_n)(x_n, y_n^i, x_1),$$

where $i = 1, 2, \dots, \frac{1}{6}(n-2)$; $y_j^i \neq y_j^k$ for $i \neq k$ and where the set $\{y_j^i : i = 1, 2, \dots, \frac{1}{6}(n-2)\}$ is determined as follows. Consider the $\frac{1}{3}(n-2)$ elements representative of the triples (x_j, x_{j+1}, y) containing the pair $\{x_j, x_{j+1}\}$ and where neither x_j nor x_{j+1} has been chosen. Then split these elements into two sets of cardinality $\frac{1}{6}(n-2)$: $Y_{\{x_j, x_{j+1}\}}$ and $Y'_{\{x_j, x_{j+1}\}}$. Then the set $\{y_j^i : i = 1, 2, \dots, \frac{1}{6}(n-2)\}$ is either the set $Y_{\{x_j, x_{j+1}\}}$ or $Y'_{\{x_j, x_{j+1}\}}$ according as the arc (x_j, x_{j+1}) or the arc (x_{j+1}, x_j) appears in the directed hamiltonian cycle. \square

Note added in proof

M.F. Foregger has proved Conjecture 1.8 (Hamiltonian Decompositions of Product of Cycles).

References

- [1] B.A. Anderson, A perfectly arranged room square, Proc. 4th Southeastern Conference on Combinatorics, Graph Theory, and Computing, Utilitas Math. (1973) 141-150.
- [2] B. Auerbach and R. Lasker, On decompositions of r -partite graphs into edge — disjoint hamiltonian circuits, Discrete Math. 14 (1976) 265-268.
- [3] C. Berge, Graphs and Hypergraphs (North-Holland, Amsterdam, 1973).

- [4] J.-C. Bermond and V. Faber, in: Problems of the 5th British Combinatorial Conference, Aberdeen 1975, *Utilitas Math. Congressus Numerantium XV* (1975), 695.
- [5] J.-C. Bermond and V. Faber, Decomposition of the complete directed graph into k -circuits, *J. Combinatorial Theory*, 21(B) (1976) 146–155.
- [6] J.-C. Bermond and D. Sotteau, Graph Decompositions and G -designs, *Proc. 5th Combinatorial Conference, Aberdeen 1975, Utilitas Math. Congressus Numerantium XV* (1975) 53–72.
- [7] J.-C. Bermond, A. Germa, M.-C. Heydemann and D. Sotteau, Hypergraphes hamiltoniens, in: *Coll. Int. C.N.R.S., Problèmes Combinatoires et Théorie des Graphes*, Orsay, 1976.
- [8] J. Denes and A.D. Keedwell, *Latin squares and their applications*. (Akademiai Kiado, Budapest and English University Press, London, 1974).
- [9] C. Huang and A. Rosa, On sets of orthogonal hamiltonian circuits, *Proc. 2nd Manitoba Conference on Numerical Mathematics, Utilitas Math. Congressus Numerantium VII* (1972), 327–332.
- [10] A.D. Keedwell, Some problems concerning complete latin squares. *Combinatorics, Proc. British Combinatorial Conference, 1973, L.M.S. Notes 13*, Cambridge University (1974) 89–96.
- [11] A. Kotzig, *Problem, Theory of Graphs and Its Applications* (Academic Press, New York, 1964) 162 and 63–82.
- [12] A. Kotzig, Every cartesian product of two circuits is decomposable into two hamiltonian circuits, *Centre de Recherches Mathematiques, Montreal* (1973).
- [13] R. Laskar, Decomposition of some composite graphs into hamiltonian cycles, *Proc. 5th Hungarian Colloquium, Keszthely, 1976*, (to appear).
- [14] N.S. Mendelsohn, Hamiltonian decomposition of the complete directed n -graph, *Theory of Graphs* (Akademiai Kiado, Budapest, 1968) 237–241.
- [15] B.R. Myers, Hamiltonian factorization of the product of a complete graph with itself, *Networks* 2 (1972) 1–9.
- [16] G. Ringel, Über drei kombinatorische Probleme am n -dimensionalen Würfel und Würfelgitter, *Abh. Math. Sem. Hamburg* 20 (1954) 10–19.
- [17] T. Tillson, A hamiltonian decomposition of K_{2m}^* , $2m \geq 8$, *J. Combinatorial Theory* (B) (to appear).
- [18] L.L. Wang, A test for sequencing a class of finite groups with two generators, *Notices Am. Math. Soc.* 20 (1973) A632.