



HAL
open science

On a Conjecture of Chvátal on m -Intersecting Hypergraphs

Jean-Claude Bermond, P. Frankl

► **To cite this version:**

Jean-Claude Bermond, P. Frankl. On a Conjecture of Chvátal on m -Intersecting Hypergraphs. Bulletin of the London Mathematical Society, 1977, 9 (3), pp.310-312. 10.1112/blms/9.3.310. hal-02321636

HAL Id: hal-02321636

<https://hal.science/hal-02321636>

Submitted on 21 Oct 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON A CONJECTURE OF CHVÁTAL ON m -INTERSECTING HYPERGRAPHS

J. C. BERMOND AND P. FRANKL

Let X be a finite set of cardinality n . A hypergraph (X, \mathcal{F}) is called m -intersecting if for any $F_1, \dots, F_m \in \mathcal{F}$ the condition $F_1 \cap \dots \cap F_m \neq \emptyset$ holds.

An (n, h, m) -hypergraph is a hypergraph $H = (X, \mathcal{F})$ satisfying

(i) $|X| = n$;

(ii) H is h -uniform, that is, \mathcal{F} consists of h -subsets of X ;

(iii) every m -intersecting partial hypergraph of H is $(m+1)$ -intersecting.

Let $f(n, h, m)$ denote the maximum number of edges in an (n, h, m) -hypergraph.

Using this terminology the Erdős–Ko–Rado theorem says that

$$f(n, h, 1) = \binom{n-1}{h-1} \text{ whenever } n \geq 2h.$$

Erdős [3] conjectured that

$$f(n, h, 2) = \binom{n-1}{h-1} \text{ whenever } 3 \leq h \leq 2n/3.$$

In [1] Chvátal made the more general conjecture:

$$f(n, h, m) = \binom{n-1}{h-1} \text{ whenever } m < h \leq mn/(m+1).$$

In [2] he proved this conjecture for $h = m+1$.

The aim of this paper is to prove this conjecture in some special cases.

DEFINITION. A $(v, k, 1)$ t -design (called also a Steiner system $S_1(t, k, v)$) \mathcal{D} is a set of

$$n = \frac{v(v-1) \dots (v-t+1)}{k(k-1) \dots (k-t+1)}$$

different subsets, called blocks, of a set Y of v elements, such that:

(i)' $|Y| = v$.

(ii)' For any $a \in Y$, there are exactly h blocks of \mathcal{D} containing a .

(iii)' For any t -tuple of elements $\{a_1, \dots, a_t\}$ of Y , there is exactly one block of \mathcal{D} containing $\{a_1, \dots, a_t\}$.

(iv)' Every block of \mathcal{D} consists of k elements.

Received 24 January, 1977; revised 9 July, 1977.

LEMMA. Suppose that there exists a $(v, k, 1)$ t -design for some v, k, t . Let

$$|X| = n = \frac{v(v-1) \dots (v-t+1)}{k(k-1) \dots (k-t+1)} \quad \text{and} \quad h = \frac{(v-1) \dots (v-t+1)}{(k-1) \dots (k-t+1)}.$$

Then there exists a family \mathcal{A} of subsets of X such that

- (i) $|\mathcal{A}| = v$.
- (ii) for $A \in \mathcal{A}$, we have $|A| = h$.
- (iii) For $A_1, \dots, A_t \in \mathcal{A}$, we have $|A_1 \cap \dots \cap A_t| = 1$.
- (iv) For $x \in X$, there are exactly k members of \mathcal{A} which contain x .

Proof. Let $\mathcal{D} = \{X_1, \dots, X_n\}$ be a $(v, k, 1)$ t -design on a set $Y = \{a_1, \dots, a_v\}$. Let us define the dual system $\mathcal{A} = \{A_1, \dots, A_v\}$ on the set $X = \{x_1, \dots, x_n\}$ by $x_i \in A_j$ if and only if $a_j \in X_i$. Then it follows from conditions (i)', (ii)', (iii)', (iv)' that \mathcal{A} satisfies the four conditions of the lemma.

Remark. The condition (ii) (resp. (ii)') is a consequence of the three other conditions.

THEOREM *If for some v, k, t , there exists a $(v, k, 1)$ t -design (called also Steiner system $S_1(t, k, v)$), then the conjecture of Chvátal is true for the triple*

$$n = \frac{v(v-1) \dots (v-t+1)}{k(k-1) \dots (k-t+1)}; \quad h = \frac{(v-1) \dots (v-t+1)}{(k-1) \dots (k-t+1)}; \quad m = t.$$

Proof of the theorem. Let (X, \mathcal{F}) be an (n, h, m) -hypergraph. We apply a method of proof due to G. O. H. Katona [6]. Suppose that there exists a $(v, k, 1)$ t -design and let \mathcal{A} be the family defined in the lemma. If P is a permutation of the elements of X , then the system $P(\mathcal{A}) = \{P(A_1), \dots, P(A_v)\}$ satisfies the conditions (i), (ii), (iii), (iv) as well. ($P(A)$ is the set consisting of the images of the elements of A by the permutation P .) We count the number of pairs (P, F) , where P is a permutation of the elements of X , F an element of \mathcal{F} and $F \in P(\mathcal{A})$. From properties (i) to (iv) it is clear that $P(\mathcal{A})$ is m -intersecting and that if A_1, \dots, A_q is an $(m+1)$ -intersecting sub-family of $P(\mathcal{A})$ then $A_1 \cap \dots \cap A_q \neq \emptyset$, entailing $q \leq k$. Thus the number of pairs (P, F) is at most $kn!$.

On the other hand to any $F \in \mathcal{F}$ and any $A \in \mathcal{A}$, there are exactly $h!(n-h)!$ different permutations P such that $F = P(A)$. Hence the number of pairs (P, F) is exactly: $h!(n-h)!v|\mathcal{F}|$. Thus we have $|\mathcal{F}|h!(n-h)!v \leq kn!$. Thus taking into

account that $nk = vh$, we obtain $|\mathcal{F}| \leq \binom{n-1}{h-1}$.

COROLLARY. *The conjecture of Chvátal is true for the following triples of integers:*

- (a) m arbitrary, $n = \binom{v}{m}$, $h = \binom{v-1}{m-1}$ for any integer $v > m$.

- (b) $m = 2, h = p^\alpha + 1, n = h^2 - h + 1$ or $n = h^2 - h$ (p is a prime and α an integer).
- (c) $m = 2, h$ and $n = h(2h+1)/3$ ($h \geq 3$)
- (d) $m = 2, h$ and $n = h(3h+1)/4$ ($h \geq 4$)
- (e) $m = 2, h$ and $n = h(4h+1)/5$ ($h \geq 5$)
- (f) $m = 2, h$ and $n = h\left(\frac{(k-1)h+1}{k}\right)$ ($h \geq h_0(k)$)
- (g) $m = 3, n = \binom{v}{4}, h = \binom{v-1}{3}$ for any integer $v \equiv 2$ or $4 \pmod{6}, v > 4$.

Proof. It follows from the existence of: for (a) trivial t -designs; for (b), (c), (d), (e), (f), known $(v, k, 1)$ 2-designs (see [4] or [7]): (b) corresponds to projective and affine planes, (c), (d), (e) to $k = 3, 4, 5$ and (f) follows from the results of Wilson [7] for large v . (g) is just the case $k = 4, t = 3$ of [3] and follows from [5].

References

1. V. Chvátal, *Problem 6*, in *Hypergraph Seminar, Lecture Notes in Math.* 411 (Springer Verlag 1974), 279–280.
2. V. Chvátal, "An extremal set-intersection theorem", *J. London Math. Soc.*, 9 (1974), 355–359.
3. P. Erdős, *Topics in combinatorial analysis*, Proc. of The Second Louisiana Conference on Combinatorics, Graph Theory and Computing (Baton Rouge, 1971), 2–20.
4. H. Hanani, "Balanced incomplete block designs and related designs", *Discrete Maths.*, 11 (1975), 255–369.
5. H. Hanani, "On quadruple systems", *Canadian J. Math.*, 12 (1960), 145–157.
6. G. O. H. Katona, "Extremal problems among subsets of a finite set", in *Combinatorics* (Eds. M. Hall and G. H. Van Lint), Math. Centre Tracts 50 (Amsterdam, 1974), 13–42.
7. R. M. Wilson, "An existence theory for pairwise balanced designs III: Proof of the existence conjectures", *J. Combinatorial Theory*, 18 (1975), 71–79.

CMS, 54 Bd. Raspail, 75006 Paris.
 Mathematical Institute,
 1053 Budapest, Realtanoda 13–15,
 Hungary.