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# ON A CONJECTURE OF CHVÁTAL ON $m$ -INTERSECTING HYPERGRAPHS

J. C. BERMOND AND P. FRANKL

Let  $X$  be a finite set of cardinality  $n$ . A hypergraph  $(X, \mathcal{F})$  is called  $m$ -intersecting if for any  $F_1, \dots, F_m \in \mathcal{F}$  the condition  $F_1 \cap \dots \cap F_m \neq \emptyset$  holds.

An  $(n, h, m)$ -hypergraph is a hypergraph  $H = (X, \mathcal{F})$  satisfying

- (i)  $|X| = n$ ;
- (ii)  $H$  is  $h$ -uniform, that is,  $\mathcal{F}$  consists of  $h$ -subsets of  $X$ ;
- (iii) every  $m$ -intersecting partial hypergraph of  $H$  is  $(m+1)$ -intersecting.

Let  $f(n, h, m)$  denote the maximum number of edges in an  $(n, h, m)$ -hypergraph.

Using this terminology the Erdős–Ko–Rado theorem says that

$$f(n, h, 1) = \binom{n-1}{h-1} \text{ whenever } n \geq 2h.$$

Erdős [3] conjectured that

$$f(n, h, 2) = \binom{n-1}{h-1} \text{ whenever } 3 \leq h \leq 2n/3.$$

In [1] Chvátal made the more general conjecture:

$$f(n, h, m) = \binom{n-1}{h-1} \text{ whenever } m < h \leq mn/(m+1).$$

In [2] he proved this conjecture for  $h = m+1$ .

The aim of this paper is to prove this conjecture in some special cases.

DEFINITION. A  $(v, k, 1)$   $t$ -design (called also a Steiner system  $S_1(t, k, v)$ )  $\mathcal{D}$  is a set of

$$n = \frac{v(v-1) \dots (v-t+1)}{k(k-1) \dots (k-t+1)}$$

different subsets, called blocks, of a set  $Y$  of  $v$  elements, such that:

- (i)'  $|Y| = v$ .
- (ii)' For any  $a \in Y$ , there are exactly  $h$  blocks of  $\mathcal{D}$  containing  $a$ .
- (iii)' For any  $t$ -tuple of elements  $\{a_1, \dots, a_t\}$  of  $Y$ , there is exactly one block of  $\mathcal{D}$  containing  $\{a_1, \dots, a_t\}$ .
- (iv)' Every block of  $\mathcal{D}$  consists of  $k$  elements.

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LEMMA. Suppose that there exists a  $(v, k, 1)$   $t$ -design for some  $v, k, t$ . Let

$$|X| = n = \frac{v(v-1) \dots (v-t+1)}{k(k-1) \dots (k-t+1)} \quad \text{and} \quad h = \frac{(v-1) \dots (v-t+1)}{(k-1) \dots (k-t+1)}.$$

Then there exists a family  $\mathcal{A}$  of subsets of  $X$  such that

- (i)  $|\mathcal{A}| = v$ .
- (ii) for  $A \in \mathcal{A}$ , we have  $|A| = h$ .
- (iii) For  $A_1, \dots, A_t \in \mathcal{A}$ , we have  $|A_1 \cap \dots \cap A_t| = 1$ .
- (iv) For  $x \in X$ , there are exactly  $k$  members of  $\mathcal{A}$  which contain  $x$ .

*Proof.* Let  $\mathcal{D} = \{X_1, \dots, X_n\}$  be a  $(v, k, 1)$   $t$ -design on a set  $Y = \{a_1, \dots, a_v\}$ . Let us define the dual system  $\mathcal{A} = \{A_1, \dots, A_v\}$  on the set  $X = \{x_1, \dots, x_n\}$  by  $x_i \in A_j$  if and only if  $a_j \in X_i$ . Then it follows from conditions (i)', (ii)', (iii)', (iv)' that  $\mathcal{A}$  satisfies the four conditions of the lemma.

*Remark.* The condition (ii) (resp. (ii)') is a consequence of the three other conditions.

THEOREM If for some  $v, k, t$ , there exists a  $(v, k, 1)$   $t$ -design (called also Steiner system  $S_1(t, k, v)$ ), then the conjecture of Chvátal is true for the triple

$$n = \frac{v(v-1) \dots (v-t+1)}{k(k-1) \dots (k-t+1)}; \quad h = \frac{(v-1) \dots (v-t+1)}{(k-1) \dots (k-t+1)}; \quad m = t.$$

*Proof of the theorem.* Let  $(X, \mathcal{F})$  be an  $(n, h, m)$ -hypergraph. We apply a method of proof due to G. O. H. Katona [6]. Suppose that there exists a  $(v, k, 1)$   $t$ -design and let  $\mathcal{A}$  be the family defined in the lemma. If  $P$  is a permutation of the elements of  $X$ , then the system  $P(\mathcal{A}) = \{P(A_1), \dots, P(A_v)\}$  satisfies the conditions (i), (ii), (iii), (iv) as well. ( $P(A)$  is the set consisting of the images of the elements of  $A$  by the permutation  $P$ .) We count the number of pairs  $(P, F)$ , where  $P$  is a permutation of the elements of  $X$ ,  $F$  an element of  $\mathcal{F}$  and  $F \in P(\mathcal{A})$ . From properties (i) to (iv) it is clear that  $P(\mathcal{A})$  is  $m$ -intersecting and that if  $A_1, \dots, A_q$  is an  $(m+1)$ -intersecting sub-family of  $P(\mathcal{A})$  then  $A_1 \cap \dots \cap A_q \neq \emptyset$ , entailing  $q \leq k$ . Thus the number of pairs  $(P, F)$  is at most  $kn!$ .

On the other hand to any  $F \in \mathcal{F}$  and any  $A \in \mathcal{A}$ , there are exactly  $h!(n-h)!$  different permutations  $P$  such that  $F = P(A)$ . Hence the number of pairs  $(P, F)$  is exactly:  $h!(n-h)!v|\mathcal{F}|$ . Thus we have  $|\mathcal{F}|h!(n-h)!v \leq kn!$ . Thus taking into

account that  $nk = v$ , we obtain  $|\mathcal{F}| \leq \binom{n-1}{h-1}$ .

COROLLARY. The conjecture of Chvátal is true for the following triples of integers:

- (a)  $m$  arbitrary,  $n = \binom{v}{m}$ ,  $h = \binom{v-1}{m-1}$  for any integer  $v > m$ .

- (b)  $m = 2, h = p^\alpha + 1, n = h^2 - h + 1$  or  $n = h^2 - h$  ( $p$  is a prime and  $\alpha$  an integer).
- (c)  $m = 2, h$  and  $n = h(2h+1)/3$  ( $h \geq 3$ )
- (d)  $m = 2, h$  and  $n = h(3h+1)/4$  ( $h \geq 4$ )
- (e)  $m = 2, h$  and  $n = h(4h+1)/5$  ( $h \geq 5$ )
- (f)  $m = 2, h$  and  $n = h\left(\frac{(k-1)h+1}{k}\right)$  ( $h \geq h_0(k)$ )
- (g)  $m = 3, n = \binom{v}{4}, h = \binom{v-1}{3}$  for any integer  $v \equiv 2$  or  $4 \pmod{6}, v > 4$ .

*Proof.* It follows from the existence of: for (a) trivial  $t$ -designs; for (b), (c), (d), (e), (f), known  $(v, k, 1)$  2-designs (see [4] or [7]): (b) corresponds to projective and affine planes, (c), (d), (e) to  $k = 3, 4, 5$  and (f) follows from the results of Wilson [7] for large  $v$ . (g) is just the case  $k = 4, t = 3$  of [3] and follows from [5].

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