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Louis Esperet, Matěj Stehlík

## To cite this version:

Louis Esperet, Matěj Stehlík. Bipartite complements of circle graphs. Discrete Mathematics, 2020, 343 (6), pp.111834. 10.1016/j.disc.2020.111834 . hal-02321500

## HAL Id: hal-02321500 <br> https://hal.science/hal-02321500

Submitted on 7 Mar 2022

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# BIPARTITE COMPLEMENTS OF CIRCLE GRAPHS 

LOUIS ESPERET AND MATĚJ STEHLÍK


#### Abstract

Using an algebraic characterization of circle graphs, Bouchet proved in 1999 that if a bipartite graph $G$ is the complement of a circle graph, then $G$ is a circle graph. We give an elementary proof of this result.


A graph is a circle graph if it is the intersection graph of the chords of a circle. Using an algebraic characterization of circle graphs proved by Naji [6] (as the class of graphs satisfying a certain system of equalities over GF(2)), Bouchet proved the following result in [1].
Theorem 1 (Bouchet [1]). If a bipartite graph $G$ is the complement of a circle graph, then $G$ is a circle graph.

The known proofs of Naji's theorem are fairly involved [3, 4, 6, 7], and Bouchet [1] (see also [2]) asked whether, on the other hand, Theorem 1 has an elementary proof. The purpose of this short note is to present such a proof.

We will need two simple lemmas. Given a finite set of points $X \subset \mathbb{R}^{2}$ of even cardinality, a line $\ell$ bisects the set $X$ if each open half-plane defined by $\ell$ contains precisely $|X| / 2$ points. The following lemma is an immediate consequence of the 2 -dimensional discrete ham sandwich theorem (see e.g. [5, Corollary 3.1.3]), and is equivalent to the necklace splitting problem with two types of beads. In order to keep this note self-contained, we include a short proof.
Lemma 2. Let $X, Y \subset \mathbb{R}^{2}$ be disjoint finite point sets of even cardinality on a circle $C$. Then there exists a line $\ell$ simultaneously bisecting both $X$ and $Y$.

Proof. Let $p_{0}, \ldots, p_{2 n-1}$ be the points of $X \cup Y$ in cyclic order along $C$. For $0 \leq i \leq 2 n-1$ we denote by $I_{i}$ the set $\left\{p_{i}, p_{i+1} \ldots, p_{i+n-1}\right\}$ (here and in the remainder of the proof, all indices are considered modulo $2 n$ ). Clearly, for every $0 \leq i \leq n-1$, there exists a line $\ell_{i}$ in $\mathbb{R}^{2}$ bisecting the points of $X \cup Y$, with $I_{i}$ on one side of $\ell_{i}$ and $I_{i+n}$ on the other side. For $0 \leq i \leq 2 n-1$, define $f(i)=\left|X \cap I_{i}\right|-\frac{1}{2}|X|$. Note that since $X$ has even cardinality, each $f(i)$ is an integer.

To prove the lemma, it suffices to show that $f(i)=0$ for some $0 \leq i \leq$ $n-1$, for then $\left|X \cap I_{i}\right|=\frac{1}{2}|X|$ and $\left|Y \cap I_{i}\right|=\frac{1}{2}(|X|+|Y|)-\left|X \cap I_{i}\right|=\frac{1}{2}|Y|$. If $f(0)=0$ then we are done, so let us assume that $f(0) \neq 0$. Without loss of generality $f(0)<0$, and hence $f(n)=-f(0)>0$. Since $f(i+1)-f(i) \in$

Date: January 17, 2020.
Partially supported by ANR Projects GATO (ANR-16-CE40-0009-01) and GrR (ANR-18-CE40-0032).
$\{-1,0,1\}$ for all $0 \leq i \leq n-1$, there exists $1 \leq i \leq n-1$ such that $f(i)=0$, as required.

Lemma 3. Consider a set of pairwise intersecting chords $c_{1}, \ldots, c_{n}$ of $a$ circle $C$, with pairwise distinct endpoints. Then any line $\ell$ that bisects the $2 n$ endpoints of the chords intersects all the chords $c_{1}, \ldots, c_{n}$.

Proof. Assume for the sake of contradiction that some chord $c_{i}$ does not intersect $\ell$. Then $c_{i}$ lies in one of the two open half-planes defined by $\ell$, say to the left of $\ell$. Since $\ell$ bisects the $2 n$ endpoints of the chords, it follows that there is another chord $c_{j}$ that does not intersect $\ell$ and which lies in the half-plane to the right of $\ell$. This implies that $c_{i}$ and $c_{j}$ do not intersect, which is a contradiction.

We are now ready to prove Theorem 1.
Proof of Theorem 1. Consider a bipartite graph $G$ such that its complement $\bar{G}$ is a circle graph. In particular, for any vertex $v_{i}$ of $\bar{G}$ there is a chord $c_{i}$ of some circle $C$ such that any two vertices $v_{i}$ and $v_{j}$ are adjacent in $\bar{G}$ (equivalently, non-adjacent in $G$ ) if and only if the chords $c_{i}$ and $c_{j}$ intersect. Since $G$ is bipartite, the vertices $v_{1}, \ldots, v_{n}$ (and the corresponding chords $\left.c_{1}, \ldots, c_{n}\right)$ can be colored with colors red and blue such that any two chords of the same color intersect. We can assume without loss of generality that the endpoints of the $n$ chords are pairwise distinct, so the coloring of the chords also gives a coloring of the $2 n$ endpoints with colors red or blue (with an even number of blue endpoints and an even number of red endpoints). Since the $2 n$ endpoints lie on the circle $C$, it follows from Lemma 2 that there exists a line $\ell$ simultaneously bisecting the set of blue endpoints and the set of red endpoints.

On one side of $\ell$, reverse the order of the endpoints of the chords $c_{1}, \ldots, c_{n}$ along the circle $C$. Observe that crossing chords intersecting $\ell$ become noncrossing, and vice versa. By Lemma 3 , $\ell$ intersects all the chords $c_{1}, \ldots, c_{n}$, and thus the resulting circle graph is precisely $G$. It follows that $G$ is a circle graph, as desired.

Acknowledgments. The authors would like to thank András Sebő for his remarks on an early version of the draft.

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Laboratoire G-SCOP (CNRS, Univ. Grenoble Alpes), Grenoble, France
Email address: louis.esperet@grenoble-inp.fr
Laboratoire G-SCOP, Univ. Grenoble Alpes, France
Email address: matej.stehlik@grenoble-inp.fr

