Bipartite complements of circle graphs

Louis Esperet, Matěj Stehlík

BIPARTITE COMPLEMENTS OF CIRCLE GRAPHS

LOUIS ESPERET AND MATĚJ STEHLÍK

ABSTRACT. Using an algebraic characterization of circle graphs, Bouchet proved in 1999 that if a bipartite graph G is the complement of a circle graph, then G is a circle graph. We give an elementary proof of this result.

A graph is a *circle graph* if it is the intersection graph of the chords of a circle. Using an algebraic characterization of circle graphs proved by Naji [6] (as the class of graphs satisfying a certain system of equalities over GF(2)), Bouchet proved the following result in [1].

Theorem 1 (Bouchet [1]). *If a bipartite graph G is the complement of a circle graph, then G is a circle graph.*

The known proofs of Naji's theorem are fairly involved [3, 4, 6, 7], and Bouchet [1] (see also [2]) asked whether, on the other hand, Theorem 1 has an elementary proof. The purpose of this short note is to present such a proof.

We will need two simple lemmas. Given a finite set of points $X \subset \mathbb{R}^2$ of even cardinality, a line ℓ *bisects* the set X if each open half-plane defined by ℓ contains precisely $|X|/2$ points. The following lemma is an immediate consequence of the 2-dimensional *discrete ham sandwich theorem* (see e.g. [5, Corollary 3.1.3]), and is equivalent to the *necklace splitting problem* with two types of beads. In order to keep this note self-contained, we include a short proof.

Lemma 2. *Let $X, Y \subset \mathbb{R}^2$ be disjoint finite point sets of even cardinality on a circle C. Then there exists a line ℓ simultaneously bisecting both X and Y.*

Proof. Let $p_0, \dots, p_{2n-1}$ be the points of $X \cup Y$ in cyclic order along C. For $0 \leq i \leq 2n-1$ we denote by I_i the set $\{p_i, p_{i+1} \dots, p_{i+n-1}\}$ (here and in the remainder of the proof, all indices are considered modulo $2n$). Clearly, for every $0 \leq i \leq n-1$, there exists a line ℓ_i in $\mathbb{R}^2$ bisecting the points of $X \cup Y$, with I_i on one side of ℓ_i and I_{i+n} on the other side. For $0 \leq i \leq 2n-1$, define $f(i) = |X \cap I_i| - \frac{1}{2}|X|$. Note that since X has even cardinality, each $f(i)$ is an integer.

To prove the lemma, it suffices to show that $f(i) = 0$ for some $0 \leq i \leq n-1$, for then $|X \cap I_i| = \frac{1}{2}|X|$ and $|Y \cap I_i| = \frac{1}{2}(|X| + |Y|) - |X \cap I_i| = \frac{1}{2}|Y|$. If $f(0) = 0$ then we are done, so let us assume that $f(0) \neq 0$. Without loss of generality $f(0) < 0$, and hence $f(n) = -f(0) > 0$. Since $f(i+1) - f(i) \in$

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$\{-1, 0, 1\}$ for all $0 \leq i \leq n-1$, there exists $1 \leq i \leq n-1$ such that $f(i) = 0$, as required. □

Lemma 3. *Consider a set of pairwise intersecting chords $c_1, \ldots, c_n$ of a circle C, with pairwise distinct endpoints. Then any line ℓ that bisects the $2n$ endpoints of the chords intersects all the chords $c_1, \ldots, c_n$.*

Proof. Assume for the sake of contradiction that some chord c_i does not intersect ℓ. Then c_i lies in one of the two open half-planes defined by ℓ, say to the left of ℓ. Since ℓ bisects the $2n$ endpoints of the chords, it follows that there is another chord c_j that does not intersect ℓ and which lies in the half-plane to the right of ℓ. This implies that c_i and c_j do not intersect, which is a contradiction. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. Consider a bipartite graph G such that its complement $\overline{G}$ is a circle graph. In particular, for any vertex v_i of $\overline{G}$ there is a chord c_i of some circle C such that any two vertices v_i and v_j are adjacent in $\overline{G}$ (equivalently, non-adjacent in G) if and only if the chords c_i and c_j intersect. Since G is bipartite, the vertices $v_1, \ldots, v_n$ (and the corresponding chords $c_1, \ldots, c_n$) can be colored with colors red and blue such that any two chords of the same color intersect. We can assume without loss of generality that the endpoints of the n chords are pairwise distinct, so the coloring of the chords also gives a coloring of the $2n$ endpoints with colors red or blue (with an even number of blue endpoints and an even number of red endpoints). Since the $2n$ endpoints lie on the circle C, it follows from Lemma 2 that there exists a line ℓ simultaneously bisecting the set of blue endpoints and the set of red endpoints.

On one side of ℓ, reverse the order of the endpoints of the chords $c_1, \ldots, c_n$ along the circle C. Observe that crossing chords intersecting ℓ become non-crossing, and vice versa. By Lemma 3, ℓ intersects all the chords $c_1, \ldots, c_n$, and thus the resulting circle graph is precisely G. It follows that G is a circle graph, as desired. □

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Laboratoire G-SCOP (CNRS, Univ. Grenoble Alpes), Grenoble, France
Email address: louis.esperet@grenoble-inp.fr

Laboratoire G-SCOP, Univ. Grenoble Alpes, France
Email address: matej.stehlik@grenoble-inp.fr