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Jonah Gaster, Brice Loustau, Leonard Monsaingeon. Computing harmonic maps between Riemannian manifolds. 2019. hal-02320952

**HAL Id: hal-02320952**

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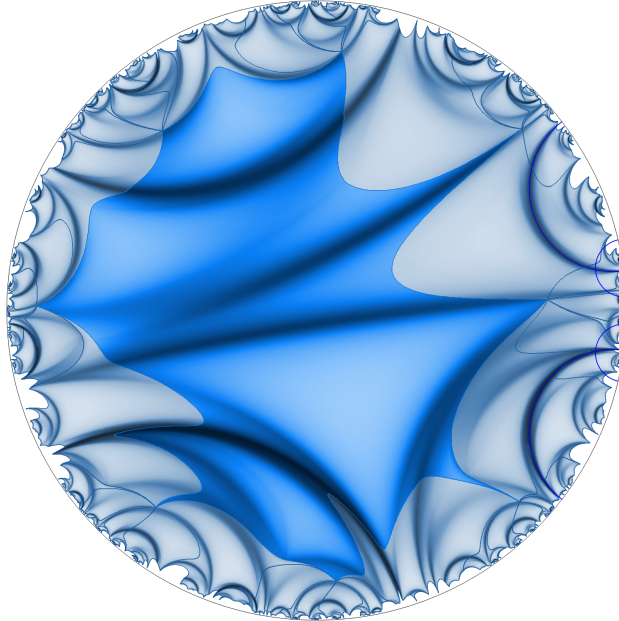
Preprint submitted on 19 Oct 2019

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# Computing harmonic maps between Riemannian manifolds

Jonah Gaster\*, Brice Loustau† and Léonard Monsaingeon‡



## Abstract

In [GLM18], we showed that the theory of harmonic maps between Riemannian manifolds may be discretized by introducing triangulations with vertex and edge weights on the domain manifold. In the present paper, we study convergence of the discrete theory to the smooth theory when taking finer and finer triangulations. We present suitable conditions on the weighted triangulations that ensure convergence of discrete harmonic maps to smooth harmonic maps. Our computer software *Harmony* implements these methods to compute equivariant harmonic maps in the hyperbolic plane.

**Key words and phrases:** Discrete differential geometry · Harmonic maps · Geometric analysis · Convexity · Riemannian optimization · Numerical PDEs · Mathematical software

**2000 Mathematics Subject Classification:** Primary: 58E20; Secondary: 53C43 · 65D18

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## Introduction

Let  $M$  be a compact Riemannian manifold and let  $N$  be a complete Riemannian manifold. A harmonic map  $f: M \rightarrow N$  is a critical point of the energy functional

$$E(f) = \frac{1}{2} \int_M \|df\|^2 dv.$$

Equivalently,  $f$  has vanishing tension field  $\tau(f) = 0$ , a nonlinear generalization of the Laplace operator. When  $N$  is compact and has negative sectional curvature, there exists a unique harmonic map  $M \rightarrow N$  in any homotopy class of smooth maps. This foundational result due to Eells-Sampson [ES64] and Hartman [Har67] can be understood in terms of the convexity properties of the energy functional.

In our previous work [GLM18], which specialized to surfaces, we showed that one can discretize the theory appropriately by meshing the domain manifold with a triangulation and assigning weights on vertices and edges. One of the main results of [GLM18] is the strong convexity of the discrete energy functional, from which we derive convergence of the discrete heat flow to the unique discrete harmonic map. (The second focus of [GLM18] is on center of mass methods, which we do not discuss in the present paper.)

While our previous paper studies a fixed discretization, the present paper initiates a study of the convergence of the discrete theory to the smooth theory when one takes finer and finer meshes. We introduce conditions on weighted triangulations in order to adequately capture the local geometry on the manifold. In particular, we discuss “Laplacian” systems of weights, which aim to produce a good approximation of the Laplacian (i.e. tension field) by the discrete Laplacian.

In the context of a fine sequence of meshes, we study the approximation of the relevant smooth objects by their discrete counterparts. We show convergence for the discrete volume form, tension field, energy density, and energy to their smooth counterparts. If the discrete energy is sufficiently convex, and the sequence of meshes has sufficiently strong Laplacian qualities, we prove that the center of mass interpolations of the discrete harmonic maps converge to the unique smooth harmonic map in  $L^2$ ,  $L^\infty$ , and in energy. Furthermore, we show that the discrete heat flow from any discretized map converges to the smooth harmonic map when both the time index and the space index run to  $+\infty$ , provided a CFL-type condition is satisfied. This theorem may be seen as a constructive implementation of the theorem of Eells-Sampson and Hartman.

The final section of the paper describes explicit sequences of meshes on surfaces, which we show are fine and crystalline, and almost asymptotically Laplacian. These are constructed by midpoint geodesic subdivision of triangulations and using the preferred “volume weights” on vertices and “cotangent weights” on edges. This construction is implemented in our freely available computer software *Harmony*, which is briefly presented in our previous paper [GLM18]. *Harmony* computes the unique harmonic map from the hyperbolic plane to itself that is equivariant with respect to the actions of two Fuchsian groups of the same genus.

Much of the theory and techniques that we develop are well-known in the Euclidean setting, such as the discrete heat flow method or the cotangent weights popularized by Pinkall-Polthier [PP93]. This paper attempts an approximation of the Euclidean theory using fine meshes on Riemannian manifolds. However, there are notable differences from the Euclidean setting: First, the Laplace equation is linear in the Euclidean setting, allowing finite element methods. Second, we restrict to compact manifolds without boundary, in contrast to Euclidean domains where boundary conditions are prescribed. Finally, there are important consequences of negative curvature, including the strong convexity of the energy functional and the uniqueness of harmonic maps, that we exploit in the present project.

The program to discretize the theory of harmonic maps between Riemannian manifolds, and, especially, to obtain convergence back to the smooth theory, is still quite unfinished. Celebrated work on the discretized theory includes [BS07, EF01, KS97], while convergence to the smooth harmonic map has been analyzed for submanifolds of  $\mathbb{R}^n$  notably by Bartels [Bar10]. The present paper seems to have some overlap with Bartels' framework, though we emphasize that our setting is more intrinsic and geometric in nature. Interestingly, Bartels also encounters some subtle difficulties with convergence that arise as technical conditions required of the meshes in the domain (which he calls *logarithmically right-angled* [Bar10, Thm. II, p. 4]). More delicate analysis of the harmonic theory in the setting of fine meshes is still needed to tease out the right conditions to obtain convergence.

*A note to the reader:* Although this paper is the sequel of [GLM18], the two papers can be read independently. We also point out that § 4 and § 5 in this paper can be read independently.

## Acknowledgments

The authors wish to thank David Dumas for his extensive advice and support with the mathematical content and the development of *Harmony*.

The first two authors gratefully acknowledge research support from the NSF Grant DMS1107367 *RNMS: GEometric structures And Representation varieties* (the *GEAR Network*). The third author was partially supported by the Portuguese Science Foundation FCT through grant PTDC/MAT-STA/0975/2014 *From Stochastic Geometric Mechanics to Mass Transportation Problems*.

## 1 Setup

Throughout the paper, let  $(M, g)$  and  $(N, h)$  be smooth connected complete Riemannian manifolds. These will be our domain and target respectively. We will typically assume that  $M$  is compact and oriented, and that  $N$  is Hadamard (complete, simply connected, with nonpositive sectional curvature). Although most of the paper holds in this generality, we are especially interested in the case where  $S = M$  is 2-dimensional.

### 1.1 Discretization setup

Our discretization setup is the following (also see [GLM18] for more details). A *mesh* on  $M$  is any topological triangulation; we denote by  $\mathcal{G}$  the embedded graph that is the 1-skeleton. A mesh (or its underlying graph) is called *geodesic* if all edges are embedded geodesic segments.

Denote  $\mathcal{V} = \mathcal{G}^{(0)}$  and  $\mathcal{E} = \mathcal{G}^{(1)}$  the set of vertices and (unoriented) edges of  $\mathcal{G}$ . We shall equip  $\mathcal{G}$  with vertex weights  $(\mu_x)_{x \in \mathcal{V}}$  and edge weights  $(\omega_{xy})_{\{x,y\} \in \mathcal{E}}$ . For now, these weights are two arbitrary and independent collections of positive numbers. Such a *biweighted graph* allows one to develop a discrete theory of harmonic maps  $M \rightarrow N$  as follows:

- The system of vertex weights defines a discrete measure  $\mu_{\mathcal{G}} = (\mu_x)_{x \in \mathcal{V}}$  on  $\mathcal{V}$ . Since  $\mathcal{G}$  is embedded in  $M$ ,  $\mu_{\mathcal{G}}$  can also be seen as a discrete measure on  $M$  supported by the set of vertices.

- A *discrete map from  $M$  to  $N$  along  $\mathcal{G}$*  is a map  $\mathcal{V} \rightarrow N$ . The space  $\text{Map}_{\mathcal{G}}(M, N)$  of such maps is a smooth finite-dimensional manifold with tangent space

$$T_f \text{Map}_{\mathcal{G}}(M, N) = \Gamma(f^* TN) := \bigoplus_{x \in \mathcal{V}} T_{f(x)} N.$$

It carries a smooth  $L^2$ -Riemannian metric given by:

$$\langle V, W \rangle = \int_M \langle V_x, W_x \rangle d\mu(x) := \sum_{x \in \mathcal{V}} \mu_x \langle V_x, W_x \rangle$$

and an associated  $L^2$  distance given by

$$d(f, g)^2 = \int_M d(f(x), g(x))^2 d\mu(x) = \sum_{x \in \mathcal{V}} \mu_x d(f(x), g(x))^2$$

where  $d(f(x), g(x))$  denotes the Riemannian distance in  $N$ .

- The *discrete energy density* of a discrete map  $f \in \text{Map}_{\mathcal{G}}(M, N)$  is the discrete nonnegative function  $e_{\mathcal{G}}(f) \in \text{Map}_{\mathcal{G}}(M, \mathbb{R})$  defined by

$$e_{\mathcal{G}}(f)_x = \frac{1}{4\mu_x} \sum_{y \sim x} \omega_{xy} d(f(x), f(y))^2.$$

- The *discrete energy functional* on  $\text{Map}_{\mathcal{G}}(M, N)$  is the map  $E_{\mathcal{G}}: \text{Map}_{\mathcal{G}}(M, N) \rightarrow \mathbb{R}$  given by

$$\begin{aligned} E_{\mathcal{G}}(f) &= \int_M e_{\mathcal{G}}(f) d\mu_{\mathcal{G}} \\ &= \frac{1}{2} \sum_{x \sim y} \omega_{xy} d(f(x), f(y))^2. \end{aligned} \tag{1}$$

Note that it does not depend on the choice of vertex weights. A *discrete harmonic map* is a critical point of  $E_{\mathcal{G}}$ .

- The *discrete tension field* of  $f \in \text{Map}_{\mathcal{G}}(M, N)$  is  $\tau_{\mathcal{G}}(f) \in \Gamma(f^* TN)$  defined by

$$\tau_{\mathcal{G}}(f)_x = \frac{1}{\mu(x)} \sum_{y \sim x} \omega_{xy} \overrightarrow{f(x)f(y)}$$

where we denote  $\overrightarrow{f(x)f(y)} := \exp_{f(x)}^{-1}(f(y))$  and  $\exp$  indicates the Riemannian exponential map. In [GLM18, Prop. 2.21] we show the discrete variational formula:

$$\tau_{\mathcal{G}}(f) = -\text{grad } E_{\mathcal{G}}(f).$$

In particular,  $f$  is harmonic if and only if  $\tau_{\mathcal{G}}(f) = 0$ . This is equivalent to the property that for all  $x \in \mathcal{V}$ ,  $f(x)$  is the center of mass of its neighbors values (more precisely of the system  $\{f(y), \omega_{xy}\}$  for  $y$  adjacent to  $x$  [GLM18, Prop. 2.22]).

- Given  $u_0 \in \text{Map}_{\mathcal{G}}(M, N)$  and  $t > 0$ , the *discrete heat flow with fixed stepsize  $t$*  is the sequence  $(u_n)_{n \geq 0}$  defined by

$$u_{n+1} = \exp(t \tau_{\mathcal{G}}(u_n)).$$

The discrete heat flow is precisely the fixed stepsize gradient descend method for the discrete energy functional  $E_{\mathcal{G}}$ .

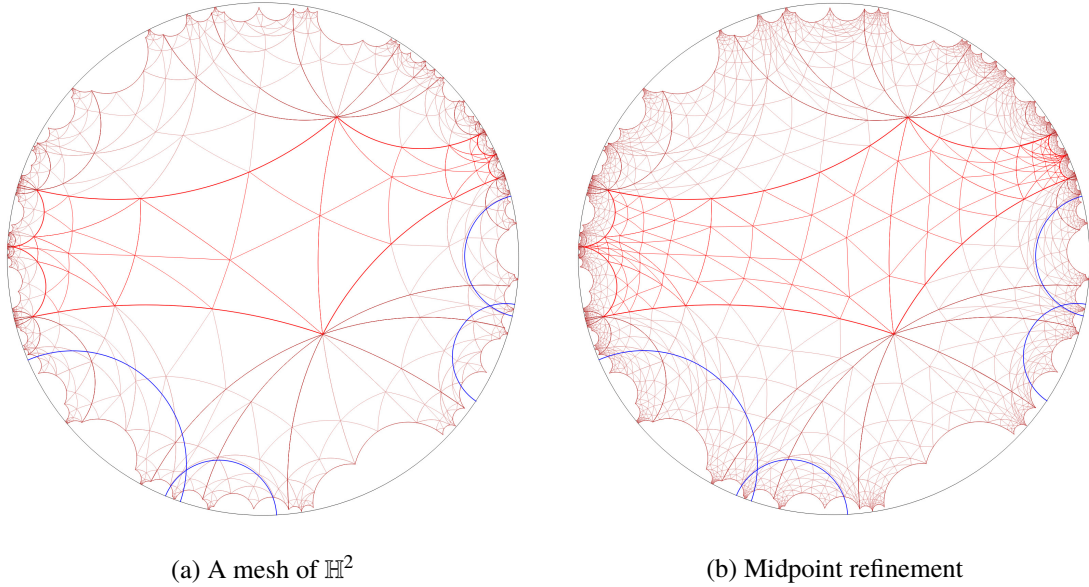


Figure 1: A mesh of the Poincaré disk model of  $\mathbb{H}^2$  on the left, its midpoint refinement on the right. Both are invariant under the action of a Fuchsian group  $\Gamma$ , yielding meshes on a closed hyperbolic surface  $S$  of genus 2. The brighter central region is a fundamental domain. The blue circle arcs are the axes of the generators of  $\Gamma \approx \pi_1 S$ .

One of the main theorems of [GLM18] is that if  $S = M$  and  $N$  are closed oriented surfaces of negative Euler characteristics and  $u_0$  has nonzero degree, then the discrete heat flow converges to the unique minimizer of  $E_{\mathcal{G}}$  in the same homotopy class with exponential convergence rate. See [GLM18, Theorem 4.5] for more details.

## 1.2 Midpoint subdivision of a mesh

Assume  $(M, g)$  is equipped with a geodesic mesh and denote by  $\mathcal{G}$  the associated graph. One can define a new mesh called the *midpoint subdivision* (or *refinement*) as follows. For comfort, let us assume  $M = S$  is 2-dimensional; the definition is easily generalized. Define a new geodesic graph  $\mathcal{G}'$  by adding to the vertex set of  $\mathcal{G}$  all the midpoints of edges of  $\mathcal{G}$ , and adding new edges so that every triangle in  $\mathcal{G}$  is subdivided as 4 triangles in  $\mathcal{G}'$  (see [GLM18, Definition 2.2]). This clearly defines a new geodesic triangulation of  $S$  whose 1-skeleton is  $\mathcal{G}'$ . See Figure 1 for an illustration of an invariant mesh in  $\mathbb{H}^2$  and its refinement generated by the software Harmony.

Evidently, this subdivision process may be iterated, thus one can define the *refinement of order  $n$*  of a geodesic mesh. Meshes obtained by successive midpoint refinements will be our standard support for approximating a smooth manifold by discrete data. Properties of such meshes will be further discussed in § 5.



## 1.3 Interpolation

### 1.3.1 Generalities

Assume  $(M, g)$  is equipped with a geodesic mesh and denote by  $\mathcal{G}$  the associated graph. A continuous map  $f: M \rightarrow N$  is *piecewise smooth along  $\mathcal{G}$*  if  $f$  is smooth in restriction to any simplex of the mesh.

Note that there is a forgetful (restriction) map

$$\pi_{\mathcal{G}}: C(M, N) \rightarrow \text{Map}_{\mathcal{G}}(M, N)$$

which assigns to any continuous map  $f: M \rightarrow N$  its restriction to the vertex set of  $\mathcal{G}$ . A first definition of an *interpolation scheme* would be a right inverse  $\iota_{\mathcal{G}}$  of the map  $\pi_{\mathcal{G}}$ .

Of course, a natural requirement to add of  $\iota_{\mathcal{G}}$  is that it is a continuous map whose image is contained in the subspace of piecewise smooth maps along  $\mathcal{G}$ . In the Euclidean setting, there is one canonical choice for interpolation, namely linear interpolation. In the general Riemannian setting there is no such obvious choice. For our purposes we will view center of mass interpolation as the canonical interpolation, though there are other natural options (e.g. harmonic interpolation), which we will not discuss.

There is a subtle deficiency in the above definition of interpolation scheme when  $N$  is not simply connected: one would like to require that  $\iota_{\mathcal{G}} \circ \pi_{\mathcal{G}}$  preserves homotopy classes of maps, but that is not possible. This problem can be solved by defining an interpolation scheme as attached to the choice of a homotopy class:

**Definition 1.1.** Let  $C$  be a connected component of  $C(M, N)$ . An interpolation scheme  $\iota_{\mathcal{G}}$  is a continuous right inverse of  $\pi_{\mathcal{G}}$  restricted to  $C$ , whose image consists of piecewise smooth maps along  $\mathcal{G}$ .

Note that this definition still does not allow one to define the homotopy class of a discrete map. A more elegant way to deal with deficiency, which we favored in [GLM18], is to work equivariantly in the universal covers.

### 1.3.2 Working equivariantly

Fix a homotopy class  $C$  of a continuous map  $M \rightarrow N$ , which induces a group homomorphism  $\rho: \pi_1 M \rightarrow \pi_1 N$ . Recall that any  $f \in C$  admits a  $\rho$ -equivariant lift between universal covers  $\tilde{f}: \tilde{M} \rightarrow \tilde{N}$ . The mesh  $\mathcal{M}$  on  $M$  also lifts to a  $\pi_1 M$ -invariant geodesic mesh  $\tilde{\mathcal{M}}$  of  $\tilde{M}$ . As usual, one has to take more care with basepoints on  $M$  and  $N$ —and use more notation—to make this story complete.

**Definition 1.2.** The discrete homotopy class  $C_{\mathcal{G}} := \text{Map}_{\tilde{\mathcal{G}}, \rho}(\tilde{M}, \tilde{N})$  is defined as the space of  $\rho$ -equivariant discrete maps  $\tilde{M} \rightarrow \tilde{N}$  along  $\tilde{\mathcal{G}}$ .

One can then define an interpolation theme as a continuous right inverse of  $\pi_{\mathcal{G}}$  on  $C_{\mathcal{G}}$ . For the purposes of this paper, however, all of the convergence analysis can be performed on the quotient manifolds. The presentation is chosen with ease in mind, and so we overlook the subtlety above. Nevertheless, we point out that there are other benefits to the equivariant setting:

- It allows one to consider equivariance with respect to group homomorphisms  $\rho: \pi_1 M \rightarrow \text{Isom}(\tilde{N})$  that are not necessarily induced by continuous maps from  $M$  to a quotient of  $\tilde{N}$ , e.g. non-discrete representations  $\rho$ .
- Computationally, it is easier to work in the universal covers. This is the point of view that we chose when coding the software `Harmony`.

This explains our present change in perspective from the equivariance throughout [GLM18].



### 1.3.3 Center of mass interpolation

We refer to [GLM18, §5.1] for generalities on centers of mass, also called barycenters, in metric spaces and Riemannian manifolds.

For comfort, let us assume that  $S = M$  is 2-dimensional; it is quite straightforward to generalize what follows to higher dimensions. First we describe interpolation between triples of points. Let  $A, B, C$  be three points on the surface  $(S, g)$ . We assume that these three points are sufficiently close, more precisely that they lie in a strongly convex geodesic ball  $B$ , i.e. any two points of  $B$  are joined by a unique minimal geodesic segment in  $S$  and this segment is contained in  $B$ . In particular, there is a uniquely defined triangle  $T \subseteq S$  with vertices  $A, B, C$  and with geodesic boundary. Any point  $P \in T$  can uniquely be written as the center of mass of  $\{(A, \alpha), (B, \beta), (C, \gamma)\}$ , where  $\alpha, \beta, \gamma \in [0, 1]$  and  $\alpha + \beta + \gamma = 1$ . Let similarly  $A', B', C'$  be three sufficiently close points in the Riemannian manifold  $(N, h)$ . Then there is a unique *center of mass interpolation* map  $f: ABC \rightarrow N$  such that for any point  $P \in T$  as above,  $f(P)$  is the center of mass of  $\{(A', \alpha), (B', \beta), (C', \gamma)\}$ . In other words,  $f$  is the identity map in barycentric coordinates.

Clearly, given a discrete map  $f \in \text{Map}_{\mathcal{G}}(S, N)$ , one can define its center of mass interpolation triangle by triangle following the procedure above. Although there seems to be a restriction on the size of the triangles in  $S$  and their images by  $f$  in  $N$  for the interpolation to be well-defined, one can work equivariantly in the universal covers as explained in § 1.3.2 and the restriction disappears as long as  $S$  has nonpositive sectional curvature, or  $\mathcal{G}$  is sufficiently fine i.e. has small maximum edge length, and  $N$  has nonpositive sectional curvature.

**Definition 1.3.** Assume  $(M, g)$  has nonpositive sectional curvature, or  $\mathcal{G}$  is sufficiently fine, and  $N$  has nonpositive sectional curvature. The discussion above yields a *center of mass interpolation* scheme

$$\iota_{\mathcal{G}}: \text{Map}_{\mathcal{G}}(M, N) \rightarrow C(M, N).$$

We denote  $\widehat{f} := \iota_{\mathcal{G}}(f)$  the center of mass interpolation of a discrete map  $f \in \text{Map}_{\mathcal{G}}(M, N)$ .

**Theorem 1.4.** Assume  $M$  has nonpositive sectional curvature, or  $\mathcal{G}$  is sufficiently fine, and  $N$  has nonpositive sectional curvature. Then

- (i) For any  $f \in \text{Map}_{\mathcal{G}}(M, N)$ , the interpolation  $\widehat{f}$  maps each edge of  $\mathcal{G}$  to a geodesic segment in  $M$  (and does so with constant speed).
- (ii) For any  $f \in \text{Map}_{\mathcal{G}}(M, N)$ , the interpolation  $\widehat{f}$  is piecewise smooth along  $\mathcal{G}$ .
- (iii) The map  $\iota_{\mathcal{G}}: \text{Map}_{\mathcal{G}}(M, N) \rightarrow C(M, N)$  is 1-Lipschitz for the  $L^\infty$  distance on both spaces.

*Proof.* For comfort, let us write the proof when  $M = S$  is 2-dimensional. The proof of (i) is immediate. For (ii), recall that the center of mass  $P$  as above is characterized by

$$\alpha \overrightarrow{PA} + \beta \overrightarrow{PB} + \gamma \overrightarrow{PC} = \vec{0}$$

(see [GLM18, Eq. (37)]), where we denote  $\overrightarrow{PA} := \exp_p^{-1}(A)$  etc. It follows from the implicit function theorem that  $(\alpha, \beta, \gamma)$  provide smooth *barycentric coordinates* on  $T$  (resp.  $T'$ ). Conclude by observing that  $\widehat{f}$  is the identity map in barycentric coordinates.

The proof of (iii) is a little more delicate, and crucially relies on  $N$  having nonpositive sectional curvature. Let  $f_1, f_2 \in \text{Map}_{\mathcal{G}}(S, N)$ , we want to show that  $d_\infty(\widehat{f}_1, \widehat{f}_2) \leq d_\infty(f_1, f_2)$ . Consider any triangle in  $\mathcal{G}$  with vertices  $A, B, C \in S$ . Let  $p \in S$  be any point inside or on the boundary of the triangle  $ABC \subseteq S$ . We denote  $A_i = f_i(A)$ ,  $B_i = f_i(B)$ ,  $C_i = f_i(C)$ ,  $P_i = \widehat{f}_i(p)$  for  $i \in \{1, 2\}$ . Since  $p$  is an arbitrary point on  $S$ , we win if we show that  $d(P_1, P_2) \leq d_\infty(f_1, f_2)$ . By definition of the center of mass interpolation,  $P_i$  is the center of mass of  $\{(A_i, \alpha), (B_i, \beta), (C_i, \gamma)\}$ , where  $\alpha, \beta, \gamma \in [0, 1]$  is some triple with  $\alpha + \beta + \gamma = 1$  (namely, the unique triple such that  $M$  is the center of mass of

$\{(A, \alpha), (B, \beta), (C, \gamma)\}$ . Let  $\vec{V}_i = \alpha \overrightarrow{P_i A_i} + \beta \overrightarrow{P_i B_i} + \gamma \overrightarrow{P_i C_i}$  and let  $\vec{W} = \alpha \overrightarrow{P_1 A_2} + \beta \overrightarrow{P_1 B_2} + \gamma \overrightarrow{P_1 C_2}$ , where we denote  $\overrightarrow{P_i A_i} = \exp_{P_i}^{-1}(A_i)$ , etc. By definition of the center of mass  $\vec{V}_i = \vec{0}$ , so we can write  $\vec{W} = \vec{W} - \vec{V}_1$ :

$$\vec{W} = \alpha \left( \overrightarrow{P_1 A_2} - \overrightarrow{P_1 A_1} \right) + \beta \left( \overrightarrow{P_1 B_2} - \overrightarrow{P_1 B_1} \right) + \gamma \left( \overrightarrow{P_1 C_2} - \overrightarrow{P_1 C_1} \right) \quad (2)$$

Since  $N$  has nonpositive sectional curvature, the exponential map  $\exp_{P_1} : T_{P_1}N \rightarrow N$  is distance nondecreasing (for this argument to be completely rigorous, we may need to pass to universal covers), so that  $\|\overrightarrow{P_1 A_2} - \overrightarrow{P_1 A_1}\| \leq d(A_1, A_2)$ , etc. Using the triangle inequality in (2) we find  $\|\vec{W}\| \leq d_\infty(f_1, f_2)$ . This shows that  $d(P_1, P_2) \leq d_\infty(f_1, f_2)$  by [GLM18, Lemma 5.3].  $\square$

## 2 Systems of weights

We follow the discretization setup of § 1 and seek systems of vertex and edge weights on  $\mathcal{G}$  that adequately capture the local geometry of  $M$ , in the sense that they ensure a good approximation of the theory of smooth harmonic maps from  $M$  to any other Riemannian manifold.

Throughout this section  $(M, g)$  is any Riemannian manifold equipped with a geodesic mesh. We denote as usual  $\mathcal{G}$  the associated graph.

### 2.1 Laplacian weights

**Definition 2.1.** A system of vertex weights  $(\mu_x)_{x \in \mathcal{V}}$  and edge weights  $(\omega_{xy})_{\{x, y\} \in \mathcal{E}}$  on the graph  $\mathcal{G} \subseteq S$  is called *Laplacian (to third order)* at a vertex  $x \in \mathcal{V}$  if, for any linear form  $L \in T_x^*M$ :

(1)

$$\frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} \vec{xy} = 0$$

(2)

$$\frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} L(\vec{xy})^2 = 2\|L\|^2$$

(3)

$$\frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} L(\vec{xy})^3 = 0$$

The biweighted graph  $(\mathcal{G}, (\mu_x), (\omega_{xy}))$  is called *Laplacian* if it is Laplacian at any vertex.

Recall that we denote  $\vec{xy} := \exp_x^{-1} y \in T_x M$ .

*Remark 2.2.* A biweighted graph being Laplacian to first order, i.e. satisfying condition (1), is equivalent to the fact that each vertex of  $\mathcal{G}$  is the weighted barycenter of its neighbors. [Theorem 2.3](#) provides many examples of Laplacian graphs to first order.

**Theorem 2.3.** *Assume  $S$  has nonpositive sectional curvature. Any biweighted graph  $\mathcal{G}$  underlying a topological triangulation of  $S$  admits a unique map to  $S$  that is Laplacian to first order, i.e. whose image graph equipped with the same weights is Laplacian to first order.*

*Proof.* Note that a map  $f : \mathcal{G} \rightarrow S$  being Laplacian to first order is equivalent to  $f$  having zero discrete tension field, i.e.  $f$  being discrete harmonic. By [GLM18, Theorem 3.20], the discrete energy functional in this setting is strongly convex, in particular it has a unique critical point.  $\square$

The following seemingly stronger characterization of Laplacian weights is immediate:

**Proposition 2.4.** *A system of weights on  $\mathcal{G}$  is Laplacian at  $x \in \mathcal{V}$  if and only if for any finite-dimensional vector space  $W$ :*

(1) *For any linear map  $L: T_x M \rightarrow W$ :*

$$\sum_{y \sim x} \omega_{xy} L(\vec{xy}) = 0.$$

(2) *For any quadratic form  $q$  on  $T_x M$  with values in  $W$ :*

$$\frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} q(\vec{xy}) = 2 \operatorname{tr} q.$$

(3) *For any cubic form  $\sigma$  on  $T_x M$  with values in  $W$ :*

$$\sum_{y \sim x} \omega_{xy} \sigma(\vec{xy}) = 0.$$

Note that we use the metric (inner product) in  $T_x M$  to define  $\operatorname{tr} q$ . By definition,  $\operatorname{tr} q$  is the trace of the self-adjoint endomorphism associated to  $q$ .

## 2.2 Preferred vertex weights: the volume weights

In this paper we favor one system of vertex weights associated to any mesh of any Riemannian manifold, the so-called *volume weights*.

For comfort assume  $(M, g) = S$  is 2-dimensional, although what follows is evidently generalized to higher dimensions. Let  $x$  be a vertex of the triangulation and consider the polygon  $P_x \subseteq S$  equal to the union of the triangles adjacent to  $x$ . We define the weight of the vertex  $x$  by

$$\mu_x := \frac{1}{3} \operatorname{Area}(P_x)$$

where  $\operatorname{Area}(P_x)$  denotes the Riemannian volume (area) of  $P_x$ . This clearly defines a system of positive vertex weights  $\mu_{\mathcal{G}} := (\mu_x)_{x \in \mathcal{V}}$ . We alternatively see  $\mu_{\mathcal{G}}$  as a discrete measure on  $S$  supported by the set of vertices, which is meant to approximate the volume density  $\nu_g$  of the Riemannian metric: see § 3.2. Note that the choice of the constant  $\frac{1}{1+\dim M} = \frac{1}{3}$  in the definition of  $\mu_x$  is motivated by the fact that each triangle is counted 3 times when integrating over  $S$ . The next proposition is almost trivial:

**Proposition 2.5.** *Let  $(M, g)$  be a closed manifold with an embedded graph  $\mathcal{G}$  associated to a geodesic mesh. Let  $\mu_{\mathcal{G}}$  be the discrete measure on  $S$  defined by the volume weights. Then*

$$\sum_{x \in \mathcal{V}} \mu_x = \int_M d\mu_{\mathcal{G}} = \int_M d\nu_g = \operatorname{Vol}(M, g).$$

Recall that any system of vertex weights endows the space of discrete maps  $\operatorname{Map}_{\mathcal{G}}(M, N)$  with an  $L^2$  distance (see § 1.1).

**Theorem 2.6.** *Let  $N$  be any Riemannian manifold of nonpositive sectional curvature. Equip the space of discrete maps  $\operatorname{Map}_{\mathcal{G}}(M, N)$  with the  $L^2$  distance associated to the volume weights. Then the center of mass interpolation map  $\iota_{\mathcal{G}}: \operatorname{Map}_{\mathcal{G}}(M, N) \rightarrow C(M, N)$  is  $L$ -Lipschitz with respect to the  $L^2$  distance on both spaces, with  $L = \sqrt{1 + \dim M}$ . When  $M$  is Euclidean (flat), the Lipschitz constant can be upgraded to  $L = 1$ .*

*Proof.* Let us assume  $M = S$  is 2-dimensional for comfort. Let  $f, g \in \text{Map}_{\mathcal{G}}(M, N)$ , denote by  $\widehat{f} := \iota_{\mathcal{G}}(f)$  and  $\widehat{g} := \iota_{\mathcal{G}}(g)$  their center of mass interpolations. By definition of the  $L^2$  distance on  $\mathcal{C}(M, N)$ ,

$$d(\widehat{f}, \widehat{g})^2 = \int_M d(\widehat{f}(x), \widehat{g}(x))^2 \, dv_g(x).$$

Denote by  $\mathcal{T}$  the set of triangles in the mesh. The integral is rewritten

$$d(\widehat{f}, \widehat{g})^2 = \sum_{T \in \mathcal{T}} \int_T d(\widehat{f}(x), \widehat{g}(x))^2 \, dv_g(x). \quad (3)$$

Let  $T = ABC$  be any triangle in  $\mathcal{T}$ . Following the proof of [Theorem 1.4 \(iii\)](#), for all  $x \in T$  there exists  $\alpha, \beta, \gamma \in [0, 1]$  such that  $\alpha + \beta + \gamma = 1$  and

$$d(\widehat{f}(x), \widehat{g}(x)) \leq \alpha d(f(A), g(A)) + \beta d(f(B), g(B)) + \gamma d(f(C), g(C)).$$

By convexity of the square function, it follows

$$d(\widehat{f}(x), \widehat{g}(x))^2 \leq \alpha d(f(A), g(A))^2 + \beta d(f(B), g(B))^2 + \gamma d(f(C), g(C))^2 \quad (4)$$

hence

$$d(\widehat{f}(x), \widehat{g}(x))^2 \leq d(f(A), g(A))^2 + d(f(B), g(B))^2 + d(f(C), g(C))^2. \quad (5)$$

Therefore we may derive from [\(3\)](#)

$$\begin{aligned} d(\widehat{f}, \widehat{g})^2 &\leq \sum_{T \in \mathcal{T}} [d(f(A), g(A))^2 + d(f(B), g(B))^2 + d(f(C), g(C))^2] \text{Area}(T) \\ &\leq \sum_{x \in \mathcal{V}} \sum_{T \in \mathcal{T}_x} d(f(x), g(x))^2 \text{Area}(T_x) \end{aligned}$$

where  $\mathcal{T}_x$  denotes the set of triangles adjacent to  $x$ . Finally this is rewritten

$$d(\widehat{f}, \widehat{g})^2 \leq \sum_{x \in \mathcal{V}} 3\mu_x d(f(x), g(x))^2$$

where  $\mu_x$  is the volume weight at  $x$ , i.e.  $d(\widehat{f}, \widehat{g})^2 \leq 3d(f, g)^2$ .

If  $M$  is Euclidean (flat), the proof can be upgraded to obtain a Lipschitz constant  $L = 1$  by keeping the finer estimate [\(4\)](#) instead of [\(5\)](#), and computing the triangle integral.  $\square$

### 2.3 Preferred edge weights: the cotangent weights

We also have a favorite system of edge weights, the so-called *cotangent weights*, although they have the following restrictions:

- (1) We only define them for 2-dimensional Riemannian manifolds, though they have higher-dimensional analogs.
- (2) They are only positive for triangulations having the ‘‘Delaunay angle property’’. (This includes any acute triangulation.)

These weights have a simple definition in terms of the cotangents of the (Riemannian) angles between edges in the triangulation, and coincide with the weights of Pinkall-Polthier [\[PP93\]](#) in the Euclidean case. For more background on the cotangent weights in the Euclidean setting and a formula for their higher-dimensional analogs, please see [\[Cra19\]](#).

The following result noticed by Pinkall-Polthier [\[PP93\]](#) is an elementary exercise of plane Euclidean geometry:

**Lemma 2.7.** Let  $T = ABC$  and  $T' = A'B'C'$  be triangles in the Euclidean plane. Denote by  $f := T \rightarrow T'$  the unique affine map such that  $f(A) = A'$ , etc. Then the energy of  $f$  is given by

$$\begin{aligned} E(f) &:= \frac{1}{2} \int_T \|df\|^2 dv \\ &= \frac{1}{4} (a'^2 \cot \alpha + b'^2 \cot \beta + c'^2 \cot \gamma) \end{aligned}$$

where  $\alpha, \beta, \gamma$  denote the unoriented angles of the triangle  $ABC$  and  $a', b', c'$  denote the side lengths of the triangle  $A'B'C'$  as in Figure 2.

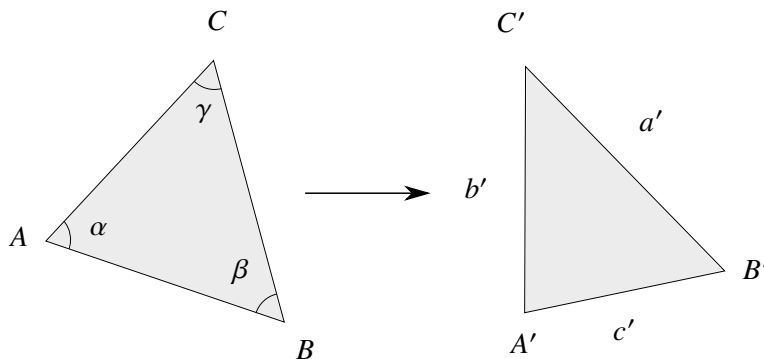


Figure 2: A triangle map in  $\mathbb{R}^2$ .

In view of Lemma 2.7, given a surface  $(S, g)$  equipped with a geodesic mesh, we define the weight of an edge  $e$  by considering the two angles  $\alpha$  and  $\beta$  opposite to  $e$  in the two triangles adjacent to  $e$  (see Figure 3), and we put

$$\omega_e := \frac{1}{2} (\cot \alpha + \cot \beta). \quad (6)$$

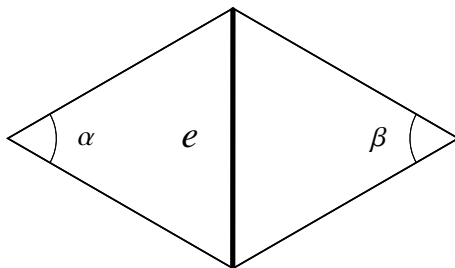


Figure 3: The weight  $\omega_e$  of the edge  $e$  is defined in terms of the opposite angles  $\alpha$  and  $\beta$ .

Note that we use the Riemannian metric  $g$  to define the geodesic edges of the graph and the angles between edges.

**Definition 2.8.** Let  $(S, g)$  be a Riemannian surface equipped with a geodesic mesh with underlying graph  $\mathcal{G}$ . The edge weights on  $\mathcal{G}$  defined as in (6) are the system of *cotangent weights*.

As a direct application of Lemma 2.7, we obtain:

**Proposition 2.9.** Let  $(S, g)$  be a flat surface with a geodesic mesh. Let  $\mathcal{G}$  be the underlying graph equipped with the cotangent edge weights. For any piecewise affine map  $f : S \rightarrow \mathbb{R}^n$ , the smooth energy  $E(f) := \frac{1}{2} \int_T \|df\|^2 dv$  coincides with the discrete energy  $E_{\mathcal{G}}(f)$  defined in (1).

Note that a priori, the cotangent weights are not necessarily positive. Clearly, they are positive for acute triangulations (all of whose triangles are acute). More generally, the cotangent weights are positive if and only if the triangulation has the property that, for any edge  $e$ , the two opposite angles add to less than  $\pi$ . This is simply because

$$\omega_e = \frac{1}{2}(\cot \alpha + \cot \beta) = \frac{\sin(\alpha + \beta)}{2 \sin \alpha \sin \beta}.$$

We call this the *Delaunay angle property*. In the Euclidean setting (for a flat surface), this property is equivalent to the triangulation being *Delaunay*, i.e. the circumcircle of any triangle does not contain any vertex in its interior. This is proven in [BS07, Lemma 9, Prop. 10]. Let us record the previous observation:

**Proposition 2.10.** *Let  $(S, g)$  be a Riemannian surface. A triangulation of  $S$  has positive cotangent edge weights if and only if it has the Delaunay angle property.*

## 2.4 Laplacian qualities of cotangent weights

In the 2-dimensional Euclidean setting, in addition to satisfying Proposition 2.9, the cotangent weights enjoy some good (but not great) Laplacian properties, although this is far less obvious.

**Proposition 2.11.** *Suppose that  $(S, g)$  is a flat surface. Then the cotangent weights associated to any triangulation of  $S$  are Laplacian to first order.*

*Proof.* Let  $x$  be a vertex and consider the polygon  $P = P_x$  equal to the union of the triangles adjacent to  $x$ . Since in the flat case the exponential map  $\exp_x$  is a local isometry, without loss of generality we can assume that  $P$  is contained in the Euclidean plane  $T_x S \approx \mathbb{R}^2$  and  $x = O$ .

Suppose that the vertices of  $P$  are given in cyclic order by  $(A_i)$ , and that we have angles  $\alpha_i, \beta_i, \gamma_i$  as in Figure 4. By definition, the weight of the edge  $OA_i$  is given by  $\omega_i := \frac{1}{2}(\cot \beta_{i-1} + \cot \gamma_i)$ .

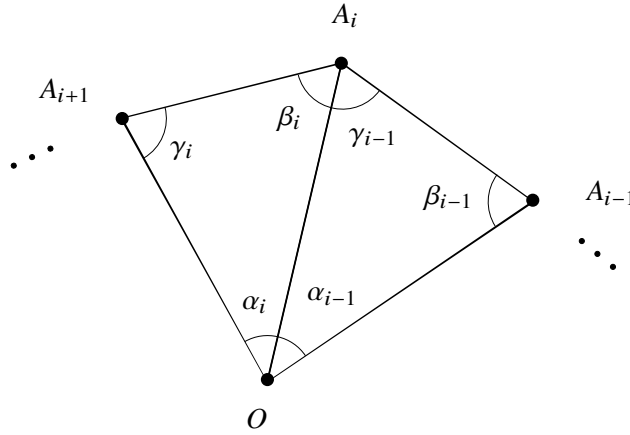


Figure 4: The triangles of  $P$  at  $O$ .

Now consider the identity map  $f: P \rightarrow \mathbb{R}^2$ . It has constant energy density  $e(f) = 2$ , therefore the total energy of  $f$  is  $E = 2 \text{Area}(P)$ . On the other hand,  $E$  is the sum of the energies of  $f$  in restriction to the triangles forming  $P$ . By Lemma 2.7 this is

$$E = \frac{1}{4} \sum_i \left[ \cot \alpha_i \|\overrightarrow{A_i A_{i+1}}\|^2 + \cot \beta_i \|\overrightarrow{O A_{i+1}}\|^2 + \cot \gamma_i \|\overrightarrow{O A_i}\|^2 \right]. \quad (7)$$

So far we assumed that  $O$  is the origin in  $\mathbb{R}^2$ , but of course the argument is valid if  $O$  is any point. In fact, let us see the energy  $E$  above as a function of  $O \in \mathbb{R}^2$  when all the other points  $A_i \in \mathbb{R}^2$  are fixed. We compute the infinitesimal variation of  $E$  under a variation of  $O$ . On the one hand,  $\dot{E}(O) = 0$  since  $E(O) = 2 \text{Area}(P)$  is constant. On the other hand, (7) yields

$$\begin{aligned} \dot{E}(O) = & -\frac{1}{4} \sum_i \left[ \frac{\dot{\alpha}_i}{\sin^2 \alpha_i} \|\overrightarrow{A_i A_{i+1}}\|^2 + \frac{\dot{\beta}_i}{\sin^2 \beta_i} \|\overrightarrow{O A_{i+1}}\|^2 + \frac{\dot{\gamma}_i}{\sin^2 \gamma_i} \|\overrightarrow{O A_i}\|^2 \right] \\ & - \frac{1}{2} \sum_i \left\langle \dot{O}, \cot \beta_i \overrightarrow{O A_{i+1}} + \cot \gamma_i \overrightarrow{O A_i} \right\rangle. \end{aligned} \quad (8)$$

We claim that the first sum in (8) vanishes. Indeed, first observe that the law of sines yields

$$\frac{\|\overrightarrow{A_i A_{i+1}}\|^2}{\sin^2 \alpha_i} = \frac{\|\overrightarrow{O A_{i+1}}\|^2}{\sin^2 \beta_i} = \frac{\|\overrightarrow{O A_i}\|^2}{\sin^2 \gamma_i} = \frac{1}{D^2}$$

where  $D$  is the diameter of the triangle  $O A_i A_{i+1}$ 's circumcircle, so the first sum is rewritten

$$\sum_i \left[ \frac{1}{D^2} (\dot{\alpha}_i + \dot{\beta}_i + \dot{\gamma}_i) \right]$$

and  $\dot{\alpha}_i + \dot{\beta}_i + \dot{\gamma}_i = 0$  since  $\alpha_i + \beta_i + \gamma_i = \pi$  is constant. Thus (8) is rewritten

$$\begin{aligned} \dot{E}(O) = & -\frac{1}{2} \sum_i \left\langle \dot{O}, \cot \beta_i \overrightarrow{O A_{i+1}} + \cot \gamma_i \overrightarrow{O A_i} \right\rangle \\ = & - \left\langle \dot{O}, \sum_i \omega_i \overrightarrow{O A_i} \right\rangle. \end{aligned}$$

In other words:  $\text{grad } E(O) = - \sum_i \omega_i \overrightarrow{O A_i}$ . Since this must be zero (recall that  $E(O)$  is constant),  $O$  is indeed the barycenter of its weighted neighbors  $\{A_i, \omega_i\}$ .  $\square$

It is not true in general that cotangent weights are Laplacian to second or third order for arbitrary triangulations. However, for triangulations obtained by midpoint refinement, it is true for all *interior vertices*, i.e. vertices that are not located on edges of the initial triangulation:

**Proposition 2.12.** *Let  $T$  be a Euclidean triangle, and let  $\mathcal{G}_n$  denote the graph given by  $n$ -th refinement of  $T$  (see § 1.2). Equip  $\mathcal{G}_n$  with the area vertex weights and the cotangent edge weights. Then  $\mathcal{G}_n$  is Laplacian at any interior vertex.*

The proof of this proposition is based on the observation that any interior vertex satisfies a strong symmetry condition, which we call hexaparallelism: We say that a plane Euclidean graph is *hexaparallel* at vertex  $x$  if  $x$  has valence six, and the set of vectors  $\{\vec{x}y : y \sim x\}$  is in the  $\text{GL}(2, \mathbb{R})$ -orbit of  $\{\pm(1, 0), \pm(1, 1), \pm(0, 1)\}$ . Equivalently, the neighbors of  $x$  are the vertices of a hexagon whose opposite sides are pairwise parallel and of the same length, and whose diagonals meet at  $x$ . See Figure 5.

It is straightforward to check by induction that the graph of Proposition 2.12 is hexaparallel at interior vertices, therefore the proposition reduces to:

**Lemma 2.13.** *Any geodesic graph  $\mathcal{G}$  in  $\mathbb{R}^2$  equipped with the area vertex weights and cotangent edge weights is Laplacian to third order at any vertex  $x$  with hexaparallel symmetry.*



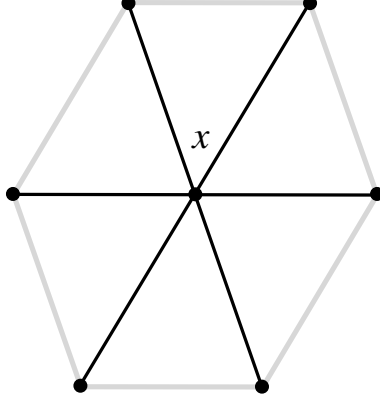


Figure 5: A vertex with hexaparallel symmetry.

*Proof.* The first-order and third-order conditions are trivial due to central symmetry of the neighbors around the vertex  $x$  and the fact that linear and cubic functions are odd. (Alternatively, the first-order condition holds by [Proposition 2.11](#).) It remains to show the second-order condition:

$$\frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} q(y - x) = 2 \operatorname{tr} q$$

for any quadratic form  $q$ . Note that the left-hand side of this equation is invariant when the embedding of  $\mathcal{G}$  is scaled, so after rotating and scaling we may assume coordinates on  $\mathbb{R}^2 \approx \mathbb{C}$  so that  $x = 0$  and a given neighbor is 1. Let the neighbor immediately following 1 in cyclic order be given by the complex coordinate  $z = a + bi$ . The hexaparallel condition implies that the other neighbors are  $z - 1$ ,  $-1$ ,  $-z$ , and  $1 - z$ . Let the oriented angles  $\angle(1, z)$ ,  $\angle(z, z - 1)$ , and  $\angle(z - 1, -1)$  be denoted by  $\alpha$ ,  $\beta$ , and  $\gamma$ , respectively. Given any complex number  $w$ , we have  $\cot(\arg w) = \frac{\operatorname{Re}(w)}{\operatorname{Im}(w)}$ . Therefore we may compute:

$$\begin{aligned} \cot \alpha &= \frac{\operatorname{Re}}{\operatorname{Im}}(z) = \frac{a}{b} \\ \cot \beta &= \frac{\operatorname{Re}}{\operatorname{Im}}\left(\frac{z - 1}{z}\right) = \frac{a^2 + b^2 - a}{b} \\ \cot \gamma &= \frac{\operatorname{Re}}{\operatorname{Im}}\left(\frac{1}{1 - z}\right) = \frac{1 - a}{b}. \end{aligned}$$

Since  $\mu_x = \frac{1}{3}(6 \cdot b/2) = b$ , we get

$$\begin{aligned} \frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} q(y - x) &= \frac{2}{b} (\cot \alpha \cdot q(z - 1) + \cot \beta \cdot q(1) + \cot \gamma \cdot q(z)) \\ &= \frac{2}{b} \left( \frac{a}{b} \cdot q(z - 1) + \frac{a^2 + b^2 - a}{b} \cdot q(1) + \frac{1 - a}{b} \cdot q(z) \right). \end{aligned}$$

It is straightforward to check that the latter is equal to 2, 0, and 2 when  $q = dx^2$ ,  $dx dy$ , or  $dy^2$ , respectively, as desired.  $\square$

*Remark 2.14.* We shall see in [§ 5](#) that in the general Riemannian setting, the cotangent weights will satisfy similar Laplacian properties asymptotically for very fine meshes.

*Remark 2.15.* Let us emphasize that the cotangent weights, while being the best choice of edge weights, will generally not satisfy the second-order Laplacian condition at vertices with no hexaparallel symmetry. Taking finer and finer triangulations will not help with this defect. At such

vertices, which must generically exist for topological reasons, the discrete Laplacian of a smooth function will *not* approximate its Laplacian. This is an intrinsic difficulty to the task of discretizing the Laplacian. Showing that discrete harmonic maps nevertheless converge to smooth harmonic maps under suitable assumptions is one of the points of this paper.

### 3 Sequences of meshes

In this section, we enhance the previous section by considering sequences of meshes on a Riemannian manifold  $(M, g)$ . The idea is to capture the local geometry of  $M$  sufficiently well provided the mesh is sufficiently fine. This allows a relaxation of the Laplacian weights conditions, which are too stringent for a fixed mesh of an arbitrary Riemannian manifold. We introduce the notion of asymptotically Laplacian and almost asymptotically Laplacian systems of weights, with the aim that these weakened conditions will be sufficient for the convergence theorems that we are after.

#### 3.1 Fine and crystalline sequences of meshes

Let  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  be a sequence of geodesic meshes of a Riemannian manifold  $(M, g)$ . Denote by  $r_n$  the “mesh size”, i.e. the longest edge length of  $\mathcal{M}_n$ . Following [dSG19], we define:

**Definition 3.1.** The sequence  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  is called *fine* provided  $\lim_{n \rightarrow +\infty} r_n = 0$ .

Given a bounded subset  $D \subseteq M$ , one calls:

- *diameter of  $D$*  the supremum of the distance between two points of  $D$ , denoted  $\text{diam}(D)$ .
- *radius of  $D$*  the distance from the center of mass of  $D$  to its boundary, denoted  $\text{radius}(D)$ .
- *thickness of  $D$*  the ratio of its radius and diameter, denoted  $\text{thick}(D)$ :

$$\text{thick}(D) := \frac{\text{radius}(D)}{\text{diam}(D)}.$$

**Definition 3.2.** The sequence  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  is called *crystalline* if there exists a uniform lower bound for the thickness of simplices in  $\mathcal{M}_n$ .

*Example 3.3.* In [Theorem 5.6](#), we will show that any sequence of meshes obtained by midpoint subdivision is fine and crystalline, a crucial fact for the strategy of this paper.

**Proposition 3.4.** Let  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  be a fine sequence of meshes. The following are equivalent:

- (i) The sequence  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  is crystalline.
- (ii) There exists a uniform positive lower bound for all angles between adjacent edges in  $\mathcal{M}_n$ .
- (iii) There exists a uniform positive lower bound for the ratio of any two edge lengths in  $\mathcal{M}_n$ .

*Proof sketch.* For brevity, we only sketch the proof; the detailed proof would include proper Riemannian estimates: see [Appendix A](#).

First one checks that (i)  $\Leftrightarrow$  (ii) in the Euclidean setting. This is an elementary calculation: for a single triangle (or  $n$ -simplex), one can bound its radius in terms of its smallest angle. One then generalizes to an arbitrary Riemannian manifold  $M$  by arguing that a very small triangle (or  $n$ -simplex) in  $M$  has almost the same radius and angles as its Euclidean counterpart in a normal chart. The fact that we only consider fine sequences of meshes means that we can assume that all simplices are arbitrarily small, making the previous argument conclusive. The proof of (ii)  $\Leftrightarrow$  (iii) is conducted similarly.  $\square$

**Theorem 3.5.** Assume that  $M$  is compact and the sequence of meshes  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  on  $M$  is fine and crystalline. Denote by  $\mathcal{G}_n$  the graph underlying  $\mathcal{M}_n$  and  $r_n$  its maximum edge length.

- (i) The volume vertex weights  $\mu_{x,n}$  of  $\mathcal{G}_n$  are  $\Theta(r_n^{\dim M})$  (uniformly in  $x$ ).
- (ii) The number of vertices of  $\mathcal{G}_n$  is  $|\mathcal{V}_n| = \Theta(r_n^{-\dim M})$ . More generally, the number of  $k$ -simplices of  $\mathcal{G}_n$  is  $\Theta(r_n^{-\dim M})$ .
- (iii) The combinatorial diameter of the graph  $\mathcal{G}_n$  is  $\Theta(r_n^{-1})$ .
- (iv) The combinatorial injectivity radius (see below) of the graph  $\mathcal{G}_n$  is  $\Theta(r_n^{-1})$ .

The *injectivity radius* at a vertex  $x$  of a graph  $\mathcal{G}$  is the smallest integer  $k \in \mathbb{N}$  such that there exists a vertex at combinatorial distance  $k$  from  $x$  all of whose neighbors are at combinatorial distance  $\leq k$  from  $x$ . The *injectivity radius of the graph*  $\mathcal{G}$  is the minimum of its injectivity radii over all vertices.

*Remark 3.6.* In this paper, we use the notation  $f = O(g)$  and  $f = o(g)$  in the usual sense, we use the notation  $f = \Omega(g)$  for  $g = O(f)$ , and  $f = \Theta(g)$  for  $[f = O(g) \text{ and } f = \Omega(g)]$ .

*Proof of Theorem 3.5.* For (i), recall that the volume vertex weight at  $x$  is the sum of the volumes of the simplices adjacent to  $x$  (divided by  $\dim M$ ). Since the sequence is fine, the diameter of all simplices is going to 0 uniformly in  $x$ . On first approximation, the volume of any such vertex is approximately equal to its Euclidean counterpart (say, in a normal chart). Since the lengths of all edges are within  $[\alpha r_n, r_n]$  for some constant  $\alpha > 0$  and all angles are bounded below by Proposition 3.4, this volume is  $\Theta(r_n^{\dim M})$ .

For (ii), simply notice that  $\sum_{x \in \mathcal{V}_n} \mu_{x,n} = \text{Vol}(M)$  by Proposition 2.5 and use (i). The generalization to  $k$ -simplices is immediate since the total number of  $k$ -simplices is clearly  $\Theta(|\mathcal{V}_n|)$ .

For (iii), let us first show that  $\text{diam } \mathcal{G}_n = \Omega(r_n^{-1})$ . Let  $x$  and  $y$  be two fixed points in  $M$  and denote  $L$  the distance between them. For all  $n \in \mathbb{N}$ , there exists vertices  $x_n$  and  $y_n$  in  $\mathcal{V}_n$  that are within distance  $r_n$  of  $x$  and  $y$  respectively, so their distance in  $M$  is  $d(x_n, y_n) \geq L - 2r_n$ . Denoting  $k_n$  the combinatorial distance between  $x_n$  and  $y_n$ , one has  $d(x_n, y_n) \leq k_n r_n$  by the triangle inequality. We thus find that  $k_n r_n \geq L - 2r_n$ , hence  $\text{diam } \mathcal{G}_n \geq k_n \geq L r_n^{-1} - 2$  so that  $\text{diam } \mathcal{G}_n = \Omega(r_n^{-1})$ . Finally, let us show that  $\text{diam } \mathcal{G}_n = O(r_n^{-1})$ . Let  $x_n$  and  $y_n$  be two vertices that achieve  $\text{diam } \mathcal{G}_n$ . Let  $\gamma_n$  be a length-minimizing geodesic from  $x_n$  to  $y_n$ . Of course, the length of  $\gamma_n$  is bounded above by the diameter of  $M$ . There is a sequence of simplices  $\Delta_1, \dots, \Delta_{k_n}$  such that  $x \in \Delta_1$ ,  $y \in \Delta_{k_n}$ , and any two consecutive simplices are adjacent. Since the valence of any vertex is uniformly bounded (because of a lower bound on all angles), the number of simplices within a distance  $\leq r_{\min}$  of any point of  $M$  is bounded above by a constant  $C$ . This implies  $k_n \leq CL(\gamma)/r_{\min}$ , so that  $k_n \leq C(\text{diam } M)\alpha r_n^{-1}$ . Following edges along the simplices  $\Delta_i$ , one finds a path of length  $(\dim M - 2)k_n$  from  $x$  to  $y$ , therefore  $\text{diam } \mathcal{G}_n \leq (\dim M - 2)C(\text{diam } M)\alpha r_n^{-1}$ .

The proof of (iv) follows similar lines and is left to the reader.  $\square$

For a continuous map  $f: M \rightarrow \mathbb{R}$ , denote  $f_n := \pi_n(w) \in \text{Map}_{\mathcal{G}_n, N}$  the discretization of  $f$ : this is just the restriction of  $f$  to the vertex set of  $\mathcal{G}_n$ . As in [dSG19] we have:

**Lemma 3.7.** If  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  is a sequence of meshes that is fine and crystalline, then for any piecewise smooth function  $f: M \rightarrow \mathbb{R}$ , the center of mass interpolation  $\widehat{f}_n$  converges to  $f$  for the piecewise  $C^1$  topology.

*Proof sketch.* As for Proposition 3.4, the proof can be conducted in two steps: First in the Euclidean setting, where the center of mass interpolation  $\widehat{f}_n$  is just the piecewise linear approximation of  $f_n$ . This proof is done in e.g. [dSG19]. One then generalizes to an arbitrary Riemannian manifold  $M$  by arguing that for very fine triangulations, the center of mass interpolation  $\widehat{f}_n$  is very close to the piecewise linear approximation of  $f_n$  in a normal chart.  $\square$

*Remark 3.8.* Any interpolation scheme satisfying [Lemma 3.7](#), as well as [Theorem 1.4](#) and [Theorem 2.6](#) (or asymptotic versions thereof), would make the machinery work to prove our upcoming main theorems. One could therefore enforce these properties as the definition of a good sequence of interpolation schemes.

**Corollary 3.9.** *Let  $f: M \rightarrow N$  be a  $C^1$  map between Riemannian manifolds. Assume that  $M$  is compact and equipped with a fine and crystalline sequence of meshes  $(\mathcal{M}_n)_{n \in \mathbb{N}}$ . The center of mass interpolation  $\widehat{f}_n$  converges to  $f$  in  $L^\infty(M, N)$  and  $E(f) = \lim_{n \rightarrow +\infty} E(\widehat{f}_n)$ .*

*Remark 3.10.* One would like to say that  $\widehat{f}_n$  converges to  $f$  in the Sobolev space  $H^1(M, N)$ , but this space is not well-defined. Note however that we can say something in that direction:  $\widehat{f}_n \rightarrow f$  in  $L^2(M, N)$  and  $E(\widehat{f}_n) \rightarrow E(f)$ . One should think of the energy as the  $L^2$  norm of the derivative, but this “norm” does not induce a distance.

The following lemma will be useful in [§ 3.3](#) and again in [§ 3.4](#).

**Lemma 3.11.** *Assume that the sequence of meshes  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  on  $M$  is fine and crystalline. If  $\mathcal{G}_n$  is equipped with a system of vertex and edge weights that is Laplacian at some vertex  $x$ , then*

$$\frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} = O(r_n^{-2}).$$

*Remark 3.12.* Before writing the proof, let us clarify the quantifiers in [Lemma 3.11](#) (as well as [Theorem 3.14](#) and [Theorem 3.16](#)): The statement is that there exists a constant  $M > 0$  independent of  $n$  such that at any vertex  $x$  of  $\mathcal{G}_n$  where the system of weights is Laplacian,  $\frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} \leq M r_n^{-2}$ .

*Proof.* Apply condition (2) of [Proposition 2.4](#) to the quadratic form  $q = \|\cdot\|^2$ :

$$\frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} d(x, y)^2 = 2m \tag{9}$$

where  $m = \dim M$ . The fact that the sequence of meshes is fine and crystalline implies that there exists a uniform lower bound for the ratio of lengths in the triangulation. Thus there exists a constant  $\alpha > 0$  such that for any neighbor vertices  $x$  and  $y$  in  $\mathcal{G}_n$ :

$$\alpha r_n \leq d(x, y) \leq r_n. \tag{10}$$

It follows from (9) and (10) that

$$\frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} \leq \frac{2m}{\alpha^2 r_n^2}.$$

□

## 3.2 Convergence of the volume form

Let  $(M, g)$  be a Riemannian manifold, let  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  be a sequence of meshes with the underlying graphs  $(\mathcal{G}_n)_{n \in \mathbb{N}}$ . We equip  $\mathcal{G}_n$  with the volume vertex weights defined in [§ 2.2](#). These define a discrete measure  $\mu_n$  on  $M$  supported by the set of vertices  $\mathcal{V}_n = \mathcal{G}_n^{(0)}$ .

**Theorem 3.13.** *If  $M$  is any Riemannian manifold and  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  is any fine sequence of meshes, then the measures  $(\mu_n)_{n \in \mathbb{N}}$  on  $M$  defined by the volume vertex weights converge weakly (some would say weak\*-ly) to the volume density on  $M$ :*

$$\int_M f \, d\mu_n \xrightarrow{n \rightarrow +\infty} \int_M f \, d\mu$$

for any  $f \in C_c^0(M, \mathbb{R})$  (continuous function with compact support), where  $\mu$  denotes the measure on  $M$  induced by the volume form  $v_g$ .

*Proof.* Recall that a continuity set  $A \subseteq M$  is a Borel set such that  $\mu(\delta A) = 0$ . Since any compact set has finite  $\mu$ -measure, it is well-known that the weak convergence of  $\mu_n$  to  $\mu$  is equivalent to

$$\mu_n(A) \xrightarrow{n \rightarrow +\infty} \mu(A)$$

for any bounded continuity set  $A$ . Let thus  $A$  be any bounded continuity set. Denote by  $B_n$  the union of all simplices that are entirely contained in  $A$ , and by  $C_n$  the union of all simplices that have at least one vertex in  $A$ . We obviously have  $B_n \subseteq A \subseteq C_n$ , and by definition of  $\mu_n$  we have:

$$\mu(B_n) \leq \mu_n(A) \leq \mu(C_n) \quad (11)$$

On the other hand, clearly we have  $C_n - B_n \subseteq N_{\varepsilon_n}(\delta A)$ , where we have denoted  $N_{\varepsilon_n}(\delta A)$  the  $\varepsilon_n$ -neighborhood of  $\delta A$ , with  $\varepsilon_n = 2r_n$  here. (As usual we denote  $r_n$  the maximal edge length in  $\mathcal{M}_n$ .) By continuity of the measure  $\mu$ , we know that  $\lim_{n \rightarrow +\infty} \mu(N_{\varepsilon_n}(\delta A)) = \mu(\delta A) = 0$ . Note that we used the boundedness of  $A$ , which guarantees that  $\mu(N_{\varepsilon_n}(\delta A)) < +\infty$ . It follows:

$$\lim_{n \rightarrow +\infty} \mu(C_n - B_n) = 0. \quad (12)$$

Since  $B_n \subseteq A \subseteq C_n$ , (12) implies that  $\lim_{n \rightarrow +\infty} \mu(B_n) = \lim_{n \rightarrow +\infty} \mu(C_n) = \mu(A)$ , and we conclude with (11) that  $\lim_{n \rightarrow +\infty} \mu_n(A) = \mu(A)$ .  $\square$

### 3.3 Convergence of the tension field

Now we consider another Riemannian manifold  $N$  and a smooth function  $f: M \rightarrow N$ .

Consider a sequence of meshes  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  on  $M$ , which we will assume fine and crystalline. We denote  $\mathcal{G}_n$  the underlying graph and  $r = r_n$  the mesh size (maximum edge length).

**Theorem 3.14.** *Assume that the sequence of meshes  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  on  $M$  is fine and crystalline. If  $\mathcal{G}_n$  is equipped with a system of vertex and edge weights that is Laplacian at some vertex  $x$ , then*

$$\tau_{\mathcal{G}}(f)_x - \tau(f)_x = O(r^2). \quad (13)$$

Note that we abusively drop the dependence in  $n$  when writing  $\mathcal{G} = \mathcal{G}_n$  and  $r = r_n$ , to alleviate notations. We also write  $\tau_{\mathcal{G}}(f)$  instead of  $\tau_{\mathcal{G}}(f_{\mathcal{G}})$ , where  $f_{\mathcal{G}} := \pi_{\mathcal{G}}(f)$  is the discretization of  $f$  (i.e. restriction to the vertex set).

*Remark 3.15.* The proof below shows that in (13), the  $O(r^2)$  function depends on  $f$ , but may be chosen independent of  $x$  if  $M$  is compact.

*Proof.* Consider  $F := \exp_{f(x)}^{-1} \circ f \circ \exp_x : T_x M \rightarrow T_{f(x)} N$ . For  $y \sim x$ , denote  $v = v_y := \exp_x^{-1} y$ . By Taylor's theorem we have

$$\exp_{f(x)}^{-1} f(y) = F(v) = (dF)_{|0}(v) + \frac{1}{2}(d^2 F)_{|0}(v, v) + \frac{1}{6}(d^3 F)_{|0}(v, v, v) + O(r^4). \quad (14)$$

This implies

$$\begin{aligned} \tau_{\mathcal{G}}(f)(x) &= \frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} \exp_{f(x)}^{-1} f(y) \\ &= \frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} (dF)_{|0}(v) + \frac{1}{2\mu_x} \sum_{y \sim x} \omega_{xy} (d^2 F)_{|0}(v, v) \\ &\quad + \frac{1}{6\mu_x} \sum_{y \sim x} \omega_{xy} (d^3 F)_{|0}(v, v, v) + \frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} O(r^4) \end{aligned}$$

By conditions (1) and (3) of [Proposition 2.4](#), the first and third sums above vanish, while the second sum is rewritten with condition (2):

$$\tau_{\mathcal{G}}(f_{\mathcal{G}})(x) = \operatorname{tr} \left( d^2 F|_0 \right) + \frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} O \left( r^4 \right).$$

Note that  $\operatorname{tr} \left( d^2 F|_0 \right) = \operatorname{tr} \left( \nabla^2 f|_x \right) = \tau(f)(x)$ , and conclude with [Lemma 3.11](#).  $\square$

### 3.4 Convergence of the energy

We keep the setting of [§ 3.3](#):  $f: M \rightarrow N$  is a smooth function between Riemannian manifolds, and  $M$  is equipped with a sequence of meshes  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  that is fine and crystalline.

#### 3.4.1 Convergence of the energy density

**Theorem 3.16.** *Assume that the sequence of meshes  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  on  $M$  is fine and crystalline. If  $\mathcal{G} = \mathcal{G}_n$  is equipped with a system of vertex and edge weights that is Laplacian at some vertex  $x$ , then*

$$e_{\mathcal{G}}(f)_x = e(f)_x + O \left( r^2 \right)$$

*Remark 3.17.* [Remark 3.15](#) holds again for [Theorem 3.16](#).

*Proof.* Using (14) again, denoting  $v_y = \exp_x^{-1} y$ , we find that

$$\begin{aligned} e_{\mathcal{G}}(f)_x &= \frac{1}{4\mu_x} \sum_{y \sim x} \omega_{xy} \|F(v_y)\|^2 \\ &= \frac{1}{4\mu_x} \sum_{y \sim x} \omega_{xy} \left\| (dF)|_0(v_y) + \frac{1}{2} (d^2 F)|_0(v_y) + O \left( r^3 \right) \right\|^2 \\ &= \frac{1}{4\mu_x} \sum_{y \sim x} \omega_{xy} \|(dF)|_0(v_y)\|^2 + \frac{1}{4\mu_x} \sum_{y \sim x} \omega_{xy} \langle (dF)|_0(v_y), (d^2 F)|_0(v_y) \rangle \\ &\quad + \frac{1}{4\mu_x} \sum_{y \sim x} \omega_{xy} O \left( r^4 \right). \end{aligned}$$

Condition (3) of [Proposition 2.4](#) implies that the second sum vanishes. [Lemma 3.11](#) implies that the third sum is  $O \left( r^2 \right)$ . By condition (2) of [Proposition 2.4](#), the remaining first sum is rewritten

$$\frac{1}{4\mu_x} \sum_{y \sim x} \omega_{xy} \|(dF)|_0(v_y)\|^2 = \frac{1}{2} \|(dF)|_0\|^2 = e(f)_x$$

since  $\operatorname{tr}(L^2) = \|L\|^2$  for any linear form  $L$ . We thus get

$$e_{\mathcal{G}}(f)_x = e(f)_x + O \left( r^2 \right).$$

$\square$

### 3.4.2 Convergence of the energy

Recall that the energy is  $E(f) := \int_M e(f) d\mu$ . The convergence of the discrete energy is now an easy consequence of the weak convergence of measures  $\mu_n \rightarrow \mu$  and the uniform convergence of the energy densities  $e_n(f) \rightarrow e(f)$ . This is the classical combination of weak convergence and strong convergence.

**Definition 3.18.** Let  $(M, g)$  be a Riemannian manifold. Consider a sequence of geodesic meshes  $(\mathcal{M}_n)_{n \in \mathbb{N}}$ , and equip the underlying graphs  $\mathcal{G}_n$  with a system of positive vertex and edge weights. We call the sequence of biweighted graphs  $(\mathcal{G}_n)_{n \in \mathbb{N}}$  *Laplacian* provided that:

- (i) The sequence of meshes  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  is fine and crystalline.
- (ii) For every  $n \in \mathbb{N}$ , the vertex weights on  $\mathcal{G}_n$  are given by the volume weights (see § 2.2).
- (iii) For every  $n \in \mathbb{N}$ , the system of vertex and edge weights on  $\mathcal{G}_n$  is Laplacian.

**Theorem 3.19.** Let  $M$  be a Riemannian manifold and let  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  be a Laplacian sequence of meshes. For any smooth  $f: M \rightarrow \mathbb{R}$  with compact support:

$$\lim_{n \rightarrow +\infty} E_{\mathcal{G}_n}(f) = E(f).$$

*Proof.* By Theorem 3.13,

$$E(f) = \lim_{n \rightarrow +\infty} \int_M e(f) d\mu_n.$$

By Theorem 3.16, on the support of  $\mu_n$ ,  $e(f) = e_n(f) + O(r_n^2)$ . It follows that

$$E(f) = \lim_{n \rightarrow +\infty} \int_M e_n(f) d\mu_n,$$

in other words  $E(f) = \lim_{n \rightarrow +\infty} E_{\mathcal{G}_n}(f)$ . □

*Remark 3.20.* The proof of Theorem 3.19 hints that  $E(f) = E_{\mathcal{G}_n}(f) + O(r_n^2)$ , provided that the convergence of  $\mu_n$  to  $\mu$  is sufficiently fast. Improvements of this estimate can occur in more restricted situations: for instance, when both the target and the domain are hyperbolic surfaces:

$$E(f) = E_{\mathcal{G}_n}(f) + O(r_n^4).$$

This can be proven by carrying out involved calculations in the hyperbolic plane, which we spare.

### 3.5 Weak Laplacian conditions

It is clear from the proofs of the main results in the previous subsections that the Laplacian conditions for sequences of meshes can be weakened and still produce the same results, or at least some of them, with minimal changes in the proofs. This is a useful generalization, for it is very stringent to require a sequence of weighted graphs  $(\mathcal{G}_n)$  to be Laplacian for all  $n$ . Instead we start by asking that the sequence is merely asymptotically Laplacian in the following sense.

**Definition 3.21.** Let  $M$  be a Riemannian manifold. Consider a sequence of geodesic meshes  $(\mathcal{M}_n)_{n \in \mathbb{N}}$ , and equip the underlying graphs  $\mathcal{G}_n$  with a system of positive vertex weights  $\{\mu_{x,n}\}$ . We call the sequence of weight systems  $(\{\mu_{x,n}\})_{n \in \mathbb{N}}$  *asymptotic volume weights* provided that:

$$\mu_{x,n} = (1 + o(1)) \widehat{\mu}_{x,n}$$

for some function  $o(1)$  independent of  $x$ , where  $\widehat{\mu}_{x,n}$  denote the volume weights (see § 2.2).



The following proposition is an immediate consequence of [Theorem 3.13](#):

**Proposition 3.22.** *If  $M$  is any Riemannian manifold and  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  is any fine sequence of meshes, then the measures  $(\mu_n)_{n \in \mathbb{N}}$  on  $M$  defined by any system of asymptotic volume vertex weights converge weakly to the volume density on  $M$ .*

It is immediate to show that for asymptotic volume weight, [Theorem 2.6](#) holds with a Lipschitz constant  $L_n = \sqrt{1 + \dim M} + o(1)$ . Although this is sufficient for the needs of this paper (see [Lemma 4.3](#)), let us state that the result can be improved to  $L_n = 1 + o(1)$ . The proof follows from [Theorem 2.6](#) by writing an expansion of the volume form in normal coordinates, we skip it for brevity.

**Theorem 3.23.** *Let  $M$  be a compact Riemannian manifold and let  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  be a fine sequence of meshes equipped with a system of asymptotic volume vertex weights. For any complete Riemannian manifold  $N$  of nonpositive sectional curvature, the center of mass interpolation map  $\iota_n: \text{Map}_{\mathcal{G}_n}(M, N) \rightarrow C(M, N)$  is  $L_n$ -Lipschitz with respect to the  $L^2$  distance on both spaces, with  $L_n = 1 + o(1)$ .*

**Definition 3.24.** Let  $M$  be a Riemannian manifold. Consider a sequence of geodesic meshes  $(\mathcal{M}_n)_{n \in \mathbb{N}}$ , and equip the underlying graphs  $\mathcal{G}_n$  with a system of positive vertex and edge weights. We call the sequence of biweighted graphs  $(\mathcal{G}_n)_{n \in \mathbb{N}}$  *asymptotically Laplacian* provided that:

- (i) The sequence of meshes  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  is fine and crystalline.
- (ii) The vertex weights are asymptotic volume weights (see [Definition 3.21](#)).
- (iii) For every  $n \in \mathbb{N}$ , the system of vertex and edge weights on  $\mathcal{G}_n$  is Laplacian up to  $O(r_n^2)$  at all vertices.

Explicitly, (iii) means that for all  $x \in \mathcal{V}_n$  and  $L \in T_x M$ :

(1)

$$\frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} \vec{xy} = O(r_n^2)$$

(2)

$$\frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} L(\vec{xy})^2 = 2\|L\|^2 \left(1 + O(r_n^2)\right)$$

(3)

$$\frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} L(\vec{xy})^3 = \|L\|^3 O(r_n^2)$$

The  $O(r_n^2)$  functions above should be independent of  $x$  and  $L$ . Note that to alleviate notations, we drop the dependence in  $n$  when writing  $\mu_x$  and  $\omega_{xy}$ .

It is immediate to check that the proofs of [Theorem 3.14](#), [Theorem 3.16](#), and [Theorem 3.19](#) apply to asymptotically Laplacian sequences of graphs. We proceed to further weaken [Definition 3.24](#), which will lead to slightly weaker results.

Consider a fine sequence of meshes with underlying sequence of graphs  $(\mathcal{G}_n)_{n \in \mathbb{N}}$ , and assume these are vertex-weighted: we have a measure  $\mu_n$  on the vertex set  $\mathcal{V}_n$  of each graph  $\mathcal{G}_n$ . We say that a sequence of sets  $X_n \subseteq \mathcal{V}_n$  is  $O(r^a)$ -negligible, where  $a > 0$ , if  $\mu_n(X_n) = O(r_n^a)$ .

**Definition 3.25.** We say that the sequence of biweighted graphs  $(\mathcal{G}_n)_{n \in \mathbb{N}}$  is  $(a, b)$ -almost asymptotically Laplacian, where  $(a, b) \in \mathbb{R}_{>0} \times (-\infty, 2]$ , if it satisfies conditions (i) and (ii) of [Definition 3.24](#), and the modified version of (iii):

- (iii) The system of vertex and edge weights on  $\mathcal{G}_n$  is Laplacian up to  $O(r_n^2)$  everywhere except on a  $O(r^a)$ -negligible set of vertices, where it is Laplacian up to  $O(r_n^b)$ .

Note that the case  $b = 2$  (and any value of  $a$ ) includes asymptotically Laplacian sequences of meshes as a particular case of almost asymptotically Laplacian.

The following theorems are generalized or weakened versions of [Theorem 3.14](#), [Theorem 3.16](#), and [Theorem 3.19](#).

**Theorem 3.26.** *Let  $M$  be a compact Riemannian manifold. Consider a sequence of geodesic meshes and equip the underlying graphs  $\mathcal{G}_n$  with a system of vertex and edge weights. Let  $f: M \rightarrow N$  be any smooth map to another Riemannian manifold.*

(1) *If  $(\mathcal{G}_n)_{n \in \mathbb{N}}$  is Laplacian or asymptotically Laplacian, then*

$$\|\tau(f) - \tau_{\mathcal{G}_n}(f)\|_{\infty} = O\left(r_n^2\right).$$

*A fortiori,  $\|\tau(f) - \tau_{\mathcal{G}_n}(f)\|_2 = O\left(r_n^2\right)$ .*

(2) *If  $(\mathcal{G}_n)_{n \in \mathbb{N}}$  is  $(a, b)$ -almost asymptotically Laplacian, then*

$$\|\tau(f) - \tau_{\mathcal{G}_n}(f)\|_2 = O\left(r_n^q\right)$$

*where  $q = \min\left(2, \frac{a}{2} + b\right)$ .*

Note that we use the discrete measure  $\mu_n$  on the vertex set of  $\mathcal{G}_n$  in order to define the  $L^2$ -norm on spaces of discrete maps along  $\mathcal{G}_n$ .

*Proof.* When  $(\mathcal{G}_n)_{n \in \mathbb{N}}$  is Laplacian, (1) is an immediate consequence of [Theorem 3.14](#). If  $(\mathcal{G}_n)_{n \in \mathbb{N}}$  is merely asymptotically Laplacian, then it is easy to check that the proof of [Theorem 3.14](#) is still valid. For the proof of (2), let  $X_n \subseteq \mathcal{V}_n$  be the set of vertices where  $\mathcal{G}_n$  is not Laplacian up to  $O\left(r_n^2\right)$ . By definition of almost asymptotically Laplacian,  $\mu_n(X_n) = O\left(r_n^a\right)$  and  $\mathcal{G}_n$  is Laplacian up to  $O\left(r_n^b\right)$  on  $X_n$ . By tracing the proof of [Theorem 3.14](#), one quickly sees that  $\tau(f) = \tau_{\mathcal{G}_n}(f) + O\left(r_n^b\right)$  on  $X_n$ , and  $\tau(f) = \tau_{\mathcal{G}_n}(f) + O\left(r_n^2\right)$  on  $\mathcal{V}_n - X_n$ . Hence

$$\begin{aligned} \|\tau(f) - \tau_{\mathcal{G}_n}(f)\|_2^2 &= \int_{\mathcal{V}_n - X_n} \|\tau(f) - \tau_{\mathcal{G}_n}(f)\|_2^2 d\mu_n + \int_{X_n} \|\tau(f) - \tau_{\mathcal{G}_n}(f)\|_2^2 d\mu_n \\ &= O(1) \cdot O\left(r_n^4\right) + O\left(r_n^a\right) \cdot O\left(r_n^{2b}\right) \\ &= O\left(r_n^4\right) + O\left(r_n^{a+2b}\right) \end{aligned}$$

and the conclusion follows. □

**Theorem 3.27.** *Let  $M$  be a compact Riemannian manifold. Consider a sequence of geodesic meshes and equip the underlying graphs  $\mathcal{G}_n$  with a system of vertex and edge weights. Let  $f: M \rightarrow N$  be any smooth map to another Riemannian manifold.*

(1) *If  $(\mathcal{G}_n)_{n \in \mathbb{N}}$  is Laplacian or asymptotically Laplacian, then*

$$\|e(f) - e_{\mathcal{G}_n}(f)\|_{\infty} = O\left(r_n^2\right).$$

(2) *If  $(\mathcal{G}_n)_{n \in \mathbb{N}}$  is  $(a, b)$ -almost asymptotically Laplacian, then*

$$\|e(f) - e_{\mathcal{G}_n}(f)\|_{\infty} = O\left(r_n^2\right)$$

*everywhere outside of a  $O\left(r^a\right)$ -negligible set, and on that set*

$$\|e(f) - e_{\mathcal{G}_n}(f)\|_{\infty} = O\left(r_n^b\right).$$

*Proof.* The proof is easily adapted from the proof of [Theorem 3.16](#).  $\square$

**Theorem 3.28.** *Let  $M$  be a compact Riemannian manifold. Consider a sequence of geodesic meshes and equip the underlying graphs  $\mathcal{G}_n$  with a system of vertex and edge weights. Let  $f: M \rightarrow N$  be any smooth map to another Riemannian manifold. If  $(\mathcal{G}_n)_{n \in \mathbb{N}}$  is  $(a, b)$ -almost asymptotically Laplacian with  $a + b > 0$ , then*

$$\lim_{n \rightarrow +\infty} E_{\mathcal{G}_n}(f) = E(f) .$$

*Remark 3.29.* Of course, [Theorem 3.28](#) also holds for Laplacian and asymptotically Laplacian sequences of meshes, given the hierarchy between these conditions.

*Proof of Theorem 3.28.* By definition of almost asymptotically Laplacian, the sequence of measures  $(\mu_n)_{n \in \mathbb{N}}$  converges weakly to the measure  $\mu$  on  $M$ , therefore

$$\begin{aligned} E(f) &= \int_M e(f) \, d\mu \\ &= \lim_{n \rightarrow +\infty} \int_M e(f) \, d\mu_n . \end{aligned} \tag{15}$$

Let  $X_n \subseteq \mathcal{V}_n$  be the set of vertices where  $\mathcal{G}_n$  is not Laplacian up to  $O(r_n^2)$ . By definition of almost asymptotically Laplacian,  $\mu_n(X_n) = O(r_n^a)$  and  $\mathcal{G}_n$  is Laplacian up to  $O(r_n^b)$  on  $X_n$ . By [Theorem 3.27](#),

$$\begin{aligned} \int_M e(f) \, d\mu_n &= \int_{\mathcal{V}_n - X_n} e(f) \, d\mu_n + \int_{X_n} e(f) \, d\mu_n \\ &= \int_{\mathcal{V}_n - X_n} \left( e_{\mathcal{G}_n}(f) + O(r_n^2) \right) \, d\mu_n + \int_{X_n} \left( e_{\mathcal{G}_n}(f) + O(r_n^b) \right) \, d\mu_n . \end{aligned}$$

It follows:

$$\begin{aligned} \int_M e(f) \, d\mu_n &= \int_M e_{\mathcal{G}_n}(f) \, d\mu_n + O(1) \cdot O(r_n^2) + O(r_n^a) O(r_n^b) \\ &= E_{\mathcal{G}_n}(f) + O(r_n^2) + O(r_n^{a+b}) . \end{aligned}$$

In particular, we find that  $\int_M e(f) \, d\mu_n = E_{\mathcal{G}_n}(f) + o(1)$ . Injecting this into (15) yields the desired result  $E(f) = \lim_{n \rightarrow +\infty} E_{\mathcal{G}_n}(f)$ .  $\square$

## 4 Convergence to smooth harmonic maps

Let  $(M, g)$  be a compact Riemannian manifold and let  $(N, h)$  be Riemannian manifold of nonpositive sectional curvature which does not contain any flats (totally geodesic flat submanifolds). Consider a connected component  $C$  of the space of smooth maps  $C^\infty(M, N)$  that does not contain any map of rank everywhere  $\leq 1$  — for instance, take any connected component of maps whose topological degree is nonzero when  $\dim M = \dim N$ . When  $N$  is compact, a celebrated theorem of Eells-Sampson implies that  $C$  contains a harmonic map  $w$  [[ES64](#)], and by Hartman [[Har67](#)] the harmonic map  $w$  is unique.

In this section we show that one can obtain the harmonic map  $w \in C$  as the limit of discrete harmonic maps  $u_n$  along a sequence of meshes  $(\mathcal{M}_n)_{n \in \mathbb{N}}$ , provided that:

- (i) The discrete energy functional  $E_n$  is sufficiently convex on the discrete homotopy class  $C_n$ . We expect that this is the case when  $N$  is compact and has negative sectional curvature, and have showed it in special cases in our previous work [GLM18].
- (ii) The sequence of meshes is Laplacian (Definition 3.18), or one of the weaker versions (Definition 3.24, Definition 3.25). In the next and final section § 5, we discuss examples where these conditions apply.

We then show convergence of the discrete heat flow  $u_{kn}$  to the smooth harmonic map  $w$ , when the time and space discretization indices  $k$  and  $n$  simultaneously run to  $+\infty$ , provided the adequate CFL condition is satisfied.

## 4.1 Strong convexity of the discrete energy

Please refer to [GLM18, §3.1] for the definition of convex, strictly convex, and strongly convex functions on Riemannian manifolds. In a nutshell, these notions are properly defined by restricting to geodesics, and the convexity [resp.  $\alpha$ -strong convexity] of a smooth function is characterized by its Hessian being  $\geq 0$  [resp.  $\geq \alpha g$  where  $g$  is the Riemannian metric].

Keeping the same setup as above, assume moreover that  $N$  is compact and has negative sectional curvature. In this case, we expect that the discrete energy functional  $E_{\mathcal{G}}: C_{\mathcal{G}} \rightarrow \mathbb{R}$  is  $\alpha_{\mathcal{G}}$ -strongly convex for any biweighted graph  $\mathcal{G}$  on  $M$  underlying a mesh, for some  $\alpha_{\mathcal{G}} > 0$ . When  $M$  and  $N$  are both 2-dimensional, we proved this in our previous paper: see [GLM18, Theorem 3.20].

Moreover we conjecture that the smooth energy  $E: C \rightarrow \mathbb{R}$  is  $\alpha$ -strongly convex itself for some  $\alpha > 0$  (see [GLM18, §3.2] for a discussion), and we expect that given any asymptotically Laplacian sequence of meshes  $(\mathcal{M}_n)_{n \in \mathbb{N}}$ ,  $\alpha = \lim_{n \rightarrow +\infty} \alpha_n$ . In particular, the sequence  $(\alpha_n)_{n \in \mathbb{N}}$  is  $\Omega(1)$  (see Remark 3.6 for the notations  $\Omega$  and  $\Theta$ ). A neat way to show our conjecture that  $E: C \rightarrow \mathbb{R}$  is strongly convex would consist in first showing that  $(\alpha_n)_{n \in \mathbb{N}}$  is  $\Omega(1)$  for some asymptotically Laplacian sequence. However this does not seem easy to achieve. In our previous work, we show that when both  $M$  and  $N$  are hyperbolic surfaces, we have at least  $\alpha_n = \Omega(r_n^c)$  with  $c = 1$ , but we believe this can be improved. We will see that the finer estimate we have for  $\alpha_n$  (i.e. the smaller value for  $c$ ), the better convergence results we can prove.

## 4.2 $L^2$ convergence

**Theorem 4.1.** *Let  $M$  be a compact Riemannian manifold and let  $N$  be a complete Riemannian manifold of nonpositive sectional curvature. Consider a sequence of meshes  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  of  $M$ . Let  $C$  be a connected component of the space of smooth maps  $C^\infty(M, N)$  containing a harmonic map  $w$ . Assume that*

- (i) *The sequence of meshes  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  is  $(a, b)$ -almost asymptotically Laplacian, and*
- (ii) *The discrete energy  $E_n := \text{Map}_{\mathcal{G}_n}(M, N)$  is  $\alpha_n$ -strongly convex on  $C_n$  with  $\alpha_n = \Omega(r_n^c)$ .*

*Denote by  $v_n$  the minimizer of  $E_n$  and  $\widehat{v}_n$  its center of mass interpolation. Let  $q := \min(2, \frac{a}{2} + b)$ . If  $q - c > 0$ , then  $\widehat{v}_n \rightarrow w$  in  $L^2(M, N)$  when  $n \rightarrow +\infty$ .*

Note that in particular,  $w$  must be the unique harmonic map in  $C$  under the assumptions of Theorem 4.1, and is the minimizer of the energy.

*Proof of Theorem 4.1.* The proof is a combination of a few key ideas that we emphasize using in-proof lemmas. The bulk of the hard work has been done in the previous sections, which we will refer to for the proof of these lemmas.

Let  $u_n := \pi_n(w) \in \text{Map}(\mathcal{G}_n, N)$  denote the discretization of  $w$  (restriction of  $w$  to the vertex set of  $\mathcal{G}_n$ ). We also denote  $\widehat{w}_n$  the center of mass interpolation of  $w_n$ .

**Lemma 4.2.** *We have  $\widehat{w}_n \rightarrow w$  in  $L^2(M, N)$  when  $n \rightarrow +\infty$ , moreover  $E(w_n) \rightarrow E(w)$ .*

*Proof of Lemma 4.2.* This is an immediate consequence of [Corollary 3.9](#), which we can invoke since  $M$  is compact and the sequence of meshes  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  is fine and crystalline.  $\square$

**Lemma 4.3.** *There exists a constant  $L > 0$  such that*

$$d(\widehat{w}_n, \widehat{v}_n) \leq L d(w_n, v_n)$$

where  $d(\widehat{w}_n, \widehat{v}_n)$  and  $d(w_n, v_n)$  indicate the  $L^2$  distances in  $C(M, N)$  and  $\text{Map}_{\mathcal{G}_n}(M, N)$ .

*Proof of Lemma 4.3.* This follows immediately from [Theorem 3.23](#).  $\square$

**Lemma 4.4.** *Let  $R$  be a complete Riemannian manifold and  $F: R \rightarrow \mathbb{R}$  be an  $\alpha$ -strongly convex function. Then  $F$  has a unique minimizer  $x^*$  and for all  $x \in R$ ,*

$$d(x, x^*) \leq \frac{\|\text{grad } F(x)\|}{\alpha} \quad (16)$$

and

$$0 \leq F(x) - F(x^*) \leq \frac{\|\text{grad } F(x)\|^2}{\alpha}. \quad (17)$$

*Proof of Lemma 4.4.* Recall that on a complete Riemannian manifold  $R$ , there exists a length-minimizing geodesic between any two points. It is not hard to show that a strongly convex function on a complete (finite-dimensional) Riemannian manifold is proper, hence existence of the minimizer, and uniqueness follows from strict convexity. The first inequality (16) is easy to prove for a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  by integrating  $f''(x) \geq \alpha$ , and for the general case, take a length-minimizing unit geodesic  $\gamma: \mathbb{R} \rightarrow F$  with  $\gamma(0) = x^*$  and  $\gamma(L) = x$ , and apply the previous result to  $f = F \circ \gamma$ . For the second inequality (17), proceed likewise: the one-dimensional case is readily obtained via the mean value theorem, and the general case quickly follows.  $\square$

**Lemma 4.5.** *We have*

$$d(w_n, v_n) \leq \frac{\|\tau_n(w_n)\|_2}{\alpha_n}.$$

where  $d$  denotes the  $L^2$  distance in  $\text{Map}_{\mathcal{G}_n}(M, N)$ .

*Proof of Lemma 4.5.* Apply [Lemma 4.4](#) (17) to  $R = \text{Map}_{\mathcal{G}_n}(M, N)$  and  $F = E_n$ .  $\square$

**Lemma 4.6.** *We have*

$$\|\tau_n(w_n)\|_2 = O(r_n^q)$$

where  $q = \min(2, \frac{a}{2} + b)$ .

*Proof of Lemma 4.3.* This is just an application of [Theorem 3.26](#), recalling that  $\tau(w) = 0$  since  $w$  is smooth harmonic.  $\square$

We can now smoothly wrap up the proof of [Theorem 4.1](#): write

$$\begin{aligned}
d(\widehat{v}_n, w) &\leq d(\widehat{v}_n, \widehat{w}_n) + d(\widehat{w}_n, w) && \text{(triangle inequality)} \\
&\leq L d(w_n, v_n) + o(1) && \text{(by [Lemma 4.3](#) and [Lemma 4.2](#))} \\
&\leq \|\tau_n(w_n)\|_2 O(r_n^{-c}) + o(1) && \text{(by [Lemma 4.5](#))} \\
&\leq O(r_n^{q-c}) + o(1) && \text{(by [Lemma 4.6](#))}
\end{aligned}$$

Since  $q - c > 0$  by assumption, we conclude that  $d(\widehat{v}_n, w) \leq o(1)$ , in other words  $\widehat{v}_n \rightarrow w$  in  $C(M, N)$  for the  $L^2$  topology.  $\square$

### 4.3 $L^\infty$ convergence

Under somewhat stronger assumptions, we are able to prove uniform convergence, simply by comparing the  $L^2$  and  $L^\infty$  distances on the space of discrete maps  $\text{Map}_{\mathcal{G}_n}(M, N)$  (and using [Corollary 3.9](#)).

#### 4.3.1 Proof of $L^\infty$ convergence

**Theorem 4.7.** *In the setup of [Theorem 4.1](#), if  $q - c - \frac{\dim M}{2} > 0$ , then  $\widehat{v}_n \rightarrow w$  in  $L^\infty(M, N)$ .*

*Proof.* Write

$$d_\infty(\widehat{v}_n, w) \leq d_\infty(\widehat{v}_n, \widehat{w}_n) + d_\infty(\widehat{w}_n, w).$$

The second term  $d_\infty(\widehat{w}_n, w)$  converges to zero by [Corollary 3.9](#). It remains to show that  $d_\infty(\widehat{v}_n, \widehat{w}_n) \rightarrow 0$ . By [Theorem 1.4 \(iii\)](#),  $d_\infty(\widehat{v}_n, \widehat{w}_n) \leq d_\infty(v_n, w_n)$ . Now, clearly the  $L^2$  the  $L^\infty$  distances verify  $d^2 \geq m_{\mathcal{G}} d_\infty^2$ , where  $m_{\mathcal{G}} := \min_{x \in \mathcal{V}} \mu_x$  (see [Appendix B](#) for more details). Here the vertex weights are asymptotic volume weights, and since the sequence of meshes is fine and crystalline the vertex weights are all  $\Theta(r^{\dim M})$  by [Proposition 3.4](#). Therefore we find  $d_\infty^2 \leq O(r_n^{-\dim M}) d^2$ . The proof of [Theorem 4.1](#) shows that  $d(v_n, w_n) = O(r_n^{q-c})$ , therefore we find that  $d_\infty(v_n, w_n) = O\left(r_n^{q-c-\frac{\dim M}{2}}\right)$ .

We conclude that  $d_\infty(v_n, w_n) = o(1)$  since  $q - c - \frac{\dim M}{2} > 0$  by assumption.  $\square$

#### 4.3.2 The equality case

Our proof of [Theorem 4.7](#) only works when  $q - c - \frac{\dim M}{2} > 0$ , where  $q = \min\left(2, \frac{a}{2} + b\right)$ . Using a technical argument, the proof can be refined to include the equality case  $q - c - \frac{\dim M}{2} = 0$ . This may sound like an insignificant improvement, but it is exactly the case we find ourselves in if we consider an asymptotically Laplacian sequence of meshes on a surface ( $q = 2$ ,  $\dim M = 2$ ) when  $c = 1$ , which is precisely our estimate when  $M$  and  $N$  are hyperbolic surfaces (see [§ 4.1](#)).

The idea of this improvement is simple: instead of using the comparison  $d_\infty(u, v) \leq m^{-1/2} d(u, v)$  where  $m := \min_{x \in \mathcal{V}} \mu_x$ , which is true for any discrete maps  $u$  and  $v$ , we use a slight improvement when  $v$  is the discrete energy minimizer. In order to avoid burdening our exposition, we relegate this technical estimate to [Appendix B](#).

**Theorem 4.8.** *[Theorem 4.7](#) still holds in the equality case  $q - c - \frac{\dim M}{2} = 0$ .*

*Proof.* At the end of the proof of [Theorem 4.7](#), the inequality  $d_\infty(v_n, w_n) = O\left(r_n^{q-c-\frac{\dim M}{2}}\right)$  can be upgraded to  $d_\infty(v_n, w_n) = o\left(r_n^{q-c-\frac{\dim M}{2}}\right)$  by [Corollary B.4](#). In particular, we still find  $d_\infty(v_n, w_n) \rightarrow 0$  when  $q - c - \frac{\dim M}{2} = 0$  and can conclude all the same.  $\square$

Our main application of [Theorem 4.8](#) is given in [§ 4.6](#).

#### 4.4 Convergence of the energy

One would like to say that (or discuss whether) the discrete minimizer  $\widehat{v}_n$  converges to the smooth harmonic map  $w$  in the Sobolev space  $H^1(M, N)$ , say, under the assumptions of [Theorem 4.1](#), but this function space is not well-defined. It is however still reasonable to ask whether the energy of  $v_n$  converges to the energy of  $w$ .

As we shall see, it does not cost much to show that the discrete energy  $E_n(v_n)$  does converge to the energy  $E(w)$ . However, it is much harder to argue that the energy of the interpolation  $E(\widehat{v}_n)$  also converges to  $E(w)$ . While we believe it is true that  $E(v_n)$  and  $E(\widehat{v}_n)$  are asymptotic, proving it is hard. We do state such a convergence result, but under assumptions that are probably far too strong.

*Remark 4.9.* The obstacle to show that  $E(v_n)$  and  $E(\widehat{v}_n)$  are asymptotic would be lifted by showing that the sequence  $(\widehat{v}_n)_{n \in \mathbb{N}}$  has a uniformly bounded Lipschitz constant, but we are unable to establish that result in our present work. In fact, showing it would enable us to prove [Theorem 4.1](#) for any asymptotically Laplacian sequence of meshes (with  $a + b > 0$ , no assumption involving  $c$ ) with a completely different method involving a Rellich–Kondrachov theorem. In the smooth setting, a uniform Lipschitz bound is achieved by using the Bochner formula and Moser’s Harnack inequality (see e.g. [[Jos84](#)], [[Lou19](#), §2.2.2]), so perhaps a discrete version of this argument would get the job done. While developing a discrete Bochner formula and a discrete Moser’s Harnack inequality is probably a worthwhile project, it is also a substantial one.

**Theorem 4.10.** *In the setup of [Theorem 4.1](#), if  $2q - c > 0$ , then  $E_n(v_n) \rightarrow E(w)$ . If moreover  $q - c - \frac{\dim M}{2} - 1 \geq 0$ , then we also have  $E(\widehat{v}_n) \rightarrow E(w)$ .*

*Proof.* For the first claim, first write that  $E(w) = \lim_{n \rightarrow +\infty} E_n(w_n)$  by [Theorem 3.28](#). Thus it is sufficient to show that  $E_n(w_n)$  and  $E_n(v_n)$  are asymptotic. By [Lemma 4.4 \(17\)](#) applied to  $F = E_n$ , we find that

$$0 \leq E_n(w_n) - E_n(v_n) \leq \frac{\|\tau_n(w_n)\|^2}{\alpha_n}$$

so with [Lemma 4.6](#) we find that  $|E_n(w_n) - E_n(v_n)| = O\left(r_n^{2q-c}\right)$  and the first claim follows.

For the second claim, first write that  $E(w) = \lim_{n \rightarrow +\infty} E(\widehat{w}_n)$  by [Corollary 3.9](#). Thus it is sufficient to show that  $E(\widehat{w}_n)$  and  $E(\widehat{v}_n)$  are asymptotic. One can derive from [Theorem 1.4 \(iii\)](#) and [Proposition 3.4 \(ii\)](#) that for a fine and crystalline sequence of meshes,

$$\left| \|\widehat{d}\widehat{f}(x)\| - \|\widehat{d}\widehat{g}(x)\| \right| = O\left(\frac{d_\infty(f, g)}{r}\right).$$

uniformly in  $f, g \in \text{Map}_{\mathcal{G}_n}(M, N)$  and in  $x \in M$  in the interior of the triangulation, from which it follows  $|E(\widehat{f}) - E(\widehat{g})| = O\left(\frac{d_\infty(f, g)}{r}\right)$ . In our case this gives  $|E(\widehat{w}_n) - E(\widehat{v}_n)| = O\left(\frac{d_\infty(w_n, v_n)}{r_n}\right)$ . As



in the proof of [Theorem 4.8](#) we have  $d_\infty(w_n, v_n) = o\left(r_n^{q-c-\frac{\dim M}{2}}\right)$ , so we find that

$$|E(\widehat{w}_n) - E(\widehat{v}_n)| = o\left(r_n^{q-c-\frac{\dim M}{2}-1}\right)$$

hence  $|E(\widehat{w}_n) - E(\widehat{v}_n)| = o(1)$  when  $q - c - \frac{\dim M}{2} - 1 \geq 0$ .  $\square$

## 4.5 Convergence in time and space of the discrete heat flow

We turn to more practical considerations about how to compute harmonic maps. In the previous subsections, we established that, under suitable assumptions, the discrete harmonic map  $v_n$  converges to the smooth harmonic map  $w$ . In our previous work [\[GLM18\]](#), we showed that for each fixed  $n \in \mathbb{N}$ ,  $v_n$  may be computed as the limit of the discrete heat flow  $u_{k,n}$  when  $k \rightarrow +\infty$ . While this is relatively satisfactory, in practice one cannot wait for the discrete heat flow to converge for each  $n$ . Hence it is preferable to let both indices  $k$  and  $n$  run to  $+\infty$  simultaneously and hope to approximate the smooth harmonic map  $w$ . In the theory of PDEs, this situation with a double discretization in time and space is typical, and one expects convergence to the solution provided that the time step and the space step satisfy a constraint, called a *CFL condition*. We are happy to report a similar result in our setting.

We keep the same setting as in the beginning of the section. Let  $u \in C$  be a smooth map, denote by  $u_n \in \text{Map}_{\mathcal{G}_n}(M, N)$  its discretization. For each  $n \in \mathbb{N}$ , denote by  $(u_{k,n})_{k \in \mathbb{N}}$  the sequence in  $\text{Map}_{\mathcal{G}_n}(M, N)$  obtained by iterating the discrete heat flow from the initial map  $u_{0,n} = u_n$ . We recall that the discrete heat flow is defined by

$$u_{k+1,n} = u_{k,n} + t_n \tau_n(u_{k,n})$$

where  $t_n$  is a suitably chosen time step and we use the notation  $x + v$  as an alias for the Riemannian exponential map  $\exp_x(v)$  in  $N$ . We point out that the discrete heat flow is just a fixed stepsize gradient descent method for the discrete energy functional  $E_n$  on the Riemannian manifold  $\text{Map}_{\mathcal{G}_n}(M, N)$ . In particular, strong convexity of the  $E_n$  implies convergence of the discrete heat flow to the unique discrete harmonic map  $v_n$  with exponential convergence rate. Please refer to [\[GLM18\]](#) for more details.

**Theorem 4.11.** *Consider the same setup and assumptions as [Theorem 4.1](#). Also assume that for any constant  $K > 0$ , the discrete energy  $E_n$  has Hessian bounded above by  $\beta_{n,K} = O(r_n^{-d})$  on its sublevel set  $\{E_n \leq K\}$ , where  $d \geq 0$  is independent of  $K$ . Then*

$$\widehat{u_{k,n}} \rightarrow w$$

in  $L^2(M, N)$  when  $k, n \rightarrow +\infty$ , provided the CFL condition:

$$k = \Omega\left(\frac{\log(r_n^{-1})}{r_n^{c+d}}\right). \quad (18)$$

*Remark 4.12.* The assumption on the upper bound of the Hessian is very reasonable: we typically expect that  $\beta_{n,K} \rightarrow \beta_K$  where  $\beta_K$  is an upper bound for the Hessian of the smooth energy  $E$  on its sublevel set  $\{E \leq K\}$ , in particular  $\beta_{n,K} = O(1)$ . In the case where  $N$  is a hyperbolic surface we find  $\beta_{n,K} = O(r_n^{-2})$  by [\[GLM18, Prop. 3.17\]](#), which satisfies the above assumption but is surely not optimal.

*Remark 4.13.* The CFL condition (18) that we give is most likely very far from optimal.

*Proof of Theorem 4.11.* Let us break the proof into a few key steps.

**Lemma 4.14.** *There exists a constant  $K > 0$  such that*

$$E_n(u_{k,n}) \leq K$$

for all  $k, n \in \mathbb{N}$ .

*Proof of Lemma 4.14.* The proof of this lemma is a favorite of ours. For each fixed  $n \in \mathbb{N}$ , the discrete energy  $E_n(u_{k,n})$  is nonincreasing with  $k$ , since the discrete heat flow is a gradient descent for the discrete energy. In particular  $E_n(u_{k,n}) \leq E_n(u_{0,n})$ . To conclude, we must argue that the sequence  $(E_n(u_n))_{n \in \mathbb{N}}$  is bounded. This is true since it converges to  $E(u)$  by Theorem 3.28.  $\square$

**Lemma 4.15.** *For every  $k, n$ , we have*

$$d(u_{k,n}, v_n) \leq c_n q_n^n$$

where  $c_n = O(r_n^{-c/2})$  and  $q_n = 1 - Cr^{c+d} + o(r^{c+d})$  with  $C > 0$ .

*Proof of Lemma 4.15.* This is an immediate consequence of [GLM18, Theorem 4.1]. Note that for the estimate of  $c_n$ , we need to use the fact that  $E_n(u_{0,n}) = O(1)$ , which we showed in Lemma 4.14.  $\square$

We now finish the proof of Theorem 4.11. For every  $k, n \in \mathbb{N}$ , we have

$$d(\widehat{u_{k,n}}, w) \leq d(\widehat{u_{k,n}}, \widehat{v_n}) + d(\widehat{v_n}, w).$$

The second term  $d(\widehat{v_n}, w)$  converges to zero by Theorem 4.1. As for the first term, we have  $d(\widehat{u_{k,n}}, \widehat{v_n}) \leq L d(u_{k,n}, v_n)$  for some constant  $L > 0$  by Theorem 3.23. Thus it is enough to show that  $d(u_{k,n}, v_n) \rightarrow 0$  under the appropriate CFL condition.

Let  $(\varepsilon_n)_{n \in \mathbb{N}}$  be a sequence of positive real numbers converging to zero to be chosen later. Since  $(u_{k,n})$  converges to  $v_n$  when  $k \rightarrow +\infty$ , there exists  $k_0(n)$  such that  $d(u_{k,n}, v_n) \leq \varepsilon_n$  for all  $k \geq k_0(n)$ . Note that the inequality  $k \geq k_0(n)$  is the CFL condition that we are after, for a/any choice of  $(\varepsilon_n)$ . It is possible to compute  $k_0(n)$  explicitly with Lemma 4.15; one finds that

$$k_0(n) = \frac{\log(c_n) + \log(\varepsilon_n^{-1})}{\log(q_n^{-1})}$$

is sufficient. With our estimates we get  $\log(c_n) = \Theta(\log(r_n^{-1}))$  and  $\log(q_n^{-1}) \sim Cr_n^{c+d}$ . It is easy to choose  $\varepsilon_n$  so that  $\log(\varepsilon_n^{-1})$  is negligible compared to  $\log(r_n^{-1})$ , e.g.  $\varepsilon_n = \log(r_n^{-1})$ . We thus find as desired

$$k_0(n) = \Theta\left(\frac{\log(r_n^{-1})}{r_n^{c+d}}\right).$$

$\square$

*Remark 4.16.* Of course, we could similarly show  $L^\infty$  convergence (resp. convergence of the energy) of  $\widehat{u_{k,n}}$  to  $w$  under the assumptions of Theorem 4.7 (resp. Theorem 4.10) and suitable CFL conditions, but we spare the details.

## 4.6 Application to surfaces

When  $M$  and  $N$  are both 2-dimensional, our previous work [GLM18] gives estimates for the strong convexity of the discrete energy. More precisely, consider the following setup:

Let  $S = M$  and  $N$  be closed Riemannian surfaces of negative Euler characteristic. Assume  $N$  has negative sectional curvature. Assume that  $S$  is equipped with a fine and crystalline sequence of meshes  $(\mathcal{M}_n)_{n \in \mathbb{N}}$ , equipped with asymptotic volume weights and positive edge weights such that the ratio of any two edge weights is uniformly bounded. Consider a homotopy class of maps  $C \subset C^\infty(M, N)$  of nonzero degree, and its discretization  $C_n$  along each mesh.

**Lemma 4.17.** *The discrete energy functional  $E_n: C_n \rightarrow \mathbb{R}$  has Hessian bounded below by  $\alpha_n$  and above by  $\beta_{n,K}$  on any sublevel set  $\{E_n \leq K\}$ , with*

$$\begin{aligned}\alpha_n &= \Omega(r_n) \\ \beta_{n,K} &= O(r_n^{-2}).\end{aligned}$$

*Proof.* The estimate for  $\alpha_n$  is an immediate consequence of [GLM18, Theorem 3.20]. The estimate for  $\beta_n$  is an immediate consequence of [GLM18, Prop. 3.17]. Note that [GLM18, Prop. 3.17] is only stated for a hyperbolic metric, but it can be extended to any Riemannian metric of curvature bounded below, which is always the case on a compact manifold.  $\square$

*Remark 4.18.* The estimate  $\alpha_n = \Omega(r_n)$  based on [GLM18, Theorem 3.20] only assumes that  $N$  has nonpositive sectional curvature. When  $N$  has negative curvature (bounded away from zero by compactness), we expect that a better bound  $\alpha_n = \Omega(r_n^c)$  with  $c < 1$  is possible to achieve, in fact we conjecture that  $\alpha_n = \Omega(1)$ .

As a consequence of Lemma 4.17 and the previous theorems of this section, we obtain the following theorem for surfaces.

**Theorem 4.19.** *If the sequence of meshes  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  is asymptotically Laplacian, then the sequence of discrete harmonic maps interpolates  $(\widehat{v}_n)_{n \in \mathbb{N}}$  converges to the unique harmonic map  $w \in C$  uniformly, hence also in  $L^2(M, N)$ , and  $E(w) = \lim_{n \rightarrow +\infty} E_n(v_n)$ .*

*Furthermore, the discrete heat flow  $(\widehat{u}_{k,n})_{k,n \in \mathbb{N}}$  from any initial condition  $u \in C$  converges to  $w$  in  $L^2(M, N)$  when both  $k, n \rightarrow +\infty$ , provided the CFL condition  $k = \Omega(\log(r_n^{-1})r_n^{-3})$  holds.*

*If the sequence  $(\mathcal{M}_n)$  is merely  $(1, 0)$ -almost asymptotically Laplacian, and we have the improved convexity estimate  $\alpha_n = \Omega(r_n^c)$  with  $c < \frac{1}{2}$  (instead of  $c = 1$ ), then the convergence of  $(v_n)$  to  $w$  in  $L^2(M, N)$  still holds as well as the convergence of the discrete heat flow  $(\widehat{u}_{k,n})$ .*

*Remark 4.20.* Theorem 4.19 could be considered one of the main results of both our previous paper [GLM18] and the present paper combined, except for the fact that we have yet to produce such sequences of meshes on surfaces. This is the goal of the final section § 5. As we shall see, in general we only produce a  $(0, 1)$ -almost asymptotically Laplacian sequence of meshes, and we have not been able to prove that  $c < \frac{1}{2}$  holds (although we believe it is true, see Remark 4.18), so that we unfortunately fall just short of the assumptions of Theorem 4.19.

## 5 Construction of Laplacian sequences

Most of our convergence theorems in § 3 and § 4 require having a Laplacian sequence of meshes (Definition 3.18), or one of the weaker variants (Definition 3.24, Definition 3.25). Indeed, one should only expect convergence for weighted graphs that reasonably capture the geometry of  $M$ .

In this section, we construct a sequence of weighted meshes on any Riemannian surface and prove that it is always almost asymptotically Laplacian, and discuss cases where more can be said. This construction is very explicit: in fact, it is implemented in our software `Harmony` in the case of hyperbolic surfaces. The construction can simply be described: take a sequence of meshes obtained by midpoint subdivision (§ 1.2) and equip it with the volume vertex weights (§ 2.2) and the cotangent weights (§ 2.3).

*Remark 5.1.* It is possible to generalize this construction to higher-dimensional manifolds: see [Cra19] for a formula for the cotangent weights in higher dimensions.

## 5.1 Description

Let  $S = M$  be a 2-dimensional compact Riemannian manifold. One could consider complete metrics with punctures and/or geodesic boundary, but for simplicity we assume  $S$  is closed.

Consider a sequence of meshes  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  with underlying graphs  $(\mathcal{G}_n)_{n \in \mathbb{N}}$  defined by:

- $\mathcal{M}_0$  is any acute triangulation.
- $\mathcal{M}_{n+1}$  is obtained from  $\mathcal{M}_n$  by midpoint subdivision (see § 1.2).

Furthermore, equip  $\mathcal{G}_n$  with the volume vertex weights (§ 2.2) and the cotangent weights (§ 2.3).

*Remark 5.2.* Finding an initial triangulation of  $S$  that is acute is far from an easy task, even for a flat surface. The reader may refer to [Zam13] for more background on this active subject.

**Definition 5.3.** A  $\Delta$ -sequence is a sequence of meshes  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  with the associated biweighted graphs  $(\mathcal{G}_n)_{n \in \mathbb{N}}$  constructed as above.

*Remark 5.4.* We think of “ $\Delta$ ” here as standing for either “Laplacian” or “simplex”.

## 5.2 Angle properties

In order for  $\Delta$ -sequences to be crystalline and have reasonable edge weights systems, we need to address some questions about the behavior of angles when iterating midpoint subdivision:

- (1) Do all the angles of the triangulation remain bounded away from zero?
- (2) Do all angles remain acute?
- (3) Do all angles remain bounded away from  $\frac{\pi}{2}$ ?

These questions, which are surprisingly hard to answer, are crucial since: (1) is necessary and sufficient for the sequence of meshes to be crystalline (see Proposition 3.4), (2) is sufficient for the edge weights to remain positive, and (3) is necessary for the ratio of any two edge weights to remain uniformly bounded, a requirement to apply Theorem 4.19 (see § 4.6).

**Lemma 5.5.** *Let  $(M, g)$  be a compact Riemannian manifold of dimension  $m$ . Let  $(\Delta_n)_{n \in \mathbb{N}}$  be a sequence of simplices with geodesic edges such that for every  $n \in \mathbb{N}$ ,  $\Delta_{n+1}$  is one of the  $2^m$  simplices obtained from  $\Delta_n$  by midpoint subdivision. Then all edge lengths of  $\Delta_n$  are  $\Theta(2^{-n})$ .*

*Proof.* To avoid burdening our presentation with technical Riemannian geometry estimates, we postpone this proof to the appendix: see Proposition A.11 in § A.2.  $\square$

Now we can answer yes to question (1):

**Theorem 5.6.** *Let  $(M, g)$  be a compact Riemannian manifold. Any sequence of meshes  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  obtained by geodesic subdivision is fine and crystalline.*

*Proof.* By compactness  $\mathcal{M}_0$  has a finite number of simplices, therefore it is sufficient to show the theorem for the iterated refinements of a single simplex. By [Proposition 3.4](#) it is enough to show that the ratio of any two edge lengths of  $\mathcal{M}_n$  is uniformly bounded. This follows from [Lemma 5.5](#).  $\square$

The answer to questions (2) and (3) is more nuanced: it is not true that refinements of an acute triangulation stay acute, even for fine triangulations in  $\mathbb{H}^2$ . However, refinements of a sufficiently fine and sufficiently acute triangulation do remain acute with angles bounded away from  $\frac{\pi}{2}$ .

**Theorem 5.7.** *Let  $(M, g)$  be a compact Riemannian manifold. Let  $\delta > 0$ . The iterated refinements of any sufficiently fine initial triangulation of  $M$  whose angles are all  $\leq \frac{\pi}{2} - \delta$  remain acute and with angles bounded away from  $\frac{\pi}{2}$ .*

*Remark 5.8.* In [Theorem 5.7](#), we are only referring to angles between adjacent edges of the triangulation.

*Proof.* Again, since  $\mathcal{M}_0$  has a finite number of simplices, it is sufficient to show the theorem for the iterated refinements of a single simplex. We postpone this proof to the appendix: see [Proposition A.13](#) in [§ A.2](#).  $\square$

We say a sequence of acute triangulations is *strongly acute* if the angles remain uniformly bounded away from  $\frac{\pi}{2}$ . In particular, any sequence of triangulations obtained from iterated refinement as in [Theorem 5.7](#) is strongly acute.

**Proposition 5.9.** *Let  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  be a strongly acute  $\Delta$ -sequence in  $(S, g)$ . All edge weights are  $\Theta(1)$ .*

*Proof.* By definition of the cotangent weights, all edge weights are  $O(1)$ , since all angles are bounded away from 0 by [Theorem 5.6](#) and [Theorem 3.5](#). They are also all  $\Omega(1)$  since all angles are bounded away from  $\frac{\pi}{2}$  by definition of strongly acute.  $\square$

### 5.3 Laplacian qualities

Let  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  be a  $\Delta$ -sequence in  $(S, g)$ , denote  $(\mathcal{G}_n)_{n \in \mathbb{N}}$  the underlying graphs.

**Definition 5.10.** Let  $x \in \mathcal{V}_n \subseteq S$  be a vertex of  $\mathcal{G}_n$ . We call  $x$  a *boundary vertex* if it belongs to an edge of the initial triangulation  $\mathcal{M}_0$ , and we call  $x$  an *interior vertex* otherwise.

It is immediate to check by induction that there are  $\Theta(2^n)$  boundary vertices, i.e.  $\Theta(r_n^{-1})$ , on a total of  $\Theta(2^{2n}) = \Theta(r_n^{-2})$  vertices. Since the sequence of meshes is fine and crystalline, the vertex weights in this setting are all  $\Theta(r_n^2)$  ([Theorem 3.5](#)). We thus find:

**Lemma 5.11.** *The set of boundary vertices is  $O(r)$ -negligible.*

In [§ 2.3](#) we have already studied Laplacian properties of a fixed mesh in the Euclidean setting. In terms of  $\Delta$ -sequences, we have:

**Theorem 5.12.** *Let  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  be a  $\Delta$ -sequence in a Euclidean (flat) surface  $(S, g)$ , denote  $(\mathcal{G}_n)_{n \in \mathbb{N}}$  the underlying graphs. Then for every  $n \in \mathbb{N}$ ,  $\mathcal{G}_n$  is Laplacian to first order at any vertex. Moreover,  $\mathcal{G}_n$  is Laplacian at interior vertices, and Laplacian up to  $O(1)$  at boundary vertices. In particular, the sequence is  $(1, 0)$ -almost asymptotically Laplacian.*

*Proof.* The only claim left to prove is that  $\mathcal{G}_n$  is Laplacian up to  $O(1)$  at boundary vertices. The first-order condition is satisfied since  $\mathcal{G}_n$  is Laplacian to first order at all vertices. Now we examine the quadratic condition. Let

$$\varepsilon_n := \max_{x \in \mathcal{V}_n} \sup_{L \in \mathcal{L}(\mathbb{R}^2, \mathbb{R})} \varepsilon_{n,x,L}$$

where

$$\varepsilon_{n,x,L} := \frac{1}{\|L\|^2} \frac{1}{\mu_{x,n}} \sum_{y \sim x} \omega_{xy,n} L(\vec{xy})^2$$

By definition, we win if we show that  $\varepsilon_n = O(1)$ . Let us show that  $(\varepsilon_n)$  is actually constant. Let  $x \in S$  be a vertex of  $\mathcal{G}_{n+1}$ . It is easy to see that either  $x$  is also a vertex of  $\mathcal{G}_n$ , or  $x$  has a neighbor  $x'$  which is a vertex of  $\mathcal{G}_n$  and such that there exists a translation taking  $x$  and its neighbors to  $x'$  and its neighbors. In the latter case,  $\mathcal{G}_n$  has the same weights at  $x'$  as at  $x$ , so that  $\varepsilon_{n,x',L} = \varepsilon_{n,x,L}$ . It is therefore sufficient to consider the case where  $x$  is also a vertex of  $\mathcal{G}_n$ . By considering the normal chart at  $x$ , we can assume  $S = \mathbb{R}^2$  and  $x = 0$ . Denote by  $y_1, \dots, y_k \in \mathbb{R}^2$  the neighbors of  $x$  in  $\mathcal{G}_n$ . Then the neighbors of  $x$  in  $\mathcal{G}_{n+1}$  are simply  $y_1/2, \dots, y_k/2$ . One can then easily calculate that the weights at  $x$  in  $\mathcal{G}_{n+1}$  are  $\mu_{x,n+1} = \mu_{x,n}/4$  and  $\omega_{xy_j,n+1} = \omega_{xy_j,n}$ . It easily follows  $\varepsilon_{n+1,x,L} = \varepsilon_{n,x,L}$  for any  $L \in \mathcal{L}(\mathbb{R}^2, \mathbb{R})$ . We conclude that  $\varepsilon_{n+1} = \varepsilon_n$ .

For the cubic condition, proceed likewise until the end of the proof, where we find  $\varepsilon_{n+1,x,L} = \varepsilon_{n,x,L}/2$ , therefore  $\varepsilon_{n+1} = \varepsilon_n/2$ . Thus  $\varepsilon_n = 2^{-n} \varepsilon_0$  is clearly bounded.  $\square$

*Remark 5.13.* The proof of [Theorem 5.12](#) shows that the cubic condition actually holds up to  $O(r)$  everywhere. We would similarly find that the order 4 condition holds up to  $O(r^2)$ , etc. The same argument gives  $O(r^{-1})$  for the first-order condition, which is not good enough. It is actually the case that  $\varepsilon_n = 2^n \varepsilon_0$ , but we are saved by the fact that  $\varepsilon_0 = 0$  is always true: it is a small miracle that Euclidean triangulations with cotangent weights are Laplacian to first order ([Proposition 2.11](#)).

*Example 5.14.* If  $S$  admits a triangulation  $\mathcal{M}_0$  that is hexaparallel at every vertex, then the  $\Delta$ -sequence built from  $\mathcal{M}_0$  is an actual Laplacian sequence. For example, if  $S$  is a flat torus—the only closed example—one can take a triangulation of  $S$  by six isometric triangles with three hexaparallel vertices. Precisely, every flat torus can be formed by gluing a hexagon in the  $\text{GL}(2, \mathbb{R})$ -orbit of  $\{\pm(1, 0), \pm(1, 1), \pm(0, 1)\}$ , and a hexaparallel triangulation is formed when a vertex is added in the center, along with six edges to the vertices of the hexagon.

Let us now examine the general Riemannian case, where  $(S, g)$  is not necessarily flat. The bottom line is that by using fine Riemannian geometry estimates, one can show that the same results as in the Euclidean case hold asymptotically. Specifically, we are going to prove the theorem:

**Theorem 5.15.** *Any strongly acute  $\Delta$ -sequence in a Riemannian surface  $(S, g)$  is  $(1, 0)$ -almost asymptotically Laplacian.*

Let us break down the proof into two lemmas, distinguishing between all vertices (including boundary vertices) and interior vertices. We denote by  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  the  $\Delta$ -sequence and  $(\mathcal{G}_n)$  the associated biweighted graphs.

**Lemma 5.16.**  *$\mathcal{G}_n$  is Laplacian up to  $O(1)$  at any vertex  $x$ .*

*Proof.* The second and third order condition come easily with crude estimates. For instance, for the second order condition, we need to show that

$$\frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} L(\vec{xy})^2 = \|L\|^2 O(1)$$

for any linear form  $L$ . This is obtained quickly by writing  $|L(\vec{xy})| \leq \|L\| \|\vec{xy}\|$ , and recalling that  $\|\vec{xy}\| = O(r)$ ,  $\mu_x = \Theta(r^2)$  (Theorem 3.5), and  $\omega_{xy} = \Theta(1)$  (Proposition 5.9). We similarly find that the third condition holds up to  $O(r)$  (notice this is better than the required  $O(1)$ ).

For the first order condition, we use some Riemannian estimates, but it comes easily. Let us work in the normal chart at  $x$ . Look at the subgraph consisting of  $x$  and its neighbors in  $T_x M$ , and consider its Euclidean analog (same vertices, edges “straightened up” in  $T_x M$ ). Denote by  $\omega_{xy}^E$  the Euclidean cotangent weights. Standard Riemannian estimates give  $\omega_{xy} = \omega_{xy}^E + O(r^2)$  (Proposition A.4). Now write

$$\frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} \vec{xy} = \frac{1}{\mu_x} \sum_{y \sim x} (\omega_{xy} - \omega_{xy}^E) \vec{xy} + \frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy}^E \vec{xy}$$

On the right-hand side, the first sum is  $O(r)$ , since  $\omega_{xy} - \omega_{xy}^E = O(r^2)$ ,  $\|\vec{xy}\| = O(r)$ , and  $\mu_x = \Theta(r^2)$ . The second sum is zero by Proposition 2.11. We thus find that the first order condition holds up to  $O(r)$  (notice this is better than the required  $O(1)$ ).  $\square$

**Lemma 5.17.**  $\mathcal{G}_n$  is Laplacian up to  $O(r^2)$  at any interior vertex  $x$ .

*Proof.* When  $x$  is an interior vertex, the graph is “almost” hexaparallel at  $x$ , so we should be able to conclude by comparing to the Euclidean situation using the Riemannian estimates of Appendix A. Let us make this more precise. By construction, there exists a triangle  $ABC$  in  $\mathcal{G}_{n-2}$ , whose midpoints  $I := m(A, B)$ ,  $J := m(B, C)$ ,  $K := m(C, A)$  are vertices of  $\mathcal{G}_{n-1}$ , such that  $x$  is the midpoint of two midpoints, e.g.  $x := m(J, K)$ . The neighbors of  $x$ , in cyclic order, are then  $J$ ,  $S := m(J, C)$ ,  $P := m(C, K)$ ,  $K$ ,  $Q := m(K, I)$ ,  $R := m(I, J)$ . Please refer to Figure 6.

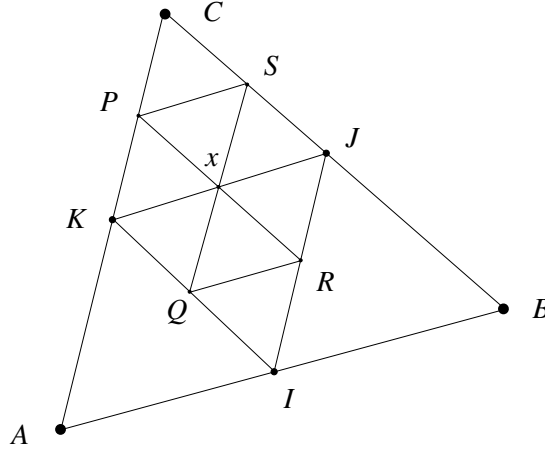


Figure 6: Interior vertex  $x$

Let us work in the normal chart at  $x$ . We thus consider the points  $A, B, C$  in the Euclidean plane  $T_x M$ . Recreate the Euclidean analog of our configuration by taking the second order refinement of  $ABC$  in the Euclidean plane  $T_x M$ . In other words, take the Euclidean midpoints  $I_E := m_E(A, B) = \frac{A+B}{2}$ ,  $J_E := m_E(B, C)$ , etc.

Observe that by construction, the Euclidean analog  $x_E$  of  $x$  has hexaparallel symmetry, so that  $x$  satisfies the Laplacian conditions *exactly*, provided one uses the Euclidean volume vertex weight  $\mu_{x_E}$  and the Euclidean cotangent weights  $\omega_{x_E y_E}$ , by Lemma 2.13.

On the other hand, the repeated application of Proposition A.7 clearly yields that all points under consideration are within  $O(r^3)$  of their Euclidean counterparts. Using Proposition A.6, it follows that the vectors  $\vec{x_E y_E}^E$  from  $x$  to its neighbors each are within  $O(r^3)$  of their Euclidean counterparts. Similarly, using Proposition A.4 we find that all weights are within  $O(r^2)$  of their Euclidean



counterparts (note that we have to use the fact that the sequence is crystalline and strongly acute to apply [Proposition A.4](#)). Finally, we derive from [Proposition A.10](#) that  $\mu_x = \mu_{x_E} (1 + O(r^2))$ . From these estimates one quickly sees that the second-order condition holds up to  $O(r^2)$ : write

$$\begin{aligned} \frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} L(\vec{x}\vec{y})^2 &= \frac{1}{\mu_x} \sum_{y \sim x} (\omega_{xy} - \omega_{xy}^E) L(\vec{x}\vec{y})^2 + \frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy}^E L(\vec{x}\vec{y} - \vec{x}_E\vec{y}_E) L(\vec{x}\vec{y} + \vec{x}_E\vec{y}_E) \\ &\quad + \frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy}^E L(\vec{x}_E\vec{y}_E)^2 \end{aligned}$$

By examining each factor, we see that the and second terms  $\|L\|^2 O(r^2)$ . As for the third term, it is rewritten

$$\begin{aligned} \frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy}^E L(\vec{x}_E\vec{y}_E)^2 &= \frac{\mu_{x_E}}{\mu_x} \frac{1}{\mu_{x_E}} \sum_{y \sim x} \omega_{xy}^E L(\vec{x}_E\vec{y}_E)^2 \\ &= \frac{\mu_{x_E}}{\mu_x} 2\|L\|^2 \\ &= 2\|L\|^2 (1 + O(r^2)) \end{aligned}$$

where we have used the second-order Laplacian condition at a hexaparallel vertex in the Euclidean plane. Gathering all three sums, we find

$$\frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} L(\vec{x}\vec{y})^2 = 2\|L\|^2 (1 + O(r^2))$$

as desired. One proceeds similarly to check the third-order Laplacian condition.

Finally, it remains to prove the first-order condition, which can be stated:

$$\frac{1}{\mu_x} \sum_{y \sim x} \omega_{xy} \vec{x}\vec{y} = O(r^4).$$

Again, the strategy is to compare this sum to the Euclidean analog, which we know is zero. Unfortunately, proceeding as above with second-order expansions is not sufficient, we would only find  $O(r^3)$ . We thus need to keep track of the third-order terms when using the Riemannian estimates [Proposition A.7](#), [Proposition A.6](#), and [Proposition A.4](#). This is a tedious computation but it succeeds: the order-three terms end up cancelling out due to the symmetry of the configuration of points. We spare the lengthy details.  $\square$

## A Riemannian estimates

Many proofs in this paper can be summarized in two steps: First, the claim is shown to be true in the Euclidean (flat) setting, by direct proof. Subsequently, it is also true in the Riemannian setting on first approximation (e.g., provided the mesh is fine). The moral justification for the second step is that locally, a Riemannian manifold looks Euclidean. Of course one should not take this aphorism too seriously, since there are local Riemannian invariants such as curvature. In some cases, one can make this type of proof rigorous with a soft argument using only first-order approximation. In others, one should be more cautious and examine the next order terms, which involve curvature.

A standard way to compute estimates in Riemannian geometry is to write Taylor expansions in normal coordinates, i.e. using the exponential map at some point as a chart, and picking an

orthonormal basis of the tangent space to have an  $n$ -tuple of coordinates. For example, the Taylor expansion of the Riemannian metric in normal coordinates reads

$$g_{ij} = \delta_{ij} - \frac{1}{3} R_{ijkl} x^k x^l + O(r^3) \quad (19)$$

where  $R_{ijkl}$  is the Riemann curvature tensor. This foundational fact of Riemannian geometry goes back to Riemann's 1854 habilitation [Rie13]. From this estimate, many other geometric quantities can be similarly approximated: distances, angles, geodesics, volume, etc.

In § A.1, we establish Riemannian estimates of the most relevant geometric quantities. These are used implicitly or explicitly throughout the paper, especially § 5.3. In § A.2, we study iterated midpoint subdivisions of a simplex in a Riemannian manifold, proving two key lemmas for § 5.2.

## A.1 Riemannian expansions in a normal chart

Let  $(M, g)$  be a Riemannian manifold and let  $x_0 \in M$ . We consider the normal chart given by the exponential map  $\exp_{x_0}: T_{x_0}M \rightarrow M$ , which is well-defined and a diffeomorphism near the origin. We do not favor the unnecessary introduction of local coordinates, so we will abstain from choosing an orthonormal basis of  $T_{x_0}M$  (in other words fixing an identification  $T_xM \approx \mathbb{R}^m$ ), and instead work in the Euclidean vector space  $(T_{x_0}M, \langle \cdot, \cdot \rangle_E)$  where the inner product  $\langle \cdot, \cdot \rangle_E$  is just  $g_x$ .

We implicitly identify objects in  $M$  and in  $T_{x_0}M$  via the exponential map  $\exp_{x_0}$ , e.g.  $x_0 = 0$ , and tangent vectors to some point  $x \in M$  to vectors (or points) in  $T_{x_0}M$  via the derivative of the exponential map. Let  $r > 0$ . In what follows, all points considered (typically denoted  $x, A, B$ ) are within distance  $O(r)$  of  $x_0$ . With this setup, (19) is written:

**Theorem A.1** (Second-order expansion of the metric.). *Let  $u, v$  be tangent vectors at some point  $x \in M$ . Then*

$$\langle u, v \rangle = \langle u, v \rangle_E - \frac{1}{3} \langle R(u, x)x, v \rangle_E + O\left(r^3 \|u\|_E \|v\|_E\right).$$

where  $R$  is the Riemann curvature tensor at  $x_0 = 0$ .

Note that when writing  $R(u, x)x$ , we think of the point  $x$  as an element of  $T_{x_0}M$ . From this fundamental estimate, it is elementary to show the following series of estimates.

**Proposition A.2** (Second-order expansion of the norm).

$$\begin{aligned} \|u\|^2 &= \|u\|_E^2 - \frac{1}{3} \langle R(u, x)x, u \rangle + O\left(r^3 \|u\|^2\right) \\ \|u\| &= \|u\|_E - \frac{1}{6} \frac{\langle R(u, x)x, u \rangle}{\|u\|_E^2} + O\left(r^3\right) \end{aligned}$$

**Proposition A.3** (Second-order expansion of cosine).

$$\cos \angle(u, v) = \cos \angle_E(u, v) \left[ 1 + \frac{\langle R(u, x)x, u \rangle_E}{6\|u\|_E^2} + \frac{\langle R(v, x)x, v \rangle_E}{6\|v\|_E^2} - \frac{\langle R(u, x)x, v \rangle_E}{3\langle u, v \rangle_E} + O\left(\frac{r^3}{\cos \angle_E(u, v)}\right) \right]$$

The previous proposition implies the less accurate estimates:

**Proposition A.4** (First-order expansions of angles).

$$\cos \angle(u, v) = \cos \angle_E(u, v) + O\left(r^2\right)$$

If  $\angle(u, v)$  (equivalently  $\angle_E(u, v)$ ) is bounded away from 0 and  $\frac{\pi}{2}$  modulo  $\pi$ , then

$$\begin{aligned}\sin \angle(u, v) &= \sin \angle_E(u, v) + O(r^2) \\ \cot \angle(u, v) &= \cot \angle_E(u, v) + O(r^2)\end{aligned}$$

Let  $A, B$  be points in our normal chart: they can either be thought of as elements of  $M$  or  $T_{x_0}M$ . We denote as usual  $\overrightarrow{AB}$  the vector  $\exp_A^{-1}(B)$ , which is an element of  $T_A M$ , or of  $T_{x_0}M$  via our chart. We also denote  $\overrightarrow{AB}^E$  the Euclidean vector  $B - A \in T_{x_0}M$ .

**Proposition A.5** (Geodesic through two points). *Let  $\gamma$  be the geodesic with  $\gamma(0) = A$  and  $\gamma(1) = B$ .*

$$\gamma(t) = \gamma_E(t) + \frac{t(t-1)}{3} R(A, B) \overrightarrow{AB}^E + O(tr^4)$$

**Proposition A.6** (Vector between two points).

$$\overrightarrow{AB} = \overrightarrow{AB}^E + \frac{1}{3} R(A, B) \overrightarrow{AB}^E + O(r^4)$$

**Proposition A.7** (Midpoint). *Let  $I$  denote be the midpoint of midpoint of  $A$  and  $B$  in  $M$ , and let  $I_E = \frac{A+B}{2}$  denote their Euclidean midpoint in  $T_{x_0}M$ .*

$$I = I_E + \frac{1}{12} R(A, B) \overrightarrow{AB}^E + O(r^4)$$

**Proposition A.8** (Distance between two points).

$$d(A, B)^2 = d_E(A, B)^2 - \frac{1}{3} \langle R(B, A) A, B \rangle + O(r^5)$$

*Remark A.9.* Note that  $\langle R(B, A) A, B \rangle = K \|B \wedge A\|^2$  where  $K$  is the sectional curvature at  $x_0 = 0$ . In particular, we see from [Proposition A.8](#) that  $d > d_E$  near  $x_0$  if and only if  $M$  has negative sectional curvature at  $x_0$ , which should be expected.

Finally, we recover the well-known expansion of the volume density:

**Proposition A.10** (Volume density). *The volume density at  $x$  is given by*

$$v_g(x) = v_E \left( 1 - \frac{\text{Ric}(x, x)}{6} + O(r^3) \right)$$

where  $v_E$  is the Euclidean volume density in  $T_x M$  and  $\text{Ric}$  is the Ricci curvature tensor at  $x_0$ .

## A.2 Iterated subdivision of a simplex

In this subsection, we estimate the edge lengths and angles in the iterated midpoint subdivision (see [§ 1.2](#)) of a simplex in a Riemannian manifold. We prove two propositions, which are the key to [Theorem 5.6](#) and [Theorem 5.7](#) respectively.

**Proposition A.11.** *Let  $(M, g)$  be a compact Riemannian manifold of dimension  $m$ . Let  $(\Delta_n)_{n \in \mathbb{N}}$  be a sequence of simplices with geodesic edges such that for every  $n \in \mathbb{N}$ ,  $\Delta_{n+1}$  is one of the  $2^m$  simplices obtained from  $\Delta_n$  by midpoint subdivision. Then all edge lengths of  $\Delta_n$  are  $\Theta(2^{-n})$ .*

*Remark A.12.* In [Proposition A.11](#), the  $\Theta(2^{-n})$  function is uniform in the choice of the sequence  $(\Delta_n)$ : more precisely, there exists constants  $C_1, C_2 > 0$  depending only  $(M, g)$  such that any edge length  $x_n$  of the triangulation obtained by  $n$ -th refinement of  $\Delta_0$  satisfies  $C_1 2^{-n} \leq x_n \leq C_2 2^{-n}$ .

*Proof.* For comfort, we write the proof when  $\dim M = 2$ , but it works in any dimensions. We thus have a sequence of geodesic triangles  $\Delta_n$  in a Riemannian surface  $(S, g)$ . Choose a labelling of the side lengths of  $\Delta_n$  by  $a_n, b_n, c_n$ . Given the labelling of  $\Delta_0$ , there is a unique sensible way to do this for all  $n$  so that  $\Delta_{n+1}$  is “similar” to  $\Delta_n$ . For instance, in the Euclidean setting, one should have  $a_n = 2^{-n} a_0$ , etc. In order to show that  $a_n, b_n$ , and  $c_n$  are  $\Theta(2^{-n})$ , we would like to use Riemannian estimates, but we must first show that  $\text{diam}(\Delta_n)$  converges to zero.

Let us prove the stronger claim that  $r_n \rightarrow 0$ , where  $r_n$  is the maximum edge length of whole triangulation obtained by  $n$ -th refinement of  $\Delta_0$ . Notice that  $(r_n)$  is nonincreasing: this follows easily from the triangle inequality in each simplex. Moreover  $r_n < r_{n+1}$  unless one of the simplices is reduced to a point, which cannot happen unless  $\Delta_0$  is a point. One can conclude that  $r_n \rightarrow 0$  by compactness: if not, we could find a converging sequence of simplices with diameter bounded below, etc.

Now we can use the estimates of [§ A.1](#). It is not hard to derive from [Proposition A.7](#) and [Proposition A.8](#) that

$$|a_n - 2a_{n+1}| = O(r_n^3) \quad (20)$$

and we have similar estimates for  $b_n$  and  $c_n$ . This means that there exists a constant  $B > 0$  such that for all  $n$  sufficiently large,  $|a_n - 2a_{n+1}| \leq B r_n^3$ . Applying this inequality repeatedly, we find

$$\begin{aligned} |a_n - 2^k a_{n+k}| &= |(a_n - 2a_{n+1}) + 2(a_{n+1} - 2a_{n+2}) + \dots + 2^{k-1}(a_{n+k-1} - 2a_{n+k})| \\ &\leq B \left( r_n^3 + 2r_{n+1}^3 + \dots + 2^{k-1} r_{n+k-1}^3 \right). \end{aligned}$$

Now, note that  $r_n$  must satisfy the same inequality [\(20\)](#), so in particular

$$2r_{n+1} \leq r_n + B r_n^2 \leq C r_n$$

for any constant  $C > 1$  chosen in advance, provided  $n$  is sufficiently large. Therefore we obtain

$$|a_n - 2^k a_{n+k}| \leq B r_n^3 \left( 1 + \frac{C^3}{2} + \dots + \left( \frac{C^3}{2} \right)^{k-1} \right).$$

Provided we chose  $1 < C^3 < 2$ , the sum  $1 + \frac{C^3}{2} + \dots + \left( \frac{C^3}{2} \right)^{k-1}$  is bounded, as a truncated convergent geometric series. In particular, we find that the sequence  $(2^k a_{n+k})_{k \in \mathbb{N}}$  is bounded, in other words  $a_{n+k} = O(2^{-k})$ . Of course this is the same as saying that  $a_n = O(2^{-n})$ . We similarly show the other inequality  $a_n = \Omega(2^{-n})$ , and conclude that  $a_n = \Theta(2^{-n})$ . Obviously, the same argument works for  $(b_n)$  and  $(c_n)$ .

Note that the claim of [Remark A.12](#) is justified by the fact that the sequence  $(r_n)$  and the constant  $C$  are independent of the choice of the sequence  $(\Delta_n)$ .  $\square$

**Proposition A.13.** *Let  $(M, g)$  be a compact Riemannian manifold of dimension  $m$ . Let  $\delta > 0$ . There exists  $R > 0$  and  $\eta > 0$  such that the following holds. Let  $(\Delta_n)_{n \in \mathbb{N}}$  be a sequence of simplices with geodesic edges where for every  $n \in \mathbb{N}$ ,  $\Delta_{n+1}$  is one of the  $2^m$  simplices obtained from  $\Delta_n$  by midpoint subdivision. If the longest edge length of  $\Delta_0$  is  $\leq R$  and all angles of  $\Delta_0$  are  $\leq \frac{\pi}{2} - \delta$ , then all angles of  $\Delta_n$  are  $\leq \frac{\pi}{2} - \eta$  for all  $n \in \mathbb{N}$ .*

*Proof.* We have seen in [Proposition A.13](#) that the diameter of  $\Delta_n$  is  $\leq r_n$ , with  $r_n = \Theta(2^{-n})$ . In particular,  $r_n \rightarrow 0$  so we can use the Riemannian estimates of [§ A.1](#).

Label  $\alpha_n$ ,  $\beta_n$ , and  $\gamma_n$  the angles of  $\Delta_n$ . Of course, one should do this labelling in the only sensible way: for instance in the Euclidean setting we should have  $\alpha_n = \alpha_{n+1}$ , etc. It is not hard to derive from [Proposition A.7](#) and [Proposition A.4](#) that for all  $n \in \mathbb{N}$ ,

$$\cos \alpha_{n+1} = \cos \alpha_n + O(r_n)^2,$$

in other words there exists a constant  $C$  depending only on  $(M, g)$  such that

$$|\cos \alpha_{n+1} - \cos \alpha_n| \leq Cr_0 2^{-2n}.$$

Using a telescopic sum, we find that

$$\begin{aligned} |\cos \alpha_n - \cos \alpha_0| &\leq \sum_{k=0}^{n-1} |\cos \alpha_{k+1} - \cos \alpha_k| \\ &\leq Cr_0 \sum_{k=0}^{n-1} 2^{-2k} \leq Cr_0 \sum_{k=0}^{\infty} 2^{-2k} = Cr_0 \frac{4}{3}. \end{aligned}$$

We therefore have the bound

$$\cos \alpha_n \geq \cos \alpha_0 - C'r_0$$

where  $C' = 4C/3$ . By assumption,  $\cos \alpha_0 \geq \cos(\pi/2 - \delta) = \sin(\delta)$ . Clearly  $\cos \alpha_n$  is bounded away from zero if  $r_0$  is sufficiently small, for instance  $r_0 \leq \frac{\sin \delta}{2C'}$  yields  $\cos \alpha_n \geq \frac{\sin \delta}{2}$ . It follows that  $\alpha_n$  is bounded away from  $\pi/2$ .  $\square$

## B Comparing the discrete $L^2$ and $L^\infty$ distances

Let  $M$  be compact Riemannian manifold, let  $N$  be a complete Riemannian manifold of nonpositive sectional curvature. Let  $\mathcal{M}$  be a mesh on  $M$  and equip the underlying graph  $\mathcal{G}$  with vertex weights  $(\mu_x)_{x \in \mathcal{V}}$  and  $(\omega_{xy})_{x \sim y}$ . Recall the  $L^2$  distance on the space of discrete maps  $\text{Map}_{\mathcal{G}}(M, N)$ :

$$d(u, v)^2 = \sum_{x \in \mathcal{V}} \mu_x d(u(x), v(x))^2$$

while the  $L^\infty$  distance is

$$d_\infty(u, v) = \max_{x \in \mathcal{V}} d(u(x), v(x)).$$

Clearly, these distances satisfy the inequality  $md_\infty^2 \leq d^2 \leq Vd_\infty^2$ , where  $m := \min_{x \in \mathcal{V}} \mu_x$  is the minimum vertex weight and  $V := \sum_{x \in \mathcal{V}} \mu_x$  is the sum of the vertex weights. Typically,  $V$  is equal to  $\text{Vol}(M)$  or asymptotic to it for a fine mesh, so the second inequality is fairly robust. On the other hand, the first inequality  $md_\infty^2 \leq d^2$ , which we rewrite

$$d_\infty(u, v) \leq m^{-1/2} d(u, v), \tag{21}$$

is less attractive since typically  $m \rightarrow 0$  for a fine mesh. This should be expected though, as the  $L^2$  and  $L^\infty$  distances are not equivalent on the space of continuous maps  $M \rightarrow N$ . The goal of this section is to find an improvement of (21) when  $v$  is a discrete harmonic map. This allows us to upgrade [Theorem 4.7](#) to [Theorem 4.8](#).

**Proposition B.1.** Let  $\mathcal{G}$  be a biweighted graph and  $N$  be a complete Riemannian manifold of nonpositive sectional curvature. Let  $v \in \text{Map}_{\mathcal{G}}(M, N)$  be a minimizer of the discrete energy. Denote by  $r$  the maximum edge length in  $\mathcal{G}$ ,  $V$  the maximum valence of a vertex,  $m = \min_{x \in \mathcal{V}} \mu_x$  the smallest vertex weight, and  $\omega = \frac{\omega_{\max}}{\omega_{\min}}$  the ratio of the largest and smallest edge weights. Let  $L > 0$ . There exists constants  $A = A(\omega) > 0$  and  $B = B(\omega, L) \in \mathbb{R}$  such that for any  $L$ -Lipschitz map  $u \in \text{Map}_{\mathcal{G}}(M, N)$ :

$$d_{\infty}(u, v) \leq \max \left\{ (Cm)^{-1/2} d(u, v), r^{1/2} \right\}$$

with  $C := \min (A \log (r^{-1}) + B, \text{inj } \mathcal{G} - 1)$ .

We recall that the combinatorial injectivity radius  $\text{inj } \mathcal{G}$  is defined below [Theorem 3.5](#).

*Proof.* Let  $\rho := Lr$ . Notice that  $\rho$  is an upper bound for the length of any edge in  $N$  that is the image of an edge of  $\mathcal{G}$  by an  $u$ . [Proposition B.1](#) is a consequence of the following ‘‘bootstrapping’’ lemma: if some distance  $d(u(x), v(x))$  is large, then  $d(u(y), v(y))$  will also be large, for many vertices  $y$  that are near  $x$ . More precisely:

**Lemma B.2.** Let  $x_0$  be a vertex which achieves  $d_{\infty}(u, v) =: D$ . Let  $K$  be given by

$$K := \min \left\{ \left\lfloor \log_{\delta}(D/\rho) \right\rfloor, \text{inj}(\mathcal{G}) \right\},$$

where  $\delta = 2(1 + \omega V)$ . For each  $k = 1, 2, \dots, K$  there exists a vertex  $x_k$  satisfying:

- (1) The combinatorial distance in  $\mathcal{G}_n$  is given by  $d_{\mathcal{G}_n}(x, x_k) = k$ , and
- (2)  $d(u(x_k), v(x_k)) \geq D - \delta^{k-1}\rho$ .

*Remark B.3.* The  $\log$  above is the cutoff function  $\log_b(x) := \max\{\log_b x, 0\}$ .

Let us postpone the proof of [Lemma B.2](#) until after the end of this proof. Now we find

$$\begin{aligned} d(u, v)^2 &= \sum_{x \in \mathcal{V}} \mu_x d(u(x), v(x))^2 \geq m \sum_{k=0}^K d(u(x_k), v(x_k))^2 \\ &\geq mD^2 + m \sum_{k=0}^{K-1} (D - \delta^k \rho)^2 \\ &\geq mD^2(K+1) - 2mD\rho\delta^K \\ &\geq mD^2(K+1) - 2mD\rho\delta^{\log_{\delta}(D/\rho)} = mD^2(K-1). \end{aligned}$$

The conclusion follows by noting that if  $D \leq r^{1/2}$  i.e.  $d_{\infty}(u, v) \leq r^{1/2}$ , then we are done, and if  $D \geq r^{1/2}$  then  $D/\rho \geq r^{-1/2}/L$ , therefore  $K-1 \geq \min (A \log (r^{-1}) + B, \text{inj } \mathcal{G} - 1)$  where  $A = \frac{1}{2 \log \delta}$  and  $B = -\log_{\delta}(L) - 1$ .  $\square$

*Proof of Lemma B.2.* We make repeated use of the following fact (see [\[GLM18, Prop. 2.22\]](#)): since  $v$  is a discrete harmonic map its discrete tension field is zero:

$$\sum_{y \sim x} \omega_{xy} \overrightarrow{v(x)v(y)} = 0.$$

In other words  $v(x)$  is the weighted barycenter of its neighbor values in  $N$ . We refer to this as the *balanced condition* of  $v$  at  $x$ .

We prove [Lemma B.2](#) by induction on  $k$ . For the base case  $k = 1$ , consider the unit geodesic  $\gamma$  through  $v(x_0)$  and  $u(x_0)$ , parametrized with a coordinate  $t$  chosen by requiring  $\gamma(0) = v(x_0)$  and  $\gamma(-D) = u(x_0)$ . Define the orthogonal projection  $pr_{\gamma}$  as a map  $T_{v(x_0)}N \rightarrow \gamma \approx \mathbb{R}$ . If  $pr_{\gamma}(v(y)) < 0$

for all  $y \sim x_0$  then  $v$  would not be balanced at  $x_0$ , therefore there exists some neighbor vertex  $x_1 \sim x_0$  so that  $pr_\gamma(v(x_1)) \geq 0$ . Moreover, by assumption  $u(x_1)$  is within  $\rho$  of  $u(x_0)$ , so that  $pr_\gamma(u(x_1)) \leq pr_\gamma(u(x_0)) + \rho = -D + \rho$ . We conclude that  $d(v(x_1), u(x_1)) \geq pr_\gamma(v(x_1)) - pr_\gamma(u(x_0)) \geq D - \rho$ .

For the inductive step, we follow the above argument with  $x_k$  in place of  $x_0$ . That is, we have the unit geodesic  $\gamma$  through  $u(x_k)$  and  $v(x_k)$ , with  $\gamma(0) = v(x_k)$ ,  $\gamma(t) = u(x_k)$  for some  $t < 0$ , and the projection  $pr_\gamma : T_{v(x_k)}N \rightarrow \gamma \approx \mathbb{R}$ . Split up the neighbors of  $x_k$  into  $A$ , those vertices at combinatorial distance at most  $k$  from  $x_0$  in  $\mathcal{G}$ , and  $B$ , those vertices at distance  $k + 1$  from  $x_0$ . For each of the vertices  $y \in A$ , observe that  $pr_\gamma(v(y)) \leq -d(v(x_k), u(x_k)) + \rho + D \leq (1 + \delta^{k-1})\rho$ . Now the balanced condition for  $v$  at  $x_k$  gives

$$\begin{aligned} 0 &= \sum_{y \sim x_k} \omega_{x_k y} pr_\gamma \left( \overrightarrow{v(x_k)v(y)} \right) \\ &= \sum_{y \in A} \omega_{x_k y} pr_\gamma \left( \overrightarrow{v(x_k)v(y)} \right) + \sum_{y \in B} \omega_{x_k y} pr_\gamma \left( \overrightarrow{v(x_k)v(y)} \right) \\ &\leq \omega_{\max} \sum_{y \in A} (1 + \delta^{k-1})\rho + \sum_{y \in B} \omega_{x_k y} \max_{y' \in B} pr_\gamma \left( \overrightarrow{v(x_k)v(y')} \right). \end{aligned}$$

If  $pr_\gamma \left( \overrightarrow{v(x_k)v(y)} \right) \geq 0$  for some  $y \in B$ , then  $d(v(y), u(y)) \geq d(v(x_k), u(x_k)) - \rho$ , so we may let  $x_{k+1} = y$ . Otherwise, each of these coordinates are negative, and we have

$$0 < \omega_{\max} V(1 + \delta^{k-1})\rho + \omega_{\min} \max_{y \in B} pr_\gamma \left( \overrightarrow{v(x_k)v(y)} \right). \quad (22)$$

Let  $x_{k+1} \in B$  satisfy  $pr_\gamma \left( \overrightarrow{v(x_k)v(x_{k+1})} \right) = \max_{y \in B} pr_\gamma \left( \overrightarrow{v(x_k)v(y)} \right)$ . Rearranging (22),

$$pr_\gamma \left( \overrightarrow{v(x_k)v(x_{k+1})} \right) > -\omega V(1 + \delta^{k-1})\rho.$$

Because  $u(x_{k+1})$  is within  $\rho$  of  $u(x_k)$ , we find that  $pr_\gamma(u(x_{k+1})) \leq pr_\gamma(u(x_k)) + \rho$ . By the induction hypothesis,

$$\begin{aligned} d(v(x_{k+1}), u(x_{k+1})) &\geq pr_\gamma \left( \overrightarrow{v(x_k)v(x_{k+1})} \right) - (u(x_k) + \rho) \\ &> -\omega V(1 + \delta^{k-1})\rho + d(u(x_k), v(x_k)) - \rho \\ &\geq D - \rho(1 + \omega V)(1 + \delta^{k-1}). \end{aligned}$$

Finally, we have

$$\begin{aligned} (1 + \omega V)(1 + \delta^{k-1}) &= \frac{\delta}{2} = (1 + \delta^{k-1}) = \frac{\delta}{2} + \frac{\delta^k}{2} \\ &\leq \frac{\delta^k}{2} + \frac{\delta^k}{2} = \delta^k \end{aligned}$$

so that we conclude  $d(v(x_{k+1}), u(x_{k+1})) \geq D - \delta^k \rho$ .  $\square$

As an application of [Proposition B.1](#) we get:

**Corollary B.4.** *Let  $M$  be a compact manifold and let  $N$  be a complete manifold of nonpositive sectional curvature. Equip  $M$  with a sequence of meshes  $(\mathcal{M}_n)_{n \in \mathbb{N}}$  that is fine and crystalline, and equip the underlying graphs  $\mathcal{G}_n$  with asymptotic vertex weights and positive edge weights. Assume that there is a uniform upper bound for the ratio of any two edge weights. Let  $w : M \rightarrow N$  be a smooth map, denote by  $w_n$  its discretization along  $\mathcal{G}_n$ , and let  $v_n$  be a discrete harmonic map. Then*

$$d_\infty(w_n, v_n) \leq \max \left\{ o \left( r^{-\dim M/2} d(w_n, v_n) \right), r_n^{1/2} \right\}.$$



*Proof.* Note that since  $w$  is  $C^1$  on a compact manifold, it must be  $L$ -Lipschitz for some  $L > 0$ , and for all  $n \in \mathbb{N}$  the discretization  $w_n$  is also  $L$ -Lipschitz. [Proposition B.1](#) yields

$$d_\infty(w_n, v_n) \leq \max \left\{ (C_n m_n)^{-1/2} d(w_n, v_n), r^{1/2} \right\}.$$

In our setting,  $m_n = \Theta(r^{\dim M})$  by [Theorem 3.5 \(i\)](#), since we have asymptotic vertex weights. Thus we win if we can show that  $C_n \rightarrow +\infty$ , where  $C_n = \min(A \log(r_n^{-1}) + B, \text{inj } \mathcal{G}_n - 1)$ . Clearly this is the case since  $A$  and  $B$  are independent of  $n$  and  $r_n \rightarrow 0$ , also  $\text{inj } \mathcal{G}_n = \Theta(r_n^{\dim M}) \rightarrow +\infty$  by [Theorem 3.5 \(iv\)](#).  $\square$

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