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# Asymptotic analysis of an optimal control problem connected to the human locomotion

Terence Bayen, Yacine Chitour, Frédéric Jean and Paolo Mason

**Abstract**—The article is devoted to the analysis of two optimal control problems. We first consider a model proposed by Arechavaleta et al. (see [3]) describing the goal-oriented locomotion, for which the control on the derivative of the curvature  $\dot{\kappa}$  along the trajectory is supposed bounded. Necessary conditions on optimal trajectories are given. We then investigate an extension of this model obtained by removing the boundedness assumption on  $\dot{\kappa}$ . In this framework several properties of the optimal trajectories are detected and in particular we determine an asymptotic behavior of the initial value of the associated covector with respect to the final point.

## I. INTRODUCTION

This paper deals with the geometric shape of locomotor trajectories on the ground level. A person walking in an empty room from an initial point to a final point has several possible trajectories to perform this task (see figure 1) and will choose one of them, based on partially unconscious non trivial criteria. We present in this paper the mathematical study of a model based on the experiments performed in [2] to understand the goal-oriented locomotor trajectories. The approach that has been chosen is macroscopic, in particular, it does not refer to biomechanical motor controls generating the motion or anatomical parameters, contrary to studies of the human walking performed by neuroscientists (see [1]). To address a model, we will mainly take advantage of the shape of trajectories and of an optimization principle (see [9]). The use of such a principle is now being common for describing the generation of motion by the human body (e.g. the problem of arm pointing [5], or the control of eye movement [10]).

To model the human locomotion, Laumond has suggested the use of a dynamical system used in the field of mobile robotics. This choice has been motivated by the analysis of a large number of data obtained monitoring the paths followed by several subjects (see [2], [3]). Subjects have been asked to walk from a pre-defined position (crossing an initial porch with an initial direction) to a final position (crossing a final porch with a final direction). Figure 1 illustrates three possible trajectories to connect a point  $I$  to a point  $F$ .

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The subject starts with a fixed initial direction  $\theta_0 \in [0, 2\pi]$  and ends with a fixed final direction  $\theta_1$ . The numerical records performed in [2] show in particular that the trunk can be considered as a steering wheel, that is, it satisfies a nonholonomic constraint. Let  $(x, y)$  denotes the trunk position in the plane and let  $\theta$  be the tangential direction of the speed vector  $\vec{v}$  with respect to a fixed direction, that is,  $(\cos \theta, \sin \theta)$  is parallel to  $\vec{v}$ . The nonholonomic constraint satisfied by the trunk writes:

$$\dot{x} \sin \theta - \dot{y} \cos \theta = 0. \quad (\text{I.1})$$

Notice that sideways walking is prohibited by the dynamical constraint above. In addition, the constraint (I.1) does not allow a subject to turn his shoulder while the body is fixed. This remark suggests that certain real trajectories may not correspond exactly to predicted trajectories (in particular if the target is located behind the initial position or close to it). A possible approach to describe locomotor trajectories consists in interpreting them as solutions of a suitable optimal control problem related to a control scheme taking into account the previous dynamical constraint. A subject is viewed as a controlled system described by a nonholonomic system. Using the coordinates  $(x, y, \theta)$  of the trunk and (I.1), a first approach to describe the human locomotion is to consider the following differential system (called *Dubin-System*):

$$\begin{cases} \dot{x} = u_1 \cos \theta, \\ \dot{y} = u_1 \sin \theta, \\ \dot{\theta} = u_2. \end{cases} \quad (\text{I.2})$$

The function  $u_1$  is the linear speed and the function  $u_2$  is the angular velocity. As the speed is bounded, it can be assumed that the function  $u := (u_1, u_2)$  takes its values within a certain compact set  $\mathcal{U}$ , that is  $(u_1, u_2) \in L^\infty([0, T], \mathcal{U})$ . Such a system is often used to describe a wheeled vehicle controlled by its linear and angular velocity (see Dubin's car model, [11]), the speed vector being tangent to  $(\cos \theta, \sin \theta)$ .

Nevertheless, system (I.2) is not well adapted to describe locomotor trajectories. Indeed, it has been pointed out experimentally in [2] that the curvature along a locomotor trajectory is continuous. Since the continuity of the curvature is not guaranteed if  $u \in L^\infty$  in (I.2), it has been suggested in [2] to control the variation of curvature instead of the curvature itself. The locomotor trajectories will be then described by the following extension of Dubin's model in  $\mathbb{R}^4$  (called *Dubin-Markov-System*), keeping the nonholonomic

constraint

$$\begin{cases} \dot{x} = u_1 \cos \theta, \\ \dot{y} = u_1 \sin \theta, \\ \dot{\theta} = u_1 \kappa, \\ \dot{\kappa} = u_2. \end{cases} \quad (\text{I.3})$$

We assume that the linear velocity  $u_1$  takes values within an interval  $[a, b]$ , with  $0 < a < b$  ( $a$  being the lowest walking speed and  $b$  the fastest one) and the function  $u_2$  takes values on an interval  $[-c, c]$ , where  $c > 0$ .

For simplicity, we will denote  $X := (x, y, \theta, \kappa) \in \mathbb{R}^2 \times \mathcal{S}^1 \times \mathbb{R}$ . According to the experiments described in [2], [3], in which the subject is asked to start walking straight ahead in the direction  $\theta_0$  at  $t = 0$  from an initial porch located at  $(x_0, y_0)$ , we assume that  $\kappa(0) = 0$ . Similarly, we will assume that  $\kappa = 0$  at the final time  $T_u$ . Therefore, without loss of generality, we consider initial and final conditions of the following form:

$$\begin{cases} X(0) = X_0, \\ X(T_u) = X_1, \end{cases} \quad (\text{I.4})$$

where  $X_0 = (0, 0, \frac{\pi}{2}, 0)$ ,  $X_1 = (x_1, y_1, \theta_1, 0)$ . This initial-final boundary condition will be kept through the rest of the paper. Notice that the final time  $T_u$  is not fixed in advance, but it depends on  $u(\cdot)$ .

To walk from the initial point to the final point, we now suppose that the human brain minimizes a certain energy. Several costs can be found in the literature to model the generation of motion of the human body: the minimum time or the jerk (see [7], [9], [8] for similar assumptions). In this paper, following Arechavaleta et al. [2], [3], we assume that the cost to be minimized along the locomotor trajectories steering  $X_0$  to  $X_1$  in time  $T_u$  is given by:

$$C_u(T_u) := \frac{1}{2} \int_0^{T_u} (u_1^2(t) + u_2^2(t)) dt. \quad (\text{I.5})$$

Clearly,  $C_u(T_u)$  takes into account the kinetic energy of the subject on the interval  $[0, T_u]$  to steer  $X_0$  to  $X_1$ . The first term is the linear kinetic energy, the second one can be viewed as the angular kinetic energy, and we argue in particular that the variation of curvature of locomotor trajectories is minimized. A person naturally reduces the variation of curvature and takes advantage of straight lines avoiding sharp bends. Other costs are possible to model the locomotion, in particular a compromise between the kinetic and angular energy could be studied. From now on, we suppose that the cost  $C_u(T_u)$  given by (I.5) is minimized along locomotor trajectories.

In view of the previous comments in this paper we will first investigate the following optimal control problem.

(OCP1): Find all the trajectories of (I.3) defined on  $[0, T_u]$ , with  $u_1(\cdot) \in L^\infty([0, T_u], [a, b])$  for some  $0 < a < b$  and  $u_2(\cdot) \in L^\infty([0, T_u], [-c, c])$  for a  $c > 0$ , such that  $X(0) = X_0$  to  $X(T_u) = X_1$  and minimizing (I.5).

Notice that system I.3 has been studied by Sussmann (see [13]) in the case of the minimum time with  $u_1 = 1$  and  $u_2 \in$

$[-1, 1]$ , who showed in particular that an optimal trajectory has infinite chattering (Fuller phenomenon). In the model above, the cost to minimize takes into account the energy and it prevents this phenomenon.

A second optimal control problem, obtained by removing the boundedness assumptions on  $u_2$ , will be then considered.

(OCP2): Find all the trajectories of (I.3) defined on  $[0, T_u]$ , with  $u_1(\cdot) \in L^\infty([0, T_u], [a, b])$  for some  $0 < a < b$  and  $u_2(\cdot) \in L^2([0, T_u])$ , such that  $X(0) = X_0$  to  $X(T_u) = X_1$  and minimizing (I.5).

Notice that both our optimization models are reasonable only when the final point is far enough from the origin. To tackle this problem a more sophisticated model taking into account the possibility of holonomic motion (such as sideways or oblique steps) has been recently proposed in [6].

In the following sections we will state qualitative properties of optimal trajectories of (OCP1) and (OCP2) and we will describe the asymptotic behaviour of optimal trajectories for (OCP2) as the final point  $(x_1, y_1)$  goes to infinity. A more detailed description with complete proofs of such results (that for reasons of space cannot be included in the present paper) can be found in [4].

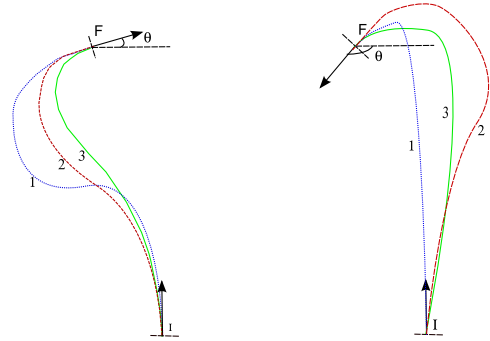


Fig. 1. The subject walks from point  $I$  with fixed initial direction  $\theta_0 = \frac{\pi}{2}$  to point  $F$  with fixed final direction  $\theta \in [0, 2\pi]$ . Three possible trajectories have been plotted in two cases for  $F$  and  $\theta$ .

## II. GENERAL RESULTS

In this section, general properties of both optimal control models (OCP1) and (OCP2) are addressed. The Pontryagin maximum principle (PMP) is applied to obtain necessary conditions satisfied by optimal trajectories.

We first set some notation. Recall that  $X = (x, y, \theta, \kappa) \in \mathbb{R}^2 \times \mathcal{S}^1 \times \mathbb{R}$ , and let  $(F_1, F_2, F_3, F_4)$  be the vector fields defined on  $\mathbb{R}^2 \times \mathcal{S}^1 \times \mathbb{R}$  by

$$\begin{cases} F_1(X) = (\cos(\theta), \sin(\theta), \kappa, 0), \\ F_2(X) = (0, 0, 0, 1), \\ F_3(X) = (0, 0, -1, 0), \\ F_4(X) = (-\sin(\theta), \cos(\theta), 0, 0). \end{cases}$$

The system (I.3) reads as follows:

$$\dot{X} = u_1 F_1(X) + u_2 F_2(X), \quad (\text{II.1})$$

with  $u_1 \in [a, b]$  and  $u_2 \in [-c, c]$  for (OCP1),  $u_2 \in \mathbb{R}$  for (OCP2).

#### A. Existence, controllability and reduction of the system

We investigate in this section the controllability of (I.3) and the existence of an optimal control for (OCP1) and (OCP2). We also show that it is possible to simplify these problems by assuming that  $u_1 \equiv a$ .

*Proposition 2.1:* The system (I.3) is controllable, provided that  $u_1 \in [a, b]$  and  $u_2 \in [-c, c]$ , for any choice of  $0 < a < b$  and  $0 < c \leq +\infty$ .

The following result shows that it is always possible to simplify the optimal control problem by setting  $u_1 \equiv a$ . More precisely, if we have existence of an optimal control, then it can be selected among the controls satisfying  $u_1 \equiv a$ .

*Theorem 2.2:* Let  $(X_0, X_1)$  be given in  $\mathbb{R}^2 \times \mathcal{S}^1 \times \mathbb{R}$ . Let  $u(\cdot) = (u_1(\cdot), u_2(\cdot))$  be a control function steering (I.3) from  $X_0$  to  $X_1$ . Then there always exists a control  $\tilde{u}$  of the form  $\tilde{u}(\cdot) = (a, \tilde{u}_2(\cdot))$  such that  $\|\tilde{u}_2(\cdot)\|_\infty \leq \|u_2(\cdot)\|_\infty$ , the corresponding trajectory  $\tilde{X}(\cdot)$  solution of (I.3) connects  $X_0$  and  $X_1$  and the cost associated to  $\tilde{u}$  is lower than the one associated to  $u$ .

In view of the previous result and for simplicity reasons from now on we will always assume  $u_1 \equiv a = 1$  and we will simply denote  $u_2$  by  $u$ , so that the initial system is rewritten as:

$$\begin{cases} \dot{x} = \cos \theta, \\ \dot{y} = \sin \theta, \\ \dot{\theta} = \kappa, \\ \dot{\kappa} = u. \end{cases} \quad (\text{II.2})$$

The cost of an admissible trajectory becomes

$$C_u(T) = \frac{1}{2}T + \frac{1}{2} \int_0^T u^2, \quad (\text{II.3})$$

and it is the sum of two terms: the first one corresponds to the minimum time problem and the second one  $\frac{1}{2} \int_0^T u^2$  can be considered as the rotational kinetic energy. For special cases it is easy to provide optimal trajectories. For instance we have the following straightforward result.

*Proposition 2.3:* The points  $X_0$  and  $X_1$  are connected optimally by a segment if and only if  $x_1 = 0$ ,  $y_1 \geq 0$ ,  $\theta_1 = \theta_0 = \frac{\pi}{2}$ . The cost associated to this trajectory is  $C_u(T) = \frac{y_1}{2}$ .

In the general case the existence of optimal trajectories is provided by the following proposition.

*Proposition 2.4:* For every choice of  $X_0$  and  $X_1$  in  $\mathbb{R}^2 \times \mathcal{S}^1 \times \mathbb{R}$  there exists an optimal trajectory  $\bar{X}(\cdot)$  defined on  $[0, \bar{T}]$  and associated to the control  $\bar{u}(\cdot)$  such that  $\bar{X}(0) = X_0$  and  $\bar{X}(\bar{T}) = X_1$ .

In order to apply the PMP to our optimal control problems we will need to ensure that any optimal control  $\bar{u}(\cdot)$  is bounded in the  $L^\infty$  topology. For Problem (OCP2) we have the following proposition which is a direct application of Theorem 1 of [12].

*Proposition 2.5:* Assume there is no a priori bound on  $u$  and let  $\bar{X}(\cdot)$  be an optimal trajectory defined on  $[0, \bar{T}]$

associated to the control  $\bar{u}(\cdot)$ . Then this trajectories satisfies the PMP. (More precisely, either it is an abnormal extremal or  $\bar{u} \in L^\infty([0, \bar{T}])$  which again implies that it is an extremal, i.e. a solution of the PMP.)

#### B. Fundamental bounds

By comparison with specific trajectories of (I.3) it is possible to deduce explicit bounds related to optimal trajectories  $X(\cdot)$  defined on  $[0, T]$ , corresponding to the control  $u(\cdot)$  and connecting  $X_0$  to  $X_1$ . In particular we have the following result.

*Proposition 2.6:* Provided that  $|(x_1, y_1)| \geq 4\sqrt{3\pi}$  and if  $X(\cdot)$  is a solution of (OCP2) or (OCP1) with  $c \geq 1$  then

$$|(x_1, y_1)| \leq T \leq 2\mathcal{C}_{u(\cdot)}(T) \leq |(x_1, y_1)| + 12\sqrt{3\pi}, \quad (\text{II.4})$$

and, consequently,

$$\int_0^T u^2(t) dt \leq 12\sqrt{3\pi}. \quad (\text{II.5})$$

#### C. Application of the Pontryagin maximum principle

In this section, we apply the PMP to (OCP1) and (OCP2) and we derive the first consequences. The Hamiltonian of system (II.2) is:

$$\begin{aligned} H &= H(X, p, u, \nu) \\ &= p_1 \cos \theta + p_2 \sin \theta + p_3 \kappa + p_4 u - \frac{\nu}{2}(1 + u^2), \end{aligned} \quad (\text{II.6})$$

where  $X = (x, y, \theta, \kappa)$  is the state variable,  $p = (p_1, p_2, p_3, p_4) \in \mathbb{R}^4$  is the covector (or adjoint vector),  $u \in [-c, c]$  is the control,  $0 < c \leq +\infty$ , and  $\nu \in \mathbb{R}$  (in particular Problem (OCP2) corresponds to  $c = +\infty$ ). The PMP writes as follows. Let  $u$  be an optimal control defined on the interval  $[0, T]$  and  $X(\cdot)$  the corresponding optimal trajectory. Then, there exists an absolutely continuous function  $p : [0, T] \rightarrow \mathbb{R}^4$  and  $\nu \leq 0$  such that the pair  $(p(\cdot), \nu)$  is non-trivial, and such that we have:

$$\begin{cases} \dot{X}(t) = \frac{\partial H}{\partial p}(X(t), p(t), \nu, u(t)), \\ \dot{p}(t) = -\frac{\partial H}{\partial X}(X(t), p(t), \nu, u(t)). \end{cases} \quad (\text{II.7})$$

Since the system is autonomous, the Hamiltonian is conserved along extremal trajectories. The maximization condition writes:

$$H(X(t), p(t), u(t), \nu) = \max_{v \in [-c, c]} H(X(t), p(t), v, \nu) \quad (\text{II.8})$$

for a.e.  $t \in [0, T]$ . As the final time is free, the Hamiltonian is zero (see [14]):

$$H(X(t), p(t), u(t), \nu) = 0, \quad \forall t \in [0, T]. \quad (\text{II.9})$$

The equation (II.7) is the state-adjoint equation. We say that  $X(\cdot)$  is an *extremal trajectory* of the optimal control problem if it can be augmented to a quadruplet  $(X(t), p(t), u(t), \nu)$  satisfying (II.7), (II.8), such that  $(p(\cdot), \nu)$  is nontrivial and  $\nu \leq 0$ . The dual equation on the covector becomes:

$$\begin{cases} \dot{p}_1 = 0, \\ \dot{p}_2 = 0, \\ \dot{p}_3 = p_1 \sin \theta - p_2 \cos \theta, \\ \dot{p}_4 = -p_3. \end{cases} \quad (\text{II.10})$$

Along an extremal trajectory,  $p_1$  and  $p_2$  are two constant and  $p_4$  satisfies:

$$\dot{p}_4 = -p_1 \sin \theta + p_2 \cos \theta. \quad (\text{II.11})$$

Note that the PMP gives necessary conditions satisfied by optimal trajectories.

Extremal trajectories are of two kinds: abnormal and normal trajectories. An *abnormal trajectory* is independent of the cost function, that is  $\nu = 0$ . A *normal trajectory* satisfies  $\nu < 0$ , and by homogeneity, we may assume  $\nu = -1$ . Let  $\gamma$  be an extremal trajectory of the system, a point  $\tau \in ]0, T[$  is called a *switching point* if for every  $\varepsilon > 0$  such that  $[\tau - \varepsilon, \tau + \varepsilon] \subset ]0, T[$ , the control  $u(\cdot)$  associated to  $\gamma$  is non-constant on  $[\tau - \varepsilon, \tau + \varepsilon]$ . An extremal trajectory corresponding to a piecewise constant control is called *bang-bang*. We now come to the study of abnormal and normal trajectories.

#### D. Abnormal trajectories

For an abnormal extremal the maximization condition trivially gives  $u = c \text{sign}(p_4)$ . Notice that there are no abnormal extremals for (OCP2). Indeed in this case the maximization condition of the PMP would imply that  $p_4 \equiv 0$ , and therefore  $p_3 \equiv 0$ . This, together with (II.10) and the fact that  $H = 0$  along any extremal, would imply  $p_1 = p_2 = 0$  leading to a contradiction.

#### E. Normal trajectories

In this section, we study the structure of normal extremal trajectories. The real  $\nu$  is nonzero in this case, and, since  $H$  is homogeneous with respect to  $(p, \nu)$ , it can be chosen equal to 1. The Hamiltonian is conserved along the extremal  $X(\cdot)$  and it writes:

$$\begin{aligned} H(X(t), p(t), u(t)) &= \\ &= p_1 \cos \theta + p_2 \sin \theta + p_3 \kappa + p_4 u - \frac{1}{2}(1 + u^2) \equiv 0. \end{aligned} \quad (\text{II.12})$$

By (II.8),  $u$  is given for a.e.  $t \in [0, T]$  by:

$$u(t) = \operatorname{argmax}_{v \in [-c, c]} \left\{ p_4(t)v - \frac{1}{2}v^2 \right\},$$

that is  $u(t)$  maximizes a quadratic function within a segment  $[-c, c]$ . Therefore, we easily get:

$$\begin{cases} p_4(t) \geq c \iff u(t) = +c, \\ p_4(t) \leq -c \iff u(t) = -c, \\ p_4(t) \in [-c, c] \iff u(t) = p_4(t). \end{cases}$$

If  $|p_4(t)| < c$ , on a subinterval  $I$  of  $[0, T]$ , then a straightforward computation shows that  $\theta$  satisfies on  $I$  the following differential equation:

$$\theta^{(4)} = -p_1 \sin \theta + p_2 \cos \theta. \quad (\text{II.13})$$

Though straight lines are optimal, more complicated optimal trajectories cannot contain segments. This can be easily verified for abnormal extremals, while for normal ones it is stated by the following proposition.

*Proposition 2.7:* Let us assume that  $X_1 \notin \{(0, y_1, \frac{\pi}{2}, 0), y_1 \in \mathbb{R}_+\}$ , and let  $X(\cdot)$  be an extremal normal trajectory connecting  $X_0$  and  $X_1$ . Then,  $X(\cdot)$  does not contain a segment.

### III. ANALYSIS OF (OCP2)

The aim of this section is to provide qualitative properties of the solutions of (OCP2), mainly in the particular case in which the final point  $(x_1, y_1)$  is far from the origin.

Recall that for the optimal control problem (OCP2) the maximization condition of the PMP gives  $u(t) = p_4(t)$ . Consequently the Hamiltonian function becomes

$$H = p_1 \cos \theta + p_2 \sin \theta + p_3 \kappa + \frac{1}{2}p_4^2 - \frac{1}{2} = 0 \quad (\text{III.1})$$

We begin this section by providing a simple but useful lemma, obtained by integrating (II.6).

*Lemma 3.1:* For an optimal trajectory  $X(\cdot)$  corresponding to the control  $u(\cdot)$  connecting  $X_0$  to  $X_1$  in time  $T$  we have the following equality:

$$\langle (p_1, p_2), (x_1, y_1) \rangle = \frac{1}{2}T - \frac{3}{2} \int_0^T u^2 dt. \quad (\text{III.2})$$

In view of the bounds (II.4), (II.5) an important consequence of the previous lemma is that when  $|(x_1, y_1)| \geq 4\sqrt{3\pi}$  we have

$$\begin{aligned} \frac{1}{2}|(x_1, y_1)| - 12\sqrt{3\pi} &\leq \langle (p_1, p_2), (x_1, y_1) \rangle \\ &\leq \frac{1}{2}|(x_1, y_1)| + 6\sqrt{3\pi}, \end{aligned} \quad (\text{III.3})$$

which can also be written as

$$\begin{aligned} \frac{1}{2} - \mathcal{O}\left(\frac{1}{|(x_1, y_1)|}\right) &\leq \langle (p_1, p_2), \frac{(x_1, y_1)}{|(x_1, y_1)|} \rangle \\ &\leq \frac{1}{2} + \mathcal{O}\left(\frac{1}{|(x_1, y_1)|}\right), \end{aligned} \quad (\text{III.4})$$

and in particular this implies that for every  $\varepsilon > 0$  there exists  $R_\varepsilon$  such that

$$|(x_1, y_1)| \geq R_\varepsilon \implies |(p_1, p_2)| \geq \frac{1}{2} - \varepsilon. \quad (\text{III.5})$$

#### A. Some preliminary result

Let  $\alpha \in [0, 2\pi]$  be such that  $(x_1, y_1) = |(x_1, y_1)|(\cos \alpha, \sin \alpha)$  and let us write as  $(p_1, p_2) = \rho(\cos \phi, \sin \phi)$  the first two components of the covector associated to an optimal trajectory and by  $\theta(\cdot)$  the corresponding angle. Notice from (III.4), (III.5) that we can assume  $\rho > \frac{1}{4}$  and  $|\phi - \alpha| < \frac{\pi}{2}$  if  $|(x_1, y_1)|$  is large enough.

We have the following lemma, which express the fact that the set of times such that  $\theta(t)$  is ‘‘far from  $\alpha$ ’’ has uniformly bounded measure (independently of the final point).

*Lemma 3.2:* Given  $\varepsilon > 0$  and given an optimal trajectory we define the set

$$J_\varepsilon = \{\tau \in [0, T] : |\alpha - \theta(\tau)| \geq \varepsilon\}.$$

Then for every  $\varepsilon > 0$  there exists  $\mathcal{T}_\varepsilon > 0$  such that for every optimal trajectory  $m(J_\varepsilon) \leq \mathcal{T}_\varepsilon$ , where  $m(\cdot)$  denotes the Lebesgue measure in  $\mathbb{R}$ .

The next lemma relates the angles  $\alpha$  and  $\phi$ .

*Lemma 3.3:* For every  $\eta > 0$  there exists  $R_\eta > 0$  such that  $|(x_1, y_1)| \geq R_\eta$  implies  $|\phi - \alpha| \leq \eta$ .

### B. Main qualitative asymptotic results

The following result, obtained by combining Lemma 3.3 with (III.4), is interesting for numerical simulations.

*Proposition 3.4:* For every  $\eta > 0$  there exists  $R_\eta > 0$  such that if  $|(x_1, y_1)| > R_\eta$  then

$$\left| (p_1, p_2) - \frac{(x_1, y_1)}{2|(x_1, y_1)|} \right| < \eta.$$

In particular  $|(p_1, p_2)| < \frac{1}{2} + \eta$ .

By applying the previous results one can prove the existence of a uniform bound on the control for optimal trajectories as stated in the following theorem.

*Theorem 3.5:* There exists a constant  $C > 0$  such that, for some  $R > 0$  we have that, if  $|(x_1, y_1)| > R$ , then  $\|u\|_{W^{1,\infty}} \leq C$ .

An important consequence of the previous result is that the optimal control problems (OCP1) and (OCP2) are equivalent out of a neighborhood of the origin, up to choosing  $c$  large enough. Also, the exclusion of a neighborhood of the origin is not crucial, since our nonholonomic model is appropriate only far from the origin.

*Remark 3.6:* It is not true that for every  $\varepsilon > 0$  there exists  $R_\varepsilon$  such that  $\|u\|_\infty \leq \varepsilon$  for every optimal triple  $(X(\cdot), u(\cdot), T)$  with  $|(x_1, y_1)| > R_\varepsilon$ . Indeed  $H \equiv 0$  implies  $u(0) = \sqrt{1 - 2p_2}$ . However if  $\alpha$  is defined as in Section III-A and  $|(x_1, y_1)|$  is large enough the previous results say that  $p_2$  is arbitrarily close to  $\frac{1}{2} \sin \alpha$  (which in general is different from  $\frac{1}{2}$ ).

*Remark 3.7:* The control function  $u(\cdot)$  associated to optimal trajectories reaching points in a neighborhood of the origin is not uniformly bounded. More precisely it is possible to find a sequence of points  $X_1^{(n)} = (x_1^{(n)}, y_1^{(n)}, \theta_1^{(n)}, \kappa_1^{(n)})$  with  $\lim_{n \rightarrow \infty} |(x_1^{(n)}, y_1^{(n)})| = 0$  such that the optimal controls  $u^{(n)}(\cdot)$  steering the system from  $X_0$  to  $X_1^{(n)}$  satisfy  $\lim_{n \rightarrow \infty} \|u^{(n)}(\cdot)\|_\infty = \infty$ .

The following important result states that, far from the origin and the final point, an optimal trajectory is similar to a segment.

*Theorem 3.8:* Given  $\eta > 0$  there exist  $\tau_\eta > 0$  and  $\sigma_\eta > 2\tau_\eta$  such that, for every optimal trajectory with final time  $T > \sigma_\eta$ , one has  $|\theta(t) - \alpha| < \eta$  for  $t \in [\tau_\eta, T - \tau_\eta]$ .

From the previous theorem and Equation (II.13) it is easy to get the following result.

*Corollary 3.9:* Let us associate to  $\theta(\cdot)$  the function  $Z(t) = (\theta(t), \dot{\theta}(t), \ddot{\theta}(t), \theta^{(3)}(t))$ . Given  $\nu > 0$  there exist  $\bar{\tau}_\nu > 0$  and  $\bar{\sigma}_\nu > 2\bar{\tau}_\nu$  such that, for every optimal trajectory with final time  $T > \bar{\sigma}_\nu$ , one has  $|Z(t) - (\alpha, 0, 0, 0)| < \nu$  for  $t \in [\bar{\tau}_\nu, T - \bar{\tau}_\nu]$ .

### C. Numerical study of the asymptotic behaviour of optimal trajectories and of the corresponding value of $p_3(0)$ .

While the previous results clarify some important properties of optimal trajectories and of the associated covectors it

is nevertheless clear that a complete qualitative description of optimal trajectories is still missing. In particular the previous results allow to determine, when the final point  $(x_1, y_1)$  is far from the origin, an approximate value of  $(p_1, p_2)$  and consequently, from the equation  $H = 0$ , of  $|p_4(0)|$ . No information however is known about the value  $p_3(0)$ , which is important since the corresponding trajectory turns out to be very sensitive with respect to changes of the latter. Also, the shape of the optimal trajectories close to the origin and close to the final point is not known.

To tackle these issues it is possible to proceed as follows. The asymptotic behaviour pointed out by the previous results can be interpreted at the light of the fourth order equation satisfied by  $\theta$ :

$$\theta^{(4)}(t) = -\rho \sin(\theta(t) - \phi).$$

An equilibrium for this equation is given by  $(\theta, \dot{\theta}, \ddot{\theta}, \theta^{(3)}) = (\phi, 0, 0, 0)$  and the results stated above show that, for optimal trajectories with  $(x_1, y_1)$  far enough from the origin, the values of  $(\theta(\cdot), \dot{\theta}(\cdot), \ddot{\theta}(\cdot), \theta^{(3)}(\cdot))$  are close to this equilibrium on some interval  $[\tau, T - \tau]$ . This behaviour seems to suggest some stability property of the system at the equilibrium. It is actually easy to see that  $(\phi, 0, 0, 0)$  is not a stable equilibrium of the system, since two of the associated eigenvalues of the linearized system have positive real part  $\sqrt{2}\rho^{1/4}/2$  while the other two eigenvalues have negative real part  $-\sqrt{2}\rho^{1/4}/2$ . The stable behaviour of the optimal trajectories must therefore be interpreted by assuming that, approaching the equilibrium, these trajectories remain very close to the stable manifold associated to it. Let us recall that, when the linearized system has no eigenvalues with zero real part, the stable (resp. unstable) manifold is a smooth submanifold of the state space having the same dimension of the stable (resp. unstable) subspace of the linearized system, i.e. the subspace spanned by the eigenvectors associated to the eigenvalues with negative (resp. positive) real part. Moreover the stable (resp. unstable) manifold is tangent to the stable (resp. unstable) subspace of the linearized system. The property characterizing the stable manifold is that it is invariant under the flow of the system and every trajectory lying inside it converges exponentially fast to the equilibrium. On the other hand the trajectories lying inside the unstable manifold diverge exponentially fast from the equilibrium and all the trajectories which are not contained in the stable and in the unstable manifold and starting on a neighborhood of the equilibrium diverge exponentially fast from it.

In view of the previous remarks and of the continuity of the stable subspace with respect to the parameters  $\rho$  and  $\phi$  it is interesting to consider the limit case in which  $\rho = \frac{1}{2}$ . On a neighborhood of the equilibrium we consider values  $(\theta, \dot{\theta}, \ddot{\theta}, \theta^{(3)})$  close to the stable manifold. If we follow backwards in time the corresponding trajectories we know that these will keep close to some trajectories contained inside the stable manifold. Note that the set of trajectories converging to the equilibrium consists of a one dimensional family of trajectories, since the dimension of the stable

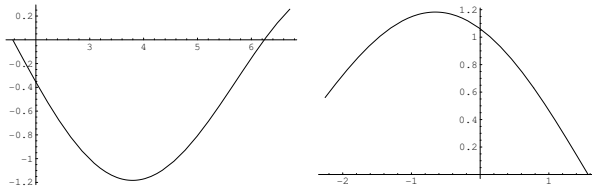


Fig. 2. The value of  $p_3(0)$  in terms of the angle  $\phi$  as  $p_4(0) > 0$  (left picture) and  $p_4(0) < 0$  (right picture).

manifold is two. Therefore to describe them it is enough to consider starting points belonging to a closed curve around the equilibrium and close to the stable manifold. Knowing that the stable subspace is generated by the vectors  $v_1 = (2^{5/4}, 0, -2^{3/4}, 2)$  and  $v_2 = (2^{5/4}, -2\sqrt{2}, 2^{3/4}, 0)$  we consider the closed curve around the equilibrium, which, up to a translation, is assumed to be the origin,  $\gamma(s) = \varepsilon(v_1 \cos s + v_2 \sin s)$ ,  $s \in [0, 2\pi]$ . This curve is close to the stable manifold if  $\varepsilon > 0$  is small. By following backwards in time the trajectories passing through the points  $\gamma(s)$  we can recover the points for which  $\kappa = 0$ , the corresponding angle  $\theta$  and the value of  $p_3$ . Since it must be  $\theta(0) = \pi/2$ , by a suitable translation of the angular variable we can therefore associate to each value of  $\phi$  the corresponding approximate value of  $p_3(0)$ . The graph of the resulting map, obtained numerically, is depicted in Figure 2.

Summing up, we have shown that when  $(x_1, y_1)$  is far from the origin, it is possible to determine approximately the value of the covector  $(p_1, p_2, p_3(0), p_4(0))$  associated to the corresponding optimal trajectory and, as a consequence, the shape of this trajectory is approximately determined by the direction of the vector  $(x_1, y_1)$  (see Figure 3).

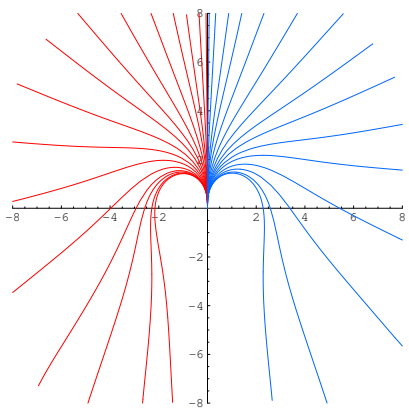


Fig. 3. Asymptotical behaviour of optimal trajectories with final point far from the origin.

#### IV. CONCLUSION

In this paper we have studied the solutions of an optimal control problem modeling the human locomotion. Qualitative results have been found. In particular we gave a precise description of optimal trajectories as the final point goes to

infinity, and this description qualitatively matches the experimental results. Moreover the possible asymptotic values of the adjoint vector at  $t = 0$  are precisely characterized and this allows to improve the shooting algorithms that determine numerically the optimal trajectories. Finally, the methods developed in this paper are likely to be applicable to more general classes of optimal control problems. Therefore future work will aim at investigating the optimal control models that best match the experimental results.

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