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Hardy spaces on Riemannian manifolds with quadratic curvature decay

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Abstract

Let (M, g) be a complete Riemannian manifold. Assume that the Ricci curvature of M has quadratic decay and that the volume growth is strictly faster than quadratic. We establish that the Hardy spaces of exact 1-differential forms on M , introduced in [4], coincide with the closure in L^p of $\mathcal{R}(d) \cap L^p(\Lambda^1 T^*M)$ when $1 < p < \nu$, where $\nu > 2$ is related to the volume growth. The range of p is optimal. This result applies, in particular, when M has a finite number of Euclidean ends.

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1 Introduction

1.1 Motivation

Let $n \geq 1$ be an integer. It is a well-known fact that, for all $j \in \llbracket 1, n \rrbracket$, the Riesz transform $\partial_j(-\Delta)^{-1/2}$ is $H^1(\mathbb{R}^n)$ -bounded, where $H^1(\mathbb{R}^n)$ denotes the real Hardy space. If one seeks for a version of this result in a complete Riemannian manifold (M, g) endowed with its Riemannian measure μ , one has to take into account that the Riesz transform, given by $d\Delta^{-1/2}$ in this context, is 1-form valued. Motivated by this observation and relying on the connection between Hardy spaces and tent spaces ([13]), Auscher, McIntosh and the second author introduced, in [4], a family of Hardy spaces of exact (resp. co-exact) differential forms on M , namely $H_d^p(\Lambda^k T^*M)$ (resp. $H_{d^*}^p(\Lambda^k T^*M)$) for $1 \leq p \leq \infty$ and $0 \leq k \leq \dim M$. Denote $\Delta_k = dd^* + d^*d$ the Hodge Laplacian acting on differential forms of degree k ; in particular, $\Delta_0 = \Delta$, the usual Laplacian acting on scalar functions on M . In [4], it was proved that, under a doubling volume condition for geodesic balls of M , the Riesz transform $d\Delta_k^{-1/2}$ acting on exact differential k -forms is bounded from $H_{d^*}^p(\Lambda^k T^*M)$ to $H_d^p(\Lambda^{k+1} T^*M)$ for all $k \in \llbracket 0, \dim M - 1 \rrbracket$ and all $1 \leq p \leq \infty$.

With the issue of L^p -boundedness of the Riesz transform in mind, one may wonder if $H_d^p(\Lambda^k T^*M)$ coincides with the closure in L^p of $\mathcal{R}(d) \cap L^p(\Lambda^k T^*M)$ for $1 < p < \infty$, as well as the corresponding statement for $H_{d^*}^p(\Lambda^k T^*M)$, as in the Euclidean case. In the case of 0-forms (that is, for functions), it was proved in [4, Theorem 8.5] that the answer is positive for $H_{d^*}^p(\Lambda^0 T^*M)$ if the heat kernel associated with the Laplace-Beltrami operator satisfies Gaussian pointwise upper estimates. A similar statement holds for $H_d^p(\Lambda^1 T^*M)$ if one assumes analogous Gaussian bounds for the heat kernel associated with Δ_1 , the Hodge Laplacian on 1-forms; this is however a much stronger assumption. In particular, it implies the L^p -boundedness of $d\Delta^{-1/2}$ for all $1 < p < +\infty$ ([14, 23]). Gaussian bounds for the heat kernels associated with Δ_0 and Δ_1 hold, in particular, if (M, g) has nonnegative Ricci curvature ([22, 5, 6]). In the present work we want to compare H^p and L^p , avoiding the use of Gaussian bounds for the heat kernel on 1-forms. A general fact proved in [4] is that if the measure μ is doubling, then for all $p \geq 2$ and all $0 \leq k \leq \dim M$, the closure in L^p of $\mathcal{R}(d) \cap L^p(\Lambda^k T^*M)$ is included in $H_{d^*}^p(\Lambda^k T^*M)$. But the inclusion may be strict, as the following example demonstrates: consider the manifold M made of the connected sum of two copies of \mathbb{R}^n . It is well-known that the heat kernel of Δ_0 on M has Gaussian estimates, but that the heat kernel of Δ_1 does not, despite M having vanishing Riemannian curvature outside a compact set. If $n \geq 3$ (resp. $n = 2$), it was

proved in [10] that $d\Delta^{-1/2}$ is L^p -bounded if and only if $1 < p < n$ (resp. $1 < p \leq 2$). and it follows that, on M , $H_d^p(\Lambda^1 T^*M)$ and the closure in L^p of $\mathcal{R}(d) \cap L^p(\Lambda^1 T^*M)$ never coincide if $p \geq n$ (resp. $p > 2$). However, as a consequence of the main result in the present paper, we shall prove that for the connected sum of two copies of \mathbb{R}^n , $n \geq 3$, $H_d^p(\Lambda^1 T^*M)$ is *equal* to the closure in L^p of $\mathcal{R}(d) \cap L^p(\Lambda^1 T^*M)$ for all $p \in (1, n)$. Thus, in this particular example, $H_d^p(\Lambda^1 T^*M)$ is equal to the closure in L^p of $\mathcal{R}(d) \cap L^p(\Lambda^1 T^*M)$, if and only if $p \in (1, n)$.

More generally, following [9], we consider complete Riemannian manifolds (M, g) with a quadratic decay of the Ricci curvature, and, under suitable assumptions on the volume growth of balls in M , we prove that $H_d^p(\Lambda^1 T^*M)$ and the closure in L^p of $\mathcal{R}(d) \cap L^p(\Lambda^1 T^*M)$ coincide for $1 < p < \nu$, where ν is an exponent related to the volume growth of balls in M . In particular, if $n := \dim M > 2$ and M has a finite number of Euclidean ends, the conclusion holds with $\nu = n$. Moreover, in the latter situation, we also prove that, for $p \geq n$, the closure in L^p of $\mathcal{R}(d) \cap L^p(\Lambda^1 T^*M)$ is a strict subspace of $H_d^p(\Lambda^1 T^*M)$, unless M has only one end, in which case the two spaces are equal.

1.2 The geometric context

Throughout the paper, if two quantities $A(f)$ and $B(f)$ depend on a function f ranging over some space L , the notation $A(f) \lesssim B(f)$ means that there exists $C > 0$ such that $A(f) \leq CB(f)$ for all $f \in L$. Moreover, $A(f) \simeq B(f)$ means that $A(f) \lesssim B(f)$ and $B(f) \lesssim A(f)$.

Let (M, g) be a complete Riemannian manifold. Denote by μ the Riemannian measure and by d the Riemannian distance. For all $x \in M$ and all $r > 0$, $B(x, r)$ stands for the open geodesic ball with center x and radius r , and $V(x, r) := \mu(B(x, r))$. We assume that the measure μ is doubling: for all $x \in M$ and all $r > 0$,

$$V(x, 2r) \lesssim V(x, r). \quad (\text{D})$$

By iteration, this condition implies at once that there exists $D > 0$ such that for all $x \in M$ and all $0 < r < R$,

$$V(x, R) \lesssim \left(\frac{R}{r}\right)^D V(x, r). \quad (\text{VD})$$

We also consider a reverse doubling volume condition: there exists $\nu > 0$ such that, for all $x \in M$ and all $0 < r < R$,

$$\left(\frac{R}{r}\right)^\nu V(x, r) \lesssim V(x, R). \quad (\text{RD})$$

When M is connected, (RD) follows from (D) (see [18, Chapter 15, p. 412]); furthermore, the exponent ν is related to lower bounds for the p -capacity of geodesic balls, see [17, Theorem 5.6]. Fix $o \in M$ and set $r(x) := d(o, x)$ for all $x \in M$. We make the following assumption on the Ricci curvature of M : denoting Ric_x the Ricci tensor at

the point x and g_x the Riemannian metric at x , we assume that there is $\eta \geq 0$ such that

$$\text{Ric}_x \geq -\frac{\eta^2}{r^2(x)}g_x, \quad \forall x \in M \quad (\text{QD})$$

in the sense of quadratic forms. We say that a ball $B(x, r)$ is *remote* if $r \leq \frac{r(x)}{2}$. A ball $B(o, r)$ will be called *anchored*. The assumption (QD) on the Ricci curvature implies by the Bishop-Gromov theorem and a simple scaling argument that if $B(x, 2r)$ is remote, then $V(x, 2r) \lesssim V(x, r)$; hence, by [19, Prop. 4.7], (D) holds if and only if M satisfies the so-called *volume comparison* condition, which writes as follows: for every $x \in M$,

$$V(o, r(x)) \lesssim V(x, \frac{r(x)}{2}). \quad (\text{VC})$$

We notice also (see [9]) that (D) implies that M has a finite number of ends. Moreover, according to [8], (QD) implies that remote balls satisfy the scale invariant L^1 Poincaré inequality: if B is remote and has radius r then

$$\|f - f_B\|_{L^1(B)} \lesssim r \|\nabla f\|_{L^1(B)}, \quad f \in C^\infty(B), \quad (P_1)$$

where f_B denotes the average of f on B , that is $f_B := V(B)^{-1} \int_B f$.

For $0 \leq k \leq \dim M$, denote by $\Delta_k = dd^* + d^*d$ the Hodge-Laplacian acting on k -forms (here d stands for the exterior differential and d^* for its adjoint). Recall that $-\Delta_k$ generates a holomorphic semigroup on $L^2(\Lambda^k T^*M)$, and the associated heat kernel, namely the kernel of $e^{-t\Delta_k}$, is denoted by p_t^k . One denotes $p_t(x, y)$ the scalar heat kernel, i.e. the kernel of $e^{-t\Delta_0}$. We consider the Gaussian upper-bounds for the heat kernel:

$$p_t(x, y) \lesssim \frac{1}{V(x, \sqrt{t})} \exp\left(-\frac{d^2(x, y)}{ct}\right), \quad \forall t > 0, \forall x, y \in M. \quad (\text{UE})$$

Under (QD) and (VC), according to [9] there is a simple geometric condition ensuring that (UE) holds:

Definition 1.1. *We say that (M, g) with a finite number of ends satisfies the Relative Connectedness in the Ends (RCE) condition, if there is a constant $\theta \in (0, 1)$ such that for any point x with $r(x) \geq 1$, there is a continuous path $c : [0, 1] \rightarrow M$ satisfying*

- $c(0) = x$.
- *the length of c is bounded by $\frac{r(x)}{\theta}$.*
- $c([0, 1]) \subset B(o, \theta^{-1}r(x)) \setminus B(o, \theta r(x))$.
- *there is a geodesic ray $\gamma : [0, +\infty) \rightarrow M \setminus B(o, r(x))$ with $\gamma(0) = c(1)$.*

In simple words, the condition (RCE) says that any point x in M can be connected to an end by a path staying at distance approximately $r(x)$ from the origin o . With this definition, [9, Theorem 2.4] asserts that under (QD), (VC) and (RCE), the Gaussian upper-estimate (UE) for the scalar heat kernel holds.

1.3 Tent and Hardy spaces

Let us briefly recall here the definitions of Hardy spaces of differential forms on (M, g) introduced in [4]. These definitions rely on tent spaces, which we first present. For all $x \in M$ and $\alpha > 0$, the cone of aperture α and vertex x is the set

$$\Gamma_\alpha(x) = \{(y, t) \in M \times (0, +\infty); y \in B(x, \alpha t)\}.$$

When $\alpha = 1$, we write $\Gamma(x)$ instead of $\Gamma_1(x)$. For any closed set $F \subset M$, let $\mathcal{R}(F)$ be the union of all cones with aperture 1 and vertices in F . Finally, if $O \subset M$ is an open set and $F = M \setminus O$, the tent over O , denoted by $T(O)$, is the complement of $\mathcal{R}(F)$ in $M \times (0, +\infty)$.

Let $F = (F_t)_{t>0}$ be a family of measurable functions on M . Write $F(y, t) := F_t(y)$ for all $y \in M$ and all $t > 0$ and assume that F is measurable on $M \times (0, +\infty)$. Define then, for all $x \in M$,

$$\mathcal{S}F(x) = \left(\iint_{\Gamma(x)} |F(y, t)|^2 \frac{dy}{V(x, t)} \frac{dt}{t} \right)^{1/2}$$

(here and after, integration with respect to μ will often be denoted by dx, dy, \dots instead of $d\mu(x), d\mu(y), \dots$) and, if $1 \leq p < +\infty$, say that $F \in T^{p,2}(M)$ if

$$\|F\|_{T^{p,2}(M)} := \|\mathcal{S}F\|_{L^p(M)} < +\infty.$$

Denote by d the exterior differentiation and by d^* its adjoint. Define

$$H_d^2(\Lambda^1 T^* M) := \overline{\{du \in L^2(\Lambda^1 T^* M); u \in L^2(M)\}}.$$

The definition of $H_d^p(\Lambda^1 T^* M)$ for $p \neq 2$ relies on two operators, which we now present:

Definition 1.2. Let $N \geq 1$ be an integer.

1. For all $F \in T^{2,2}(M)$, let

$$\mathcal{S}_d(F) := \int_0^{+\infty} t d e^{-t\Delta} F_t \frac{dt}{t} \in L^2(\Lambda^1 T^* M).$$

2. For all $\omega \in L^2(\Lambda^1 T^* M)$ and all $t > 0$, let

$$(\mathcal{Q}_{d^*}^N \omega)_t := t d^*(t^2 \Delta_1)^N e^{-t^2 \Delta_1} \omega \in T^{2,2}(M).$$

The spectral theorem shows that, on $H_d^2(\Lambda^1 T^* M)$,

$$\mathcal{S}_d \mathcal{Q}_{d^*}^N = c \text{Id} \tag{1.1}$$

for some constant $c > 0$.

We now turn to the definitions of Hardy spaces:

Definition 1.3. Let $N \geq 1$ be an integer and $p \in (1, \infty)$.

1. Define

$$E_d^p(\Lambda^1 T^* M) := \{\omega \in H_d^2(\Lambda^1 T^* M); td^*(t^2 \Delta_1)^N e^{-t^2 \Delta_1} \omega \in T^{p,2}(M)\},$$

equipped with the norm

$$\|\omega\|_{H_d^p(\Lambda^1 T^* M)} = \left\| td^*(t^2 \Delta_1)^N e^{-t^2 \Delta_1} \omega \right\|_{T^{p,2}(M)}.$$

2. Let $H_d^p(\Lambda^1 T^* M)$ be the completion of $E_d^p(\Lambda^1 T^* M)$ under the norm $\|\cdot\|_{H_d^p(\Lambda^1 T^* M)}$.

Remark 1. As shown in [4, Section 5], these spaces do not depend on N provided that N is large enough (only depending on the parameter D in (VD)). Actually, we would still get the same spaces if $(t^2 \Delta_1)^N e^{-t^2 \Delta_1}$ was replaced by a more general function of Δ_1 .

1.4 Statement of the results

With these definitions settled, our main result states as follows:

Theorem 1.4. *Assume that (M, g) satisfies (QD), (VC), (RCE) and (RD) with some $\nu > 2$. Then, $H_d^p(\Lambda^1 T^* M) \subset L^p(\Lambda^1 T^* M)$ for all $p \in (1, \nu)$.*

Corollary 1.5. *Let M be a complete Riemannian manifold of dimension $n > 2$ with a finite number of Euclidean ends. Then, $H_d^p(\Lambda^1 T^* M) \simeq \overline{\mathcal{R}(d) \cap L^p(\Lambda^1 T^* M)}^{L^p}$ for all $p \in (1, n)$. Moreover, if $p \in [n, +\infty)$, then $H_d^p(\Lambda^1 T^* M) \simeq \overline{\mathcal{R}(d) \cap L^p(\Lambda^1 T^* M)}^{L^p}$ if M has only one end, and $\overline{\mathcal{R}(d) \cap L^p(\Lambda^1 T^* M)}^{L^p} \subsetneq H_d^p(\Lambda^1 T^* M)$ if M has two or more ends.*

Proof. The statement for $2 < p < n$ follows from Theorem 1.4 and [4, Corollary 1.2]. The statement for $1 < p \leq 2$ follows from (UE) and [4, Theorem 8.5]. It thus remains to discuss the case $p \in [n, +\infty)$. For $p \geq n$, according to [4, Corollary 1.2], the inclusion

$$\overline{\mathcal{R}(d) \cap L^p(\Lambda^1 T^* M)}^{L^p} \subset H_d^p(\Lambda^1 T^* M)$$

holds true. Moreover, by [4, Theorem 5.16], the Riesz transform is bounded from $H_{d^*}^p(\Lambda^0 T^* M)$ to $H_d^p(\Lambda^1 T^* M)$. By the argument in [16, p. 12-13], (D) and (UE) imply that $H_{d^*}^p(\Lambda^0 T^* M) \simeq L^p(M)$, hence the Riesz transform $d\Delta_0^{-1/2}$ is bounded from $L^p(M)$ to $H_d^p(\Lambda^1 T^* M)$. Since it is known that $d\Delta_0^{-1/2}$ is *not* bounded on L^p , $p \geq n$, in the case M has several Euclidean ends (see [10]), one concludes that in this case, for $p \in [n, +\infty)$,

$$\overline{\mathcal{R}(d) \cap L^p(\Lambda^1 T^* M)}^{L^p} \subsetneq H_d^p(\Lambda^1 T^* M).$$

If M has only one end, (RCE is the more familiar (RCA) condition (Relative Connectedness of Annuli) from [19], hence by [19, Corollary 5.4] M satisfies the scaled

L^2 Poincaré inequalities. According to [10], the Riesz transform on M is bounded on L^p , for every $p \in (1, +\infty)$; hence, by [16], for every $p \in [n, +\infty)$,

$$H_d^p(\Lambda^1 T^* M) \simeq \overline{\mathcal{R}(d) \cap L^p(\Lambda^1 T^* M)}^{L^p}.$$

□

More generally, for manifolds with conical ends one can fully answer the question whether H^p is equal to the closure in L^p of $\mathcal{R}(d) \cap L^p$ for the Hardy spaces of exact differential 1-forms. Recall that an end E of M is called *conical*, if there exists a compact Riemannian manifold (Σ, g_Σ) and $R > 0$, such that E is isometric to $(R, +\infty) \times \Sigma$ endowed with the metric

$$g = dr^2 + r^2 g_\Sigma.$$

The precise result writes as follows:

Corollary 1.6. *Let M be a complete Riemannian manifold of dimension $n > 2$ with a finite number of conical ends. Define a number p_* as follows: p_* is equal to n if M has two ends or more, whereas if M has only one end which is isometric to $[R, +\infty) \times \Sigma$, one lets*

$$p_* = \frac{n}{\frac{n}{2} - \sqrt{\lambda_1 + \left(\frac{n-2}{2}\right)^2}} > n,$$

where $\lambda_1 > 0$ is the first non-zero eigenvalue of the Laplacian on Σ (by convention, $p_* = +\infty$ if $\lambda_1 \geq n - 1$). Then, for all $p \in (1, p_*)$,

$$H_d^p(\Lambda^1 T^* M) \simeq \overline{\mathcal{R}(d) \cap L^p(\Lambda^1 T^* M)}^{L^p},$$

whereas for all $p \in [p_*, +\infty)$,

$$\overline{\mathcal{R}(d) \cap L^p(\Lambda^1 T^* M)}^{L^p} \subsetneq H_d^p(\Lambda^1 T^* M).$$

Remark 2. The same result holds for *asymptotically conical* manifolds in the sense of [20].

Proof. The proof is the same as for Corollary 1.5, taking into account that the Riesz transform on M is bounded on L^p , if and only if $p < p_*$ (see [20]). □

1.5 Strategy of the proof

Our strategy is as follows. Observe first that the conclusion of Theorem 1.4 is already known when $1 < p \leq 2$ ([4, Corollary 6.3]). We will therefore assume that $2 < p < \nu$. The duality of Hardy spaces implies:

Proposition 1.7. *Let $p \in (2, \infty)$, and denote $q = p'$ the conjugate exponent of p . Then, $H_d^p(\Lambda^1 T^* M) \subset L^p(\Lambda^1 T^* M)$ if the following inequality is satisfied:*

$$\left\| t d^* e^{-t^2 \Delta_1} \omega \right\|_{T^{q,2}(M)} \lesssim \|\omega\|_q, \quad \forall \omega \in L^q(\Lambda^1 T^* M), \quad (1.2)$$

Proof. Assume that (1.2) holds. Let $p \in (2, \nu)$, $\eta \in H_d^p(\Lambda^1 T^* M) \cap H_d^2(\Lambda^1 T^* M)$ and $\omega \in L^q(\Lambda^1 T^* M) \cap H_d^2(\Lambda^1 T^* M)$. Then, using the duality pairing between $T^{p,2}$ and $T^{q,2}$ ([13, Section 5, Theorem 2]) and (1.1), we get

$$\begin{aligned} \int_M \langle \eta(x), \omega(x) \rangle d\mu(x) &= c \int_M \langle \mathcal{S}_d \mathcal{Q}_d^N \eta(x), \omega(x) \rangle d\mu(x) \\ &= c \int \int_{M \times (0, +\infty)} \langle \mathcal{Q}_d^N \eta(x), \mathcal{Q}_d^* \omega(x) \rangle d\mu(x) \frac{dt}{t} \\ &\leq c \left\| td^*(t^2 \Delta_1)^N e^{-t^2 \Delta_1} \eta \right\|_{T^{p,2}} \left\| td^* e^{-t^2 \Delta_1} \omega \right\|_{T^{q,2}} \\ &\lesssim \|\eta\|_{H_d^p(\Lambda^1 T^* M)} \|\omega\|_{L^q(\Lambda^1 T^* M)}, \end{aligned}$$

where we have used the hypothesis, as well as Definition (1.3). Dividing both sides by $\|\omega\|_{L^{p'}(\Lambda^1 T^* M)}$ and taking the supremum in $\omega \neq 0$ belonging to $L^{p'}(\Lambda^1 T^* M)$, one obtains that for every $\eta \in H_d^p(\Lambda^1 T^* M) \cap H_d^2(\Lambda^1 T^* M)$,

$$\|\eta\|_{L^p(\Lambda^1 T^* M)} \lesssim \|\eta\|_{H_d^p(\Lambda^1 T^* M)}.$$

By density, this inequality extends to all $\eta \in H_d^p(\Lambda^1 T^* M)$, which in turns implies that $H_d^p(\Lambda^1 T^* M) \subset L^p(\Lambda^1 T^* M)$. □

According to Proposition 1.7, in order to prove Theorem 1.4, it is enough to establish that for all $p \in (\frac{\nu}{\nu-1}, 2)$,

$$\left\| td^* e^{-t^2 \Delta_1} \omega \right\|_{T^{p,2}(M)} \lesssim \|\omega\|_p. \quad (1.3)$$

We now introduce the inequality (1.3), in restriction to *exact* forms:

$$\left\| td^* e^{-t^2 \Delta_1} (df) \right\|_{T^{p,2}(M)} \lesssim \|df\|_p \quad (1.4)$$

The following lemma shows that (1.3) follows from (1.4) if the Riesz transform is bounded in appropriate Lebesgue spaces:

Lemma 1.8. *Let $p \in (1, 2)$, and assume that the Riesz transform is bounded on L^p and on L^q , $\frac{1}{p} + \frac{1}{q} = 1$. Then, (1.4) implies (1.3).*

Proof. It follows from the assumption that the Riesz transform and its adjoint are bounded on L^p . Therefore, the Hodge projector on exact forms,

$$\Pi = d\Delta_0^{-1}d^* = (d\Delta_0^{-1/2})(d\Delta_0^{-1/2})^*,$$

is bounded on L^p . Let $\omega \in L^p(\Lambda^1 T^* M)$, then using (1.4) with $\Pi\omega$ (which is exact) and the boundedness on L^p of Π , one obtains

$$\left\| td^* e^{-t^2 \Delta_1} \Pi\omega \right\|_{T^{p,2}(M)} \lesssim \|\Pi\omega\|_p \lesssim \|\omega\|_p.$$

Therefore, noticing that $td^* e^{-t^2 \Delta_1} \omega = td^* e^{-t^2 \Delta_1} \Pi\omega$, (1.3) holds. □

The key technical result in this work is the following:

Proposition 1.9. *Assume that (M, g) satisfies (QD), (VC), (RCE) and (RD) for some $\nu > 2$. Let $p \in (\frac{\nu}{\nu-1}, 2)$, where ν is the reverse doubling exponent from (RD). Then, (1.4) holds.*

The remaining of the article will be devoted to the proof of Proposition 1.9. Assuming for the moment the result of Proposition 1.9, let us give the proof of Theorem 1.4:

Proof of Theorem 1.4: recall ([9, Theorem A]) that, under the assumptions of Theorem 1.4, the Riesz transform $d\Delta^{-1/2}$ is L^q -bounded for all $q \in (1, \nu)$. This implies, according to Lemma 1.8, that (1.3) holds. The result then follows from Proposition 1.7. \square

In what follows, we establish (1.4).¹

Our strategy for (1.4) is as follows. This inequality amounts to

$$\|\mathcal{A}(df)\|_p \lesssim \|df\|_p, \quad (1.5)$$

where

$$\mathcal{A}\omega(x) := \left(\iint_{\Gamma(x)} \left| td^* e^{-t^2 \Delta_1} \omega(z) \right|^2 \frac{dz}{V(z, t)} \frac{dt}{t} \right)^{\frac{1}{2}}. \quad (1.6)$$

The spectral theorem implies that

$$\|\mathcal{A}\omega\|_2 \lesssim \|\omega\|_2, \quad \omega \in L^2(\Lambda^1 T^* M).$$

We express

$$td^* e^{-t^2 \Delta_1} \omega(z) = \int_M k_t(z, y) \cdot \omega(y) dy,$$

where k_t is the kernel of $td^* e^{-t^2 \Delta_1}$, and plug this expression into (1.6). Following ideas of [9], we then split the integration domain into three parts, involving different conditions on t, y, z .

The first one, called “long-to-short”, is defined by the conditions $(z, t) \in \Gamma(x)$ and $r(y) \geq \kappa r(z)$. We establish the part of inequality (1.5) corresponding to this regime thanks to pointwise bounds on $|k_t|$, which in turn follow from pointwise Gaussian type bounds on the heat kernel on functions and its gradient. More precisely, we obtain in this way a weak type $(1, 1)$ inequality, and the required L^p bound is obtained by interpolation between this weak type $(1, 1)$ inequality and a strong type $(2, 2)$ inequality.

¹Our proof of (1.4) relies on the L^p and the $L^{p'}$ boundedness of $d\Delta_0^{-1/2}$. A variation on our argument for (1.4) (with square vertical functionals instead of non-tangential ones) will show that for every $p \in (\frac{\nu}{\nu-1}, 2)$ and every function u , $\|d^* \Delta_1^{-1/2}(du)\|_p \lesssim \|du\|_p$. This is equivalent to $\|d^* \Delta_1^{-1/2} \Pi \omega\|_p \lesssim \|\Pi \omega\|_p$, where $\Pi = d\Delta_0^{-1} d^*$ is the Hodge projector. It is not clear how to get from this the boundedness of the Riesz transform on $L^{p'}$. It would be more satisfying to recover directly the boundedness of the Riesz transform from our result on $H_d^p(\Lambda^1 T^* M)$.

The second one, called “short-to-long”, is defined by the conditions $(z, t) \in \Gamma(x)$, $\kappa r(z) > r(y)$ and $d(z, y) \geq \kappa^{-1}r(z)$, and the corresponding part of (1.5) is proved by similar arguments. Note that the part of (1.4) corresponding to these two regimes holds even if the form ω is not exact.

The last part of the splitting is the so-called “diagonal regime”, defined by $(z, t) \in \Gamma(x)$ and $d(z, y) < \kappa^{-1}r(z)$. The proof of the corresponding part in (1.5) is more involved. We use a covering of M by a suitable collection of balls $(B_\alpha)_{\alpha \in A}$ which are either remote or anchored, and localize in some sense the operator \mathcal{A} in the balls B_α . When $t \geq r_\alpha$, a pointwise bound for $|k_t|$ is still sufficient. When $t < r_\alpha$, we use the fact that ω is an exact form and, writing $\omega = df$, decompose

$$\omega = \sum_{\alpha \in A} d(\chi_\alpha(f - f_{B_\alpha})) - \sum_{\alpha \in A} (f - f_{B_\alpha})d\chi_\alpha = \sum_{\alpha \in A} df_\alpha - \sum_{\alpha \in A} \eta_\alpha,$$

where $(\chi_\alpha)_{\alpha \in A}$ is a special partition of unity associated with the covering $(B_\alpha)_{\alpha \in A}$. The part corresponding to df_α is treated by arguments similar to those used in [2], and relies on $L^1 - L^2$ estimates for the heat semigroup of the Hodge-Laplacian acting on *exact* 1-forms (see Lemma 2.8 below). Roughly speaking, these estimates hold since

$$e^{-s\Delta_1}du = de^{-s\Delta_0}u$$

and pointwise estimates on the gradient of the heat kernel on functions can be used again (note that pointwise bounds on the heat kernel on 1-forms do not hold in the context of the present paper).

Finally, to treat the terms arising from η_α , we write

$$d^*e^{-s\Delta_1}\eta_\alpha = e^{-s\Delta_0}d^*\eta_\alpha,$$

and we conclude using pointwise bounds for $e^{-s\Delta_0}$, the inequality $|d\chi_\alpha| \lesssim r_\alpha^{-1}$, and the fact that, due to L^1 Poincaré inequalities on remote balls,

$$\|\eta_\alpha\|_{L^1(B_\alpha)} \lesssim \frac{1}{r_\alpha} \|f - f_{B_\alpha}\|_{L^1(B_\alpha)} \lesssim \|df\|_{L^1(B_\alpha)}.$$

The paper is organized as follows. Section 2 first presents the covering of M by remote and anchored balls, as well as the associated partition of unity. We also gather (and give proofs for) various pointwise or integrated estimates involving the heat semigroup on functions or 1-forms. The proof of (1.4) is presented in Section 3, where the three regimes are successively considered.

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2 Preliminary estimates

2.1 A good covering by admissible balls

For convenience, let us first gather definitions about balls of M (the first two ones were already introduced before):

Definition 2.1. Let $x \in M$ and $r > 0$.

1. The ball $B(x, r)$ is called *remote* if $r \leq \frac{r(x)}{2}$,
2. The ball $B(x, r)$ is called *anchored* if $x = o$,
3. The ball $B(x, r)$ is *admissible* if and only if B is remote or B is anchored and $r(B) \leq r(B_0)$, where the ball B_0 will be defined in the construction of the covering below.

We now explain how the assumption on the Ricci curvature allows one to construct a good covering of M by remote and anchored balls, as well as a good partition of unity associated to it. Following [9, Sections 4.3 and 5.3], consider a special covering of M by a countable collection of admissible balls $(B_\alpha)_{\alpha \in \mathbb{N}}$, with the finite overlap property. Let us briefly recall the construction, for the sake of completeness:

1. define $B_{0,1} := B(o, 1)$,
2. for all integer $N \geq 0$, since

$$B(o, 2^{N+1}) \setminus B(o, 2^N) \subset \bigcup_{2^N \leq r(x) < 2^{N+1}} B(x, 2^{N-13}),$$

the “5r” covering lemma ([21, Theorem 1.2]) provides a collection of points $(x_{N+1,i})_{i \in I_N} \in B(o, 2^{N+1}) \setminus B(o, 2^N)$, where the set I_N is at most countable, such that the balls $B(x_{N+1,i}, 2^{N-13})$ are pairwise disjoint and

$$B(o, 2^{N+1}) \setminus B(o, 2^N) \subset \bigcup_{i \in I_N} B(x_{N+1,i}, 2^{N-10}).$$

Since, for all $i \in I_N$, $B(x_{N+1,i}, 2^{N-13}) \subset B(o, 2^{N+2})$ and the balls $B(x_{N+1,i}, 2^{N-13})$ are pairwise disjoint, the doubling property shows that, for all finite subset $J \subset I_N$,

$$(\sharp J)V(o, 2^{N+2}) \leq \sum_{i \in J} V(x_{N+1,i}, 2^{N+3}) \lesssim \sum_{i \in J} V(x_{N+1,i}, 2^{N-13}) \leq V(o, 2^{N+2}),$$

hence the set I_N is actually finite and $\sharp I_N \leq C$ with C independent of N .

For all $N \geq 0$ and all $i \in I_N$, denoting $B_{N+1,i} = B(x_{N+1,i}, 2^{N-9})$, the balls $B_{N+1,i}$ and $7B_{N+1,i}$ are remote and satisfy

$$2^9 r(B_{N+1,i}) \leq r(x_{N+1,i}) \leq 2^{10} r(B_{N+1,i}).$$

We have constructed a countable family $(B_\alpha)_{\alpha \geq 0}$ of balls covering M ; actually the family of balls $(\frac{1}{2}B_\alpha)_{\alpha \geq 0}$ also covers M and this will be relevant later. Up to increasing the radius of B_0 and deleting balls included in B_0 , we assume that B_0 is the unique ball containing the origin o . Denoting the family of balls by $(B_\alpha)_{\alpha \in \mathbb{N}}$, by r_α the radius of B_α and by x_α its center, then for $\alpha \neq 0$,

$$2^{-10} r(x_\alpha) \leq r_\alpha \leq 2^{-9} r(x_\alpha). \quad (2.7)$$

In particular, for $\alpha \neq 0$, the balls B_α and $7B_\alpha$ are remote. Also, note that by construction, if $\alpha \neq \beta$ such that $B_\alpha \cap B_\beta \neq \emptyset$, then

$$r_\alpha \simeq r_\beta. \quad (2.8)$$

Another consequence of the construction is that there exists $C \geq 1$ such that, for all $x \in M$,

$$\sharp \{ \alpha \in \mathbb{N}; x \in B_\alpha \} \leq C.$$

In the sequel, if $B \subset M$ is a ball with radius $r(B)$, say that B is admissible if and only if B is remote or B is anchored and $r(B) \leq r(B_0)$. We also state for future use (see (P_1)):

Lemma 2.2. *For all admissible balls $B \subset M$ with radius $r(B)$ and all C^∞ functions $u \in L^1(B)$:*

1. *if B is remote,*

$$\|u - u_B\|_{L^1(B)} \lesssim r(B) \|du\|_{L^1(B)}, \quad (2.9)$$

2. *if B is anchored, (2.9) holds, as well as*

$$\|u - u_{2B}\|_{L^1(2B)} \lesssim r(B) \|du\|_{L^1(2B)}$$

for all C^∞ functions $u \in L^1(2B)$.

Let us now construct a suitable partition of unity adapted to the covering $(B_\alpha)_{\alpha \in \mathbb{N}}$.

Lemma 2.3. *There is a partition of unity $(\chi_\alpha)_{\alpha \in A}$ subordinate to $(B_\alpha)_{\alpha \in \mathbb{N}}$, satisfying, for $\alpha \in \mathbb{N}$,*

$$|d\chi_\alpha| \lesssim \frac{1}{r_\alpha + 1}, \quad |\Delta\chi_\alpha| \lesssim \frac{1}{r_\alpha^2 + 1}. \quad (2.10)$$

Proof. It is clearly enough to prove the estimates (2.10) only for $\alpha \neq 0$. By [11, Theorem 6.33] and a scaling argument, for every α , there exists a smooth function $\varphi_\alpha : M \rightarrow [0, 1]$ such that:

$$(i) \quad \varphi_\alpha|_{\frac{1}{2}B_\alpha} \equiv 1,$$

$$(ii) \quad \text{The support of } \varphi_\alpha \text{ is included in the (remote) ball } B_\alpha,$$

$$(iii) \quad |\nabla\varphi_\alpha| \lesssim \frac{1}{r_\alpha},$$

$$(iv) \quad |\Delta\varphi_\alpha| \lesssim \frac{1}{r_\alpha^2}.$$

Let

$$\varphi := \sum_{\alpha} \varphi_\alpha,$$

then $\varphi \geq 1$ on M since the family of balls $(B_\alpha)_{\alpha \geq 0}$ covers M . As a consequence of (2.8), of the fact that the covering has the finite overlap property, and of (iii) and (iv) above,

$$|\nabla \varphi| \lesssim r_\alpha^{-1}, \quad |\Delta \varphi| \lesssim r_\alpha^{-2} \quad \text{on } B_\alpha. \quad (2.11)$$

We let

$$\chi_\alpha := \frac{\varphi_\alpha}{\varphi}.$$

Obviously, $\sum_\alpha \chi_\alpha \equiv 1$, and the support of χ_α is included in B_α . Hence, $(\chi_\alpha)_{\alpha \in A}$ is a partition of unity, subordinate to $(B_\alpha)_{\alpha \in A}$. Let us check that χ_α has the desired properties. One has

$$\nabla \chi_\alpha = -\frac{\varphi_\alpha \nabla \varphi}{\varphi^2} + \frac{\nabla \varphi_\alpha}{\varphi},$$

which implies that $|\nabla \chi_\alpha| \lesssim r_\alpha^{-1}$ by using (2.11) and $\varphi \geq 1$, $0 \leq \varphi_\alpha \leq 1$. Next,

$$\begin{aligned} \Delta \chi_\alpha &= \frac{\Delta \varphi_\alpha}{\varphi} + \varphi_\alpha \Delta(\varphi^{-1}) + 2 \frac{\nabla \varphi_\alpha \cdot \nabla \varphi}{\varphi^2} \\ &= \frac{\Delta \varphi_\alpha}{\varphi} + \varphi_\alpha \left(-\frac{\Delta \varphi}{\varphi^2} + 4 \frac{|\nabla \varphi|^2}{\varphi^3} \right) + 2 \frac{\nabla \varphi_\alpha \cdot \nabla \varphi}{\varphi^2}, \end{aligned}$$

and it follows from (2.11) and $\varphi \geq 1$, $0 \leq \varphi_\alpha \leq 1$ that $|\Delta \chi_\alpha| \lesssim r_\alpha^{-2}$. \square

2.2 Heat kernel estimates

Recall that p_t denotes the kernel of $e^{-t\Delta_0}$. The hypothesis (QD) on the Ricci curvature together with the Li-Yau gradient estimate imply the following estimate for the gradient of p_t (see [9, Section 3.2-3.3]):

Lemma 2.4. *Assume that (QD) holds. Then,*

$$\|\nabla_x p_t(x, y)\| \lesssim \left(\frac{1}{\sqrt{t}} + \frac{1}{r(x) + 1} \right) \frac{1}{V(x, \sqrt{t})} e^{-c \frac{d^2(x, y)}{t}}, \quad t > 0, x, y \in M.$$

(for $r(x) \lesssim 1$, we use the fact that the Ricci curvature is bounded from below on M). By duality, Lemma 2.4 has consequences for the heat kernel on 1-forms; let $k_t(x, y)$ be the kernel of $td^*e^{-t^2\Delta_1}$. Then,

Lemma 2.5. *One has, for all $t > 0$ and all $x, y \in M$,*

$$|k_t(x, y)| \lesssim \frac{1}{V(y, t)} \left(1 + \frac{t}{r(y) + 1} \right) e^{-c \frac{d^2(x, y)}{t^2}}.$$

Proof. For all $g \in C_0^\infty(\Lambda^1 T^* M)$ and $h \in C_0^\infty(M)$,

$$\begin{aligned}
\left| \int_M t d^* e^{-t^2 \Delta_1} g(x) h(x) dx \right| &= \left| \int_M e^{-t^2 \Delta_1} g(x) \cdot t dh(x) dx \right| \\
&= \left| \int_M g(x) \cdot t e^{-t^2 \Delta_1} dh(x) dx \right| \\
&= \left| \int_M g(x) \cdot t d e^{-t^2 \Delta_0} h(x) dx \right| \\
&\lesssim \iint_M |g(x)| \frac{1}{V(x, t)} \left(1 + \frac{t}{r(x) + 1} \right) e^{-c \frac{d^2(x, y)}{t^2}} \\
&\quad \times |h(y)| dy dx,
\end{aligned}$$

where the last line follows from Lemma 2.4. \square

The following lemma deals with heat kernel estimates for complex time. Before stating the result, define, for all $\theta \in (0, \frac{\pi}{2})$,

$$\Sigma_\theta := \{z \in \mathbb{C}; |\arg z| < \theta\}.$$

Lemma 2.6. *Let $\theta < \frac{\pi}{4}$ and $\delta > 0$. The operator $V(\cdot, |z|)^\delta e^{-z^2 \Delta_0} V(\cdot, |z|)^{-\delta}$ has $L^2 \rightarrow L^2$ off-diagonal estimates for $z \in \Sigma_\theta$. More precisely, for every $x, y \in M$, and every $z \in \Sigma_\theta$,*

$$\left\| \chi_{B(y, |z|)} V(\cdot, |z|)^\delta e^{-z^2 \Delta_0} V(\cdot, |z|)^{-\delta} \chi_{B(x, |z|)} \right\|_{2 \rightarrow 2} \lesssim e^{-C \frac{d^2(x, y)}{|z|^2}}.$$

Proof. For a fixed $z \in \Sigma_\theta$, let us consider a covering of M by balls $B_i := B(x_i, |z|)$, $i \in \mathbb{N}$ with the following property: there exists $N \geq 1$ independent of z such that, for all $x \in M$, at most N balls B_i intersect $B(x, |z|)$.²

Denote $d_{ij} := d(x_i, x_j)$, and $\chi_i := \chi_{B_i}$. Then, by the properties of the covering, it is easy to see that it is enough to prove:

$$\left\| \chi_i V(\cdot, |z|)^\delta e^{-z^2 \Delta_0} V(\cdot, |z|)^{-\delta} \chi_j \right\|_{2 \rightarrow 2} \lesssim e^{-C \frac{d_{ij}^2}{|z|^2}}.$$

By doubling and Davies-Gaffney estimates for complex times (see [2, Prop 2.1], the proof of which only relies on uniform ellipticity of the operator under consideration),

²Indeed, by the “5r” covering theorem, for any fixed z there is a covering of M with balls $B_i = B(x_i, |z|)$ such that the balls $\frac{1}{5}B_i$ are pairwise disjoint. Now, if $x \in M$, $I_x := \{i; B_i \cap B(x, |z|) \neq \emptyset\}$ and $i \in I_x$, then by doubling $V(x, |z|) \simeq V(B_i)$. Thus, if we call $N_x := \#I_x$, then

$$N_x V(x, |z|) \leq C \sum_{i \in I_x} V(B_i) \leq C \sum_{i \in I_x} V\left(\frac{1}{5}B_i\right) \leq C' V(x, |z|),$$

where the constant $C > 0$ only depends on the doubling constants. It follows that $N_x \leq C'$.

$$\begin{aligned}
\left\| \chi_i V(\cdot, |z|)^\delta e^{-z^2 \Delta_0} V(\cdot, |z|)^{-\delta} \chi_j \right\|_{2 \rightarrow 2} &\lesssim \left(\frac{V(x_i, |z|)}{V(x_j, |z|)} \right)^\delta \left\| \chi_i e^{-z^2 \Delta_0} \chi_j \right\|_{2 \rightarrow 2} \\
&\lesssim \left(\frac{V(x_j, |z| + d(x_i, x_j))}{V(x_j, |z|)} \right)^\delta e^{-c \frac{d_{ij}^2}{|z|^2}} \\
&\lesssim \left(1 + \frac{d_{ij}^2}{|z|^2} \right)^{\delta D} e^{-c \frac{d_{ij}^2}{|z|^2}} \\
&\lesssim e^{-C \frac{d_{ij}^2}{|z|^2}}
\end{aligned}$$

□

We now turn to a lemma concerning Gaussian kernels. Let

$$K_t(x, y) := \frac{1}{V(x, t)} e^{-c \frac{d^2(x, y)}{t^2}}$$

be a Gaussian kernel, and K_t be the associated integral operator

$$K_t v(x) := \int_M K_t(x, y) v(y) dy,$$

defined for all measurable functions v such that the integral converges.

Lemma 2.7. *Let $1 \leq p \leq q \leq +\infty$, and denote $\gamma_{p,q} = \frac{1}{p} - \frac{1}{q}$. Let E and F be two measurable sets in M . Then, for some positive constants c_1 and c_2 , independent of the sets E and F , and for all $t > 0$,*

$$e^{c_1 \frac{d^2(E, F)}{t^2}} \|V(\cdot, t)^{\gamma_{p,q}} K_t\|_{L^p(E) \rightarrow L^q(F)} \leq c_2$$

as well as

$$e^{c_1 \frac{d^2(E, F)}{t^2}} \|K_t V(\cdot, t)^{\gamma_{p,q}}\|_{L^p(E) \rightarrow L^q(F)} \leq c_2.$$

Proof. We first claim that K_t satisfies

$$\sup_{t>0} \|V(\cdot, t)^{\gamma_{p,q}} K_t\|_{p \rightarrow q} < +\infty, \quad (2.12)$$

Indeed, let us denote $A(x, t, 0) = B(x, t)$ and $A(x, t, k) = B(x, (k+1)t) \setminus B(x, kt)$, $k \geq 1$. Let $x_0 \in M$. Then, for all $k \geq 2$, all measurable functions v supported in $A(x_0, t, k)$, and $x \in B(x_0, t)$, one has by doubling and Hölder

$$\begin{aligned}
|K_t v(x)| &= \int_{A(x_0, t, k)} \frac{e^{-c \frac{d^2(x, y)}{t^2}}}{V(x, t)} |v(y)| dy \\
&\lesssim \frac{1}{V(x_0, t)} e^{-ck^2} \mu(A(x_0, t, k))^{1-1/p} \|v\|_p \\
&\lesssim \frac{1}{V(x_0, t)^{1/p}} e^{-Ck^2} \|v\|_p.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|K_t v\|_{L^q(B(x_0, t))} &\leq V(x_0, t)^{1/q} \|K_t v\|_{L^\infty(B(x_0, t))} \\
&\lesssim \frac{1}{V(x_0, t)^{\gamma_{p,q}}} e^{-Ck^2} \|v\|_p.
\end{aligned}$$

Consequently,

$$\|\chi_{B(x_0, t)} K_t \chi_{A(x_0, t, k)}\|_{p \rightarrow q} \lesssim \frac{e^{-Ck^2}}{V(x_0, t)^{\gamma_{p,q}}}.$$

Hence, the proof of [1, Prop. 2.9] applies, and gives (2.12). This implies the result, in the case $d(E, F) = 0$. If now $d(E, F) > 0$, then for every u with support in E and every $x \in F$,

$$\begin{aligned}
|K_t u(x)| &\leq e^{-\frac{c}{2} \frac{d^2(F, E)}{t^2}} \int_E \frac{1}{V(x, t)} e^{-\frac{c}{2} \frac{d^2(x, y)}{t^2}} |u(y)| dy \\
&= e^{-\frac{c}{2} \frac{d^2(F, E)}{t^2}} \int_E \tilde{K}_t(x, y) |u(y)| dy,
\end{aligned}$$

where

$$\tilde{K}_t(x, y) = \frac{1}{V(x, t)} e^{-\frac{c}{2} \frac{d^2(x, y)}{t^2}}$$

is a Gaussian kernel. By the above argument, the associated operator \tilde{K}_t satisfies (2.12), hence with $C = c/2$,

$$\sup_{t>0} e^{C \frac{d^2(E, F)}{t^2}} \|V(\cdot, t)^{\gamma_{p,q}} K_t\|_{L^p(E) \rightarrow L^q(F)} < +\infty.$$

Finally, the inequality for $K_t V(\cdot, t)^{\gamma_{p,q}}$ can be proved by duality: indeed, it is equivalent to

$$\sup_{t>0} e^{C \frac{d^2(E, F)}{t^2}} \|V(\cdot, t)^{\gamma_{p,q}} K_t^*\|_{L^{q'}(E) \rightarrow L^{p'}(F)} < +\infty,$$

where p' and q' are the conjugate exponent to p and q respectively, and K_t^* is the adjoint operator to K_t .

The kernel of K_t^* is

$$K_t^*(x, y) = K_t(y, x) = \frac{1}{V(y, t)} e^{-c \frac{d^2(x, y)}{t^2}},$$

and using the inequality

$$\frac{V(x, t)}{V(y, t)} \leq \frac{V(y, t + d(x, y))}{V(y, t)} \lesssim \left(1 + \frac{d(x, y)}{t}\right)^D,$$

it is easily seen that

$$K_t^*(x, y) \lesssim \frac{1}{V(x, t)} e^{-C \frac{d^2(x, y)}{t^2}},$$

hence K_t^* is bounded by a Gaussian kernel. Therefore, the first part of the argument yields the inequality

$$\sup_{t>0} e^{C \frac{d^2(E, F)}{t^2}} \|V(\cdot, t)^{\gamma_{p, q}} K_t^*\|_{L^{q'}(E) \rightarrow L^{p'}(F)} < +\infty,$$

which implies

$$\sup_{t>0} e^{C \frac{d^2(E, F)}{t^2}} \|K_t V(\cdot, t)^{\gamma_{p, q}}\|_{L^p(E) \rightarrow L^q(F)} < +\infty.$$

□

The next two lemmata will be needed in order to control the heat kernel of the Hodge Laplacian acting on exact one-forms.

Lemma 2.8. *Let B be a ball such that $2B$ is admissible, and u be a function in $C_0^\infty(B)$. Let $F \subset M$ be such that*

$$r(B) \lesssim r(x) + 1, \quad \forall x \in F.$$

Then, for every $t > 0$,

$$\|V(\cdot, t)^{1/2} e^{-t^2 \Delta_1} (du)\|_{L^2(F)} \lesssim \left(1 + \frac{r(B)}{t}\right) e^{-\frac{cd(F, B)^2}{t^2}} \|du\|_1.$$

Proof. For every $x \in F$,

$$\begin{aligned} \left| e^{-t^2 \Delta_1} (du) \right| (x) &= |\nabla e^{-t^2 \Delta_0} u| (x) \\ &\leq \int_B |\nabla_x p_{t^2}(x, y)| |u(y)| dy \\ &\lesssim \left(\frac{1}{t} + \frac{1}{r(x) + 1} \right) \int_B K_t(x, y) |u(y)| dy \\ &\lesssim \left(\frac{1}{t} + \frac{1}{r(B)} \right) K_t(|u|)(x), \end{aligned} \tag{2.13}$$

where K_t is a Gaussian kernel and we have used the assumption on F and Lemma 2.4. According to Lemma 2.7, one gets

$$\sup_{t>0} e^{C \frac{d^2(F,B)}{t^2}} \|V^{1/2}(\cdot, t) e^{-t^2 \Delta_1}(du)\|_{L^2(F)} \lesssim \left(\frac{1}{t} + \frac{1}{r(B)} \right) \|u\|_{L^1(B)}.$$

Since $2B$ is admissible, it supports an L^1 Poincaré inequality with constant of order $r(B)$ by Lemma 2.2. Since u vanishes on $2B \setminus B$, one gets (see [7, Lemma 4.2.3])

$$\int_B |u| \lesssim r(B) \int_{2B} |\nabla u| = r(B) \int_B |\nabla u|. \quad (2.14)$$

Therefore, one arrives to

$$\sup_{t>0} e^{C \frac{d^2(F,B)}{t^2}} \|V^{1/2}(\cdot, t) e^{-t^2 \Delta_1}(du)\|_{L^2(F)} \lesssim \left(\frac{r(B)}{t} + 1 \right) \|du\|_{L^1(B)}.$$

and the result follows. \square

Lemma 2.9. *Let B be an admissible ball, and u be a function in $C_0^\infty(B)$. Let $0 < \theta < \frac{\pi}{2}$, and let Σ_θ denotes the sector of angle θ in \mathbb{C} . Let F be a measurable set in M . Then, for $z \in \Sigma_\theta$, there holds:*

$$\|V(\cdot, |z|)^{1/2} z d^* e^{-z^2 \Delta_1}(du)\|_{L^2(F)} \lesssim e^{-c \frac{d(F,B)^2}{|z|^2}} \|du\|_1,$$

where the various constants in the inequality are independent of the ball B and the function u .

Proof. Denote x_B the center of B . We start with the case $z = t > 0$ positive real number, for which there are two cases: either $t \leq r(x_B) + 1$, or $t > r(x_B) + 1$. For $t \leq r(x_B) + 1$, we proceed by duality: let $h \in L^2$ with support in F , then

$$\begin{aligned} \left| \int_F V(x, t)^{1/2} t d^* e^{-t^2 \Delta_1}(du)(x) \cdot h(x) dx \right| &= \left| \int_F V(x, t)^{1/2} t e^{-t^2 \Delta_0}(d^* du)(x) \cdot h(x) dx \right| \\ &= \left| \int_M d^* du(x) \cdot t e^{-t^2 \Delta_0} V(x, t)^{1/2} h(x) dx \right| \\ &= \left| \int_B du(x) \cdot \left(t d e^{-t^2 \Delta_0} V(\cdot, t)^{1/2} h \right)(x) dx \right|. \end{aligned}$$

However, by Lemma 2.4,

$$\left| \left(t d e^{-t^2 \Delta_0} V(\cdot, t)^{1/2} h \right)(x) \right| \lesssim \int_F \left(1 + \frac{t}{r(x) + 1} \right) \frac{e^{-c \frac{d^2(x,y)}{t^2}}}{V(x, t)} V(y, t)^{1/2} |h(y)| dy$$

Since $t \leq r(x_B) + 1$ and B is admissible, it follows that $t \lesssim r(x) + 1$ for every $x \in B$. Hence,

$$\left| \left(t d e^{-t^2 \Delta_0} V(\cdot, t)^{1/2} h \right)(x) \right| \lesssim \int K_t(x, y) V(y, t)^{1/2} |h(y)| dy, \quad x \in B,$$

where $K_t(x, y)$ is a Gaussian kernel. According to Lemma 2.7, one obtains

$$\sup_{t>0} e^{C\frac{d^2(F,B)}{t^2}} \left\| tde^{-t^2\Delta_0}V(\cdot, t)^{1/2} \right\|_{L^2(F) \rightarrow L^\infty(B)} < +\infty.$$

This implies

$$\left| \int_F V(x, t)^{1/2} td^*e^{-t^2\Delta_1}(du)(x) \cdot h(x) dx \right| \lesssim e^{-C\frac{d^2(F,B)}{t^2}} \|du\|_{L^1(B)} \cdot \|h\|_{L^2(F)},$$

hence

$$\|V(\cdot, t)^{1/2} td^*e^{-t^2\Delta_1}(du)\|_{L^2(F)} \lesssim e^{-c\frac{d(F,B)^2}{t^2}} \|du\|_1.$$

This proves the result for $z = t \leq r(x_B) + 1$. Now, we treat the case $t > r(x_B) + 1$: we write

$$td^*e^{-t^2\Delta_1}(du) = t\Delta_0e^{-t^2\Delta_0}u.$$

According to [15, Theorem 4], the kernel $s\frac{\partial}{\partial s}p_s(x, y)$ has pointwise Gaussian estimates. Applying this with $s = t^2$ and using Lemma 2.7,

$$\|V(\cdot, t)^{1/2} td^*e^{-t^2\Delta_1}(du)\|_{L^2(F)} \lesssim \frac{1}{t} e^{-c\frac{d^2(F,B)}{t^2}} \|u\|_1.$$

As in the proof of Lemma 2.8, (2.14) yields

$$\|V(\cdot, t)^{1/2} td^*e^{-t^2\Delta_1}(du)\|_{L^2(F)} \lesssim \frac{r(B)}{t} e^{-c\frac{d^2(F,B)}{t^2}} \|du\|_1.$$

Since B is admissible, $r(B) \lesssim r(x_B) + 1$, and because $t > r(x_B) + 1$, one gets that

$$\|V(\cdot, t)^{1/2} td^*e^{-t^2\Delta_1}(du)\|_{L^2(F)} \lesssim e^{-c\frac{d^2(F,B)}{t^2}} \|du\|_1.$$

This concludes the proof for $z = t > 0$ real; it remains to prove the lemma for complex z . We write $z^2 = (z')^2 + t^2$, where $z' \in \Sigma_\mu$ with $\mu > \theta$, $t > 0$, and

$$|z| \simeq |z'| \simeq t.$$

Then, one has

$$\begin{aligned} V(\cdot, |z|)^{1/2} z d^* e^{-z^2 \Delta_1} &= \frac{z}{t} \left(V(\cdot, |z|)^{1/2} e^{-(z')^2 \Delta_0} V(\cdot, |z|)^{-1/2} \right) \left(V(\cdot, |z|)^{1/2} t d^* e^{-t^2 \Delta_1} \right) \\ &\simeq \frac{z}{t} \left(V(\cdot, |z'|)^{1/2} e^{-(z')^2 \Delta_0} V(\cdot, |z'|)^{-1/2} \right) \left(V(\cdot, t)^{1/2} t d^* e^{-t^2 \Delta_1} \right). \end{aligned}$$

The term $\frac{z}{t}$ is bounded, while by Lemma 2.6 and the above, the operators $\left(V(\cdot, |z'|)^{1/2} e^{-(z')^2 \Delta_0} V(\cdot, |z'|)^{-1/2} \right)$ and $\left(V(\cdot, t)^{1/2} t d^* e^{-t^2 \Delta_1} \right)$ have respectively $L^2 \rightarrow L^2$ and $L^1 \rightarrow L^2$ off-diagonal estimates. The composition lemma (see [2, Proposition 3.1]) for the Euclidean case) yields the $L^1 \rightarrow L^2$ off-diagonal estimates for the composed operator

$$\left(V(\cdot, |z'|)^{1/2} e^{-(z')^2 \Delta_0} V(\cdot, |z'|)^{-1/2} \right) \left(V(\cdot, t)^{1/2} t d^* e^{-t^2 \Delta_1} \right),$$

hence the result. \square

3 Proof of Proposition 1.9

3.1 Splitting into three regimes

Recall that, for all $x \in M$,

$$\mathcal{A}\omega(x) := \left(\iint_{\Gamma(x)} \left| t d^* e^{-t^2 \Delta_1} \omega(z) \right|^2 \frac{dz}{V(z, t)} \frac{dt}{t} \right)^{\frac{1}{2}}.$$

The conclusion of Proposition 1.9 means that

$$\|\mathcal{A}(du)\|_p \lesssim \|du\|_p. \quad (3.15)$$

For the proof of (3.15), following [9], we fix a constant $\kappa \geq 2^{10}$ and, as explained in the introduction, decompose the integration domain in the definition of $\mathcal{A}\omega$ into three pieces or “regimes”, namely:

$$\mathcal{A}\omega \leq \mathcal{A}_l\omega + \mathcal{A}_s\omega + \mathcal{A}_d\omega, \quad (3.16)$$

where $\mathcal{A}_l\omega$ stands for the “long-to-short” regime, that is

$$\mathcal{A}_l\omega(x) := \left(\iint_{(z,t) \in \Gamma(x)} \left(\int_{r(y) \geq \kappa r(z)} k_t(z, y) \cdot \omega(y) dy \right)^2 \frac{dz}{V(z, t)} \frac{dt}{t} \right)^{\frac{1}{2}}, \quad (3.17)$$

$\mathcal{A}_s\omega$ stands for the “short-to-long” regime, that is

$$\mathcal{A}_s\omega(x) := \left(\iint_{(z,t) \in \Gamma(x)} \left(\int_{\kappa r(z) > r(y), d(z,y) \geq \kappa^{-1} r(z)} k_t(z, y) \cdot \omega(y) dy \right)^2 \frac{dz}{V(z, t)} \frac{dt}{t} \right)^{\frac{1}{2}}, \quad (3.18)$$

and $\mathcal{A}_d\omega$ stands for the “diagonal” regime, that is

$$\mathcal{A}_d\omega(x) := \left(\iint_{(z,t) \in \Gamma(x)} \left(\int_{d(z,y) < \kappa^{-1} r(z)} k_t(z, y) \cdot \omega(y) dy \right)^2 \frac{dz}{V(z, t)} \frac{dt}{t} \right)^{\frac{1}{2}}. \quad (3.19)$$

Recall that k_t is the kernel of $t d^* e^{-t^2 \Delta_1}$. Notice that, whenever $r(y) \geq \kappa r(z)$, one has

$$d(z, y) \geq r(y) - r(z) \geq (\kappa - 1)r(z) \geq \kappa^{-1} r(z),$$

which shows that the long-to-short and the short-to-long regimes cover the case where $d(z, y) \geq \kappa^{-1} r(z)$.

3.2 The “long-to-short” regime

In this section, we establish that, for all $\lambda > 0$,

$$\mu(\{x \in M; \mathcal{A}_l \omega(x) > \lambda\}) \lesssim \frac{\|\omega\|_1}{\lambda}. \quad (3.20)$$

To this purpose, we split \mathcal{A}_l into two parts, whether $t \geq r(y)$ or $t < r(y)$, that is

$$\begin{aligned} \mathcal{A}_l \omega(x) &\leq \left(\iint_{(z,t) \in \Gamma(x)} \left(\int_{t \geq r(y) \geq \kappa r(z)} |k_t(z,y)| |\omega(y)| dy \right)^2 \frac{dz}{V(z,t)} \frac{dt}{t} \right)^{\frac{1}{2}} \\ &+ \left(\iint_{(z,t) \in \Gamma(x)} \left(\int_{r(y) \geq \max(t, \kappa r(z))} |k_t(z,y)| |\omega(y)| dy \right)^2 \frac{dz}{V(z,t)} \frac{dt}{t} \right)^{\frac{1}{2}} \\ &=: \mathcal{A}_{l,1} \omega(x) + \mathcal{A}_{l,2} \omega(x). \end{aligned}$$

3.2.1 The case $t \geq r(y)$:

For this part,

$$\begin{aligned} \mathcal{A}_{l,1} \omega(x) &= \left(\iint_{d(z,x) \leq t} \left(\int_{\kappa r(z) \leq r(y) \leq t} |k_t(z,y)| |\omega(y)| dy \right)^2 \frac{dz}{V(z,t)} \frac{dt}{t} \right)^{\frac{1}{2}} \\ &\lesssim \left(\iint_{d(z,x) \leq t} \left(\int_{\kappa r(z) \leq r(y) \leq t} \frac{1}{V(y,t)} \frac{t}{r(y)} |\omega(y)| dy \right)^2 \frac{dz}{V(z,t)} \frac{dt}{t} \right)^{\frac{1}{2}}. \end{aligned}$$

Pick up a function $h \in L^2 \left(\Gamma(x), \frac{dz}{V(z,t)} \frac{dt}{t} \right)$ such that

$$\int_{\Gamma(x)} |h(z,t)|^2 \frac{dz}{V(z,t)} \frac{dt}{t} = 1. \quad (3.21)$$

Then, by Fubini,

$$\begin{aligned} &\iint_{d(z,x) \leq t} |h(z,t)| \left(\int_{\kappa r(z) \leq r(y) \leq t} \frac{1}{V(y,t)} \frac{t}{r(y)} |\omega(y)| dy \right) \frac{dz}{V(z,t)} \frac{dt}{t} \\ &= \int_M |\omega(y)| \left(\iint_{d(z,x) \leq t, \kappa r(z) \leq r(y) \leq t} |h(z,t)| \frac{1}{V(y,t)} \frac{t}{r(y)} \frac{dz}{V(z,t)} \frac{dt}{t} \right) dy. \end{aligned}$$

The Cauchy-Schwarz inequality and condition (3.21) yield

$$\begin{aligned} &\iint_{d(z,x) \leq t, \kappa r(z) \leq r(y) \leq t} |h(z,t)| \frac{1}{V(y,t)} \frac{t}{r(y)} \frac{dz}{V(z,t)} \frac{dt}{t} \\ &\leq \left(\iint_{d(z,x) \leq t, \kappa r(z) \leq r(y) \leq t} \frac{1}{V(y,t)^2} \frac{t^2}{r^2(y)} \frac{dz}{V(z,t)} \frac{dt}{t} \right)^{1/2}, \end{aligned}$$

so that

$$\begin{aligned}
& \iint_{d(z,x) \leq t} |h(z,t)| \left(\int_{\kappa r(z) \leq r(y) \leq t} \frac{1}{V(y,t)} \frac{t}{r(y)} |\omega(y)| dy \right) \frac{dz}{V(z,t)} \frac{dt}{t} \\
& \leq \int_M |\omega(y)| \left(\left(\iint_{d(z,x) \leq t, \kappa r(z) \leq r(y) \leq t} \frac{1}{V(y,t)^2} \frac{t^2}{r^2(y)} \frac{dz}{V(z,t)} \frac{dt}{t} \right)^{1/2} \right) dy \\
& = \int_{r(y) \geq r(x)} |\omega(y)| \left(\left(\iint_{d(z,x) \leq t, \kappa r(z) \leq r(y) \leq t} \frac{1}{V(y,t)^2} \frac{t^2}{r^2(y)} \frac{dz}{V(z,t)} \frac{dt}{t} \right)^{1/2} \right) dy \\
& + \int_{r(y) < r(x)} |\omega(y)| \left(\left(\iint_{d(z,x) \leq t, \kappa r(z) \leq r(y) \leq t} \frac{1}{V(y,t)^2} \frac{t^2}{r^2(y)} \frac{dz}{V(z,t)} \frac{dt}{t} \right)^{1/2} \right) dy \\
& =: I_1 + I_2.
\end{aligned} \tag{3.22}$$

Notice that

$$\begin{aligned}
\int_{\kappa r(z) \leq r(y) \leq t} \frac{1}{V(z,t)} dz &= \frac{1}{V(o,t)} \int_{\kappa r(z) \leq r(y) \leq t} \frac{V(o,t)}{V(z,t)} dz \\
&\leq \frac{1}{V(o,t)} \int_{\kappa r(z) \leq r(y) \leq t} \frac{V(z,t+r(z))}{V(z,t)} dz \\
&\lesssim \frac{V(o,r(y))}{V(o,t)}.
\end{aligned} \tag{3.23}$$

We therefore estimate the innermost integral in I_1 as follows:

$$\begin{aligned}
& \iint_{d(z,x) \leq t, \kappa r(z) \leq r(y) \leq t} \frac{1}{V(y,t)^2} \frac{t^2}{r^2(y)} \frac{dz}{V(z,t)} \frac{dt}{t} \\
&= \int_{r(y)}^{+\infty} \frac{1}{V(y,t)^2} \frac{t^2}{r^2(y)} \left(\int_{d(z,x) \leq t, \kappa r(z) \leq r(y)} \frac{dz}{V(z,t)} \right) \frac{dt}{t} \\
&\lesssim \int_{r(y)}^{+\infty} \frac{1}{V(y,t)^2} \frac{t^2}{r^2(y)} \frac{V(o,r(y))}{V(o,t)} \frac{dt}{t} \\
&= \int_{r(y)}^{+\infty} \frac{1}{V(o,t)^2} \frac{V(o,t)^2}{V(y,t)^2} \frac{t^2}{r^2(y)} \frac{V(o,r(y))}{V(o,t)} \frac{dt}{t} \\
&\lesssim \int_{r(y)}^{+\infty} \frac{1}{V(o,t)^2} \frac{t^2}{r^2(y)} \frac{V(o,r(y))}{V(o,t)} \frac{dt}{t} \\
&= \frac{1}{V(o,r(y))^2} \int_{r(y)}^{+\infty} \frac{V(o,r(y))^3}{V(o,t)^3} \frac{t^2}{r^2(y)} \frac{dt}{t} \\
&\lesssim \frac{1}{V(o,r(y))^2} \int_{r(y)}^{+\infty} \left(\frac{r(y)}{t} \right)^{3\nu-2} \frac{dt}{t} \lesssim \frac{1}{V(o,r(y))^2}.
\end{aligned}$$

where the third line holds since $r(y) \leq t$ and the fifth line follows from $V(o,t) \leq V(y,t+r(y)) \lesssim V(y,t)$ since $r(y) \leq t$. As a consequence,

$$I_1 \lesssim \int_{r(y) \geq r(x)} \frac{|\omega(y)|}{V(o,r(y))} dy \leq \frac{1}{V(o,r(x))} \|\omega\|_1. \tag{3.24}$$

For I_2 , notice first that, when $d(z,x) \leq t$ et $\kappa r(z) \leq r(y) \leq t$,

$$r(x) \leq r(z) + t \leq \frac{1}{\kappa} r(y) + t \leq 2t. \tag{3.25}$$

On the other hand,

$$d(x, y) \leq d(x, z) + d(z, y) \leq t + r(z) + r(y) \leq 3t. \quad (3.26)$$

Gathering (3.25) and (3.26) and using (3.23) again, we obtain

$$\begin{aligned} & \iint_{\substack{d(z,x) \leq t, \\ \kappa r(z) \leq r(y) \leq t}} \frac{1}{V(y,t)^2} \frac{t^2}{r^2(y)} \frac{dz}{V(z,t)} \frac{dt}{t} \\ & \leq \int_{\frac{r(x)}{2}}^{+\infty} \frac{1}{V(y,t)^2} \frac{t^2}{r^2(y)} \left(\int_{d(z,x) \leq t, \kappa r(z) \leq r(y)} \frac{dz}{V(z,t)} \right) \frac{dt}{t} \\ & \lesssim \int_{\frac{r(x)}{2}}^{+\infty} \frac{1}{V(y,t)^2} \frac{t^2}{r^2(y)} \frac{V(o, r(y))}{V(o, t)} \frac{dt}{t} \\ & \lesssim \int_{\frac{r(x)}{2}}^{+\infty} \frac{1}{V(x,t)^2} \frac{t^2}{r^2(y)} \frac{V(o, r(y))}{V(o, t)} \frac{dt}{t} \\ & \lesssim \frac{1}{V(x, r(x))^2} \int_{\frac{r(x)}{2}}^{+\infty} \frac{t^2}{r^2(y)} \frac{V(o, r(y))}{V(o, t)} \frac{dt}{t} \\ & \lesssim \frac{1}{V(o, r(x))^2} \int_{\frac{r(x)}{2}}^{+\infty} \left(\frac{r(y)}{t} \right)^{\nu-2} \frac{dt}{t} \\ & \lesssim \frac{1}{V(o, r(x))^2}, \end{aligned}$$

where the fourth line uses (3.26). As a consequence,

$$I_2 \lesssim \frac{1}{V(o, r(x))} \int_{r(y) < r(x)} |\omega(y)| dy \leq \frac{1}{V(o, r(x))} \|\omega\|_1. \quad (3.27)$$

Using (3.22), (3.24) and (3.27), we conclude that

$$\begin{aligned} & \iint_{\substack{d(z,x) \leq t \\ \mathbf{I}}} |h(z, t)| \left(\int_{\kappa r(z) \leq r(y) \leq t} \frac{1}{V(y, t)} \frac{t}{r(y)} |\omega(y)| dy \right) \frac{dz}{V(z, t)} \frac{dt}{t} \\ & \lesssim \frac{1}{V(o, r(x))} \|\omega\|_1, \end{aligned}$$

and finally, taking the supremum over all functions $h \in L^2 \left(\Gamma(x), \frac{dz}{V(z, t)} \frac{dt}{t} \right)$ satisfying (3.21),

$$\mathcal{A}_{l,1}\omega(x) \lesssim \frac{1}{V(o, r(x))} \|\omega\|_1.$$

Thus, if $\lambda > 0$ and $\mathcal{A}_{l,1}\omega(x) > \lambda$, then $V(o, r(x)) \leq \frac{\|\omega\|_1}{\lambda}$. Lemma 3.2 in the Appendix therefore yields

$$\mu(\{x \in M; \mathcal{A}_{l,1}\omega(x) > \lambda\}) \lesssim \frac{\|\omega\|_1}{\lambda}. \quad (3.28)$$

3.2.2 The case $t \leq r(y)$:

Here,

$$\begin{aligned}\mathcal{A}_{l,2}\omega(x) &= \left(\iint_{d(z,x) \leq t} \left(\int_{\max(\kappa r(z), t) \leq r(y)} |k_t(z, y)| |\omega(y)| dy \right)^2 \frac{dz}{V(z, t)} \frac{dt}{t} \right)^{\frac{1}{2}} \\ &\lesssim \left(\iint_{d(z,x) \leq t} \left(\int_{\max(\kappa r(z), t) \leq r(y)} \frac{1}{V(y, t)} e^{-c \frac{r^2(y)}{t^2}} |\omega(y)| dy \right)^2 \frac{dz}{V(z, t)} \frac{dt}{t} \right)^{\frac{1}{2}},\end{aligned}\tag{3.29}$$

where the last line holds since

$$r(y) \leq r(z) + d(z, y) \leq \frac{1}{\kappa} r(y) + d(z, y)$$

so that $r(y) \lesssim d(z, y)$. As in the previous case, we therefore have to estimate

$$\begin{aligned}& \int_{r(y) \geq r(x)} |\omega(y)| \left(\left(\iint_{d(z,x) \leq t, \max(\kappa r(z), t) \leq r(y)} \frac{1}{V(y, t)^2} e^{-c \frac{r^2(y)}{t^2}} \frac{dz}{V(z, t)} \frac{dt}{t} \right)^{1/2} \right) dy \\ &+ \int_{r(y) < r(x)} |\omega(y)| \left(\left(\iint_{d(z,x) \leq t, \max(\kappa r(z), t) \leq r(y)} \frac{1}{V(y, t)^2} e^{-c \frac{r^2(y)}{t^2}} \frac{dz}{V(z, t)} \frac{dt}{t} \right)^{1/2} \right) dy \\ &=: I_1 + I_2.\end{aligned}$$

Since $V(o, t) \leq V(z, t + r(z)) \leq V(z, t + r(y)) \lesssim V(z, t) \left(1 + \frac{r(y)}{t}\right)^D$ whenever $\kappa r(z) \leq r(y)$, we estimate the innermost integral in I_1 by

$$\begin{aligned}& \iint_{d(z,x) \leq t, \max(\kappa r(z), t) \leq r(y)} \frac{1}{V(y, t)^2} e^{-c \frac{r^2(y)}{t^2}} \frac{dz}{V(z, t)} \frac{dt}{t} \\ &= \int_0^{r(y)} \frac{1}{V(y, t)^2} e^{-c \frac{r^2(y)}{t^2}} \left(\int_{\kappa r(z) \leq r(y)} \frac{dz}{V(z, t)} \right) \frac{dt}{t} \\ &\lesssim \int_0^{r(y)} \frac{V(o, r(y))}{V(o, t)} \frac{1}{V(y, t)^2} \left(1 + \frac{r(y)}{t}\right)^D e^{-c \frac{r^2(y)}{t^2}} \frac{dt}{t} \\ &\leq \int_0^{r(y)} \left(\frac{r(y)}{t}\right)^D \frac{1}{V(y, r(y))^2} \frac{V(y, r(y))^2}{V(y, t)^2} \left(1 + \frac{r(y)}{t}\right)^D e^{-c \frac{r^2(y)}{t^2}} \frac{dt}{t} \\ &\lesssim \frac{1}{V(o, r(y))^2}.\end{aligned}$$

It follows that

$$I_1 \lesssim \int_{r(y) \geq r(x)} \frac{|\omega(y)|}{V(o, r(y))} dy \leq \frac{\|\omega\|_1}{V(o, r(x))}.\tag{3.30}$$

For the innermost integral in I_2 , since $\kappa r(z) \leq r(y) \leq r(x)$,

$$r(x) \leq r(z) + t \leq \frac{1}{\kappa} r(y) + t \leq \frac{1}{\kappa} r(x) + t,$$

one has $r(x) \lesssim t \leq r(y) \leq r(x)$, which entails

$$\begin{aligned}
& \iint_{d(z,x) \leq t, \max(\kappa r(z), t) \leq r(y)} \frac{1}{V(y, t)^2} e^{-c \frac{r^2(y)}{t^2}} \frac{dz}{V(z, t)} \frac{dt}{t} \\
&= \int_{cr(x)}^{r(y)} \frac{1}{V(y, t)^2} e^{-c \frac{r^2(y)}{t^2}} \left(\int_{B(x, t)} \frac{dz}{V(z, t)} \right) \frac{dt}{t} \\
&\lesssim \int_{cr(x)}^{r(y)} \frac{1}{V(y, t)^2} e^{-c \frac{r^2(y)}{t^2}} \frac{dt}{t} \\
&\leq \frac{1}{V(y, r(y))^2} \int_{cr(x)}^{r(y)} \left(\frac{r(y)}{t} \right)^{2D} e^{-c \frac{r^2(y)}{t^2}} \frac{dt}{t} \\
&\lesssim \frac{1}{V(y, r(y))^2} \lesssim \frac{1}{V(x, r(x))^2},
\end{aligned}$$

therefore

$$I_2 \lesssim \frac{1}{V(x, r(x))} \|\omega\|_1 \lesssim \frac{1}{V(o, r(x))} \|\omega\|_1. \quad (3.31)$$

Gathering (3.29), (3.30) and (3.31), we obtain

$$\mathcal{A}_{l,2}\omega(x) \lesssim \frac{1}{V(o, r(x))} \|\omega\|_1,$$

and, using Lemma 3.2 again, we conclude that

$$\mu(\{x \in M; \mathcal{A}_{l,2}\omega(x) > \lambda\}) \lesssim \frac{\|\omega\|_1}{\lambda}. \quad (3.32)$$

The conjunction of (3.28) and (3.32) yields (3.20).

3.3 The “short-to-long” regime

This section is devoted to the analysis of $\mathcal{A}_s\omega$. Again, we split this term into two parts: we bound $\mathcal{A}_s\omega(x)$ by the sum $\mathcal{A}_{s,1}\omega(x) + \mathcal{A}_{s,2}\omega(x)$, where

$$\mathcal{A}_{s,1}\omega(x) = \left(\iint_{(z,t) \in \Gamma(x)} \left(\int_{\kappa r(z) > r(y), d(z,y) \geq \kappa^{-1}r(z), t \geq r(y)} |k_t(z, y)| |\omega(y)| dy \right)^2 \frac{dz}{V(z, t)} \frac{dt}{t} \right)^{\frac{1}{2}}$$

and

$$\mathcal{A}_{s,2}\omega(x) = \left(\iint_{(z,t) \in \Gamma(x)} \left(\int_{\kappa r(z) > r(y), d(z,y) \geq \kappa^{-1}r(z), t < r(y)} |k_t(z, y)| |\omega(y)| dy \right)^2 \frac{dz}{V(z, t)} \frac{dt}{t} \right)^{\frac{1}{2}}$$

In this regime, we will assume and use the fact that $p > \frac{\nu}{\nu-1}$. We then intend to prove that, for all $\lambda > 0$,

$$\mu(\{x \in M; \mathcal{A}_{s,1}\omega(x) > \lambda\}) \lesssim \frac{\|\omega\|_p^p}{\lambda^p} \quad (3.33)$$

and

$$\mu(\{x \in M; \mathcal{A}_{s,2}\omega(x) > \lambda\}) \lesssim \frac{\|\omega\|_1}{\lambda}. \quad (3.34)$$

Note that, in this short-to-long regime, since $d(y, z) \geq \kappa^{-1}r(z)$,

$$|k_t(z, y)| \lesssim \frac{1}{V(z, t)} \left(1 + \frac{t}{r(y)}\right) e^{-c\frac{r^2(z)}{t^2}}. \quad (3.35)$$

Indeed, using $r(z) \lesssim d(y, z)$ and doubling, one has

$$\begin{aligned} |k_t(z, y)| &\lesssim \frac{1}{V(y, t)} \left(1 + \frac{t}{r(y)}\right) e^{-c\frac{d^2(y, z)}{t^2}} \\ &\lesssim \frac{1}{V(z, t)} \frac{V(y, t + d(y, z))}{V(y, t)} \left(1 + \frac{t}{r(y)}\right) e^{-c\frac{d^2(y, z)}{t^2}} \\ &\lesssim \frac{1}{V(z, t)} \left(1 + \frac{t}{r(y)}\right) e^{-c\frac{r^2(z)}{t^2}}. \end{aligned}$$

3.3.1 The case $t \geq r(y)$

As in the corresponding case of Section 3.2, using (3.35), one has

$$\begin{aligned} \mathcal{A}_{s,1}\omega(x) &\leq \left(\iint_{d(z,x) \leq t} \left(\int_{\kappa r(z) > r(y), d(z,y) \geq \kappa^{-1}r(z), r(y) \leq t} |k_t(z, y)| |\omega(y)| dy \right)^2 \frac{dz}{V(z, t)} \frac{dt}{t} \right)^{\frac{1}{2}} \\ &\lesssim \left(\iint_{d(z,x) \leq t} \left(\int_{\kappa r(z) > r(y), d(z,y) \geq \kappa^{-1}r(z), r(y) \leq t} \frac{1}{V(z, t)} \frac{t}{r(y)} e^{-c\frac{r^2(z)}{t^2}} |\omega(y)| dy \right)^2 \frac{dz}{V(z, t)} \frac{dt}{t} \right)^{\frac{1}{2}} \\ &\leq \left(\iint_{d(z,x) \leq t} \frac{1}{V(z, t)^2} e^{-c\frac{r^2(z)}{t^2}} \left(\int_{\kappa r(z) > r(y), d(z,y) \geq \kappa^{-1}r(z), r(y) \leq t} \frac{t}{r(y)} |\omega(y)| dy \right)^2 \frac{dz}{V(z, t)} \frac{dt}{t} \right)^{\frac{1}{2}} \end{aligned}$$

By the Hölder inequality and Lemma 3.1 from the Appendix, since $p' < \nu$,

$$\begin{aligned} \int_{\kappa r(z) > r(y), d(z,y) \geq \kappa^{-1}r(z), r(y) \leq t} \frac{t}{r(y)} |\omega(y)| dy &\leq t \|\omega\|_p \left(\int_{r(y) \leq \min(\kappa r(z), t)} \frac{1}{r(y)^{p'}} dy \right)^{\frac{1}{p'}} \\ &\lesssim t \|\omega\|_p \min(\kappa r(z), t)^{-1} V(o, \min(\kappa r(z), t))^{\frac{1}{p'}}. \end{aligned}$$

As a consequence,

$$\begin{aligned} \mathcal{A}_{s,1}\omega(x) &\lesssim \|\omega\|_p \left(\iint_{d(z,x) \leq t} \frac{t^2}{V(z, t)^3} e^{-c\frac{r^2(z)}{t^2}} (\min(\kappa r(z), t))^{-2} V(o, \min(\kappa r(z), t))^{\frac{2}{p'}} dz \frac{dt}{t} \right)^{\frac{1}{2}} \\ &\leq \|\omega\|_p \left(\iint_{d(z,x) \leq t, t \leq \frac{r(x)}{2}} \frac{t^2}{V(z, t)^3} e^{-c\frac{r^2(z)}{t^2}} (\min(\kappa r(z), t))^{-2} V(o, \min(\kappa r(z), t))^{\frac{2}{p'}} dz \frac{dt}{t} \right)^{\frac{1}{2}} \\ &\quad + \|\omega\|_p \left(\iint_{d(z,x) \leq t, t > \frac{r(x)}{2}} \frac{t^2}{V(z, t)^3} e^{-c\frac{r^2(z)}{t^2}} (\min(\kappa r(z), t))^{-2} V(o, \min(\kappa r(z), t))^{\frac{2}{p'}} dz \frac{dt}{t} \right)^{\frac{1}{2}} \\ &=: I_1 + I_2. \end{aligned} \quad (3.36)$$

When $d(z, x) \leq t$ and $t \leq \frac{r(x)}{2}$, then $r(x) \leq r(z) + t \leq r(z) + \frac{r(x)}{2}$, and $r(z) \leq r(x) + t \leq \frac{3r(x)}{2}$, so that $r(x) \simeq r(z)$ and $\min(\kappa r(z), t) \simeq t$, which shows that

$$\begin{aligned}
I_1 &\lesssim \|\omega\|_p \left(\iint_{d(z,x) \leq t, t \leq \frac{r(x)}{2}} \frac{1}{V(z, t)^3} e^{-c \frac{r^2(z)}{t^2}} V(o, t)^{\frac{2}{p'}} dz \frac{dt}{t} \right)^{\frac{1}{2}} \\
&\lesssim \|\omega\|_p \left(\int_0^{\frac{r(x)}{2}} \frac{1}{V(x, t)^2} e^{-c \frac{r^2(x)}{t^2}} V(o, t)^{\frac{2}{p'}} \frac{dt}{t} \right)^{\frac{1}{2}} \\
&\lesssim \frac{\|\omega\|_p}{V(x, r(x))} V(o, r(x))^{\frac{1}{p'}} \left(\int_0^{\frac{r(x)}{2}} \left(\frac{r(x)}{t} \right)^D e^{-c \frac{r^2(x)}{t^2}} \frac{dt}{t} \right)^{\frac{1}{2}} \\
&\lesssim \frac{\|\omega\|_p}{V(o, r(x))^{\frac{1}{p}}}.
\end{aligned} \tag{3.37}$$

When $d(z, x) \leq t$ and $t > \frac{r(x)}{2}$, then $r(z) \leq r(x) + t \leq 3t$, so that $\min(\kappa r(z), t) \simeq r(z)$. Therefore,

$$\begin{aligned}
I_2 &\lesssim \|\omega\|_p \left(\iint_{d(z,x) \leq t, t > \frac{r(x)}{2}} \frac{t^2}{V(z, t)^3} e^{-c \frac{r^2(z)}{t^2}} r(z)^{-2} V(o, r(z))^{\frac{2}{p'}} dz \frac{dt}{t} \right)^{\frac{1}{2}} \\
&\lesssim \|\omega\|_p \left(\int_{\frac{r(x)}{2}}^{+\infty} \frac{1}{V(x, t)^3} \left(\int_{d(z,x) \leq t, r(z) \leq 3t} \frac{t^2}{r(z)^2} e^{-c \frac{r^2(z)}{t^2}} V(o, r(z))^{\frac{2}{p'}} dz \right) \frac{dt}{t} \right)^{\frac{1}{2}} \\
&\lesssim \|\omega\|_p \left(\int_{\frac{r(x)}{2}}^{+\infty} \frac{1}{V(x, t)^3} \left(\int_{d(z,x) \leq t, r(z) \leq 3t} \frac{t^2}{r(z)^2} \left(\frac{r(z)}{t} \right)^{\frac{2\nu}{p'}} e^{-c \frac{r^2(z)}{t^2}} dz \right) V(o, t)^{\frac{2}{p'}} \frac{dt}{t} \right)^{\frac{1}{2}} \\
&\lesssim \|\omega\|_p \left(\int_{\frac{r(x)}{2}}^{+\infty} \frac{1}{V(x, t)^2} V(o, t)^{\frac{2}{p'}} \frac{dt}{t} \right)^{\frac{1}{2}} \\
&\lesssim \|\omega\|_p \left(\int_{\frac{r(x)}{2}}^{+\infty} V(x, t)^{-\frac{2}{p}} \frac{dt}{t} \right)^{\frac{1}{2}} \\
&\lesssim \|\omega\|_p V(x, r(x))^{-\frac{1}{p}} \left(\int_{\frac{r(x)}{2}}^{+\infty} \left(\frac{r(x)}{t} \right)^{\frac{2\nu}{p}} \frac{dt}{t} \right)^{\frac{1}{2}} \\
&\lesssim \|\omega\|_p V(x, r(x))^{-\frac{1}{p}},
\end{aligned} \tag{3.38}$$

where the third line follows from the second one since $\frac{2\nu}{p'} > 2$ (this, in turn, is due to the fact that $p > \frac{\nu}{\nu-1}$). Thus, (3.36), (3.37) and (3.38) yield

$$\mathcal{A}_{s,1}\omega(x) \lesssim \frac{\|\omega\|_p}{V(o, r(x))^{\frac{1}{p}}}.$$

Lemma 3.2 therefore ensures that (3.33) holds.

3.3.2 The case $t \leq r(y)$:

In this case, following the argument in Section 3.2.2 and using (3.35) again, one obtains

$$\begin{aligned}
\mathcal{A}_{s,2}\omega(x) &\leq \left(\iint_{d(z,x) \leq t} \left(\int_{\kappa r(z) > r(y), d(z,y) \geq \kappa^{-1}r(z), t \leq r(y)} |k_t(z,y)| |\omega(y)| dy \right)^2 \frac{dz}{V(z,t)} \frac{dt}{t} \right)^{\frac{1}{2}} \\
&\lesssim \left(\iint_{d(z,x) \leq t} \left(\int_{\kappa r(z) > r(y), d(z,y) \geq \kappa^{-1}r(z), t \leq r(y)} \frac{1}{V(z,t)} e^{-c\frac{r^2(z)}{t^2}} |\omega(y)| dy \right)^2 \frac{dz}{V(z,t)} \frac{dt}{t} \right)^{\frac{1}{2}} \\
&\lesssim \int_M |\omega(y)| \left(\int_0^{r(y)} \frac{1}{V(z,t)^3} \left(\int_{d(z,x) \leq t, \kappa r(z) > r(y), d(z,y) \geq \kappa^{-1}r(z)} e^{-c\frac{r^2(z)}{t^2}} dz \right) \frac{dt}{t} \right)^{1/2} dy \\
&\lesssim \int_M |\omega(y)| \left(\int_0^{r(y)} \frac{1}{V(x,t)^3} \left(\int_{d(z,x) \leq t, \kappa r(z) > r(y), d(z,y) \geq \kappa^{-1}r(z)} e^{-c\frac{r^2(z)}{t^2}} dz \right) \frac{dt}{t} \right)^{1/2} dy \\
&=: \int_M |\omega(y)| I(y) dy.
\end{aligned} \tag{3.39}$$

When $\frac{1}{2}r(x) \leq r(y)$, then, using the doubling property, we simply estimate

$$\begin{aligned}
I(y) &\lesssim \left(\int_0^{r(y)} \frac{1}{V(x,t)^2} e^{-c\frac{r^2(y)}{t^2}} \frac{dt}{t} \right)^{1/2} \\
&\lesssim \frac{1}{V(x, r(y))} \lesssim \frac{1}{V(x, r(x))} \lesssim \frac{1}{V(o, r(x))}.
\end{aligned} \tag{3.40}$$

If $r(y) < \frac{1}{2}r(x)$, then

$$r(x) \leq r(z) + d(x, z) \leq r(z) + t \leq r(z) + r(y) < r(z) + \frac{1}{2}r(x),$$

so that $r(x) \leq 2r(z)$. As a consequence,

$$\begin{aligned}
I(y) &\lesssim \left(\int_0^{r(y)} \frac{1}{V(x,t)^2} e^{-c\frac{r^2(x)}{t^2}} \frac{dt}{t} \right)^{1/2} \\
&\leq \left(\int_0^{r(x)} \frac{1}{V(x,t)^2} e^{-c\frac{r^2(x)}{t^2}} \frac{dt}{t} \right)^{1/2} \\
&\lesssim \frac{1}{V(x, r(x))} \lesssim \frac{1}{V(o, r(x))}.
\end{aligned} \tag{3.41}$$

Gathering (3.39), (3.40) and (3.41), we obtain

$$\mathcal{A}_{s,2}\omega(x) \lesssim \frac{1}{V(o, r(x))} \|\omega\|_1,$$

and we conclude as before that (3.34) holds.

3.4 The “diagonal” regime

We now turn to the range $d(z, y) < \kappa^{-1}r(z)$. As in [9, Sections 4.3 and 5.3], we will use the covering $(B_\alpha)_{\alpha \in \mathbb{N}}$ of M by admissible balls introduced in Section 2.1, as well as the associated partition of unity. Let $\alpha \in \mathbb{N}$ and $y \in B_\alpha$. If $d(z, y) \leq \kappa^{-1}r(z)$, then, by (2.7),

$$\begin{aligned} d(z, x_\alpha) &\leq d(z, y) + d(y, x_\alpha) \\ &\leq \kappa^{-1}r(z) + r_\alpha \\ &\leq \kappa^{-1}d(z, x_\alpha) + \kappa^{-1}r(x_\alpha) + r_\alpha \\ &\leq \kappa^{-1}d(z, x_\alpha) + (1 + 2^{10}\kappa^{-1})r_\alpha, \end{aligned}$$

and a short computation shows that since $\kappa \geq 2^{10}$ by assumption, one has $d(z, x_\alpha) \leq 4r_\alpha$, that is $z \in 4B_\alpha$. Therefore,

$$\begin{aligned} \mathcal{A}_{d\omega}(x) &\leq \sum_{\alpha} \left(\iint_{(z,t) \in \Gamma(x), z \in 4B_\alpha} \left| \int_{d(z,y) < \kappa^{-1}r(z)} k_t(z, y) \cdot (\chi_\alpha \omega)(y) dy \right|^2 \frac{dz}{V(z, t)} \frac{dt}{t} \right)^{\frac{1}{2}} \\ &=: \sum_{\alpha} \mathcal{A}_{d,\alpha}(\omega)(x). \end{aligned}$$

Fix $\alpha \in \mathbb{N}$ and split

$$\begin{aligned} \mathcal{A}_{d,\alpha}(\omega)(x) &\leq \left(\iint_{(z,t) \in \Gamma(x), z \in 4B_\alpha, t \geq r_\alpha} \left| \int_{d(z,y) < \kappa^{-1}r(z)} k_t(z, y) \cdot (\chi_\alpha \omega)(y) dy \right|^2 \frac{dz}{V(z, t)} \frac{dt}{t} \right)^{\frac{1}{2}} \\ &\quad + \left(\iint_{(z,t) \in \Gamma(x), z \in 4B_\alpha, t < r_\alpha} \left(\int_{d(z,y) < \kappa^{-1}r(z)} k_t(z, y) \cdot (\chi_\alpha \omega)(y) dy \right)^2 \frac{dz}{V(z, t)} \frac{dt}{t} \right)^{\frac{1}{2}} \\ &=: \mathcal{A}_{d,\alpha,1}\omega(x) + \mathcal{A}_{d,\alpha,2}\omega(x). \end{aligned}$$

3.4.1 The case $t \geq r_\alpha$:

We intend to prove that

$$\mu \left(\left\{ x \in M; \sum_{\alpha} \mathcal{A}_{d,\alpha,1}\omega(x) > \lambda \right\} \right) \lesssim \frac{\|\omega\|_1}{\lambda}. \quad (3.42)$$

We use the upper bound

$$|k_t(z, y)| \lesssim \frac{1}{V(y, t)} \frac{t}{r(y) + 1},$$

which follows from Lemma 2.5. Indeed, note that, for all α and all $y \in B_\alpha$, $r(y) \lesssim r_\alpha$. As a consequence, $r(y) \lesssim t$ in the range under consideration.

As in section 3.2.1, one has

$$\mathcal{A}_{d,\alpha,1}(\omega)(x) \leq \int_{y \in B_\alpha} |\omega(y)| I(y) dy,$$

where

$$I(y) \lesssim \left(\int_{d(z,x) \leq t, t \geq r_\alpha, z \in 4B_\alpha} \frac{1}{V(y,t)^3} \left(\frac{t^2}{r_\alpha^2} \right) dz \frac{dt}{t} \right)^{1/2}.$$

Noticing that, for all $y \in B_\alpha$, $4B_\alpha \subset B(y, 5r_\alpha)$, we write

$$\begin{aligned} I(y) &\lesssim \left(\int_{t \geq r_\alpha} \frac{V(y, 5r_\alpha)}{V(y,t)^3} \left(\frac{t^2}{r_\alpha^2} \right) \frac{dt}{t} \right)^{1/2} \\ &\lesssim \frac{1}{V(y, r_\alpha)} \left(\int_{r_\alpha}^{+\infty} \frac{V(y, r_\alpha)^3}{V(y,t)^3} \left(\frac{t^2}{r_\alpha^2} \right) \frac{dt}{t} \right)^{1/2} \\ &\lesssim \frac{1}{V(y, r_\alpha)} \left(\int_{r_\alpha}^{+\infty} \left(\frac{r_\alpha}{t} \right)^{3\nu-2} \frac{dt}{t} \right)^{1/2} \\ &\lesssim \frac{1}{V(y, r_\alpha)}. \end{aligned}$$

As a consequence,

$$\begin{aligned} \mathcal{A}_{d,\alpha,1}(\omega)(x) &\lesssim \int_{y \in B_\alpha} \frac{|\omega(y)|}{V(y, r_\alpha)} dy \\ &\lesssim \frac{1}{V(B_\alpha)} \|\omega\|_{L^1(B_\alpha)}, \end{aligned}$$

so that

$$\mathcal{A}_{d,\alpha,1}(\omega)(x) \lesssim \frac{1}{V(o, r(x))} \|\omega\|_{L^1(B_\alpha)}, \quad (3.43)$$

whenever $r(x) \leq 2(4 + 2^{10})r_\alpha$.

Consider now the case where $r(x) > 2(4 + 2^{10})r_\alpha$. Then, for all $z \in 4B_\alpha$, $r(z) \leq 4r_\alpha + r(x_\alpha) \leq (4 + 2^{10})r_\alpha < \frac{r(x)}{2}$, so that $d(x, z) \geq \frac{r(x)}{2}$. As a consequence,

$$\begin{aligned} I(y) &\lesssim \left(\int_{t \geq \frac{1}{2}r(x)} \frac{V(y, 5r_\alpha)}{V(y,t)^3} \left(\frac{t^2}{r_\alpha^2} \right) \frac{dt}{t} \right)^{1/2} \\ &\lesssim \left(\frac{V(y, r_\alpha)}{V(y, r(x))} \right)^{1/2} \left(\int_{t \geq \frac{1}{2}r(x)} \frac{V(y, r(x))}{V(y,t)^3} \left(\frac{t^2}{r_\alpha^2} \right) \frac{dt}{t} \right)^{1/2} \\ &\leq \left(\frac{V(y, r_\alpha)}{V(y, r(x))} \right)^{1/2} \frac{r(x)}{r_\alpha} \frac{1}{V(y, r(x))} \left(\int_{t \geq \frac{1}{2}r(x)} \frac{V(y, r(x))^3}{V(y,t)^3} \left(\frac{t^2}{r(x)^2} \right) \frac{dt}{t} \right)^{1/2} \\ &\lesssim \left(\frac{r_\alpha}{r(x)} \right)^{\frac{\nu}{2}-1} \frac{1}{V(y, r(x))} \left(\int_{t \geq \frac{1}{2}r(x)} \left(\frac{r(x)}{t} \right)^{3\nu-2} \frac{dt}{t} \right)^{1/2} \\ &\lesssim \frac{1}{V(y, r(x))} \lesssim \frac{1}{V(x, r(x))}, \end{aligned}$$

where the last inequality follows from the fact that $d(x, y) \leq r(x) + r(y) \lesssim r(x) + r_\alpha \lesssim r(x)$, which entails $V(x, r(x)) \lesssim V(y, r(x))$. As a consequence,

$$\mathcal{A}_{d,\alpha,1}(\omega)(x) \lesssim \frac{1}{V(o, r(x))} \|\omega\|_{L^1(B_\alpha)}. \quad (3.44)$$

Gathering (3.43) and (3.44), and using the fact that the balls $(B_\alpha)_\alpha$ have the finite intersection property, we obtain

$$\sum_{\alpha} \mathcal{A}_{d,\alpha,1}(\omega)(x) \lesssim \frac{1}{V(o, r(x))} \|\omega\|_1.$$

By Lemma 3.2 from Appendix A, we obtain, for all $\lambda > 0$,

$$\mu \left(\left\{ x; \sum_{\alpha} \mathcal{A}_{d,\alpha,1}(\omega)(x) > \lambda \right\} \right) \lesssim \frac{1}{\lambda} \|\omega\|_1.$$

3.4.2 The case $t < r_\alpha$:

We now turn to the case of $\mathcal{A}_{d,\alpha,2}\omega$; we wish to prove that

$$\mu \left(\left\{ x; \sum_{\alpha} \mathcal{A}_{d,\alpha,2}(\omega)(x) > \lambda \right\} \right) \lesssim \frac{1}{\lambda} \|\omega\|_1. \quad (3.45)$$

Decompose

$$\begin{aligned} \omega &= df = \sum_{\alpha \in A} \chi_\alpha df \\ &= \sum_{\alpha \in A} \chi_\alpha d(f - f_{B_\alpha}) \\ &= \sum_{\alpha \in A} d(\chi_\alpha (f - f_{B_\alpha})) - \sum_{\alpha \in A} (f - f_{B_\alpha}) d\chi_\alpha \\ &=: \sum_{\alpha \in A} df_\alpha - \sum_{\alpha \in A} \eta_\alpha. \end{aligned}$$

Note that f_α and η_α are supported in B_α . Moreover, we claim:

$$\|df_\alpha\|_{L^1} + \|\eta_\alpha\|_{L^1} + r_\alpha \|d^*\eta_\alpha\|_{L^1} \lesssim \|\omega\|_{L^1(B_\alpha)}. \quad (3.46)$$

Indeed, by Lemma 2.2,

$$\|\eta_\alpha\|_{L^1(B_\alpha)} \lesssim \frac{1}{r_\alpha} \|f - f_{B_\alpha}\|_{L^1(B_\alpha)} \lesssim \|df\|_{L^1(B_\alpha)} = \|\omega\|_{L^1(B_\alpha)}.$$

Thanks to the fact that $\chi_\alpha \omega = df_\alpha - \eta_\alpha$ this also entails that $\|df_\alpha\|_{L^1} \lesssim \|\omega\|_{L^1(B_\alpha)}$. Finally,

$$\begin{aligned} d^*\eta_\alpha &= d^*[(f - f_{B_\alpha}) d\chi_\alpha] \\ &= \langle df, d\chi_\alpha \rangle + (f - f_{B_\alpha}) \Delta \chi_\alpha, \end{aligned}$$

which, in view of (2.10) and Lemma 2.2, entails that $r_\alpha \|d^*\eta_\alpha\|_{L^1} \lesssim \|\omega\|_{L^1(B_\alpha)}$, completing the proof of (3.46).

Let T_α be the operator defined by

$$T_\alpha \omega(x) = \left(\int_{d(x,z) \leq t, t < r_\alpha} \chi_{4B_\alpha}(z) |td^*e^{-t^2\Delta_1}\omega(z)|^2 \frac{dz}{V(z,t)} \frac{dt}{t} \right)^{1/2}.$$

Note that if $d(x, z) \leq t$, $z \in 4B_\alpha$ and $t \leq r_\alpha$, then $x \in 5B_\alpha$; consequently, the support of $T_\alpha \omega$ is included in $5B_\alpha$, which is a remote ball for $\alpha \neq 0$ by (2.7).

Clearly, one has

$$\|T_\alpha\|_{2 \rightarrow 2} \leq \|\mathcal{A}\|_{2 \rightarrow 2} \leq C,$$

where C is independent of α . Also, by (3.46),

$$\mathcal{A}_{2,d,\alpha}(df)(x) \leq T_\alpha(df_\alpha)(x) + T_\alpha(\eta_\alpha)(x). \quad (3.47)$$

Since the support of T_α is included in $5B_\alpha$, the covering $(5B_\alpha)_{\alpha \geq 0}$ has the finite intersection property and in view of (3.47), in order to show (3.45), it is enough to prove the following pair of inequalities:

$$\mu(\{x \in 5B_\alpha; |T_\alpha(df_\alpha)(x)| > \lambda\}) \lesssim \frac{1}{\lambda} \|df_\alpha\|_{L^1} \quad (3.48)$$

and

$$\mu(\{x \in 5B_\alpha; |T_\alpha(\eta_\alpha)(x)| > \lambda\}) \lesssim \frac{1}{\lambda} \|\omega\|_{L^1(B_\alpha)}, \quad (3.49)$$

3.4.3 The exact diagonal part

According to Proposition 3.4 in Appendix B applied to T_α , the inequality (3.48) will follow from the following pair of inequalities: for every (admissible) sub-ball $B \subset 2B_\alpha$, and every function u supported in B ,

$$\left(\frac{1}{V(2^{j+1}B)} \int_{C_j(B) \cap 5B_\alpha} |T_\alpha(I - A_{r(B)})(du)|^2 \right)^{1/2} \leq g(j) \frac{1}{V(B)} \int_B |du|, \quad j \geq 2, \quad (3.50)$$

as well as

$$\left(\frac{1}{V(2^{j+1}B)} \int_{C_j(B) \cap 5B_\alpha} |A_{r(B)}(du)|^2 \right)^{1/2} \leq g(j) \frac{1}{V(B)} \int_B |du|, \quad j \geq 1, \quad (3.51)$$

where $C_1(B) := 4B$, $C_j(B) := 2^{j+1}B \setminus 2^jB$ for all $j \geq 2$ and $A_{r(B)}$ is a smoothing operator to be defined and $\sum_{j=1}^{\infty} g(j)2^{jD} < \infty$.

Fix $\alpha \in \mathbb{N}$, let $B \subset 2B_\alpha$ be a sub-ball, and let $r := r(B)$. Given $t > 0$, define the operator

$$\psi(\Delta_1) := e^{-t\Delta_1}(I - e^{-r^2\Delta_1})^m,$$

where m will be chosen big enough later. According to [2, Equations (2.6) and (4.3)], one has

$$\psi(\Delta_1) = \int_{\Gamma_\pm} e^{-z\Delta_1} \eta_\pm(z) dz,$$

where Γ_{\pm} is the half-ray $\mathbb{R}_+ e^{\pm i(\frac{\pi}{2}-\theta)}$ for a suitable $\theta \in (0, \frac{\pi}{2})$ and $\eta_{\pm}(z)$ is a complex function satisfying the estimate

$$|\eta_{\pm}(z)| \lesssim \frac{1}{|z|+t} \inf \left(1, \frac{r^{2m}}{(|z|+t)^m} \right).$$

Using Lemma 2.9 and following the argument in [2, p.27-28], one obtains

$$\begin{aligned} \|V(\cdot, \sqrt{t})^{1/2} \sqrt{t} d^* \psi(\Delta_1)(du)\|_{L^2(C_j(B))} &\lesssim \frac{1}{4^j m} \left(\frac{t}{4^j r^2} \right)^{D/2} \times \\ &\inf \left(\left(\frac{t}{4^j r^2} \right)^{1/2}, \left(\frac{4^j r^2}{t} \right)^{m-1/2} \right) \|du\|_{L^1(B)}, \end{aligned} \quad (3.52)$$

for every u supported in the admissible ball B . If one now lets

$$\varphi(\Delta_1) := e^{-t^2 \Delta_1} (I - e^{-r^2 \Delta_1})^m,$$

then by (3.52), one gets

$$\|V(\cdot, t)^{1/2} t d^* \varphi(\Delta_1)(du)\|_{L^2(C_j(B))} \lesssim 4^{-jm} \left(\frac{t}{2^j r} \right)^{D+1} \inf \left(1, \left(\frac{2^j r}{t} \right)^{2m} \right) \|du\|_{L^1(B)}. \quad (3.53)$$

We now define the smoothing operator $A_{r(B)}$ by

$$A_{r(B)} = I - (I - e^{-r(B)^2 \Delta_1})^m.$$

We need to check that (3.50) and (3.51) hold. In what follows, for simplicity we will simply write r instead of $r(B)$.

Proof of (3.51): this uses the estimate in Lemma 2.8. Indeed, we first notice that $A_{r(B)}$ is a linear combination of terms $e^{-kr^2 \Delta_1}$, $k = 1, \dots, m$, and it suffices to treat independently each of these terms. In what follows, we will thus fix an integer k between 1 and m . For every $j \geq 1$, letting $F = C_j(B) \cap 5B_{\alpha}$, one has

$$r(B) \lesssim r(x) + 1, \quad \forall x \in F.$$

Indeed, the inequality is trivial for $\alpha = 0$, and for $\alpha \neq 0$, $r(x) \geq r(x_{\alpha}) - 5r_{\alpha} \geq (2^9 - 5)r_{\alpha} \geq \frac{(2^9 - 5)}{2} r(B)$. Consequently, by Lemma 2.8, for every u with support in B and $du \in L^1$,

$$\|V(\cdot, r\sqrt{k})^{1/2} e^{-kr^2 \Delta_1}(du)\|_{L^2(C_j(B) \cap 5B_{\alpha})} \lesssim e^{-c4^j} \|du\|_{L^1(B)}.$$

By doubling, if $x \in C_j(B)$, then

$$\begin{aligned} \frac{V(x_B, r)}{V(x, r\sqrt{k})} &\leq \frac{V(x, r(1 + 2^{j+1}))}{V(x, r\sqrt{k})} \\ &\lesssim (1 + 2^j)^D \\ &\lesssim 2^{Dj}. \end{aligned}$$

Hence,

$$V(B)^{1/2} \|e^{-kr^2 \Delta_1}(du)\|_{L^2(C_j(B) \cap 5B_\alpha)} \lesssim 2^{jD/2} e^{-c4^j} \|du\|_{L^1(B)}.$$

Therefore,

$$V(B)^{1/2} \left(\frac{V(B)}{V(2^{j+1}B)} \right)^{1/2} \|e^{-kr^2 \Delta_1}(du)\|_{L^2(C_j(B) \cap 5B_\alpha)} \lesssim 2^{jD/2} e^{-c4^j} \|du\|_{L^1(B)},$$

which implies that (3.51) holds with $g(j) \simeq 2^{jD/2} e^{-c4^j}$.

□

Proof of (3.50): this uses (3.53). Given $j \geq 2$, we write $T_\alpha(1 - A_{r(B)})(du) \leq S_\alpha(du) + L_\alpha(du)$, where

$$S_\alpha \omega(x) = \left(\int_{d(x,z) \leq t, t \leq r_\alpha \wedge 2^{j-1}r} \chi_{4B_\alpha}(z) |td^* e^{-t^2 \Delta_1} (I - e^{-r(B)^2 \Delta_1})^m \omega(z)|^2 \frac{dz}{V(z,t)} \frac{dt}{t} \right)^{1/2}$$

and

$$L_\alpha \omega(x) = \left(\int_{d(x,z) \leq t, 2^{j-1}r \leq t \leq r_\alpha} \chi_{4B_\alpha}(z) |td^* e^{-t^2 \Delta_1} (I - e^{-r(B)^2 \Delta_1})^m \omega(z)|^2 \frac{dz}{V(z,t)} \frac{dt}{t} \right)^{1/2}$$

(of course, L_α is non-zero only if $2^{j-1}r \leq r_\alpha$). Let

$$F_t(z) := |td^* e^{-t^2 \Delta_1} (I - e^{-r(B)^2 \Delta_1})^m(du)(z)|.$$

We need to estimate

$$\begin{aligned} I &:= \left(\frac{1}{V(2^{j+1}B)} \int_{C_j(B)} |S_\alpha(du)(x)|^2 dx \right)^{1/2} \\ &\leq \left(\frac{1}{V(2^{j+1}B)} \int_{C_j(B)} \int_{d(x,z) \leq t, t \leq 2^{j-1}r} |F_t(z)|^2 \frac{dz dt}{tV(z,t)} dx \right)^{1/2} \end{aligned}$$

as well as

$$\begin{aligned} II &:= \left(\frac{1}{V(2^{j+1}B)} \int_{C_j(B)} |L_\alpha(du)(x)|^2 dx \right)^{1/2} \\ &\leq \left(\frac{1}{V(2^{j+1}B)} \int_{C_j(B)} \int_{d(x,z) \leq t, 2^{j-1}r \leq t} |F_t(z)|^2 \frac{dz dt}{tV(z,t)} dx \right)^{1/2}. \end{aligned}$$

According to (3.53), one has

$$\|V(\cdot, t)^{1/2} F_t\|_{L^2(C_j(B))} \lesssim \begin{cases} 4^{-jm} \left(\frac{t}{2^j r}\right)^{D+1} \|du\|_{L^1(B)}, & t \leq 2^j r. \\ 4^{-jm} \left(\frac{2^j r}{t}\right)^{2m-D-1} \|du\|_{L^1(B)}, & t \geq 2^j r. \end{cases}$$

Furthermore, doubling and reverse doubling imply that if $z \in C_j(B)$ and $t \geq 2^j r$, then

$$\frac{V(z, t)}{V(x_B, r)} \geq \frac{V(z, t)}{V(z, 2^{j+2} r)} \geq C \inf \left(\left(\frac{t}{2^j r}\right)^\nu, \left(\frac{t}{2^j r}\right)^D \right).$$

Hence,

$$V(B)^{1/2} \|F_t\|_{L^2(C_j(B))} \lesssim \begin{cases} 4^{-jm} \left(\frac{t}{2^j r}\right)^{\frac{D}{2}+1} \|du\|_{L^1(B)}, & t \leq 2^j r. \\ 4^{-jm} \left(\frac{2^j r}{t}\right)^{2m-D+\frac{\nu}{2}-1} \|du\|_{L^1(B)}, & t \geq 2^j r. \end{cases}$$

One deduces that

$$V(B)^{1/2} \|F_t\|_{L^2(C_j(B))} \lesssim \begin{cases} 4^{-jm} \left(\frac{t}{2^j r}\right) \|du\|_{L^1(B)}, & t \leq 2^j r. \\ 4^{-jm} \left(\frac{2^j r}{t}\right)^{2m-D+\frac{\nu}{2}-1} \|du\|_{L^1(B)}, & t \geq 2^j r. \end{cases} \quad (3.54)$$

Estimate of I : for $t \leq 2^{j-1} r$ and $x \in C_j(B)$, $d(x, z) \leq t$ implies that z belongs to $C_{j-1}(B) \cup C_j(B) \cup C_{j+1}(B)$. Furthermore, if z is fixed, then the measure of the set

$$\{x \in C_j(B); d(x, z) \leq t\}$$

is by definition at most $V(z, t)$. Therefore, by Fubini and (3.54), we obtain

$$\begin{aligned} V(B) \cdot I &\leq \left(\frac{V(B)}{V(2^{j+1} B)} \int_0^{2^{j-1} r} V(B) \|F_t\|_{L^2(C_{j-1}(B) \cup C_j(B) \cup C_{j+1}(B))}^2 \frac{dt}{t} \right)^{1/2} \|du\|_{L^1(B)} \\ &\lesssim \left(2^{-j\nu} \int_0^{2^{j-1} r} 16^{-jm} \left(\frac{t}{2^j r}\right)^2 \frac{dt}{t} \right)^{1/2} \|du\|_{L^1(B)} \\ &= C 2^{-j(2m+\nu)/2} \|du\|_{L^1(B)}. \end{aligned}$$

Therefore,

$$I \lesssim 2^{-j(2m+\nu)/2} \cdot \frac{1}{V(B)} \|du\|_{L^1(B)}.$$

Estimate of II: let $t \geq 2^{j-1}r$, and $i = i(t)$ be the lowest integer such that

$$2^i r \geq t.$$

Then, for $x \in C_j(B)$, $d(x, z) \leq t$ implies that z belongs to $2^{i+3}B$. We bound the integral

$$\int_{C_j(B)} \int_{d(x,z) \leq t} |F_t(z)|^2 dz dx$$

by

$$V(2^{j+1}B) \cdot \sum_{k \leq i+2} \|F_t\|_{L^2(C_k(B))}^2.$$

By (3.54), we get

$$\begin{aligned} \int_{C_j(B)} \int_{d(x,z) \leq t} |F_t(z)|^2 dz dx &\leq \frac{V(2^{j+1}B)}{V(B)} \sum_{k \leq i+2} V(B) \cdot \|F_t\|_{L^2(C_k(B))}^2 \\ &\lesssim \frac{V(2^{j+1}B)}{V(B)} \left(\sum_{k \leq i+2} 4^{-2km} \left(\frac{2^k r}{t} \right)^{4m-2D+\nu-2} \right) \|du\|_{L^1(B)}^2 \\ &\lesssim \frac{V(2^{j+1}B)}{V(B)} \left(\frac{r}{t} \right)^{4m-2D+\nu-2} \left(\sum_{k \leq i+2} 4^{-k(D-\frac{\nu}{2}+1)} \right) \|du\|_{L^1(B)}^2 \\ &\lesssim V(2^{j+1}B) \left(\frac{r}{t} \right)^{4m-2D+\nu-2} \frac{1}{V(B)} \cdot \|du\|_{L^1(B)}^2, \end{aligned}$$

where the last line follows from the fact that $D - \frac{\nu}{2} + 1 > 0$ (since $\nu \leq D$). Also, if $z \in C_j(B)$ and $t \geq 2^{j-1}r$, by doubling

$$\frac{V(B)}{V(z, t)} \leq \frac{V(z, 8t)}{V(z, t)} \lesssim 1.$$

Consequently,

$$\frac{1}{V(2^{j+1}B)} \int_{C_j(B)} \int_{d(x,z) \leq t} |F_t(z)|^2 \frac{dz dx}{V(z, t)} \lesssim \left(\frac{r}{t} \right)^{4m-2D+\nu-2} \frac{1}{V(B)^2} \cdot \|du\|_{L^1(B)}^2.$$

Therefore,

$$\begin{aligned}
II &\lesssim \left(\int_{2^{j-1}r}^{\infty} \left(\frac{r}{t} \right)^{4m-2D+\nu-2} \frac{dt}{t} \right)^{1/2} \frac{1}{V(B)} \cdot \|du\|_{L^1(B)} \\
&\lesssim 2^{-j(2m-D+\frac{\nu}{2}-1)} \frac{1}{V(B)} \cdot \|du\|_{L^1(B)}.
\end{aligned}$$

Collecting the estimates, and choosing $m > 2D + 1$, one gets (3.50) with $g(j) \simeq 2^{-j(2m+\nu)/2} + 2^{-j(2m-D+\frac{\nu}{2}-1)} \lesssim 2^{-jm}$. Since $m > 2D + 1$, one has

$$\sum_j g(j) 2^{jD} < \infty,$$

which concludes the proof of (3.50). □

3.4.4 The non-exact diagonal part

We now prove (3.49). One has

$$T_\alpha \eta_\alpha(x) = \left(\int_{d(x,z) \leq t < r_\alpha} \chi_{4B_\alpha}(z) |te^{-t^2 \Delta_0} d^* \eta_\alpha(z)|^2 \frac{dz}{V(z,t)} \frac{dt}{t} \right)^{1/2}.$$

Define the function g_α by

$$g_\alpha := r_\alpha d^* \eta_\alpha,$$

so that g_α is supported in B_α , and according to (3.46),

$$\|g_\alpha\|_1 \lesssim \|\omega\|_{L^1(B_\alpha)}. \quad (3.55)$$

Then,

$$\begin{aligned}
T_\alpha \eta_\alpha(x) &= \left(\int_{d(x,z) \leq t < r_\alpha} \chi_{4B_\alpha}(z) |e^{-t^2 \Delta_0} g_\alpha(z)|^2 \left(\frac{t}{r_\alpha} \right)^2 \frac{dz}{V(z,t)} \frac{dt}{t} \right)^{1/2} \\
&\leq \left(\int_{d(x,z) \leq t < r_\alpha} |e^{-t^2 \Delta_0} g_\alpha(z)|^2 \left(\frac{t}{r_\alpha} \right)^2 \frac{dz}{V(z,t)} \frac{dt}{t} \right)^{1/2}.
\end{aligned}$$

Thus, for $r > 0$, we are led to consider the following non-tangential functional:

$$R_r g(x) := \left(\int_{d(x,z) \leq t \leq r} |e^{-t^2 \Delta_0} g(z)|^2 \left(\frac{t}{r} \right)^2 \frac{dz}{V(z,t)} \frac{dt}{t} \right)^{1/2}.$$

We claim that there exists a constant C independent of $r > 0$ such that, for every $g \in L^1(M)$,

$$\mu(\{x; |R_r g(x)| > \lambda\}) \leq C \frac{1}{\lambda} \|g\|_1, \quad \forall \lambda > 0. \quad (3.56)$$

This claim, together with (3.55), readily implies (3.49).

Let us first check that R_r is bounded on L^2 , with

$$\|R_r\|_{2 \rightarrow 2} \leq C,$$

independent of $r > 0$. One has

$$\begin{aligned} \int_M |R_r g(x)|^2 dx &= \int_M \int_{d(x,z) \leq t \leq r} |e^{-t^2 \Delta_0} g(z)|^2 \left(\frac{t}{r}\right)^2 \frac{dz}{V(z,t)} \frac{dt}{t} \\ &= \int_M \int_0^r |e^{-t^2 \Delta_0} g(z)|^2 dz \left(\frac{t}{r}\right)^2 \frac{dt}{t} \\ &= \int_0^r \|e^{-t^2 \Delta_0} g\|_2^2 \left(\frac{t}{r}\right)^2 \frac{dt}{t} \\ &\leq \|g\|_2^2 \left(\int_0^r \left(\frac{t}{r}\right)^2 \frac{dt}{t} \right) \\ &= \|g\|_2^2. \end{aligned}$$

On the other hand, one notices that for every $r > 0$,

$$R_r g(x) \leq Rg(x) := \left(\int_{d(x,z) \leq t} |e^{-t^2 \Delta_0} g(z)|^2 \frac{dz}{V(z,t)} \frac{dt}{t} \right)^{1/2}.$$

This inequality, as well as the Gaussian estimates satisfied by $e^{-t^2 \Delta_0}$ allow one to show that the hypotheses of Theorem 1.1 in [2] for $p_0 = 1$ are satisfied for R_r with the choice $A_{r(B)} = I - (I - e^{-r(B)^2 \Delta_0})^m$, $m \gg 1$, and with constants that are bounded independently of $r > 0$. As a consequence of this theorem, (3.56) holds.

Appendix A: two lemmata on volume

The following two lemmata are of frequent use in the present work:

Lemma 3.1. *Let $A > 0$, $1 \leq p < \nu$. Then*

$$\int_{r(y) \leq A} \frac{1}{r(y)^p} dy \lesssim A^{-p} V(o, A).$$

Proof. Splitting the integration domain into dyadic annuli, we get

$$\begin{aligned}
\int_{r(y) \leq A} \frac{1}{r(y)^p} dy &= \sum_{j \geq 0} \int_{2^{-j-1}A < r(y) \leq 2^{-j}A} \frac{1}{r(y)^p} dy \\
&\leq A^{-p} \sum_{j \geq 0} 2^{(j+1)p} V(o, 2^{-j}A) \\
&\lesssim A^{-p} \sum_{j \geq 0} 2^{(j+1)p} 2^{-j\nu} V(o, A),
\end{aligned}$$

which yields the result since $p < \nu$. \square

Lemma 3.2. *Let $m > 0$ and $A \subset M$ be a measurable set such that, for all $x \in A$, $V(o, r(x)) \leq m$. Then $\mu(A) \lesssim m$.*

Proof. Define $t := \sup \{r > 0; V(o, r) \leq m\}$ (note that t is well-defined and $t > 0$, since $\lim_{s \rightarrow 0} V(o, s) = 0$ and $\lim_{s \rightarrow +\infty} V(o, s) = +\infty$). Since $B(o, t) = \bigcup_{k \geq 1} B\left(o, t - \frac{1}{k}\right)$, $V(o, t) \leq m$. The assumption on A means that, for all $x \in A$, $r(x) \leq t$, so that $A \subset B(o, 2t)$. Therefore, $\mu(A) \leq V(o, 2t) \lesssim m$. \square

Appendix B: A Calderón-Zygmund decomposition localized in balls

Recall that \mathcal{M} denotes the uncentered Hardy-Littlewood maximal function, given by

$$\mathcal{M}u(x) := \sup_{B \ni x} \frac{1}{V(B)} \int_B |u(y)| dy,$$

where the supremum is taken over all open balls containing x .

Lemma 3.3. *Let B be a ball in M , and $u \in C_0^\infty(B)$. Let $1 \leq q < \infty$, and assume that the Poincaré inequality with exponent q holds for any ball $7\tilde{B}$, where $\tilde{B} \subset 2B$. Then, there exists a constant $C > 0$ depending only on the doubling constant, with the*

following property: for all $\lambda > \left(\frac{C \|\nabla u\|_q^q}{V(B)}\right)^{\frac{1}{q}}$, there exists a denumerable collection of balls $(B_i)_{i \geq 1} \subset 2B$, a denumerable collection of C^1 functions $(b_i)_{i \geq 1}$ and a Lipschitz function g such that:

1. $u = g + \sum_{i \geq 1} b_i$,
2. The support of g is included in B , and $|\nabla g(x)| \lesssim \lambda$, for a.e. $x \in B$.
3. The support of b_i is included in B_i , and

$$\int_{B_i} |\nabla b_i|^q \lesssim \lambda^q V(B_i).$$

$$4. \sum_{i \geq 1} V(B_i) \lesssim \lambda^{-q} \int |\nabla u|^q.$$

5. *There is a finite upper bound N for the number of balls B_i that have a non-empty intersection.*

Proof. Define

$$\Omega := \{x \in M; \mathcal{M}(|\nabla u|^q)(x) > \lambda^q\},$$

which is an open subset of M , and set $F := M \setminus \Omega$. We first claim that $\Omega \subset 2B$. Indeed, if $x \notin 2B$, and \tilde{B} is a ball containing x and intersecting the support of ∇u (hence intersecting B), then $B \subset 3\tilde{B}$, hence $V(B) \leq cV(\tilde{B})$ by doubling. Consequently,

$$\frac{1}{V(\tilde{B})} \int_{\tilde{B}} |\nabla u|^q \leq c \frac{1}{V(B)} \int_B |\nabla u|^q \leq cC^{-q}\lambda^q,$$

hence, if $C \geq c^{1/q}$, one obtains

$$\frac{1}{V(\tilde{B})} \int_{\tilde{B}} |\nabla u|^q \leq \lambda^q.$$

Taking the supremum over all balls \tilde{B} containing x , one gets

$$\mathcal{M}(|\nabla u|^q)(x) \leq \lambda^q,$$

and consequently $x \notin \Omega$. Therefore, we have proved that $\Omega \subset 2B$.

For all $x \in \Omega$, let $r_x := \frac{1}{10}d(x, M \setminus \Omega)$ and $B_x := B(x, r_x)$, so that $B_x \subset \Omega$, and $\Omega = \bigcup_{x \in \Omega} B_x$. Since the radii of the balls B_x are uniformly bounded, there exists a denumerable collection of points $(x_i)_{i \geq 1} \in \Omega$ such that the balls B_{x_i} are pairwise disjoint and $\Omega = \bigcup_{i \geq 1} 5B_{x_i}$. For all i , write $s_i := 5r_{x_i}$ and let $B_i = B(x_i, s_i)$. Notice that $B_i \subset 2B$ for all i . Furthermore, the balls $\frac{1}{5}B_i$ being disjoint together with doubling entail that the covering by balls B_i has the finite intersection property. And by construction also, $3B_i \cap F \neq \emptyset$ for every i . Therefore, if one lets $\underline{B}_i := \frac{1}{5}B_i$ and $\bar{B}_i := 3B_i$, then the families of balls $(\underline{B}_i, B_i, \bar{B}_i)_i$ form a Whitney-type covering of Ω in the sense of Coifman and Weiss ([12]). Note that, for all $i, j \geq 1$, if $B_i \cap B_j \neq \emptyset$, then $\delta^{-1}s_i \leq s_j \leq \delta s_i$ with $\delta = 3$. Indeed, let $x \in B_i \cap B_j$. Then

$$s_i = \frac{1}{2}d(x_i, M \setminus \Omega) \leq \frac{1}{2}d(x_i, x) + \frac{1}{2}d(x, M \setminus \Omega) \leq \frac{1}{2}s_i + \frac{1}{2}d(x, M \setminus \Omega),$$

so that

$$s_i \leq d(x, M \setminus \Omega) \leq d(x, x_j) + d(x_j, M \setminus \Omega) \leq 3s_j,$$

and exchanging the roles of i and j proves the claim.

Let $(\chi_i)_{i \geq 1}$ be a partition of unity of Ω , subordinated to the covering $(B_i)_{i \geq 1}$, and such that $|\nabla \chi_i| \lesssim s_i^{-1}$. Then, define

$$b_i = (u - u_{B_i})\chi_i,$$

so that b_i has support in B_i . We also let

$$g = u - \sum_{i \geq 1} b_i.$$

According to the proof of Prop. 1.1 in [3], g is a well-defined, locally integrable function on M . The Lebesgue differentiation theorem implies that $|\nabla u| \leq \lambda$ a.e. on F . Following the proof of Prop. 1.1 in [3], and using the L^q Poincaré inequality for the balls B_i as well as $(2\delta + 1)B_i = 7B_i$, the points 4. as well as 5. are easily proved. \square

Proposition 3.4. *Let $B \subset M$ be an admissible ball. Let T be a real-valued sublinear operator of strong type $(2, 2)$. Assume that, for all balls $\tilde{B} \subset B$, all $u \in C_0^\infty(B)$, and for all $j \geq 2$,*

$$\left(\frac{1}{V(2^{j+1}\tilde{B})} \int_{C_j(\tilde{B}) \cap 5B} |T(I - A_{r(\tilde{B})})(du)|^2 \right)^{1/2} \leq g(j) \frac{1}{V(\tilde{B})} \int_{\tilde{B}} |du|, \quad j \geq 2, \quad (3.57)$$

and, for all $j \geq 1$,

$$\left(\frac{1}{V(2^{j+1}\tilde{B})} \int_{C_j(\tilde{B}) \cap 5B} |A_{r(\tilde{B})}(du)|^2 \right)^{1/2} \leq g(j) \frac{1}{V(\tilde{B})} \int_{\tilde{B}} |du|, \quad j \geq 1, \quad (3.58)$$

where $(A_r)_{r>0}$ is a collection of operators acting on 1-differential forms and $\sum_{j=1}^\infty g(j)2^{Dj} < \infty$, where $D > 0$ is the doubling exponent from (VD). Then,

$$\mu(\{x \in 5B; |T(du)(x)| > \lambda\}) \lesssim \frac{\|\nabla u\|_1}{\lambda}, \quad u \in C_0^\infty(B).$$

Proof. Let $\lambda > 0$. If $\lambda \leq \frac{C \|\nabla u\|_1}{V(B)}$, where C is given by Lemma 3.3, then by doubling

$$\mu(\{x \in 5B; |T(du)(x)| > \lambda\}) \leq V(5B) \lesssim V(B) \leq \frac{C \|\nabla u\|_1}{\lambda}.$$

Assume now that $\lambda > \frac{C \|\nabla u\|_1}{V(B)}$. Decompose $u = g + b$ with g and b given by Lemma 3.3 applied with $q = 1$. One has

$$\begin{aligned} \mu(\{x \in 5B; |T(du)(x)| > \lambda\}) &\leq \mu\left(\left\{x \in 5B; |T(dg)(x)| > \frac{\lambda}{2}\right\}\right) \\ &\quad + \mu\left(\left\{x \in 5B; |T(db)(x)| > \frac{\lambda}{2}\right\}\right). \end{aligned}$$

For the first term of the right-hand side of (3.59), write

$$\begin{aligned}
\mu \left(\left\{ x \in 5B; |T(dg)(x)| > \frac{\lambda}{2} \right\} \right) &\leq \frac{4}{\lambda^2} \|T(dg)\|_2^2 \\
&\leq \frac{C}{\lambda^2} \|dg\|_2^2 \\
&\leq \frac{C}{\lambda^2} \|dg\|_\infty \|dg\|_1 \\
&\leq \frac{C}{\lambda} \|du\|_1.
\end{aligned}$$

The second line follows from the L^2 -boundedness of T and the last line is due to property 2 of Lemma 3.3 and the fact (due in turn to items 3 and 4 of Lemma 3.3) that

$$\|dg\|_1 \leq \|du\|_1 + \|db\|_1 \leq \|du\|_1 + \sum_i \|db_i\|_1 \lesssim \|du\|_1 + \lambda \sum_i V(B_i) \lesssim \|du\|_1.$$

As far as the second term in the right-hand side of (3.59) is concerned, write

$$\left| T \left(\sum_i db_i \right) \right| \leq \sum_i |T(I - A_{r_i})db_i| + \left| T \left(\sum_i A_{r_i}db_i \right) \right|.$$

This entails that it is enough to check that

$$I := \mu \left(\left\{ x \in 5B; \sum_i |T(I - A_{r_i})db_i(x)| > \frac{\lambda}{4} \right\} \right) \leq \frac{C}{\lambda} \|du\|_1 \quad (3.59)$$

and

$$J := \mu \left(\left\{ x \in 5B; \left| T \left(\sum_i A_{r_i}db_i \right) (x) \right| > \frac{\lambda}{4} \right\} \right) \leq \frac{C}{\lambda} \|du\|_1. \quad (3.60)$$

For I ,

$$I \leq \mu \left(\bigcup_{i \geq 1} 4B_i \right) + \mu \left(\left\{ x \in 5B \setminus \bigcup_i 4B_i; \sum_{i \geq 1} |T(I - A_{r_i})db_i(x)| > \frac{\lambda}{4} \right\} \right) =: I_1 + I_2.$$

On the one hand, by doubling and property 4. of Lemma 3.3,

$$I_1 \leq C \sum_i V(B_i) \leq \frac{C}{\lambda} \|\nabla u\|_1. \quad (3.61)$$

On the other hand, the Chebyshev inequality entails

$$I_2 \leq \frac{16}{\lambda^2} \left\| \sum_i \mathbf{1}_{5B \setminus \bigcup_i 4B_i} T(I - A_{r_i})db_i \right\|_{L^2(5B)}^2. \quad (3.62)$$

Pick up a function $h \in L^2(5B)$ with $\|h\|_2 = 1$. One has

$$\left| \int_{5B} \sum_i \mathbf{1}_{5B \setminus \cup_i 4B_i} T(I - A_{r_i}) db_i(x) h(x) dx \right| \leq \sum_i \sum_{j \geq 2} A_{ij},$$

where

$$A_{ij} := \int_{5B \cap C_j(B_i)} |T(I - A_{r_i}) db_i(x)| |h(x)| dx.$$

By assumption (3.57) and property 3 of Lemma 3.3,

$$\|T(I - A_{r_i}) db_i\|_{L^2(5B \cap C_j(B_i))} \leq V(2^{j+1}B_i)^{\frac{1}{2}} g(j) \frac{1}{V(B_i)} \|db_i\|_1 \leq V(2^{j+1}B_i)^{\frac{1}{2}} g(j) \lambda.$$

Moreover, for all $y \in B_i$,

$$\|h\|_{L^2(5B \cap C_j(B_i))} \leq \|h\|_{L^2(2^{j+1}B_i)} \leq V(2^{j+1}B_i)^{\frac{1}{2}} (\mathcal{M}|h|^2(y))^{\frac{1}{2}}.$$

As a consequence,

$$A_{ij} \leq V(2^{j+1}B_i) g(j) \lambda (\mathcal{M}|h|^2(y))^{\frac{1}{2}} \leq C 2^{jD} V(B_i) g(j) \lambda (\mathcal{M}|h|^2(y))^{\frac{1}{2}},$$

and since this holds for all $y \in B_i$,

$$A_{ij} \leq C 2^{jD} g(j) \lambda \int_{B_i} (\mathcal{M}|h|^2(y))^{\frac{1}{2}} dy.$$

It follows that $|\int_{5B} \sum_i \mathbf{1}_{5B \setminus \cup_i 4B_i} T(I - A_{r_i}) db_i(x) h(x) dx|$ is bounded by

$$\begin{aligned} C \lambda \sum_{j \geq 2} 2^{jD} g(j) \left(\sum_i \int_{B_i} (\mathcal{M}|h|^2(y))^{\frac{1}{2}} dy \right) &\leq C N \lambda \int_{\cup_i B_i} (\mathcal{M}|h|^2(y))^{\frac{1}{2}} dy \\ &\leq C N \lambda \left| \bigcup_i B_i \right|^{\frac{1}{2}} \| |h|^2 \|_1^{\frac{1}{2}} \\ &\leq C \lambda^{1/2} \|\nabla u\|_1^{1/2}, \end{aligned}$$

and (3.62) shows that

$$I_2 \leq \frac{C}{\lambda} \|\nabla u\|_1. \quad (3.63)$$

Gathering (3.61) and (3.63) yields (3.59).

For J , the L^2 -boundedness of T gives

$$J \leq \frac{16}{\lambda^2} \left\| T \left(\sum_i A_{r_i} db_i \right) \right\|_{L^2(5B)}^2 \leq \frac{C}{\lambda^2} \left\| \sum_i A_{r_i} db_i \right\|_{L^2(M)}^2.$$

Pick up again a function $h \in L^2(M)$ with $\|h\|_2 = 1$. Then

$$\left| \int_M \left(\sum_i A_{r_i} db_i(x) \right) h(x) dx \right| \leq \sum_i \left(\sum_{j \geq 1} \int_{C_j(B_i)} |A_{r_i} db_i(x)| |h(x)| dx \right) := \sum_i \sum_{j \geq 1} B_{ij}.$$

For all i, j , using (3.58), one obtains

$$\begin{aligned} B_{ij} &\leq V(2^{j+1}B_i)^{1/2}g(j)\frac{1}{V(B_i)}\|db_i\|_1 \\ &\leq CV(2^{j+1}B_i)^{1/2}g(j)\lambda, \end{aligned}$$

and arguing as before, we conclude that (3.60) holds. \square

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