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NONLOCAL DISSIPATION MEASURE AND L^1 KINETIC THEORY FOR FRACTIONAL CONSERVATION LAWS

NATHAËL ALIBAUD, BORIS ANDREIANOV, AND ADAMA OUÉDRAOGO

Abstract. We introduce a kinetic formulation for scalar conservation laws with nonlocal and nonlinear diffusion terms. We deal with merely L^1 initial data, general self-adjoint pure jump Lévy operators, and locally Lipschitz nonlinearities of porous medium kind possibly strongly degenerate. The cornerstone of the formulation and the uniqueness proof is an adequate explicit representation of the nonlocal dissipation measure. This approach is inspired from the second order theory unlike the cutting technique previously introduced for bounded entropy solutions. The latter technique no longer seems to fit our setting. This is moreover the first time that the more standard and sharper tools of the second order theory are faithfully adapted to fractional conservation laws.

Contents

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1. Introduction

In this paper, we define a kinetic formulation for scalar conservation laws with nonlocal and nonlinear diffusion terms. We consider initial-value problems of the form

$$\begin{cases} \partial_t u + \nabla \cdot F(u) + g[A(u)] = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \\ u(t=0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1}$$

where $u = u(t,x)$ is the unknown function, ∇ denotes the gradient operator with respect to x and g is an integro-differential operator properly defined at least on $\mathcal{D}(\mathbb{R}^d)$ by

$$g[\varphi](x) := -P.V. \int_{\mathbb{R}^d} (\varphi(x+z) - \varphi(x))\mu(z)\, dz, \tag{2}$$

where μ is a (Borel) measure and $\mu(z)\,dz$ abusively stands for $d\mu(z)$ or $\mu(dz)$. Throughout, the initial data u_0 is assumed merely integrable, and the other data are assumed to satisfy the following conditions:

$$F \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^d), \tag{3}$$

$$A \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}) \text{ and is nondecreasing}, \tag{4}$$

$$\mu \geq 0 \text{ with } \mu(\{0\}) = 0 \text{ and } \textstyle\int_{\mathbb{R}^d} (|z|^2 \wedge 1)\mu(z)\, dz < \infty, \tag{5}$$

$$\mu \text{ is even, i.e., it is invariant by the application } z \mapsto -z. \tag{6}$$

The principal value in (2) is defined as the limit

$$g[\varphi](x) := -\lim_{r \downarrow 0} \int_{|z|>r} (\varphi(x+z) - \varphi(x))\, \mu(z)\, dz, \tag{7}$$

which makes sense because of (5)–(6). For all fixed $r > 0$, we also have

$$g[\varphi](x) = -\int_{\mathbb{R}^d} \left(\varphi(x+z) - \varphi(x) - z \cdot \nabla\varphi(x) \mathbf{1}_{|z| \leq r}\right) \mu(z)\, dz. \tag{8}$$

The operators of the form (8) correspond to generators of pure jump Lévy processes. By (6) we restrict to self-adjoint operators. They constitute a general class of nonlocal diffusive operators [47]. Their use in scalar conservation laws goes back to [46] on the Chapman-Enskog expansion involving a convolution operator corresponding to $\int \mu < \infty$. Scalar conservation laws with singular nonlocal diffusions were considered later for instance in semiconductor growth [51] or gas detonations [21]. A typical example of such a diffusion is the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$, $\alpha \in (0,2)$, corresponding to $\mu(z) = c\,|z|^{-d-\alpha}$; see [38]. More recent models led to further study scalar conservation laws with nonlocal and nonlinear diffusions; see for instance [45] on radiation hydrodynamics. Our setting covers all these problems. As a byproduct, it includes the nonlocal diffusion equation $\partial_t u + g[A(u)] = 0$ and allows for porous medium nonlinearities $A(u) = |u|^{m-1}u$ with $m \geq 1$; see [24, 25, 26, 27] for an extensive account on these PDEs. For other related PDEs with nonlocal and nonlinear diffusions, see [11, 12, 14] and the references therein.

State-of-the-art. Equation (1) may degenerate and should share properties with the scalar hyperbolic conservation law

$$\partial_t u + \nabla \cdot F(u) = 0 \tag{9}$$

and the degenerate parabolic equation

$$\partial_t u + \nabla \cdot F(u) - \triangle A(u) = 0. \tag{10}$$

Their main difficulties are the possible creation of singularities and the nonuniqueness of weak solutions. Since the fundamental work of Kruzhkov [37] who defined entropy solutions for first order equations and established well-posedness in the L^∞ framework, many other well-posedness results were obtained. We refer to Carrillo [15] for the entropy formulation of elliptic-parabolic-hyperbolic problems involving Leray-Lions type operators. The more delicate anisotropic diffusion case has been treated by Bendahmane and Karlsen in [7] using, in particular, the insight from the paper [19] of Chen and Perthame. The setting of [19] is different because the kinetic formulation is used to achieve well-posedness; this concept goes back to [13, 29] and to the classical work [40, 41] of Lions, Perthame and Tadmor. An extensive account on the kinetic formulation of conservation laws can be found in [44]. One of the advantages of the kinetic formulation is that the L^1 space is natural for both existence and uniqueness.

Several authors have extended the notion of L^∞ entropy solutions to nonlocal problems of the form (1). We refer to [33, 34, 42, 39, 49, 45] and the references therein for the case $\int \mu < \infty$. The case of singular operators is more delicate and the first results were concerned with conservation laws with memory nonlocal in time. The adequate notion of entropy solutions was introduced in [22] by Cockburn, Gripenberg and Londen; see also [30]. It was adapted later in [1] for fractional diffusions in space with a focus on the equation

$$\partial_t u + \nabla \cdot F(u) + (-\triangle)^{\frac{\alpha}{2}} u = 0. \tag{11}$$

The pioneering work on (11) goes back to Biler, Funaki and Woyczyński [10]. Now the well-posedness is well-understood: If $\alpha \geq 1$, there is a unique smooth solution [10, 28, 16, 48, 23]; if $\alpha < 1$, shocks can occur [6, 35] and weak solutions can be non-unique [2]; for any $\alpha \in (0, 2)$, there exists a unique entropy solution corresponding to the classical one when it exists as well [1]. The entropy solution theory was finally extended by Karlsen and Ulusoy [32] to pure jump Lévy operators and by Cifani and Jakobsen [20] to nonlinear diffusions such as in (1).

As concerning L^1 data, Wei, Duan and Lv recently introduced a kinetic formulation in [50] for $L^1 \cap BV$ solutions of (11) when $\alpha < 1$. This situation is very particular since $(-\triangle)^{\frac{\alpha}{2}} u \in L^1$ and this allows to treat the diffusion as a zero order term. In general, we do not know whether $A(u)$ is locally integable for unbounded solutions and non-globally Lipschitz A. We thus have difficulties to define $g[A(u)]$ as a distribution.

To conclude this state-of-the-art, note that the concept of renormalized solutions [9, 8] provides another framework for L^1 data. It was extended to nonlocal problems but essentially for elliptic PDEs as far as we know; see e.g. [3, 36].

Main contribution. The present paper extends the kinetic formulation of Lions, Pethame and Tadmor [40, 41] to (1); see Definition 8. The main results are the equivalence with entropy solutions in $L^1 \cap L^\infty$, the well-posedness in L^1 and the L^1 contraction principle; see Theorems 12 and 13. The cornerstone of the theory is a new explicit representation of the nonlocal dissipation measure in the spirit of Chen and Perthame [19]; see Theorem 6.

Technically, the uniqueness proof does not rely on the *cutting technique* used in every known proofs on L^∞ entropy solutions. This technique was based on first order like arguments which no longer seem to fit our setting. Here we faithfully adapt the more standard and sharper tools of the second order theory and this is our main technical contribution. To give more details, we need to recall some facts on entropy solutions.

Various entropy inequalities. An entropy solution to (9) is a function $u = u(t, x)$ such that for all $\xi \in \mathbb{R}$,

$$\partial_t |u - \xi| + \nabla_x \cdot \{\mathrm{sgn}(u - \xi)(F(u) - F(\xi))\} \leq 0$$

in $\mathcal{D}'((0,\infty) \times \mathbb{R}^d)$, where $\mathrm{sgn}(\cdot) := |\cdot|'$ is the sign function. The uniqueness can be achieved by using the Kruzhkov device of doubling the variables [37]. For second order equations, one needs to take into account some form of parabolic dissipation. This can be achieved in two ways. The first way was developed by Carrillo in [15] and consists in recovering such a dissipation from the entropy inequalities. The second way was introduced in [17] by Chen and DiBenedetto and consists in explicitly including a proper form of dissipation in the entropy inequalities; see also [31, 19, 8, 18]. For (10), this gives

$$\begin{aligned}\partial_t |u - \xi| &+ \nabla_x \cdot \{\mathrm{sgn}(u - \xi)(F(u) - F(\xi))\} \\ &- \triangle_x |A(u) - A(\xi)| \leq -2\delta(u - \xi) A'(u) |\nabla u|^2,\end{aligned} \tag{12}$$

where $\delta(\cdot) := \frac{1}{2}\mathrm{sgn}'(\cdot)$ is the Dirac measure at zero. In [20], Cifani and Jakobsen used the following entropy inequalities for (1):

$$\begin{aligned}&\partial_t |u - \xi| + \nabla_x \cdot \{\mathrm{sgn}(u - \xi)(F(u) - F(\xi))\} \\ &- P.V. \int_{|z| \leq r} (|A(\tau_z u) - A(\xi)| - |A(u) - A(\xi)|)\, \mu(z)\, dz \\ &- \mathrm{sgn}(u - \xi) \int_{|z| > r} (A(\tau_z u) - A(u)) \mu(z)\, dz \leq 0,\end{aligned} \tag{13}$$

for all $\xi \in \mathbb{R}$ and all $r > 0$, where $\tau_z u$ designs the function $(t, x) \mapsto u(t, x + z)$. The idea is to treat the integral in $|z| > r$ as a zero order term and neglect the other integral as $r \downarrow 0$; cf. also [22, 30, 1];

A natural question is whether it is possible to reformulate (13) in the spirit of (12). A first try was attempted by Karlsen and Ulusoy in [32] but their uniqueness proof reduces to recover (13). Here we reformulate (13) with a new proper form of nonlocal dissipation:

$$\begin{aligned}&\partial_t |u - \xi| + \nabla_x \cdot \{\mathrm{sgn}(u - \xi)(F(u) - F(\xi))\} \\ &+ g_x[|A(u) - A(\xi)|] \leq -\underbrace{\int_{\mathbb{R}^d} |A(\tau_z u) - A(\xi)| 1\!\!1_{\mathrm{conv}\{u, \tau_z u\}}(\xi) \mu(z)\, dz}_{=:n},\end{aligned} \tag{14}$$

where $1\!\!1_{\mathrm{conv}\{u,\tau_z u\}}(\cdot)$ is the characteristic function of the real interval of extremities u and $\tau_z u$. This will be the cornerstone of the kinetic theory.

A successfull kinetic formulation. Following [40, 41, 19, 44], we obtain the kinetic equation in $\chi(\xi;u) := \partial_\xi\left(|\xi| - |\xi - u|\right)/2$ by derivating (14) in ξ:

$$\partial_t \chi(\xi;u) + F'(\xi)\cdot\nabla_x \chi(\xi;u) + A'(\xi) g_x[\chi(\xi;u)] = \partial_\xi(m+n) \tag{15}$$

in $\mathcal{D}'((0,\infty)\times\mathbb{R}^{d+1})$ where $m \geq 0$ is unknown and $n \geq 0$ is defined as in (14). The advantage of (15) when u is merely L^1 is that it makes sense even if $F(u)$ and $A(u)$ are not L^1_{loc}. It seems better to consider the kinetic equation from (14), because derivating (13) would give the term $\delta(u-\xi)\int_{|z|>r}(A(\tau_z u) - A(\xi))\mu(z)\,dz$ which does not obviously make sense. Small and large jumps z are thus treated in the same way in our uniqueness proof thanks roughly speaking to the explicit dissipation in (14) and Lemma 23.

Outline of the paper. Our main results are stated in Section 2 and proved in Sections 3–5. The uniqueness is first proved formally and then rigorously in the spirit of [19, 44]. It is the core of the paper. The existence could be established without relying on entropy solutions, making the kinetic theory self-sufficient as in [19, 44], but here we use a known existence result for entropy solutions to be brief.

Reminders of our notation. The symbol ∇ is used for the gradient in x and ∇^2 for the Hessian. The symbol $\delta(\xi)$ designs the Dirac mass at $\xi = 0$. In integrals, we use the notation $\delta(\xi)\,d\xi$ for $d\delta(\xi)$ or $\delta(d\xi)$. We do the same for the other measures $\mu(z)$, $m(t,x,\xi)$, etc. Note that μ in (5) is σ-finite and Fubini's theorem applies to define its product with dx, etc., which we denote by $\mu(z)\,dx\,dz$, etc. Further notation is introduced in Section 3.

2. Entropy and kinetic solutions: Definitions and main results

Let us now give the rigorous definition of entropy and kinetic solutions and state our main results. In order to avoid unnecessary technical issues, we only use C^2 entropies $u \mapsto S(u)$.

Let us recall the notion of entropy solutions to (1) from [20], see also [22, 30, 1, 32]. Given any convex C^2 function $S : \mathbb{R} \to \mathbb{R}$, we consider

$$\eta : \mathbb{R} \to \mathbb{R}^d \quad \text{and} \quad \beta : \mathbb{R} \to \mathbb{R},$$

satisfying

$$\eta' = S'F' \quad \text{and} \quad \beta' = S'A'.$$

Throughout such a triplet (S,η,β) is refered to as an *entropy-entropy flux triple.*

Definition 1 (Entropy solutions)**.** Let $u_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$ and (3)–(6) hold. A function $u \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ is an entropy solution of (1) provided that for all entropy-entropy flux triple (S,η,β), all $r > 0$, and all nonnegative test function $\varphi \in \mathcal{D}(\mathbb{R}^{d+1})$,

$$
\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^d} \left(S(u)\partial_t\varphi + \eta(u)\cdot\nabla\varphi\right) dt\,dx \\
&+ \int_0^\infty \int_{\mathbb{R}^d} \int_{|z|>r} S'(u(t,x))\left(A(u(t,x+z)) - A(u(t,x))\right)\varphi(t,x)\mu(z)\,dt\,dx\,dz \\
&+ P.V. \int_0^\infty \int_{\mathbb{R}^d} \int_{|z|\le r} \beta(u(t,x))\left(\varphi(t,x+z) - \varphi(t,x)\right)\mu(z)\,dt\,dx\,dz \\
&+ \int_{\mathbb{R}^d} S(u_0(x))\varphi(0,x)\,dx \geq 0.
\end{aligned} \tag{16}
$$

Remark 2.

(1) The original definition of [20] was actually given with the entropies of Kruzhkov. The definition above is an equivalent reformulation already used for instance in [32, Section 6] or [5, Section 7].
(2) The principal value makes sense by (5)–(6) (see (7) and (8)).
(3) Here it may look that we are integrating a Lebesgue measurable function u with respect to the Borel measure μ. An easy way to avoid such measurability issues consists in only considering Borel representative of u.
(4) As usually, classical solutions are entropy solutions and entropy solutions are weak (distributional) solutions; see [20] for more details.

Here is the well-posedness result from [20].

Theorem 3 (Well-posedness of entropy solutions)**.** *Let* $u_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$ *and let us assume* (3)–(6)*. Then there exists a unique entropy solution* u *of* (1)*. This solution belongs to* $C([0,\infty); L^1(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ *with* $u(0,\cdot) = u_0(\cdot)$*. Moreover, we have* $\operatorname{ess\,inf} u_0 \leq u \leq \operatorname{ess\,sup} u_0$,

$$\|u(t,\cdot)\|_{L^1(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d)} \quad \textit{for all } t \geq 0,$$

and if $\tilde{u}$ *is the solution of* (1) *associated to* $\tilde{u}(0,\cdot) = \tilde{u}_0(\cdot) \in L^1 \cap L^\infty(\mathbb{R}^d)$*, then*

$$\|u(t,\cdot) - \tilde{u}(t,\cdot)\|_{L^1(\mathbb{R}^d)} \leq \|u_0 - \tilde{u}_0\|_{L^1(\mathbb{R}^d)} \quad \textit{for all } t \geq 0.$$

To motivate our kinetic formulation, we need to reformulate the entropy inequalities with a proper form of nonlocal dissipation in the spirit of [17], see also [31, 19, 8, 18, 32]. The key point is the following elementary Taylor's identity: For all $a, b \in \mathbb{R}$,

$$S'(a)(A(b) - A(a)) = \beta(b) - \beta(a) - \int_{\mathbb{R}} S''(\xi)|A(b) - A(\xi)|\mathbb{1}_{\operatorname{conv}\{a,b\}}(\xi)\,d\xi, \tag{17}$$

where throughout the paper, $\operatorname{conv}\{a,b\}$ stands for the interval

$$\operatorname{conv}\{a,b\} := \left(\min\{a,b\}, \max\{a,b\}\right),$$

and $\mathbb{1}_{\mathrm{conv}\{a,b\}}(\cdot)$ denotes its characteristic function normalized (everywhere defined) by

$$\mathbb{1}_{\mathrm{conv}\{a,b\}}(\xi) := \begin{cases} 1 & \text{if } \min\{a,b\} < \xi < \max\{a,b\}, \\ \frac{1}{2} & \text{if } \xi = a \text{ or } b, \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

Remark 4. In (18), the choice of the value $\frac{1}{2}$ at the endpoints of the interval $\mathrm{conv}\{a,b\}$ is dictated by the regularization procedure exploited in the uniqueness proof, since nonlinearities S with singular second derivative are used. In this section, this technical detail can be neglected since S is assumed to have the C^2 regularity.

Remark 5. In the sequel, we will use other characteristic functions defined as usually. To avoid confusion, they will be denoted by $\mathbf{1}$ (and not $\mathbb{1}$). For instance, if $E \subseteq \mathbb{R}$, then

$$\mathbf{1}_E(\xi) := \begin{cases} 1 & \text{if } \xi \in E, \\ 0 & \text{if not.} \end{cases}$$

The result below is a simple rewritting of the entropy inequality (16) based on the identity (17) and a passage to the limit as $r \downarrow 0$.

Theorem 6 (Explicit representation of the nonlocal dissipation)**.** *Assume* (3)–(6) *and let $u_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$. A function $u \in L^\infty\left(\mathbb{R}^+; L^1(\mathbb{R}^d)\right) \cap L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ is an entropy solution of* (1) *if and only if for all entropy-entropy flux triple* (S, η, β) *and all nonnegative $\varphi \in \mathcal{D}(\mathbb{R}^{d+1})$,*

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^d} \left(S(u)\partial_t \varphi + \eta(u) \cdot \nabla \varphi - \beta(u) g[\varphi]\right) dt\, dx \\ &+ \int_{\mathbb{R}^d} S(u_0(x))\varphi(0,x)\, dx \geq \int_0^\infty \int_{\mathbb{R}^{d+1}} S''(\xi) n(t,x,\xi)\varphi(t,x)\, dt\, dx\, d\xi, \end{aligned} \tag{19}$$

where the function $n : \mathbb{R}^+ \times \mathbb{R}^{d+1} \to [0, \infty]$ is defined by

$$n(t,x,\xi) := \int_{\mathbb{R}^d} |A(u(t,x+z)) - A(\xi)| \mathbb{1}_{\mathrm{conv}\{u(t,x),u(t,x+z)\}}(\xi)\mu(z)\, dz. \tag{20}$$

The proof of Theorem 6 is deferred to Section 4.

Remark 7. Throughout n is refered to as the *nonlocal dissipation measure.* This is a nonlocal version of the parabolic dissipation measure $2\delta(u-\xi)A'(u)|\nabla u|^2$ obtained for the degenerate parabolic equation (10). Here, we get a measure absolutely continuous with respect to the Lebesgue one, since $n \in L^1(\mathbb{R}^+ \times \mathbb{R}^{d+1})$ as a consequence of Inequality (19) with $S'' \equiv 1$.

We are now ready to define the notion of kinetic solutions. We use the framework of [40, 41, 44] and especially the insight of [19] by including the identification of the dissipation measure in the formulation. We consider the kinetic function $\chi : \mathbb{R}^2 \to \{-1, 0, 1\}$ defined by

$$\chi(\xi; u) := \begin{cases} 1 & \text{if } 0 < \xi < u, \\ -1 & \text{if } u < \xi < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

Note that $u \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^d))$ if and only if $\chi(\xi; u(t,x)) \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^{d+1}))$. Note also that one has the following simple representation:

$$S(u) - S(0) = \int_{\mathbb{R}} S'(\xi)\chi(\xi; u)\, d\xi. \tag{22}$$

These observations lead us to the definition below where Inequality (19) is roughly speaking rewritten with the χ function. Throughout the paper,

M^1 stands for the space of bounded Borel measures

and

L_0^∞ stands for the space of a.e. bounded functions vanishing at infinity.

Definition 8 (Kinetic solutions)**.** Let $u_0 \in L^1(\mathbb{R}^d)$ and (3)–(6) hold. A function $u \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^d))$ is a kinetic solution of (1) provided that there exists a nonnegative measure $m \in M^1_{\mathrm{loc}}([0,\infty) \times \mathbb{R}^{d+1})$ such that for almost all $\xi \in \mathbb{R}$,

$$\int_0^\infty \int_{\mathbb{R}^d} (m+n)(t,x,\xi)\, dt\, dx \le \nu(\xi), \tag{23}$$

for some $\nu \in L_0^\infty(\mathbb{R}_\xi)$, and for all $\varphi \in \mathcal{D}(\mathbb{R}^{d+2})$,

(24)
$$\int_0^\infty \int_{\mathbb{R}^{d+1}} \chi(\xi; u)\left(\partial_t \varphi + F'(\xi)\cdot \nabla_x \varphi - A'(\xi) g_x[\varphi]\right) dt\, dx\, d\xi$$
$$+ \int_{\mathbb{R}^{d+1}} \chi(\xi; u_0(x))\varphi(0,x,\xi)\, dx\, d\xi = \int_0^\infty \int_{\mathbb{R}^{d+1}} (m+n)(t,x,\xi)\partial_\xi \varphi(t,x,\xi)\, dt\, dx\, d\xi,$$

where n is the nonnegative function defined in (20).

Note that (23) has to be understood in the distribution sense, that is to say

$$\int_0^\infty \int_{\mathbb{R}^{d+1}} (m+n)(t,x,\xi)\varphi(\xi)\, dt\, dx\, d\xi \le \int_{\mathbb{R}} \nu(\xi)\varphi(\xi)\, d\xi, \tag{25}$$

for any nonnegative $\varphi \in \mathcal{D}(\mathbb{R}_\xi)$, where hereafter, we sometimes write $\mathbb{R}_\xi$ to highlight dependence of functions and measures on the kinetic variable ξ.

Remark 9. The measure m is referred to as the *entropy defect measure.* It is a priori unknown but a posteriori uniquenely determined (see Theorem 13); in other words, the couple (u,m) is the unknown of the kinetic formulation (24).

The next remark enumerates standard properties of kinetic solutions which remain valid in the nonlocal setting. Most of them will not be needed, so we refer to the arguments of [40, 41, 19, 44] for proofs.

Remark 10. (1) If $u_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$,

$$\operatorname{supp}(m+n) \subseteq \{\operatorname{ess\,inf} u_0 \le \xi \le \operatorname{ess\,sup} u_0\}.$$

(2) If $u_0 \in L^1 \cap L^2(\mathbb{R}^d)$,

$$\int_0^\infty \int_{\mathbb{R}^{d+1}} (m+n)(t,x,\xi)\,dt\,dx\,d\xi \le \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^d)}.$$

(3) If u_0 is merely integrable, then for almost all $\xi \in \mathbb{R}$,

$$\int_0^\infty \int_{\mathbb{R}^d} (m+n)(t,x,\xi)\,dt\,dx \le \nu(\xi),$$

where $\nu(\xi) := \|(u_0-\xi)^+\mathbf{1}_{\xi>0}\|_{L^1(\mathbb{R}^d)} + \|(u_0-\xi)^-\mathbf{1}_{\xi<0}\|_{L^1(\mathbb{R}^d)} \in L^\infty_0(\mathbb{R}_\xi)$.

(4) If $S_A(u_0) \in L^1(\mathbb{R}^d)$ where $S_A(\xi) := \int_0^\xi (A(\zeta)-A(0))\,d\zeta$,

$$\int_0^\infty \int_{\mathbb{R}^{2d}} \left(A(u(t,x+z)) - A(u(t,x))\right)^2 \mu(z)\,dt\,dx\,dz \le \|S_A(u_0)\|_{L^1(\mathbb{R}^d)}.$$

This should be compared to the usual H^1 estimate for degenerate parabolic equations [15], since for $g = (-\triangle)^{\frac{\alpha}{2}}$ the above estimate reads

$$|A(u)|_{L^2(\mathbb{R}^+;H^{\frac{\alpha}{2}}(\mathbb{R}^d))} \le \sqrt{\|S_A(u_0)\|_{L^1(\mathbb{R}^d)}}.$$

Here is another standard property of convection-diffusion conservation laws that remains valid for the nonlocal case, and which we often use throughout. It says that we can reformulate Definition 8 by expressing the initial data in the classical sense.

Proposition 11. *Assume that* (3)–(6) *hold and that* $u_0 \in L^1(\mathbb{R}^d)$. *Then* $u \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^d))$ *is a kinetic solution of* (1) *if and only if there exsits a nonnegative measure* $m \in M^1_{\text{loc}}([0,\infty)\times\mathbb{R}^{d+1})$ *such that* (23) *holds together with the following conditions:*

$$\begin{cases} \partial_t \chi(\xi;u) + F'(\xi)\cdot\nabla_x\chi(\xi;u) + A'(\xi)g_x[\chi(\xi;u)] = \partial_\xi(m+n), \\ \lim_{t\downarrow 0}\|u(t,\cdot)-u_0(\cdot)\|_{L^1(\mathbb{R}^d)} = 0, \\ \textit{and } \lim_{t\downarrow 0}\int_0^t\int_{\mathbb{R}^d}\int_{-R}^R (m+n)(s,x,\xi)\,ds\,dx\,d\xi = 0, \quad \forall R>0, \end{cases} \tag{26}$$

where the equation holds in $\mathcal{D}'((0,\infty)\times\mathbb{R}^d\times\mathbb{R}_\xi)$ *and the limits are taken in the essential sense.*

We can now state the two main results of this paper.

Theorem 12 (Equivalence between entropy and kinetic solutions)**.** *Let* (3)–(6) *hold,* $u_0 \in L^1\cap L^\infty(\mathbb{R}^d)$ *and* $u \in L^\infty(\mathbb{R}^+;L^1(\mathbb{R}^d))\cap L^\infty(\mathbb{R}^+\times\mathbb{R}^d)$. *Then u is an entropy solution of* (1) *if and only if it is a kinetic solution of* (1).

Theorem 13 (Well-posedness in the pure L^1 setting)**.** *Let* $u_0 \in L^1(\mathbb{R}^d)$ *and let us assume* (3)–(6). *Then there exists a unique kinetic solution u of* (1) *and a unique measure m satisfying Definition 8. This solution belongs to* $C([0,\infty);L^1(\mathbb{R}^d))$ *with* $u(0,\cdot) = u_0(\cdot)$. *Moreover,*

$$\|u(t,\cdot)\|_{L^1(\mathbb{R}^d)} \le \|u_0\|_{L^1(\mathbb{R}^d)} \quad \textit{for all } t\ge 0$$

and if $\tilde{u}$ is the solution of (1) *associated to* $\tilde{u}(0,\cdot) = \tilde{u}_0(\cdot) \in L^1(\mathbb{R}^d)$, *then*

$$\|u(t,\cdot)-\tilde{u}(t,\cdot)\|_{L^1(\mathbb{R}^d)} \le \|u_0-\tilde{u}_0\|_{L^1(\mathbb{R}^d)} \quad \textit{for all } t\ge 0.$$

Remark 14. More generally, we have

$$\|(u(t,\cdot)-\tilde{u}(t,\cdot))^{\pm}\|_{L^1(\mathbb{R}^d)} \leq \|(u_0-\tilde{u}_0)^{\pm}\|_{L^1(\mathbb{R}^d)},$$

so that $u_0 \leq \tilde{u}_0$ entails $u \leq \tilde{u}$.

The rest of this paper is devoted to the proofs of these results. The proof of Proposition 11 is given in Appendix B, for the sake of completeness.

3. Preliminary lemmas

In this section, we give some basic results that will be useful in the sequel.

3.1. Main properties of the kinetic function χ.

Let us begin with some properties concerning the function defined in (21). We omit the proofs which can be found in [44], for instance.

Let us first make precise the definition of the sign function used throughout:

$$\operatorname{sgn}(\xi) := \begin{cases} 1 & \text{if } \xi > 0, \\ -1 & \text{if } \xi < 0, \\ 0 & \text{if } \xi = 0. \end{cases}$$

Lemma 15. (i) *For any reals u and ξ,*

$$\operatorname{sgn}(\xi)\chi(\xi;u) = |\chi(\xi;u)| = (\chi(\xi;u))^2.$$

(ii) *For any reals u and $\tilde{u}$,*

$$|u-v| = \int_{\mathbb{R}} |\chi(\xi;u)-\chi(\xi;\tilde{u})|\,d\xi = \int_{\mathbb{R}} (\chi(\xi;u)-\chi(\xi;\tilde{u}))^2\,d\xi.$$

Remark 16. The map $u \in L^\infty(\mathbb{R}^+;L^1(\mathbb{R}^d)) \mapsto \chi(\xi;u) \in L^\infty(\mathbb{R}^+;L^1(\mathbb{R}^{d+1}))$ is thus an isometry.

3.2. Main properties of the nonlocal diffusion operator g.

Let us continue with standard results on the operator defined in (7). The proofs are gathered in Appendix A for the sake of completeness; see also [38, 47, 20, 4, 5].

Let us first precise the sense of g for sufficiently regular functions.

Lemma 17. *Under (5)–(6), g is still well defined by (7) from $W^{2,1}(\mathbb{R}^d)$ (resp. $C_b^2(\mathbb{R}^d)$) into $L^1(\mathbb{R}^d)$ (resp. $C_b(\mathbb{R}^d)$). It is moreover linear, bounded, and*

$$\int_{\mathbb{R}^d} \varphi g[f]\,dx = \int_{\mathbb{R}^d} f g[\varphi]\,dx \quad \forall f \in W^{2,1}(\mathbb{R}^d), \forall \varphi \in C_b^2(\mathbb{R}^d). \tag{27}$$

Remark 18. If f is merely integrable, we can thus define $g[f]$ in the distribution sense by $\langle g[f],\varphi\rangle_{\mathcal{D}',\mathcal{D}} := \int f g[\varphi]$.

Let us continue with another useful formula; it is interpreted as an integration by parts formula involving the square root of g.

Lemma 19 (Bilinear form)**.** *Assume (5)–(6), $f \in W^{1,1}(\mathbb{R}^d)$ and $\tilde{f} \in W^{1,\infty}(\mathbb{R}^d)$. Then*

$$\int_{\mathbb{R}^d} \tilde{f} g[f]\,dx = \frac{1}{2}\int_{\mathbb{R}^{2d}} (f(x)-f(y))(\tilde{f}(x)-\tilde{f}(y))\mu(x-y)\,dx\,dy.$$

Remark 20. (1) The Borel measure $\mu(x-y)\,dx\,dy$ is defined as the pushforward measure

$$\mu(x-y)\,dx\,dy := T_{\#}\left(\mu(z)\,dx\,dz\right)$$

associated to $T : (x,z) \in \mathbb{R}^{2d} \mapsto (x, x+z) \in \mathbb{R}^{2d}$. We can thus change variables by

$$\int\limits_{\mathbb{R}^{2d}} f(x,y)\mu(x-y)\,dx\,dy = \int\limits_{\mathbb{R}^{2d}} f(x,x+z)\mu(z)\,dx\,dz \tag{28}$$

for any Borel measurable $f : \mathbb{R}^{2d} \to [0,\infty]$.

(2) Note that $\mu(x-y)\,dx\,dy$ is σ-finite because $\mu(z)\,dz$ is σ-finite. The Fubini's theorem then applies to define $\mu(x-y)\,dx\,dy\,d\xi := (\mu(x-y)\,dx\,dy) \otimes d\xi$, etc.

Here is a version of the previous result with time-dependent functions f, as we will have to deal with such functions in the sequel.

Lemma 21. *Assume* (5)–(6) *and let* $f = f(t,x,\xi)$ *and* $\tilde{f} = \tilde{f}(t,x,\xi)$ *be such that*

$$f, \nabla_x f, \nabla_x^2 f \in C([0,\infty); L^1(\mathbb{R}^{d+1})) \quad \text{and} \quad \tilde{f}, \nabla_x \tilde{f} \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^{d+1}).$$

Then $g_x[f] \in C([0,\infty); L^1(\mathbb{R}^{d+1}))$ *and for almost any* $t \geq 0$,

$$\begin{aligned} &\int\limits_{\mathbb{R}^{d+1}} \tilde{f} g_x[f] dx\,d\xi \\ &\quad = \frac{1}{2} \int\limits_{\mathbb{R}^{2d+1}} (f(t,x,\xi) - f(t,y,\xi))(\tilde{f}(t,x,\xi) - \tilde{f}(t,y,\xi))\mu(x-y)\,dx\,dy\,d\xi. \end{aligned} \tag{29}$$

3.3. **Main properties of the nonlocal dissipation measure** n**.** Let us end up with lemmas in relation with the function defined in (20). The first one is the rigorous justification of (17).

Lemma 22. *Under the assumption* (4)*, the formula* (17) *holds for all* $S \in C^2(\mathbb{R})$ *and* $a, b \in \mathbb{R}$ *(with* $\beta' = S'A'$*).*

Proof. Setting $I := \beta(b) - \beta(a) - S'(a)(A(b) - A(a))$, we have to prove that

$$I = \int\limits_{\mathbb{R}} S''(\xi)|A(b) - A(\xi)| \mathbb{1}_{\operatorname{conv}\{a,b\}}(\xi)\,d\xi. \tag{30}$$

This relies upon the following version of the Taylor's formula:

$$\begin{aligned} \beta(b) - \beta(a) &= \beta'(a)(b-a) + \int\limits_a^b \beta''(\xi)(b-\xi)\,d\xi \\ &= S'(a)A'(a)(b-a) + \int\limits_a^b \left(S''(\xi)A'(\xi) + S'(\xi)A''(\xi)\right)(b-\xi)\,d\xi. \end{aligned}$$

We also have $A'(a)(b-a) = A(b) - A(a) - \int_a^b A''(\xi)(b-\xi)\,d\xi$; hence,

$$I = \int\limits_a^b S''(\xi)A'(\xi)(b-\xi)\,d\xi + \int\limits_a^b (S'(\xi) - S'(a))A''(\xi)(b-\xi)\,d\xi =: I_1 + I_2.$$

Let us use again Taylor to rewrite the first term:

$$\begin{aligned}
I_1 &= \int_a^b S''(\xi)\left(A(b)-A(\xi)-\int_\xi^b A''(\zeta)(b-\zeta)\,d\zeta\right)d\xi\\
&= \int_a^b S''(\xi)(A(b)-A(\xi))\,d\xi - \int_a^b\int_\xi^b S''(\xi)A''(\zeta)(b-\zeta)\,d\zeta\,d\xi\\
&=: J_1 - J_2.
\end{aligned}$$

By the monotonicity of $A(\cdot)$, we recognize that J_1 is the right-hand side of (30). It thus only remains to prove that $J_2 = I_2$. But, we can rewrite I_2 as

$$I_2 = \int_a^b\int_a^\xi S''(\zeta)A''(\xi)(b-\xi)\,d\zeta\,d\xi$$

and the fact that $I_2 = J_2$ follows by the Fubini theorem. □

The result below will be crucial in the proof of the uniqueness.

Lemma 23. *Assume* (4). *For $a,b,c,d\in\mathbb{R}$, define*

$$\begin{aligned}
F(a,b,c,d) &:= \int_{\mathbb{R}} A'(\xi)\,(\chi(\xi;a)-\chi(\xi;b))\,(\chi(\xi;c)-\chi(\xi;d))\,d\xi,\\
G(a,b,c,d) &:= |A(b)-A(c)|\mathbb{1}_{\mathrm{conv}\{a,b\}}(c)+|A(d)-A(a)|\mathbb{1}_{\mathrm{conv}\{c,d\}}(a),
\end{aligned}$$

having in mind (18). *Then*

$$\forall a,b,c,d\in\mathbb{R} \text{ there holds } F(a,b,c,d)\le G(a,b,c,d).$$

Proof. Note first that $\chi(\xi;a)-\chi(\xi;b) = \mathrm{sgn}(a-b)\mathbb{1}_{\mathrm{conv}\{a,b\}}(\xi)$ if ξ is not an extremity of $\mathrm{conv}\{a,b\}$, so that

$$F(a,b,c,d) = \int_{\mathbb{R}} A'(\xi)\mathrm{sgn}(a-b)\mathrm{sgn}(c-d)\mathbf{1}_{\mathrm{conv}\{a,b\}\cap\mathrm{conv}\{c,d\}}(\xi)\,d\xi. \tag{31}$$

Let us now argue in several cases according as the way a, b, c and d are ordered. Note first that F could be nonpositive, whereas G is always nonnegative. We thus do not need to consider the cases where $F\le 0$, which, by (31), reduces our study to

- either $a\le b$ and $c\le d$,
- or $b\le a$ and $d\le c$.

Note next the symmetry $F(a,b,c,d)=F(c,d,a,b)$ and the analogous symmetry for G. We can thus also assume without loss of generality that $a\le c$ in every cases, that is to say:

- either $a\le b$, $c\le d$ and $a\le c$,
- or $b\le a$, $d\le c$ and $a\le c$.

Moreover, we can assume that $\mathrm{conv}\{a,b\}\cap\mathrm{conv}\{c,d\}$ is neither empty nor reduced to a singleton, because F would equal zero by (31) otherwise. This allows to precise again the preceding cases by

- either $a\le c<b$ and $c<d$,

- or $b < a$ and $d < a \leq c$.

Let us finally divide these cases into the four following ones:

1. either $a < c < b$ and $c < d$,
2. or $a = c < b$ and $c < d$,
3. or $b < a$ and $d < a < c$.
4. or $b < a$ and $d < a = c$.

In both the first and the second cases, we have

$$F(a,b,c,d) = \int_{\mathbb{R}} \mathbf{1}_{(a,b)\cap(c,d)=(c,\min\{b,d\})}(\xi) A'(\xi)\, d\xi = A(\min\{b,d\}) - A(c).$$

As far as G is concerned, we have

$$G(a,b,c,d) = \begin{cases} A(b) - A(c) & \text{in the first case,} \\ \frac{1}{2}(A(b) - A(c=a)) + \frac{1}{2}(A(d) - A(a=c)) & \text{in the second one,} \end{cases}$$

taking into account the monotonicity of $A(\cdot)$ and the specific definition of $\mathbb{1}$ in (18). In both cases, the monotonicity of $A(\cdot)$ implies that $F(a,b,c,d) \leq G(a,b,c,d)$. We argue similarly for the third and fourth cases, which completes the proof. $\square$

For the accurate proof of the uniqueness, we will need to consider truncations of the preceding quadruplet, namely $T_R(a)$, $T_R(b)$, $T_R(c)$ and $T_R(d)$, where

$$T_R(u) := \begin{cases} u & \text{if } -R \leq u \leq R, \\ \pm R & \text{if } \pm u > R, \end{cases} \tag{32}$$

for any given $R > 0$. Here is the precise result that we will use.

Lemma 24. *Assume* (4) *and* $R > 0$. *For any reals* a, b *and* ξ, *we have*

$$\mathbf{1}_{(-R,R)}(\xi)\,(\chi(\xi;a) - \chi(\xi;b)) = \chi(\xi;T_R(a)) - \chi(\xi;T_R(b)) \tag{33}$$

and

$$\begin{aligned} &|A(b) - A(T_R(\xi))|\mathbb{1}_{\mathrm{conv}\{a,b\}}(T_R(\xi)) \\ &\qquad\qquad \geq |A(T_R(b)) - A(T_R(\xi))|\mathbb{1}_{\mathrm{conv}\{T_R(a),T_R(b)\}}(T_R(\xi)) \end{aligned} \tag{34}$$

(with the representation (18)*).*

Proof. The identity (33) is immediate since $\mathbf{1}_{(-R,R)}(\xi)\chi(\xi;u) = \chi(\xi;T_R(u))$ for any reals u and ξ. Let us now prove (34). To do so, note that for any reals b and ξ,

$$|A(b) - A(T_R(\xi))| \geq |A(T_R(b)) - A(T_R(\xi))|; \tag{35}$$

indeed, the monotonicity of $A(\cdot)$ implies that

$$\begin{aligned} &|A(T_R(b)) - A(T_R(\xi))| \\ &= \begin{cases} A(R) - A(T_R(\xi)) \leq A(b) - A(T_R(\xi)) & \text{if } b > R, \\ |A(b) - A(T_R(\xi))| & \text{if } -R \leq b \leq R, \\ A(T_R(\xi)) - A(-R) \leq A(T_R(\xi)) - A(b) & \text{if } b < -R, \end{cases} \end{aligned}$$

so that (35) always holds. With (35) in hands, the proof of (34) is obvious in the case where $\mathrm{conv}\{T_R(a), T_R(b)\} \subseteq \mathrm{conv}\{a,b\}$; indeed, we then have

$$\mathbb{1}_{\mathrm{conv}\{a,b\}}(T_R(\xi)) \geq \mathbb{1}_{\mathrm{conv}\{T_R(a),T_R(b)\}}(T_R(\xi))$$

(even if $T_R(\xi) = a$ or b, in which case both these functions take the value $\frac{1}{2}$). When that inclusion fails, a and b are necessarily either both greater than R or both lower than $-R$. The right-hand side of (34) thus equals zero everywhere, which completes the proof. $\square$

4. Equivalence between entropy and kinetic solutions

This section is devoted to the proof of Theorem 12. Let us first justify the reformulation of the notion of entropy solutions in terms of the nonlocal dissipation measure (14).

Proof of Theorem 6. Let us assume that $u \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^+ \cap \mathbb{R}^d)$ is an entropy solution in the sense of Definition 1, thus satisfying the inequalities (16). Let I_r denote the nonlocal term in $|z| > r$ of (16). Applying the identity (17), we have

$$\begin{aligned} I_r = &\int_0^\infty \int_{\mathbb{R}^d} \int_{|z|>r} (\beta(u(t,x+z)) - \beta(u(t,x)))\varphi(t,x)\mu(z)\,dt\,dx\,dz \\ &- \int_0^\infty \int_{\mathbb{R}^d} \int_{|z|>r} \int_{\mathbb{R}} S''(\xi)|A(u(t,x+z)) - A(\xi)|\mathbb{1}_{\mathrm{conv}\{u(t,x),u(t,x+z)\}}(\xi) \\ &\cdot \varphi(t,x)\mu(z)\,dt\,dx\,dz\,d\xi \\ =:& J_r - K_r. \end{aligned}$$

We can integrate by parts, as in the proof of Lemma 17 in Appendix A, to rewrite

$$J_r = \int_0^\infty \int_{\mathbb{R}^d} \int_{|z|>r} \beta(u(t,x)))(\varphi(t,x+z) - \varphi(t,x))\mu(z)\,dt\,dx\,dz. \tag{36}$$

We then obtain (19) by passing to the limit in (16) as $r \downarrow 0$, thanks to the monotone convergence theorem giving us that

$$\begin{aligned} \lim_{r\downarrow 0} K_r = &\int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}} S''(\xi)\varphi(t,x) \\ &\cdot \underbrace{\int_{\mathbb{R}^d} |A(u(t,x+z)) - A(\xi)|\mathbb{1}_{\mathrm{conv}\{u(t,x),u(t,x+z)\}}(\xi)\mu(z)\,dz}_{=n(t,x,\xi)}\,dt\,dx\,d\xi. \end{aligned}$$

Conversely, if we now assume that (19) holds, then we cut all the nonlocal terms according as $|z| > r$ or not. Putting all the $|z| > r$ parts at the left-hand side, we get that

$$
\begin{aligned}
&\textit{first order} + \textit{initial terms}\\
&+\int_0^\infty\int_{\mathbb{R}^d}\int_{|z|>r}(\beta(u(t,x+z))-\beta(u(t,x)))\varphi(t,x)\mu(z)\,dt\,dx\,dz\\
&-\int_0^\infty\int_{\mathbb{R}^d}\int_{|z|>r}\int_{\mathbb{R}}S''(\xi)|A(u(t,x+z))-A(\xi)|\mathbb{1}_{\operatorname{conv}\{u(t,x),u(t,x+z)\}}(\xi)\\
&\quad\cdot\varphi(t,x)\mu(z)\,dt\,dx\,dz\,d\xi\\
&+P.V.\int_0^\infty\int_{\mathbb{R}^d}\int_{|z|\le r}\beta(u(t,x)))(\varphi(t,x+z)-\varphi(t,x))\mu(z)\,dt\,dx\,dz\\
&\ge\int_0^\infty\int_{\mathbb{R}^d}\int_{|z|\le r}\int_{\mathbb{R}}S''(\xi)|A(u(t,x+z))-A(\xi)|\mathbb{1}_{\operatorname{conv}\{u(t,x),u(t,x+z)\}}(\xi)\\
&\quad\cdot\varphi(t,x)\mu(z)\,dt\,dx\,dz\,d\xi,
\end{aligned}
$$

where we have done the reverse integration by parts than in (36) to rewrite J_r in its initial form. The left-hand side is thus the same than in (16), again by (17). Since moreover the right-hand side is nonnegative (the test φ being nonnegative in our considerations), we already have (16) and the proof is complete. □

With Theorem 6 at hand, we can establish the equivalence between entropy and kinetic solutions by following standard arguments from [40, 41, 19, 44]. Let us give details for completeness. We will use Proposition 11, whose proof is postponed to Appendix B. We will also use the two following lemmas.

Lemma 25. *Let $u_0 \in L^1(\mathbb{R}^d)$, assume that* (3)–(6) *hold and suppose that the function $u \in L^\infty(\mathbb{R}; L^1(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ is a kinetic solution of* (1)*. The associated measures then satisfy* $\operatorname{supp}(m+n) \subseteq \{\operatorname{ess\,inf} u \le \xi \le \operatorname{ess\,sup} u\}$.

Proof. If $\xi \notin [\operatorname{ess\,inf} u, \operatorname{ess\,sup} u]$, $\chi(\xi; u) = 0$ by (21) and thus $\partial_\xi(m+n) = 0$ in $\mathcal{D}'((0,\infty) \times \mathbb{R}^d \times (\mathbb{R}_\xi \setminus [\operatorname{ess\,inf} u, \operatorname{ess\,sup} u]))$ by the first line of (26). The result follows from (23) and the last line of (26). □

Lemma 26. *Let* (3)–(6) *hold and consider an entropy solution u of* (1) *with initial data $u_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$. Then the associated nonlocal dissipation measure n belongs to the space $L^1(\mathbb{R}^+ \times \mathbb{R}^{d+1})$.*

Note that this is the rigorous justification of Remark 7.

Proof. Consider $S(u) = \frac{u^2}{2}$ which is integrable in x. Take associated nonlinearities vanishing at zero so that $\eta(u)$ and $\beta(u)$ are also integrable in x. Consider the test function $\varphi_k(t)\phi(x/M)$ in (19) with $0 \le \varphi_k \in \mathcal{D}([0,\infty))$ pointwise converging to 1 as $k \to \infty$, such that $\varphi_k' \le 0$, and with $0 \le \phi \in \mathcal{D}(\mathbb{R}^d)$ such that $\phi(0) = 1$. The limit $M \to \infty$ implies that

$$
\int_{\mathbb{R}^d} S(u_0(x))\varphi_k(0)\,dx \ge \int_0^\infty\int_{\mathbb{R}^{d+1}} S''(\xi)n(t,x,\xi)\varphi_k(t)\,dt\,dx\,d\xi,
$$

thanks to the fact that $S(u)\varphi_k' = u^2\varphi_k'/2 \leq 0$, to Lemma 35 in appendix and to Fatou's lemma. Since $S'' \equiv 1$, the limit $k \to \infty$ completes the proof.

□

We will finally need a classical density result recalled below.

Lemma 27. *Let $\varphi \in \mathcal{D}((0,\infty)\times\mathbb{R}^{d+1})$ be nonnegative. Then it can be approximated (for the topology of $\mathcal{D}$) by functions of the form*

$$(t,x,\xi) \mapsto \sum_{i=1}^{N} \varphi_i(t,x)\phi_i(\xi),$$

for some integer N and nonnegative $\varphi_i \in \mathcal{D}((0,\infty)\times\mathbb{R}^d)$ and $\phi_i \in \mathcal{D}(\mathbb{R}_\xi)$.

The property can be obtained by mollifying $\varphi = \varphi(t,x,\xi)$ with an approximate unit of the form $\rho_\epsilon(t,x)\theta_\epsilon(\xi)$ and discretizing the convolutions.

Proof of Theorem 12. Assume first that $u \in L^\infty(\mathbb{R}^+;L^1(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^+\times\mathbb{R}^d)$ is a kinetic solution and let us show that it is an entropy solution. Recall that $\chi(\xi;u) = 0$ if $\xi \notin [\operatorname{ess\,inf} u, \operatorname{ess\,sup} u]$ and note that $\chi(\xi;u_0) = 0$ as well by the middle line of (26). By Lemma 25, we can then choose test functions in (24) of the form

$$(t,x,\xi) \mapsto \varphi(t,x)S'(\xi),$$

with $\varphi \in \mathcal{D}(\mathbb{R}_t\times\mathbb{R}^d)$ and $S \in C^\infty(\mathbb{R}_\xi)$ convex, up to modifying S for large $|\xi|$. Using in addition the identity (22), we deduce that

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}^d} (S(u)\partial_t\varphi + \eta(u)\cdot\nabla\varphi - \beta(u)g[\varphi])\,dt\,dx\\
&-\int_0^\infty\int_{\mathbb{R}^d} (S(0)\partial_t\varphi + \eta(0)\cdot\nabla\varphi - \beta(0)g[\varphi])\,dt\,dx\\
&+\int_{\mathbb{R}^{d+1}} S(u_0(x))\varphi(0,x)\,dx - \int_{\mathbb{R}^d} S(0)\varphi(0,x)\,dx\\
&=\int_0^\infty\int_{\mathbb{R}^{d+1}} S''(\xi)(m+n)(t,x,\xi)\varphi(t,x)\,dt\,dx\,d\xi.
\end{aligned}$$

Noticing that the sum of the second and fourth integrals of the left-hand side is zero, we obtain (19) for smooth entropies. But, it is clear that S can be chosen merely C^2 by an approximation procedure.

Conversely, assume that u is an entropy solution and let us show that it is a kinetic one. We use the reformulation (26) of Proposition 11. Since we have $u \in C([0,\infty);L^1(\mathbb{R}^d))$ with $u(0,\cdot) = u_0(\cdot)$ by Theorem 3, we already know that

$$\lim_{t\downarrow 0}\|u(t,\cdot) - u_0(\cdot)\|_{L^1(\mathbb{R}^d)} = 0.$$

Let us now construct m as the distribution

$$m(t,x,\xi) := \int_{-\infty}^{\xi} \left(\partial_t \chi(\zeta;u) + F'(\zeta)\cdot\nabla_x \chi(\zeta;u) + A'(\zeta) g_x[\chi(\zeta;u)]\right) d\zeta - n(t,x,\xi). \tag{37}$$

The integral is well defined in $\mathcal{D}'((0,\infty)\times\mathbb{R}^{d+1})$ since $\chi(\xi;u)=0$ if $\xi<\operatorname{ess\,inf} u$. The distribution m is thus well defined since $n\in L^1(\mathbb{R}^+\times\mathbb{R}^{d+1})$ by Lemma 26. It satisfies the equation in (26) by construction and it only remains to show the other conditions of Proposition 11. Let us first show that

$$\operatorname{supp}(m) \subseteq \{(t,x,\xi) : \operatorname{ess\,inf} u \le \xi \le \operatorname{ess\,sup} u\}. \tag{38}$$

Firstly, it is immediate from (20) that

$$\operatorname{supp}(n) \subseteq \{(t,x,\xi) : \operatorname{ess\,inf} u \le \xi \le \operatorname{ess\,sup} u\}.$$

Secondly, (21) implies that for any locally Lipschitz $S(\cdot)$,

$$\int_{-\infty}^{\xi} S'(\zeta)\chi(\zeta;u)\, d\zeta = \begin{cases} S(u)-S(0) & \text{if } \xi > \operatorname{ess\,sup} u,\\ 0 & \text{if } \xi < \operatorname{ess\,inf} u.\end{cases}$$

Taking these facts into account in (37) with S as the identity, F and A, we find that

$$m = \begin{cases} \partial_t u + \nabla_x F(u) + g_x[A(u)] - \underbrace{(\nabla_x F(0) + g_x[A(0)])}_{=0} & \text{if } \xi > \operatorname{ess\,sup} u,\\ 0 & \text{if } \xi < \operatorname{ess\,inf} u.\end{cases}$$

But, the remaining term of the right-hand side equals zero thanks to the weak formulation of (1), see [20]. This completes the proof of (38).

The test functions of Equation (37) can thus also be taken of the form

$$(t,x,\xi)\mapsto \varphi(t,x)S''(\xi),$$

for any $0\le\varphi\in\mathcal{D}((0,\infty)\times\mathbb{R}^d)$ and $S\in C^\infty(\mathbb{R}_\xi)$ convex, up to modifying S for large $|\xi|$. This gives us that

$$\begin{aligned} &\langle m, \varphi S''\rangle_{\mathcal{D}',\mathcal{D}} \\ &= \int_0^\infty \int_{\mathbb{R}^{d+1}} \chi(\xi;u)\left(S'(\xi)\partial_t\varphi + (S'F')(\xi)\cdot\nabla\varphi - (S'A')(\xi)g[\varphi]\right) dt\, dx\, d\xi \\ &\quad - \int_0^\infty \int_{\mathbb{R}^{d+1}} S''(\xi) n(t,x,\xi)\varphi(t,x)\, dt\, dx\, d\xi, \end{aligned} \tag{39}$$

where we recognize the terms in (19) again by (22). Hence

$$\langle m, \varphi S''\rangle_{\mathcal{D}',\mathcal{D}} \ge 0,$$

for such $\varphi=\varphi(t,x)$ and $S=S(\xi)$, which implies that m is a nonnegative Radon measure on $(0,\infty)\times\mathbb{R}^{d+1}$ by the density claim of Lemma 27. To conclude, we need the result below.

Lemma 28. *The measure m thus constructed on $(0,\infty)\times\mathbb{R}^{d+1}$ satisfies, for almost every $\xi\in\mathbb{R}$,*

$$\int_0^\infty \int_{\mathbb{R}^d} (m+n)(t,x,\xi)\,dt\,dx \le \nu(\xi)$$

(in the sense of (25)) where

$$\nu(\xi) = \|(u_0-\xi)^+ \mathbf{1}_{\xi>0}\|_{L^1(\mathbb{R}^d)} + \|(u_0-\xi)^- \mathbf{1}_{\xi<0}\|_{L^1(\mathbb{R}^d)}$$

is such that $\nu\in L^\infty_0(\mathbb{R}_\xi)$.

Let us admit it for a while and complete the proof of Theorem 12. Extending m on $[0,\infty)\times\mathbb{R}^{d+1}$ by $m(\{t=0\}) := 0$, we obtain a Radon measure such that $\int_0^\infty \int_{\mathbb{R}^d} \int_{-R}^R (m+n)\,dt\,dx\,d\xi < \infty$, for any $R>0$. The last condition of (26) follows from the dominated convergence theorem. The proof is complete. □

Let us now prove the preceding lemma.

Proof of Lemma 28. Recall that before admitting Lemma 28, u was an entropy solution of (1) and $0\le m\in M^1_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^{d+1})$ was such that (38) and (39) hold. For any $t_0>0$, $0\le\varphi\in\mathcal{D}((0,\infty)\times\mathbb{R}^d)$ and $S\in C^\infty(\mathbb{R}_\xi)$ convex, we thus have

$$\begin{aligned}
&\int_{t_0}^\infty \int_{\mathbb{R}^{d+1}} \chi(\xi;u)\,(F'(\xi)\cdot\nabla_x\varphi - A'(\xi)g_x[\varphi])\,dt\,dx\,d\xi \\
&+ \int_{t_0}^\infty \int_{\mathbb{R}^d} S(u)\partial_t\varphi\,dt\,dx + \int_{\mathbb{R}^{d+1}} S(u(t_0,x))\varphi(t_0,x)\,dx \\
&= \int_{t_0}^\infty \int_{\mathbb{R}^{d+1}} S''(\xi)(m+n)(t,x,\xi)\varphi(t,x)\,dt\,dx\,d\xi.
\end{aligned}$$

As previously, we choose $\varphi(t,x) := \varphi_k(t)\phi(x/M)$ with φ_k nonincreasing on $[t_0,\infty)$, pointwise converging to 1 as $k\to\infty$, $0\le\phi\in\mathcal{D}(\mathbb{R}^d)$, and $\phi(0)=1$. Then for any $S\ge 0$ with $S(0)=0$, the successive limits $M,k\to\infty$ imply that

$$\int_{\mathbb{R}^d} S(u(t_0,x))\,dx \ge \int_{t_0}^\infty \int_{\mathbb{R}^{d+1}} S''(\xi)(m+n)(t,x,\xi)\,dt\,dx\,d\xi,$$

thanks again to the fact that $S(u)\varphi_k'\le 0$, Lemma 35, and Fatou's lemma. Now considering any arbitrarily given $0\le\psi\in\mathcal{D}(\mathbb{R})$, we can take

$$S(\xi) := \int_{\mathbb{R}} \left\{(\xi-\zeta)^+\mathbf{1}_{\zeta>0} + (\xi-\zeta)^-\mathbf{1}_{\zeta<0}\right\}\psi(\zeta)\,d\zeta$$

because it is convex, zero at zero, and nonnegative. Hence

$$\begin{aligned}
&\int_{t_0}^\infty \int_{\mathbb{R}^{d+1}} \psi(\xi)(m+n)(t,x,\xi)\,dt\,dx\,d\xi \\
&\qquad \le \int_{\mathbb{R}^{d+1}} \left\{(u(t_0,x)-\zeta)^+\mathbf{1}_{\zeta>0} + (u(t_0,x)-\zeta)^-\mathbf{1}_{\zeta<0}\right\}\psi(\zeta)\,dx\,d\zeta
\end{aligned}$$

which is the desired result for the initial time t_0. We get the result as $t_0 \downarrow 0$ by recalling that $u(t_0, \cdot) \to u_0(\cdot)$ in $L^1(\mathbb{R}^d)$, see Theorem 3. $\square$

5. Uniqueness and L^1 contraction for kinetic solutions

This section and the next one are devoted to the proof of Theorem 13. Here we focus on the L^1 contraction principle that we restate below for the reader's convenience.

Theorem 29. *Assume* (3)–(6) *and let u and $\tilde{u}$ be two kinetic solutions of* (1) *with respective initial data u_0 and $\tilde{u}_0$ belonging to $L^1(\mathbb{R}^d)$. Then,*

$$\|u(t,\cdot) - \tilde{u}(t,\cdot)\|_{L^1(\mathbb{R}^d)} \leq \|u_0 - \tilde{u}_0\|_{L^1(\mathbb{R}^d)} \quad \textit{for a.e. } t \geq 0. \tag{40}$$

As usually in the kinetic setting, we first give a formal proof which will be made rigorous later by a regularization procedure.

We will follow the guidelines of [19] without needing to regularize in ξ similarly to what is done in [44]. This simplification will be also possible in our setting roughly speaking because the nonlocal dissipation measure is absolutely continuous.

5.1. A formal proof of uniqueness.

During this formal proof of (40), $u(t, x)$ is often shortly denoted by $u(x)$; this means that we abusively omit the time variable if there is no confusion. Moreover, $\chi(\xi; u(t, x))$ is shortened to $\chi(\xi; u)$ (as we did many times already). Let now m, n, $\tilde{m}$ and $\tilde{n}$ be the measures associated to u and $\tilde{u}$, respectively. Let us recall that n and $\tilde{n}$ are given by (20), that is to say:

$$\begin{aligned} n(t,x,\xi) &= \int_{\mathbb{R}^d} |A(u(x+z)) - A(\xi)| \mathbb{1}_{\operatorname{conv}\{u(x), u(x+z)\}}(\xi) \mu(z)\, dz, \\ \tilde{n}(t,x,\xi) &= \int_{\mathbb{R}^d} |A(\tilde{u}(x+z)) - A(\xi)| \mathbb{1}_{\operatorname{conv}\{\tilde{u}(x), \tilde{u}(x+z)\}}(\xi) \mu(z)\, dz. \end{aligned} \tag{41}$$

As in [19], we introduce the following microscopic contraction functional

$$M(t,x,\xi) := \left(\chi(\xi; u(t,x)) - \chi(\xi; \tilde{u}(t,x))\right)^2 \tag{42}$$

and consider its derivative with respect to time:

$$\dot{\mathcal{M}}(t) := \frac{d}{dt} \int_{\mathbb{R}^{d+1}} M(t,x,\xi)\, dx\, d\xi.$$

Having in mind the properties given in Lemma 15, we see that on the one hand,

$$\begin{aligned} \dot{\mathcal{M}}(t) &= \int_{\mathbb{R}^{d+1}} \partial_t \left\{ |\chi(\xi; u)| + |\chi(\xi; \tilde{u})| - 2\left[\chi(\xi; u)\chi(\xi; \tilde{u})\right] \right\} dx\, d\xi \\ &= \int_{\mathbb{R}^{d+1}} \operatorname{sgn}(\xi) \left(\partial_t \chi(\xi; u) + \partial_t \chi(\xi; \tilde{u})\right) dx\, d\xi \\ &\quad - 2 \int_{\mathbb{R}^{d+1}} \left(\chi(\xi; u) \partial_t \chi(\xi; \tilde{u}) + \chi(\xi; \tilde{u}) \partial_t \chi(\xi; u)\right) dx\, d\xi \end{aligned}$$

and on the other hand,

$$\dot{\mathcal{M}}(t) = \frac{d}{dt} \|u(t,\cdot) - \tilde{u}(t,\cdot)\|_{L^1(\mathbb{R}^d)}.$$

So we reach to (40) if we get the property $\dot{\mathcal{M}}(t) \le 0$.

To do so, let us consider the equation of $\chi(\xi;u)$ written in (26). Multiplying it by $\mathrm{sgn}(\xi)$, we get

$$\partial_t|\chi(\xi;u)| + F'(\xi).\nabla_x|\chi(\xi;u)| + A'(\xi)g_x[|\chi(\xi;u)|] = \mathrm{sgn}(\xi)\partial_\xi(m+n). \tag{43}$$

In the same way, we have

$$\partial_t|\chi(\xi;\tilde{u})| + F'(\xi).\nabla_x|\chi(\xi;\tilde{u})| + A'(\xi)g_x[|\chi(\xi;\tilde{u})|] = \mathrm{sgn}(\xi)\partial_\xi(\tilde{m}+\tilde{n}). \tag{44}$$

Secondly, we multiply the equation of $\chi(\xi;u)$ by $\chi(\xi;\tilde{u})$, and do similar computations for $\tilde{u}$, to get

$$\begin{aligned}\chi(\xi;\tilde{u})\partial_t\chi(\xi;u) + F'(\xi).\chi(\xi;\tilde{u})\nabla_x\chi(\xi;u) + A'(\xi)\chi(\xi;\tilde{u})g_x[\chi(\xi;u)]& \\ = \chi(\xi;\tilde{u})\partial_\xi(m+n).&\end{aligned} \tag{45}$$

and

$$\begin{aligned}\chi(\xi;u)\partial_t\chi(\xi;\tilde{u}) + F'(\xi).\chi(\xi;u)\nabla_x\chi(\xi;\tilde{u}) + A'(\xi)\chi(\xi;u)g_x[\chi(\xi;\tilde{u})]& \\ = \chi(\xi;u)\partial_\xi(\tilde{m}+\tilde{n}).&\end{aligned} \tag{46}$$

Now we add the equalities (43) and (44) from which we subtract twice the sum of those given in (45) and (46). Then, after an integration over $\mathbb{R}^{d+1}$, we get

$$\begin{aligned}\dot{\mathcal{M}}(t) = &\int_{\mathbb{R}^{d+1}} (\mathrm{sgn}(\xi) - 2\chi(\xi;\tilde{u}))\partial_\xi(m+n)\,dx\,d\xi \\ &+ \int_{\mathbb{R}^{d+1}} (\mathrm{sgn}(\xi) - 2\chi(\xi;u))\partial_\xi(\tilde{m}+\tilde{n})\,dx\,d\xi \\ &+2\int_{\mathbb{R}^{d+1}} A'(\xi)\left\{(\chi(\xi;\tilde{u})g_x[\chi(\xi;u)] + \chi(\xi;u)g_x[\chi(\xi;\tilde{u})]\right\}dx\,d\xi \\ =:&\ I_1(t) + I_2(t) + I_3(t).\end{aligned} \tag{47}$$

Notice that we have omitted several terms because - at least formally - they are equal to zero, namely

$$\int_{\mathbb{R}^{d+1}} F'(\xi)\cdot\nabla_x|\chi(\xi;u)|\,dx\,d\xi \;=\; 0 \;=\; \int_{\mathbb{R}^{d+1}} F'(\xi)\cdot\nabla_x|\chi(\xi;\tilde{u})|\,dx\,d\xi,$$

as well as

$$\begin{aligned}&\int_{\mathbb{R}^{d+1}} F'(\xi)\cdot\{\chi(\xi;\tilde{u})\nabla_x\chi(\xi;u) + \chi(\xi;u)\nabla_x\chi(\xi;\tilde{u})\}\,dx\,d\xi \\ &\qquad = \int_{\mathbb{R}^{d+1}} F'(\xi)\cdot\nabla_x(\chi(\xi;u)\chi(\xi;\tilde{u}))\,dx\,d\xi \;=\; 0\end{aligned}$$

and

$$\int_{\mathbb{R}^{d+1}} A'(\xi)g_x[|\chi(\xi;u)|]\,dx\,d\xi \;=\; 0 \;=\; \int_{\mathbb{R}^{d+1}} A'(\xi)g_x[|\chi(\xi;\tilde{u})|]\,dx\,d\xi.$$

All these equalities stem from the use of the Fubini theorem and from the fact that, in a sense, the functions $\chi(\xi,u)$ and $\chi(\xi,\tilde{u})$ vanish as $|\xi| \to \infty$ due to their integrability. To get the last equality, we have also (formally) used Lemma 19. Now it remains to show that

$$I_1(t) + I_2(t) + I_3(t) \leq 0$$

in (47). For the first term, we use that

$$\partial_\xi \mathrm{sgn}(\xi) = 2\delta(\xi) \quad \text{and} \quad \partial_\xi \chi(\xi; \tilde{u}) = \delta(\xi) - \delta(\xi - \tilde{u}).$$

We use similar (formal) calculations for the second term and infer that

$$\begin{aligned}
&I_1(t) + I_2(t) \\
&= -2 \int\limits_{\mathbb{R}^{d+1}} \{(\delta(\xi - \tilde{u}(x))(m+n)(t,x,\xi) + \delta(\xi - u(x))(\tilde{m}+\tilde{n})(t,x,\xi)\}\,dx\,d\xi \\
&\leq -2 \int\limits_{\mathbb{R}^{2d+1}} \delta(\xi - \tilde{u}(x))|A(u(x+z)) - A(\xi)|\mathbb{1}_{\mathrm{conv}\{u(x),u(x+z)\}}(\xi)\mu(z)\,dx\,dz\,d\xi \\
&\quad - 2 \int\limits_{\mathbb{R}^{2d+1}} \delta(\xi - u(x))|A(\tilde{u}(x+z)) - A(\xi)|\mathbb{1}_{\mathrm{conv}\{\tilde{u}(x),\tilde{u}(x+z)\}}(\xi)\mu(z)\,dx\,dz\,d\xi,
\end{aligned}$$

thanks to the nonnegativity of the measures m, $\tilde{m}$ and to explicit representations (41) of the measures n, $\tilde{n}$. After the integration in ξ, we get

$$\begin{aligned}
&I_1(t) + I_2(t) \\
&\leq -2 \int\limits_{\mathbb{R}^{2d}} |A(u(y)) - A(\tilde{u}(x))|\mathbb{1}_{\tilde{u}(x)\in\mathrm{conv}\{u(x),u(y)\}}\mu(x-y)\,dx\,dy \\
&\quad - 2 \int\limits_{\mathbb{R}^{2d}} |A(\tilde{u}(y)) - A(u(x))|\mathbb{1}_{u(x)\in\mathrm{conv}\{\tilde{u}(x),\tilde{u}(y)\}}\mu(x-y)\,dx\,dy,
\end{aligned}$$

where we have also (formally) changed the variables by $x + z \mapsto y$ and used the symmetry $\mu(y - x) = \mu(x - y)$ in (6). We recognize the G-term of Lemma 23 and thus infer that

$$I_1(t) + I_2(t) \leq -2 \int\limits_{\mathbb{R}^{2d}} G(u(x), u(y), \tilde{u}(x), \tilde{u}(y))\mu(x-y)\,dx\,dy.$$

Further, we use Lemma 19 to rewrite the last term $I_3(t)$ as follows:

$$\begin{aligned}
I_3(t) = 2 \int\limits_{\mathbb{R}^{2d+1}} &A'(\xi)\,\{\chi(\xi; u(x)) - \chi(\xi; u(y))\} \\
&\cdot \{\chi(\xi; \tilde{u}(x)) - \chi(\xi; \tilde{u}(y))\}\,\mu(x-y)\,dx\,dy\,d\xi.
\end{aligned}$$

We recognize the F-term of Lemma 23. Hence

$$I_3(t) = 2 \int\limits_{\mathbb{R}^{2d}} F(u(x), u(y), \tilde{u}(x), \tilde{u}(y))\mu(x-y)\,dx\,dy$$

and finally $\dot{\mathcal{M}}(t) = I_1(t) + I_2(t) + I_3(t) \leq 0$, since

$$F(u(x), u(y), \tilde{u}(x), \tilde{u}(y)) \leq G(u(x), u(y), \tilde{u}(x), \tilde{u}(y)),$$

by Lemma 23. This completes the formal proof of the L^1 contraction.

5.2. **Accurate uniqueness and L^1 contraction proof.** Let us now give the rigorous proof of Theorem 29. For brevity, we set $\chi = \chi(t,x,\xi) := \chi(\xi; u(t,x))$ and do similarly for $\tilde{u}$. Next, we follow the regularization approach of [44] thus considering $\epsilon > 0$ and some approximate unit

$$\rho_\epsilon(t,x) := \frac{1}{\epsilon}\rho_1\left(\frac{t}{\epsilon}\right)\frac{1}{\epsilon^d}\rho_2\left(\frac{x}{\epsilon}\right),$$

with kernels satisfying

$$\begin{cases}\rho_1 \in \mathcal{D}((-1,0)),\ \rho_2 \in \mathcal{D}(\mathbb{R}^d),\\ \rho_1,\rho_2 \geq 0 \text{ and } \int_{\mathbb{R}} \rho_1 = \int_{\mathbb{R}^d} \rho_2 = 1.\end{cases}$$

Given $f \in L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^{d+1})$ (or M^1_{loc}), we denote its regularized version by

$$f_\epsilon(t,x,\xi) := (f * \rho_\epsilon)(t,x,\xi) = \int_0^\infty \int_{\mathbb{R}^d} \rho_\epsilon(t-s, x-\eta) f(s,\eta,\xi)\, ds\, d\eta.$$

This can also write

$$f_\epsilon(t,x,\xi) = \int_0^\infty \int_{\mathbb{R}^d} \rho_\epsilon(-s,-\eta) f(t+s, x+\eta, \xi)\, ds\, d\eta$$

(with the pushforward measure, cf. Remark 20). Note that the symbol '$*$' denotes the convolution in (x,t) without convoluting in ξ. We then define

$$M^\epsilon(t,x,\xi) := |\chi_\epsilon| + |\tilde{\chi}_\epsilon| - 2\chi_\epsilon\tilde{\chi}_\epsilon,$$

$$\mathcal{M}^\epsilon(t) := \int_{\mathbb{R}^{d+1}} M^\epsilon(t,x,\xi)\, dx\, d\xi \;=\; \mathcal{M}_1^\epsilon(t) + \mathcal{M}_2^\epsilon(t) + \mathcal{M}_3^\epsilon(t), \tag{48}$$

where $\mathcal{M}_1^\epsilon$, $\mathcal{M}_2^\epsilon$ and $\mathcal{M}_3^\epsilon$ correspond to the contributions of the respective terms of $M^\epsilon(t,x,\xi)$ to the integral $\mathcal{M}^\epsilon(t)$.

We shall see that this is a regularized version of the microscopic contraction functional (42). Here is the main lemma that we will have to prove.

Lemma 30. *Let the assumptions of Theorem 29 hold and let $\epsilon > 0$ be fixed. Then, we have $\mathcal{M}^\epsilon \in C^1([0,\infty))$ with $\dot{\mathcal{M}}^\epsilon(t) \leq 0$ for all $t \geq 0$.*

Let us admit this result for a while and complete the proof of Theorem 29. For that, we argue as in [44]. For the sake of completeness, we provide details.

Proof of Theorem 29. Let us recall that

$$\chi \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^{d+1})) \cap L^\infty(\mathbb{R}^+ \times \mathbb{R}^{d+1}) \tag{49}$$

by Remark 16. It is standard that $\chi_\epsilon \to \chi$ in $L^1_{\text{loc}}([0,\infty); L^1(\mathbb{R}^{d+1}))$, as $\epsilon \downarrow 0$, while remaining bounded in $L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^{d+1})) \cap L^\infty(\mathbb{R}^+ \times \mathbb{R}^{d+1})$. Recalling then that

$$M(t,x,\xi) = (\chi - \tilde{\chi})^2 = |\chi| + |\tilde{\chi}| - 2\chi\tilde{\chi}$$

and

$$\mathcal{M}(t) = \int_{\mathbb{R}^{d+1}} M(t,x,\xi)\, dx\, d\xi = \|u(t,\cdot) - \tilde{u}(t,\cdot)\|_{L^1(\mathbb{R}^d)},$$

we infer that $\mathcal{M}(\cdot)$ is the limit of $\mathcal{M}^\epsilon(\cdot)$ in $L^1_{\text{loc}}([0,\infty))$. By Lemma 30

$$t \geq 0 \mapsto \|u(t,\cdot) - \tilde{u}(t,\cdot)\|_{L^1(\mathbb{R}^d)}$$

is essentially nondecreasing and we get (40) by using that

$$\lim_{t\downarrow 0} \|u(t,\cdot)-\tilde{u}(t,\cdot)\|_{L^1(\mathbb{R}^d)} = \|u_0-v_0\|_{L^1(\mathbb{R}^d)}.$$

Let us recall that the latter limit is a consequence of Proposition 11 proved in Appendix B. This completes the proof of Theorem 29. $\square$

Let us now establish Lemma 30. Before, we need some technical results. The two first ones work as in [44]. Let us give details for completeness.

Lemma 31. *Let $u \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^d))$ and $\epsilon > 0$. Then*

$$\chi_\epsilon \in C([0,\infty); L^1(\mathbb{R}^{d+1})) \cap L^\infty(\mathbb{R}^+ \times \mathbb{R}^{d+1})$$

and all its derivatives in (t,x) satisfy the same property.

The proof is immediate from (49) since $\chi_\varepsilon = \chi * \rho_\epsilon$.

Corollary 32. *Let $u, \tilde{u} \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^d))$. Then $\mathcal{M}^\epsilon \in C^1([0,\infty))$ with*

$$\dot{\mathcal{M}}^\epsilon(t) = \int_{\mathbb{R}^{d+1}} \operatorname{sgn}(\xi)\,(\partial_t \chi_\epsilon + \partial_t \tilde{\chi}_\epsilon)\,dx\,d\xi - 2\int_{\mathbb{R}^{d+1}} \partial_t\left[\chi_\epsilon \tilde{\chi}_\epsilon\right]dx\,d\xi \quad \forall t \geq 0. \tag{50}$$

Proof. By (21) for almost every (t,x,ξ) there holds $\operatorname{sgn}(\chi(t,x,\xi)) = \operatorname{sgn}(\xi)$; this property is inherited by χ_ϵ, for all $\epsilon > 0$. Hence we have $|\chi_\epsilon| = \operatorname{sgn}(\xi)\chi_\epsilon \in C([0,\infty); L^1(\mathbb{R}^{d+1}))$ with a time distribution derivative satisfying

$$\partial_t|\chi_\epsilon| = \operatorname{sgn}(\xi)\partial_t\chi_\epsilon \in C([0,\infty); L^1(\mathbb{R}^{d+1})).$$

For any $\varphi \in \mathcal{D}((0,\infty))$ and $\phi \in \mathcal{D}(\mathbb{R}^{d+1})$, we thus have

$$\int_0^\infty \int_{\mathbb{R}^{d+1}} |\chi_\epsilon| \frac{d\varphi}{dt}(t)\phi(x,\xi)\,dt\,dx\,d\xi = -\int_0^\infty \int_{\mathbb{R}^{d+1}} \varphi(t)\phi(x,\xi)\operatorname{sgn}(\xi)\partial_t\chi_\epsilon\,dt\,dx\,d\xi.$$

Since we know that χ_ϵ and $\partial_t\chi_\epsilon$ belong to $L^1_{\text{loc}}([0,\infty); L^1(\mathbb{R}^{d+1}))$, we can take $\phi \equiv 1$ and find that

$$\frac{d}{dt}\left(t \mapsto \int_{\mathbb{R}^{d+1}} |\chi_\epsilon|\,dx\,d\xi\right) = \int_{\mathbb{R}^{d+1}} \operatorname{sgn}(\xi)\partial_t\chi_\epsilon\,dx\,d\xi \quad \text{in} \quad \mathcal{D}'((0,\infty)).$$

This gives the contribution to (50) of the term $\mathcal{M}_1^\epsilon = \int_{\mathbb{R}^{d+1}} |\chi_\epsilon|\,dx\,d\xi$ from (48). We argue in the same way for the terms $\mathcal{M}_2^\epsilon$, $\mathcal{M}_3^\epsilon$ and justify (50) in the sense of distributions. In particular $\mathcal{M}^\epsilon \in C^1([0,\infty))$ since the right-hand side of (50) is continuous by Lemma 31. $\square$

The next lemma is specific to nonlocal diffusions especially (iii). It will allow us to avoid regularization in ξ during the whole proof of uniqueness, as mentioned previously.

Lemma 33. *Assume* (3)–(6) *and u is a kinetic solution of* (1) *(for some L^1 initial data). Let m and n be the associated dissipation measures. Let $\epsilon > 0$ be fixed. Then:*

(i) *$m_\epsilon + n_\epsilon \in W^{1,\infty}_{\text{loc}}([0,\infty) \times \mathbb{R}^{d+1})$ and $\partial_\xi(m_\epsilon + n_\epsilon) \in C([0,\infty); L^1(\mathbb{R}^d \times K)$, for any compact $K \subset \mathbb{R}_\xi$,*

(ii) *there exists* $\nu_\epsilon \in C_0(\mathbb{R}_\xi)$ *such that*

$$\int\limits_{\mathbb{R}^d} (m_\epsilon + n_\epsilon)(t,x,\xi)\,dx \leq \nu_\epsilon(\xi),$$

for any $t \geq 0$ *and* $\xi \in \mathbb{R}$,

(iii) *and for any* $(t,x,\xi) \in [0,\infty) \times \mathbb{R}^{d+1}$,

$$(m_\epsilon + n_\epsilon)(t,x,\xi) \geq \int\limits_0^\infty \int\limits_{\mathbb{R}^{2d}} |A(u(t+s,x+z+\eta)) - A(\xi)| \cdot \mathbb{1}_{\mathrm{conv}\{u(t+s,x+\eta),u(t+s,x+z+\eta)\}}(\xi)\rho_\epsilon(-s,-\eta)\mu(z)\,ds\,d\eta\,dz$$

with the function $\mathbb{1}$ *everywhere defined in* (18).

Proof. First, let us prove that m_ϵ and n_ϵ and all their derivatives in (t,x) belong to $L^\infty(\mathbb{R}^+ \times \mathbb{R}^{d+1})$. We have for instance

$$\langle \partial_t m_\epsilon, \varphi\rangle_{\mathcal{D}',\mathcal{D}} = \int\limits_0^\infty \int\limits_{\mathbb{R}^{d+1}} m(s,\eta,\xi) \left(\int\limits_0^\infty \int\limits_{\mathbb{R}^d} \varphi(t,x,\xi)\partial_t \rho_\epsilon(t-s,x-\eta)\,dt\,dx \right) ds\,d\eta\,d\xi,$$

for any $\varphi \in \mathcal{D}((0,\infty)\times\mathbb{R}^{d+1})$. By (23) we deduce that

$$|\langle \partial_t m_\epsilon, \varphi\rangle_{\mathcal{D}',\mathcal{D}}| \leq \|\partial_t \rho_\epsilon\|_\infty \|\nu\|_{L^\infty(\mathbb{R})} \|\varphi\|_{L^1(\mathbb{R}^+\times\mathbb{R}^{d+1})},$$

which proves that $\partial_t m_\epsilon \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^{d+1})$. We argue the same way for the other derivatives in (t,x) of m_ϵ and n_ϵ.

Now taking the convolution of the equation satisfied by χ in (26) gives

$$\partial_t \chi_\epsilon + F'(\xi).\nabla_x \chi_\epsilon + A'(\xi) g_x[\chi_\epsilon] = \partial_\xi (m_\epsilon + n_\epsilon) \quad \text{in} \quad \mathcal{D}'((0,\infty)\times\mathbb{R}^{d+1}).$$

We deduce that $\partial_\xi(m_\epsilon + n_\epsilon) \in L^\infty_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^{d+1})$, by what precedes, which proves the first part of (i). The second part is also immediate from the above equation and Lemmas 31 and 21.

Let us now prove (ii). We use again (23) to see that

$$\begin{aligned}
&\int\limits_{\mathbb{R}^{d+1}} (m_\epsilon + n_\epsilon)(t,x,\xi)\varphi(\xi)\,dx\,d\xi \\
&= \int\limits_0^\infty \int\limits_{\mathbb{R}^{2d+1}} (m+n)(s,\eta,\xi) \underbrace{\rho_\epsilon(t-s,x-\eta)}_{=\rho_1((t-s)/\epsilon)\rho_2((x-\eta)/\epsilon)\epsilon^{-d-1}} \varphi(\xi)\,ds\,dx\,d\eta\,d\xi \\
&\leq \frac{\|\rho_1\|_\infty}{\epsilon} \int\limits_{\mathbb{R}} \nu(\xi)\varphi(\xi)\,d\xi,
\end{aligned}$$

for every $t \geq 0$ and nonnegative $\varphi \in \mathcal{D}(\mathbb{R}_\xi)$. Setting $C_\epsilon := \frac{\|\rho_1\|_\infty}{\epsilon}$, we infer

$$\int\limits_{\mathbb{R}^d} (m_\epsilon + n_\epsilon)(t,x,\xi)\,dx \leq C_\epsilon \nu(\xi),$$

for almost every ξ. We can replace the right-hand side by $\nu_\epsilon \in C_0(\mathbb{R}_\xi)$, if choosing such a ν_ϵ satisfying $\nu_\epsilon \geq C_\epsilon \nu$. This is the case if we take for instance

$$\nu_\epsilon(\xi) := \frac{2C_\epsilon}{|\xi|} \int_{|\xi|/2}^{|\xi|} \operatorname*{ess\,sup}_{|\zeta|\geq\tau} \nu(\zeta)\, d\tau$$

(recall that $\nu \in L^\infty_0(\mathbb{R}_\xi)$). The pointwise inequality in (ii) is then easily deduced from Fatou's lemma.

Let us finally prove (iii). Only at this point, we use in passing the regularization in ξ. We consider a kernel $\theta \in \mathcal{D}(\mathbb{R}_\xi)$ that we assume to be nonnegative, even, and such that $\int \theta = 1$. Let us take the approximate unit

$$\theta_\delta(\xi) := \frac{1}{\delta}\theta\left(\frac{\xi}{\delta}\right)$$

and define $(m_\epsilon + n_\epsilon)_\delta := (m_\epsilon + n_\epsilon) *_\xi \theta_\delta$. For each $(t,x,\xi) \in [0,\infty) \times \mathbb{R}^{d+1}$, we have

$$\begin{aligned}(m_\epsilon + n_\epsilon)_\delta(t,x,\xi) &= \int_0^\infty \int_{\mathbb{R}^{d+1}} (m+n)(s,\eta,\zeta)\rho_\epsilon(t-s,x-\eta)\theta_\delta(\xi-\zeta)\, ds\, d\eta\, d\zeta \\ &\geq \int_0^\infty \int_{\mathbb{R}^{d+1}} n(t+s,x+\eta,\xi+\zeta)\rho_\epsilon(-s,-\eta)\theta_\delta(\zeta)\, ds\, d\eta\, d\zeta \\ &= \int_0^\infty \int_{\mathbb{R}^{2d}} I_\delta(t,x,\xi;s,\eta,z)\rho_\epsilon(-s,-\eta)\mu(z)\, ds\, d\eta\, dz,\end{aligned}$$

where I_δ stands for the expression

$$\int_{\mathbb{R}} |A(u(t+s,x+z+\eta)) - A(\xi+\zeta)|\, \mathbb{1}_{\mathrm{conv}\{u(t+s,x+\eta),u(t+s,x+z+\eta)\}}(\xi+\zeta)\theta_\delta(\zeta)\, d\zeta.$$

Let us pass to the limit as $\delta \downarrow 0$ in order to obtain (iii). Note first that the left-hand side always converges towards $(m_\epsilon + n_\epsilon)(t,x,\xi)$ by the item (i) established above. As far as the right-hand side is concerned, we have

$$\lim_{\delta\downarrow 0} I_\delta = |A(u(t+s,x+z+\eta)) - A(\xi)|\, \mathbb{1}_{\mathrm{conv}\{u(t+s,x+\eta),u(t+s,x+z+\eta)\}}(\xi)$$

for every fixed $(s,\eta,z) \in \mathbb{R}^+ \times \mathbb{R}^{2d}$, taking into account the everywhere representation (18). Indeed, this limit exists also for ξ being an extremity of the interval, in this case, the value $\frac{1}{2}$ appears at the limit because the kernel θ is even. Fatou's lemma then completes the proof of (iii). □

We are now ready to prove Lemma 30.

Proof of Lemma 30. Similarly to the formal computations, we will show that the right-hand side of (50) is nonnegative by integrating the equations in χ_ϵ and $\tilde{\chi}_\epsilon$. Recall that

$$\partial_t \chi_\epsilon + F'(\xi).\nabla_x \chi_\epsilon + A'(\xi) g_x[\chi_\epsilon] = \partial_\xi(m_\epsilon + n_\epsilon) \tag{51}$$

(with a similar equation for $\tilde{\chi}_\epsilon$). Since the terms in $F'(\xi)$ and $A'(\xi)$ may not be integrable in ξ, we need to truncate. This amounts to rewrite (50) as

$$\dot{\mathcal{M}}^\epsilon(t) = \lim_{R\to\infty} \Bigg\{ \int_{\mathbb{R}^d} \int_{-R}^{R} (\operatorname{sgn}(\xi)\partial_t \chi_\epsilon - 2\tilde{\chi}_\epsilon \partial_t \chi_\epsilon)\, dx\, d\xi \\ + \int_{\mathbb{R}^d} \int_{-R}^{R} (\operatorname{sgn}(\xi)\partial_t \tilde{\chi}_\epsilon - 2\chi_\epsilon \partial_t \tilde{\chi}_\epsilon)\, dx\, d\xi \Bigg\}, \quad \forall t \geq 0, \tag{52}$$

and estimate the terms in brackets before passing to the limit. Note that the above limit holds by Lemma 31. In the sequel $\epsilon > 0$ is fixed.

For any $R > 0$, each term of (51) belongs to $C([0,\infty); L^1(\mathbb{R}^d \times (-R,R)))$ by Lemmas 31 and 21. In particular, we can integrate (51) in $x \in \mathbb{R}^d$ and $\xi \in (-R,R)$ for any fixed $t \geq 0$. Proceeding so by previously multiplying (51) by $\operatorname{sgn}(\xi)$ gives

$$\int_{\mathbb{R}^d} \int_{-R}^{R} \operatorname{sgn}(\xi)\partial_t \chi_\epsilon\, dx\, d\xi = \int_{\mathbb{R}^d} \int_{-R}^{R} \operatorname{sgn}(\xi)\partial_\xi (m_\epsilon + n_\epsilon)\, dx\, d\xi. \tag{53}$$

Indeed, let us make precise that the contributions to this calculation from the convection and the nonlocal diffusion terms in (51) vanish, due to the integration by parts in x. Its validity is justified, in particular, by Lemma 31 which gives us enough regularity to apply (29): We get, e.g.,

$$\int_{\mathbb{R}^d} \int_{-R}^{R} \operatorname{sgn}(\xi) A'(\xi) g_x[\chi_\epsilon]\, dx\, d\xi = \frac{1}{2} \int_{\mathbb{R}^{2d}} \int_{-R}^{R} \{\operatorname{sgn}(\xi)A'(\xi) - \operatorname{sgn}(\xi)A'(\xi)\} \\ \cdot \{\chi_\epsilon(t,x,\xi) - \chi_\epsilon(t,y,\xi)\}\, \mu(x-y)\, dx\, dy\, d\xi,$$

which indeed equals zero. Similarly, we use Lemma 31 and the Fubini theorem to show that

$$\int_{\mathbb{R}^d} \int_{-R}^{R} \operatorname{sgn}(\xi) F'(\xi) \nabla_x \chi_\epsilon\, dx\, d\xi = 0,$$

see also [44]. Let us now integrate the right-hand side of (53) in ξ first. The regularity in Lemma 33(i) justifies that for any $t \geq 0$ and almost every $x \in \mathbb{R}^d$,

$$\int_{-R}^{R} \operatorname{sgn}(\xi)\partial_\xi (m_\epsilon + n_\epsilon)(t,x,\xi)\, d\xi = -2(m_\epsilon + n_\epsilon)(t,x,0) + \sum_{\pm} (m_\epsilon + n_\epsilon)(t,x,\pm R).$$

Using Lemma 33(ii) to bound the terms in $\pm R$ after the integration in x, we conclude that for any $t \geq 0$ and $R > 0$,

$$\int_{\mathbb{R}^d} \int_{-R}^{R} \operatorname{sgn}(\xi)\partial_t \chi_\epsilon\, dx\, d\xi = -2 \int_{\mathbb{R}^d} (m_\epsilon + n_\epsilon)(t,x,0)\, dx + o_R(1), \tag{54}$$

where $o_R(1) \to 0$ as $R \to \infty$. Note that $o_R(1)$ depends on ϵ but we do not need to care about it since ϵ is fixed up to the end.

The computation in (54) will serve us to bound the limiting right-hand side of (52). Let us leave it aside for a while and do another computation that will be needed. Now let us multiply (51) by $\tilde{\chi}_\epsilon$ and integrate as before. We get

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\int_{-R}^{R} \tilde{\chi}_\epsilon \partial_t \chi_\epsilon \, dx\, d\xi \\
&= \int_{\mathbb{R}^d}\int_{-R}^{R} \tilde{\chi}_\epsilon \partial_\xi (m_\epsilon + n_\epsilon)\, dx\, d\xi - \int_{\mathbb{R}^d}\int_{-R}^{R} A'(\xi)\tilde{\chi}_\epsilon g_x[\chi_\epsilon]\, dx\, d\xi + \mathcal{R}(u,\tilde{u}) \\
&=: I_R(t) - J_R(t) - \mathcal{R}(u,\tilde{u}),
\end{aligned}
\tag{55}
$$

where $\mathcal{R}(u,\tilde{u}) := -\int_{\mathbb{R}^d}\int_{-R}^{R} F'(\xi)\tilde{\chi}_\epsilon \nabla_x \chi_\epsilon \, dx\, d\xi$. To compute the first integral, we write that

$$
I_R(t) = \int_0^\infty \int_{\mathbb{R}^{2d}} \left(\int_{-R}^{R} \tilde{\chi}(t+\tau, x+\theta, \xi)\partial_\xi (m_\epsilon + n_\epsilon)(t,x,\xi)\, d\xi \right) \rho_\epsilon(-\tau,-\theta)\, d\tau\, dx\, d\theta
$$

where we first integrate in ξ. Recalling the definition of

$$
\tilde{\chi}(t+\tau, x+\theta, \xi) = \chi(\xi; \tilde{u}(t+\tau, x+\theta))
$$

given in (21) and having in mind (22), we get that

$$
\begin{aligned}
I_R(t) = &\int_0^\infty \int_{\mathbb{R}^{2d}} (m_\epsilon + n_\epsilon)(t, x, T_R(\tilde{u}(t+\tau, x+\theta)))\rho_\epsilon(-\tau,-\theta)\, d\tau\, dx\, d\theta \\
&- \int_{\mathbb{R}^d} (m_\epsilon + n_\epsilon)(t,x,0)\, dx
\end{aligned}
\tag{56}
$$

with the truncation function $T_R(\cdot)$ defined in (32). In this reasoning, the use of (22) is justified by Lemma 33(i). We can also split the right-hand side of (56) in two integrals since the last integral is finite by (ii) of the same lemma, which implies that the first one is finite as well. Applying now the item (iii), we deduce that

$$
I_R(t) \geq I - \int_{\mathbb{R}^d} (m_\epsilon + n_\epsilon)(t,x,0)\, dx,
\tag{57}
$$

$$
\begin{aligned}
I := &\int_0^\infty \int_0^\infty \int_{\mathbb{R}^{4d}} |A(u(t+s, x+z+\eta)) - A(T_R(\tilde{u}(t+\tau, x+\theta)))| \\
&\cdot \mathbb{1}_{T_R(\tilde{u}(t+\tau,x+\theta)) \in \operatorname{conv}\{u(t+s,x+\eta),\, u(t+s,x+z+\eta)\}} \\
&\cdot \rho_\epsilon(-s,-\eta)\rho_\epsilon(-\tau,-\theta)\mu(z)\, ds\, d\tau\, dx\, d\eta\, d\theta\, dz.
\end{aligned}
$$

Let us rewrite I with the pushforward measure in (28), which amounts to change the variables by $(x, x+z) \mapsto (x, y)$. We get

$$
\begin{aligned}
I = & \int_0^\infty \int_0^\infty \int_{\mathbb{R}^{4d}} |A(u(t+s, y+\eta)) - A(T_R(\tilde{u}(t+\tau, x+\theta)))| \\
& \cdot \mathbb{1}_{T_R(\tilde{u}(t+\tau, x+\theta)) \in \operatorname{conv}\{u(t+s,x+\eta), u(t+s,y+\eta)\}} \\
& \cdot \rho_\epsilon(-s,-\eta)\rho_\epsilon(-\tau,-\theta)\mu(x-y)\, ds\, d\tau\, dx\, dy\, d\eta\, d\theta \\
= & \int_0^\infty \int_0^\infty \int_{\mathbb{R}^{4d}} |A(b) - A(T_R(c))|\, \mathbb{1}_{T_R(c) \in \operatorname{conv}\{a,b\}} \\
& \cdot \rho_\epsilon(-s,-\eta)\rho_\epsilon(-\tau,-\theta)\mu(x-y)\, ds\, d\tau\, dx\, dy\, d\eta\, d\theta,
\end{aligned}
$$

with the convenient notation

$$
\begin{cases}
a := u(t+s, x+\eta), \\
b := u(t+s, y+\eta), \\
c := \tilde{u}(t+\tau, x+\theta), \\
d := \tilde{u}(t+\tau, y+\theta);
\end{cases}
\tag{58}
$$

note that d will appear later when doing the same computations for $\tilde{u}$. Using in addition (34), we deduce from (57) that

$$
\begin{aligned}
I_R(t) \geq & \int_0^\infty \int_0^\infty \int_{\mathbb{R}^{4d}} |A(T_R(b)) - A(T_R(c))|\, \mathbb{1}_{T_R(c) \in \operatorname{conv}\{T_R(a), T_R(b)\}} \\
& \cdot \rho_\epsilon(-s,-\eta)\rho_\epsilon(-\tau,-\theta)\mu(x-y)\, ds\, d\tau\, dx\, dy\, d\eta\, d\theta \\
& - \int_{\mathbb{R}^d} (m_\epsilon + n_\epsilon)(t, x, 0)\, dx.
\end{aligned}
\tag{59}
$$

Let us now focus on the second integral in (55). Let us integrate it by parts as in (29) which is again justified by Lemma 31. We get

$$
\begin{aligned}
J_R(t) = \frac{1}{2} & \int_{\mathbb{R}^{2d}} \int_{-R}^{R} A'(\xi) \{\chi_\epsilon(t,x,\xi) - \chi_\epsilon(t,y,\xi)\} \{\tilde{\chi}_\epsilon(t,x,\xi) - \tilde{\chi}_\epsilon(t,y,\xi)\} \\
& \cdot \mu(x-y)\, dx\, dy\, d\xi.
\end{aligned}
$$

After writing the formula of the convolution products, $\chi_\epsilon = \chi * \rho_\epsilon$ and $\tilde{\chi}_\epsilon = \tilde{\chi} * \rho_\epsilon$, and using the convenient notation (58), we obtain that

$$
\begin{aligned}
J_R(t) = \frac{1}{2} & \int_0^\infty \int_0^\infty \int_{\mathbb{R}^{4d}} \int_{-R}^{R} A'(\xi) \{\chi(\xi; a) - \chi(\xi; b)\} \{\chi(\xi; c) - \chi(\xi; d)\} \\
& \cdot \rho_\epsilon(-s,-\eta)\rho_\epsilon(-\tau,-\theta)\mu(x-y)\, ds\, d\tau\, dx\, dy\, d\eta\, d\theta\, d\xi.
\end{aligned}
$$

Applying (33), we infer that

$$J_R(t)$$
$$= \frac{1}{2}\int_0^\infty\int_0^\infty\int_{\mathbb{R}^{4d}}\int_{\mathbb{R}} A'(\xi)\left\{\chi(\xi;T_R(a)) - \chi(\xi;T_R(b))\right\}\left\{\chi(\xi;T_R(c)) - \chi(\xi;T_R(d))\right\}$$
$$\cdot\, \rho_\epsilon(-s,-\eta)\rho_\epsilon(-\tau,-\theta)\mu(x-y)\,ds\,d\tau\,dx\,dy\,d\eta\,d\theta\,d\xi.$$

Now substracting twice (55) to (54), while taking into account the preceding lower bound (59) of $I_R(t)$, we finally deduce that

$$\int_{\mathbb{R}^d}\int_{-R}^{R}(\mathrm{sgn}(\xi)\partial_t\chi_\epsilon - 2\tilde{\chi}_\epsilon\partial_t\chi_\epsilon)\,dx\,d\xi \le 2\mathcal{R}(u,\tilde{u}) + o_R(1) \tag{60}$$
$$+\int_0^\infty\int_0^\infty\int_{\mathbb{R}^{4d}} P_k\,\rho_\epsilon(-s,-\eta)\rho_\epsilon(-\tau,-\theta)\mu(x-y)\,ds\,d\tau\,dx\,dy\,d\eta\,d\theta,$$

$$P_k := \int_{\mathbb{R}} A'(\xi)\left\{\chi(\xi;T_R(a)) - \chi(\xi;T_R(b))\right\}\left\{\chi(\xi;T_R(c)) - \chi(\xi;T_R(d))\right\}d\xi$$
$$-\,2\left|A(T_R(b)) - A(T_R(c))\right|\mathbb{1}_{T_R(c)\in\mathrm{conv}\{T_R(a),T_R(b)\}}.$$

Inverting the roles of u and $\tilde{u}$, we get a similar estimate of the form

$$\int_{\mathbb{R}^d}\int_{-R}^{R}(\mathrm{sgn}(\xi)\partial_t\tilde{\chi}_\epsilon - 2\chi_\epsilon\partial_t\tilde{\chi}_\epsilon)\,dx\,d\xi \le 2\mathcal{R}(\tilde{u},u) + o_R(1) \tag{61}$$
$$+\int_0^\infty\int_0^\infty\int_{\mathbb{R}^{4d}} \tilde{P}_k\,\rho_\epsilon(-s,-\eta)\rho_\epsilon(-\tau,-\theta)\mu(x-y)\,ds\,d\tau\,dx\,dy\,d\eta\,d\theta,$$

$$\tilde{P}_k := \int_{\mathbb{R}} A'(\xi)\left\{\chi(\xi;T_R(a)) - \chi(\xi;T_R(b))\right\}\left\{\chi(\xi;T_R(c)) - \chi(\xi;T_R(d))\right\}d\xi$$
$$-\,2\left|A(T_R(d)) - A(T_R(a))\right|\mathbb{1}_{T_R(a)\in\mathrm{conv}\{T_R(c),T_R(d)\}}.$$

We again recognize the F-term and G-term of Lemma 23 if adding (61) to (60), more precisely

$$P_k + \tilde{P}_k = 2F(T_R(a),T_R(b),T_R(c),T_R(d)) - 2G(T_R(a),T_R(b),T_R(c),T_R(d))$$

which is nonnegative. Injecting the sum of (60) and (61) into (52) then implies that for any $t \ge 0$,

$$\dot{\mathcal{M}}^\epsilon(t) \le \lim_{R\to\infty}\left\{2\mathcal{R}(u,\tilde{u}) + 2\mathcal{R}(\tilde{u},u) + o_R(1)\right\}$$

where $o_R(1) \to 0$ as $R \to \infty$ and

$$\mathcal{R}(u,\tilde{u}) + \mathcal{R}(\tilde{u},u) = \int_{\mathbb{R}^d}\int_{-R}^{R} F'(\xi)\nabla_x\left[\tilde{\chi}_\epsilon\chi_\epsilon\right]dx\,d\xi = 0,$$

thanks to an integration by parts in x justified by Lemma 31; see also [44]. We get that $\dot{\mathcal{M}}^\epsilon(t) \le 0$ and complete the proof. $\square$

6. Existence of kinetic solutions

Let us now prove the existence part in Theorem 13 that is to say the result below. The complete proof of Theorem 13 is given just after.

Theorem 34. *Let $u_0 \in L^1(\mathbb{R}^d)$ and assume* (3)–(6)*. Then there exists at least a kinetic solution u of* (1) *which belongs to $C([0,\infty);L^1(\mathbb{R}^d))$.*

Theorem 34 can be proven from the kinetic approach without relying on entropy solutions, in the spirit of [44]. However, in order to shorten the paper we will use the known existence result for entropy solutions [20].

Proof of Theorem 34. Let us define $u_0^k := T_k(u_0) \in L^1 \cap L^\infty(\mathbb{R}^d)$, with the truncation function of (32). We have $u_0^k \to u_0$ in $L^1(\mathbb{R}^d)$ as $k \to \infty$. Let u_k be the associated entropy solutions given by Theorem 3. By the L^1 contraction principle,

$$\|u_k - u_p\|_{C([0,T];L^1(\mathbb{R}^d))} \leq \|u_0^k - u_0^p\|_{L^1(\mathbb{R}^d)},$$

for any $T \geq 0$ and integers k,p. This Cauchy sequence thus converges towards some function $u \in C([0,\infty);L^1(\mathbb{R}^d))$ in $C([0,T];L^1(\mathbb{R}^d))$, for any $T > 0$, and almost everywhere in $\mathbb{R}^+ \times \mathbb{R}^d$ (up to some subsequence). We will show that this u is the desired kinetic solution.

By Theorem 12, each u_k is a kinetic solution with some measure m_k satisfying the estimate of Lemma 28. Since

$$(T_k(u_0(x)) - \xi)^{\pm}\mathbf{1}_{\pm\xi>0} \leq (u_0(x) - \xi)^{\pm}\mathbf{1}_{\pm\xi>0},$$

we deduce that for all integer k and almost any $\xi \in \mathbb{R}$,

$$\int_0^\infty \int_{\mathbb{R}^d} (m_k + n_k)(t,x,\xi)\,dt\,dx \leq \nu(\xi), \tag{62}$$

with the same fixed $\nu(\xi) = \|(u_0-\xi)^+\mathbf{1}_{\xi>0}\|_{L^1(\mathbb{R}^d)} + \|(u_0-\xi)^-\mathbf{1}_{\xi<0}\|_{L^1(\mathbb{R}^d)}$. By the weak compactness of measures, there is some $q \in M^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^{d+1})$ such that

$$\int_0^\infty \int_{\mathbb{R}^{d+1}} (m_k + n_k)(t,x,\xi)\varphi(t,x,\xi)\,dt\,dx\,d\xi \to \int_0^\infty \int_{\mathbb{R}^{d+1}} q(t,x,\xi)\varphi(t,x,\xi)\,dt\,dx\,d\xi,$$

for any $\varphi \in C_c(\mathbb{R}^{d+2})$ (and up to another subsequence if necessary). This is sufficient to pass to the limit in (24) since $\chi(\xi;u_k) \to \chi(\xi;u)$ in $C([0,T];L^1(\mathbb{R}^{d+1}))$ for any $T > 0$. But, we get the measure q instead of $m+n$ at the right-hand side. Nevertheless, we can rewrite q as $m+n$, for some nonnegative measure m, if we can prove that $q \geq n$. Let us do so. For any $0 \leq \varphi \in C_c(\mathbb{R}^{d+2})$,

$$\int_0^\infty \int_{\mathbb{R}^{d+1}} (m_k+n_k)\varphi\,dt\,dx\,d\xi \geq \int_0^\infty \int_{\mathbb{R}^{d+1}} n_k\varphi\,dt\,dx\,d\xi = \int_{\mathbb{R}^d} I_k(z)\mu(z)\,dz, \tag{63}$$

$$I_k(z) := \int_0^\infty \int_{\mathbb{R}^{d+1}} \underbrace{|A(u_k(t,x+z)) - A(\xi)|\mathbb{1}_{\mathrm{conv}\{u_k(t,x),u_k(t,x+z)\}}(\xi)}_{=:Q_k(t,x,z,\xi)}\,\varphi(t,x,\xi)\,dt\,dx\,d\xi.$$

Recall that $u_k(t,x) \to u(t,x)$ almost everywhere, let us say for $(t,x) \notin N$ with $N \subset \mathbb{R}^+ \times \mathbb{R}^d$ negligible. For any $z \in \mathbb{R}^d$, we thus have $u_k(t,x+z) \to u(t,x+z)$ for any (t,x) not in the negligible $N-(0,z)$. Hence

$$\lim_{k\to\infty} Q_k(t,x,z,\xi) = \underbrace{|A(u(t,x+z)) - A(\xi)| \mathbb{1}_{\mathrm{conv}\{u(t,x),u(t,x+z)\}}(\xi)}_{=:Q(t,x,z,\xi)},$$

for any $(t,x,\xi) \in \mathbb{R}^+ \times \mathbb{R}^{d+1}$ such that $(t,x) \notin N \cup (N-(0,z))$ and $\xi \neq u(t,x)$. We recognize the complementary of the graph of u; the latter has zero Lebesgue measure in $\mathbb{R}^+ \times \mathbb{R}^{d+1}$. Fatou's lemma then implies that

$$\liminf_{k\to\infty} I_k(z) \geq \int_0^\infty \int_{\mathbb{R}^{d+1}} Q(t,x,z,\xi)\varphi(t,x,\xi)\,dt\,dx\,d\xi \quad \forall z \in \mathbb{R}^d.$$

Applying again Fatou's lemma to the right-hand side of (63), we obtain $q \geq n$. We thus have reached all the conditions required in Definition 8 excepted (23), but the latter is immediate from (62). □

Proof of Theorem 13. It only remains to check the uniqueness of m. Note that the equation in (26) determines m for $t > 0$, since u is unique. But, the last condition of (26) implies moreover that $m(\{t=0\}) = 0$. This completes the proof. □

Appendix A. Proofs of Lemmas 17, 19 and 21

Proof of Lemma 17. The boundedness of g follows from the formula

$$g[f](x) = \int_{|z|>r} (f(x+z)-f(x))\mu(z)\,dz + \int_{|z|\leq r} \int_0^1 (1-\tau)\nabla^2 f(x+\tau z)z^2\mu(z)\,dz\,d\tau. \tag{64}$$

The integration by parts formula (27) follows, as $r \downarrow 0$, from the cutting

$$\begin{aligned}\int_{|z|>r} (f(x+z)-f(x))\varphi(x)\mu(z)\,dx\,dz = &\int_{|z|>r} f(x+z)\varphi(x)\,dx\,dz \\ &- \int_{|z|>r} f(x)\varphi(x)\mu(z)\,dx\,dz\end{aligned}$$

and the changes of variables $x+z \mapsto x$ and $-z \mapsto z$ in the first integral. □

Proof of Lemma 19. We have

$$\begin{aligned} I_r &:= -\int_{|z|>r} (f(x+z)-f(x))\tilde{f}(x)\mu(z)\,dx\,dz \\ &= \int_{|z|>r} f(x)\tilde{f}(x)\mu(z)\,dx\,dz - \int_{|z|>r} f(x+z)\tilde{f}(x)\mu(z)\,dx\,dz \\ &= \int_{|x-y|>r} (f(x)-f(y))\tilde{f}(x)\mu(x-y)\,dx\,dy,\end{aligned}$$

after having rewritten both the preceding integrals with the help of (28). We can exchange the roles of x and y thus getting also

$$I_r = \int\limits_{|x-y|>r} \underbrace{(f(y)-f(x))\mu(y-x)}_{=-(f(x)-f(y))\mu(x-y)} \tilde f(y)\,dx\,dy.$$

Applying each of these formulas to half of I_r, we get

$$I_r = \frac{1}{2} \int\limits_{|x-y|>r} (f(x)-f(y))(\tilde f(x)-\tilde f(y))\mu(x-y)\,dx\,dy.$$

Now we conclude by passing to the limit as $r \downarrow 0$. This is justified since

$$\begin{aligned}
&\int\limits_{\mathbb{R}^{2d}} |f(x)-f(y)||\tilde f(x)-\tilde f(y)|\mu(x-y)\,dx\,dy\\
&= \int\limits_{\mathbb{R}^{2d}} |f(x)-f(x+z)||\tilde f(x)-\tilde f(x+z)|\mu(z)\,dx\,dz \quad \text{(by (28))}\\
&\leq \|f\|_{W^{1,1}(\mathbb{R}^d)}\|\tilde f\|_{W^{1,\infty}(\mathbb{R}^d)} \int (|z|\wedge 2)(|z|\wedge 2)\mu(z)\,dz,
\end{aligned}$$

which is finite by (5). □

Proof of Lemma 21. Use Lemma 17 to show that $g_x[f] \in C([0,\infty); L^1(\mathbb{R}^{d+1}))$ and Lemma 19 for the formula (29). □

Appendix B. Proof of Proposition 11

We follow the guidelines of [19, 44]. Let us first give technical results.

Lemma 35. *Let $\phi, \varphi \in C_b^\infty(\mathbb{R}^d)$ and $\phi_M(x) := \phi(x/M)$. Then $g[\phi_M\varphi] \to \phi(0)g[\varphi]$ pointwise as $M \to \infty$ while being bounded uniformly in large M. In particular $g[\phi_M] \to 0$ pointwise.*

Proof. The uniform bound in large M holds since $g : C_b^2(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$ is bounded. For the convergence, use (64) to write

$$\begin{aligned}
&|g[\phi_M\varphi - \phi(0)\varphi](x)|\\
&\leq \int\limits_{|z|>r} |(\phi_M\varphi - \phi(0)\varphi)(x+z) - (\phi_M\varphi - \phi(0)\varphi)(x)|\,\mu(z)\,dz + C\int\limits_{|z|\leq r} z^2\mu(z)\,dz
\end{aligned}$$

with C independent of $r > 0$ and large M, and let then $M \to \infty$ and $r \downarrow 0$ successively. □

We can then use test functions as below.

Lemma 36. *Assume (3)–(6) and $u \in L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^d))$ is a kinetic solution of (1) with initial data $u_0 \in L^1(\mathbb{R}^d)$. Then, for any Lebesgue point T of the locally integrable function $t \in \mathbb{R}^+ \mapsto u(t) \in L^1$ and any $\varphi \in C_b^\infty(\mathbb{R}^d \times \mathbb{R}_\xi)$ compactly* *supported in* ξ*,*

(65)
$$\int\limits_{\mathbb{R}^{d+1}} \chi(\xi;u_0(x))\varphi(x,\xi)\,dx\,d\xi + \int\limits_0^T \int\limits_{\mathbb{R}^{d+1}} \chi(\xi;u)\left(F'(\xi)\cdot\nabla_x\varphi - A'(\xi)g_x[\varphi]\right)dt\,dx\,d\xi$$
$$= \int\limits_{\mathbb{R}^{d+1}} \chi(\xi;u(T,x))\varphi(x,\xi)\,dx\,d\xi + \int\limits_0^T \int\limits_{\mathbb{R}^{d+1}} (m+n)(t,x,\xi)\partial_\xi\varphi(x,\xi)\,dt\,dx\,d\xi.$$

Proof. Take the test $\varphi_k(t)\phi(x/M)\varphi(x,\xi)$ in (24) where $\varphi_k(t) := 1 - \int_0^t \rho_k$ for some standard mollifier $\rho_k = \rho_k(t)$ approximating $\delta(t=T)$ as $k\to\infty$, and $\phi \in \mathcal{D}(\mathbb{R}^d)$ satisfies $\phi(0) = 1$. Let then $k, M \to \infty$ successively by using Lemma 35 and the dominated convergence theorem. □

Let us continue with standard results. The first one is the de La Vallée Poussin criterion for weak compactness in L^1 (or equivalently weakly sequentially compactness by Eberlian Šmulian theorem), cf. e.g. [43, p. 19].

Lemma 37. *For any ball* $B \subset \mathbb{R}^d$*,* $\{u_k\}_k \subset L^1(B)$ *is relatively weakly compact if and only if there exists a nonnegative and convex function* Φ *such that* $\lim_{|\xi|\to\infty} \frac{\Phi(\xi)}{|\xi|} = \infty$ *and* $\{\Phi(u_k)\}_k$ *is bounded in* $L^1(B)$*.*

To establish the strong convergence, we will argue on the weak-$\star$ limit of the kinetic functions and the properties below will be needed.

Lemma 38. *Let us assume that* $u_k \rightharpoonup \overline{u}$ *in* $L^1_{\mathrm{loc}}(\mathbb{R}^d)$*–w and* $\chi(\xi;u_k(x)) \rightharpoonup \overline{\chi}(x,\xi)$ *in* $L^\infty(\mathbb{R}^{d+1})$*–w*$\star$*. Then:*

(i) $\overline{\chi} \in L^1(B\times\mathbb{R}_\xi)$ *for any ball* $B\subset\mathbb{R}^d$*,*
(ii) $\overline{u}(x) = \int \overline{\chi}(x,\zeta)\,d\zeta$ *and* $\int_{-\infty}^\xi (\overline{\chi}(x,\zeta) - \chi(\zeta;\overline{u}(x)))\,d\zeta \le 0$ *for a.e.* (x,ξ)*,*
(iii) $u_k \to \overline{u}$ *strongly in* $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ *when* $\overline{\chi} = \chi(\xi;\overline{u})$*.*

See [44, Lem 2.3.1] for the proof of (i), [44, Lem 2.3.3] for the first part of (ii) and (iii), and [44, Thm 2.2.1] for the second part of (ii). We are now in position to prove the time continuity of kinetic solutions at $t=0$ by reproducing the arguments of [19, 44].

Proof of Proposition 11. Let $u_0 \in L^1(\mathbb{R}^d)$, $u \in L^\infty(\mathbb{R}^+;L^1(\mathbb{R}^d))$ and $0 \le m \in M^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R}^{d+1})$ be such that (23) holds. We have to prove that (24) $\Longleftrightarrow$ (26), the difference being in the sense of the initial datum.

Claim (26) $\Rightarrow$ (24)**.** This follows from a standard approximation procedure of test functions in $\mathcal{D}([0,\infty)\times\mathbb{R}^{d+1})$ by test functions in $\mathcal{D}((0,\infty)\times\mathbb{R}^{d+1})$. The details are left to the reader.

Claim (24) $\Rightarrow$ (26)**:** *Strong continuity in* $L^1_{\mathrm{loc}}(\mathbb{R}^d)$. The main property to establish is the second line of (26), on which we focus now. Consider an even, nonnegative and strictly convex C^∞ function Φ such that $\lim_{|\xi|\to\infty}\frac{\Phi(\xi)}{|\xi|} = \infty$ and

$\int \Phi(u_0) < \infty$.[1]. We claim that for any Lebesgue point T of $t \mapsto u(t)$,

$$\int_{\mathbb{R}^d} \Phi(u(T,x))\,dx + \int_0^T \int_{\mathbb{R}^{d+1}} \Phi''(\xi)(m+n)(t,x,\xi)\,dt\,dx\,d\xi = \int_{\mathbb{R}^d} \Phi(u_0)\,dx. \tag{66}$$

To prove this, take $\varphi(x,\xi) := \phi(x/M)\psi(\xi/R)\Phi'(\xi)$ in (65) with $\phi \in \mathcal{D}(\mathbb{R}^d)$ and $0 \le \psi \in \mathcal{D}(\mathbb{R})$ such that ψ is even, nonincreasing on $\mathbb{R}^+$, and $\phi(0) = \psi(0) = 1$. Letting $M \to \infty$,

$$\begin{aligned}\int_{\mathbb{R}^{d+1}} \chi(\xi;u_0(x))\psi(\xi/R)\Phi'(\xi)\,dx\,d\xi &= \int_{\mathbb{R}^{d+1}} \chi(\xi;u(T,x))\psi(\xi/R)\Phi'(\xi)\,dx\,d\xi \\ &\quad + \int_0^T \int_{\mathbb{R}^{d+1}} (m+n)(t,x,\xi)\psi(\xi/R)\Phi''(\xi)\,dt\,dx\,d\xi \\ &\quad + \frac{1}{R}\int_0^T \int_{\mathbb{R}^{d+1}} (m+n)(t,x,\xi)\psi'(\xi/R)\Phi'(\xi)\,dt\,dx\,d\xi,\end{aligned}$$

thanks to Lemma 35 as well as (23) and the dominated convergence theorem. To continue, we need that to cancel the last integral as $R \to \infty$. This will be the case if $\nu\Phi' \in L^\infty_0(\mathbb{R})$ with ν from (23). For this sake, it suffices to fix a smaller Φ from the begining, if necessary, in order to have also this property. Letting $R \to \infty$ then implies (66) thanks to the monotone convergence theorem to handle all the other terms.

Consider now Lebesgue points t_k of $t \mapsto u(t)$ such that $t_k \downarrow 0$ as $k \to \infty$. By (66) and Lemma 37, $\{u_k := u(t_k,\cdot)\}_k$ is relatively weakly compact in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$. Hence

$$u_k \rightharpoonup \overline{u} \quad \text{in} \quad L^1_{\mathrm{loc}}(\mathbb{R}^d)\text{–}w, \quad \chi(\xi;u_k(x)) \rightharpoonup \overline{\chi}(x,\xi) \quad \text{in} \quad L^\infty(\mathbb{R}^{d+1})\text{–}w\star,$$

up to taking a subsequence if necessary. Taking eventually another subsequence,

$$\int_0^{t_k} m(t,x,\xi)\,dt \rightharpoonup \overline{m}(x,\xi) \quad \text{in} \quad M^1_{\mathrm{loc}}(\mathbb{R}^{d+1})\text{–}w$$

thanks to (23). Letting $T = t_k \downarrow 0$ in (65) implies that

$$\begin{aligned}\int_{\mathbb{R}^{d+1}} \chi(\xi;u_0(x))\varphi(x,\xi)\,dx\,d\xi &= \int_{\mathbb{R}^{d+1}} \overline{\chi}(x,\xi)\varphi(x,\xi)\,dx\,d\xi \\ &\quad + \int_{\mathbb{R}^{d+1}} \overline{m}(x,\xi)\partial_\xi\varphi(x,\xi)\,dx\,d\xi,\end{aligned}$$

that is $\overline{\chi}(x,\xi) - \chi(\xi;u_0(x)) = \partial_\xi \overline{m}(x,\xi)$. Since $\int_{\mathbb{R}^d} \overline{m}(x,\xi)\,dx \le \nu(\xi) \in L^\infty_0(\mathbb{R})$ by stability of (23) at the weak limit, $\overline{m}(x,\xi) = \int_{-\infty}^{\xi} (\overline{\chi}(x,\zeta) - \chi(\zeta;u_0(x)))\,d\zeta$ thanks to the item (i) of Lemma 38. The limit as $\xi \to +\infty$ then implies that $\int \overline{\chi}(x,\zeta)\,d\zeta = u_0(x)$ and it follows that $\overline{u} = u_0$ and $\overline{m} \le 0$ by (ii). This nonnegative

[1] Take e.g. a regular version of $\xi \mapsto \sum_{k\ge1}(|\xi| - r_k)^+$ where $\sum_{k\ge1}\int_{|u_0|\ge r_k} |u_0| < \infty$ for some fixed $0 = r_1 < r_2 < \ldots$

measure is thus zero which implies that $\overline{\chi}(x,\xi) = \chi(\xi; u_0(x))$ and $u_k \to u_0$ strongly in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ by (iii).

Claim (24) $\Rightarrow$ (26)**:** *Strong continuity in* $L^1(\mathbb{R}^d)$. Let $R > 0$ be arbitrarily fixed and let us prove that

$$\sup_{k\in\mathbb{N}} \int\limits_{|x|\geq M} |T_R(u_k)|\, dx \to 0 \quad \text{as} \quad M \to \infty, \tag{67}$$

with T_R from (32). Consider a regularization of $\frac{d}{d\xi}|T_R(\xi)|$ given by $S_R * \theta_\delta$ where $S_R(0) = 0$,

$$S_R'(\xi) := \begin{cases} \mathrm{sgn}(\xi) & \text{for } |\xi| \leq R, \\ \mathrm{sgn}(\xi)(R+1-|\xi|) & \text{for } R < |\xi| < R+1, \\ 0 & \text{otherwise,} \end{cases}$$

and $0 \leq \theta_\delta \in \mathcal{D}(\mathbb{R})$ is an approximate unit as $\delta \downarrow 0$. Note that

$$|\cdot| \geq S_R(\cdot) \geq |T_R(\cdot)| \quad \text{and} \quad (S_R * \theta_\delta)'' \geq -\mathbf{1}_{(-R-1,-R)\cup(-R,R+1)} * \theta_\delta. \tag{68}$$

Now choose $\varphi(x,\xi) := \phi_M(x)(S_R * \theta_\delta)'(\xi)$ in (65) where $\phi_M(x) = \phi(x/M)$ with

$$\phi(x) := \begin{cases} 0 & \text{for } |x| \leq 1/2, \\ 1 & \text{for } |x| \geq 1, \end{cases} \quad \text{and} \quad 0 \leq \phi \leq 1 \quad \text{elsewhere.}$$

Doing this with $T = t_k$, we infer that

$$\begin{aligned} &\int\limits_{|x|>\frac{M}{2}} S_R * \theta_\delta(u_0(x))\, dx + C_R \int\limits_0^{t_k} \int\limits_{\mathbb{R}^{d+1}} |\chi(\xi;u)|\,(|\nabla\phi_M| + |g[\phi_M]|)\, dt\, dx\, d\xi \\ &\geq \int\limits_{|x|>M} S_R * \theta_\delta(u(t_k,x))\, dx + \int\limits_0^{t_k} \int\limits_{\mathbb{R}^{d+1}} (m+n)(t,x,\xi)\phi_M(x)(S_R * \theta_\delta)''(\xi)\, dt\, dx\, d\xi, \end{aligned}$$

for some Lipschitz constant C_R of F and A on the support of $(S_R * \theta_\delta)'$, thus independent of small δ. Letting $\delta \to 0$ while using (68) then implies that

$$\begin{aligned} \int\limits_{|x|>M} |T_R(u_k)|\, dx \leq &\int\limits_{|x|>\frac{M}{2}} |u_0|\, dx \\ &+ C_R \int\limits_0^{t_k} \int\limits_{\mathbb{R}^{d+1}} |\chi(\xi;u)|\,(|\nabla\phi_M| + |g[\phi_M]|)\, dt\, dx\, d\xi \\ &+ \int\limits_0^{t_k} \int\limits_{|x|>\frac{M}{2}} \int\limits_{R\leq|\xi|\leq R+1} (m+n)\, dt\, dx\, d\xi. \end{aligned}$$

The claim (67) is now obtained by using Lemma 35 to cancel the penultimate integral as $M \to \infty$, as well as (23) and the dominated convergence theorem for the last integral.

Conclusion. By (67) and the previous L^1_{loc} convergence, $T_R(u_k) \to T_R(u_0)$ in $L^1(\mathbb{R}^d)$ for any $R > 0$ up to a subsequence. Hence $u_k \to u_0$ in $L^1(\mathbb{R}^d)$ since

$\lim_{R\to\infty}\sup_k \int_{|u_k|>R} |u_k| = 0$ by (66). This completes the proof of the middle line of (26). Since the first line is immediate and the last one is a consequence of (66), the whole proof is complete. □

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