



HAL
open science

AVERAGE FIELD APPROXIMATION FOR ALMOST BOSONIC ANYONS IN A MAGNETIC FIELD

Théotime Girardot

► **To cite this version:**

Théotime Girardot. AVERAGE FIELD APPROXIMATION FOR ALMOST BOSONIC ANYONS IN A MAGNETIC FIELD. 2019. ⟨hal-02320093v2⟩

HAL Id: hal-02320093

<https://hal.science/hal-02320093v2>

Preprint submitted on 11 Nov 2019 (v2), last revised 25 May 2020 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

AVERAGE FIELD APPROXIMATION FOR ALMOST BOSONIC ANYONS IN A MAGNETIC FIELD

THÉOTIME GIRARDOT

ABSTRACT. We study the ground state of a large number N of 2D anyons in an external magnetic field. We consider a scaling limit where the statistics parameter α is proportional to N^{-1} when $N \rightarrow \infty$ which allows the statistics to be seen as a “perturbation from the bosonic end”. Our model is that of bosons in a magnetic field and interacting through long-range magnetic potential generated by magnetic charges carried by each particle, smeared over discs of radius R . Our method allows to take $R \rightarrow 0$ not too fast at the same time as $N \rightarrow \infty$: $R \sim N^{-1/4+\epsilon}$. We use the information theoretic version of the de Finetti theorem of Brandão and Harrow to justify the so-called “average field approximation”: the particles behave like independent, identically distributed bosons interacting via a self-consistent magnetic field.

CONTENTS

1. Introduction	2
1.1. The model	2
1.2. Extended anyons	3
1.3. Average field approximation	4
1.4. Main results	5
2. The extended anyon Hamiltonian	6
2.1. Operator bounds for the interaction terms	7
2.2. A priori bound for the ground state	14
3. Mean-field limit	16
3.1. Energy cut-off	16
3.2. Localisation in Fock space	16
3.3. Quantum de Finetti theorem	17
3.4. Truncated Hamiltonian	18
3.5. Application of the de Finetti theorem	18
3.6. Energy bound	20
3.7. Convergence of states	22
Appendix A. Properties of the average-field functional	24
References	28

Date: october, 2019.

1. INTRODUCTION

This paper is a continuation of the quantum de Finetti-based approach to the mean-field limit of anyonic ground states of [17] with the motivation to extend its results in two different ways. The first extension concerns the physical situation. Indeed we take into account an external magnetic field \mathbf{B}_e absent in [17]. The second concerns the improvement of the range of validity of the average field approximation thanks to an information-theoretic variant of the quantum de Finetti theorem due to Brandão and Harrow [4, 15], and revisited in [24].

In lower dimensions there are possibilities for quantum statistics other than the bosonic one and the fermionic one, so called intermediate or fractional statistics. Such particles, termed anyons (as in *anything* in between bosons and fermions), could arise as effective quasiparticles in many-body quantum systems confined to lower dimensions. For an example of these intermediate ex-change quantum statistic properties see [19, 20] or [18]. These quasiparticles have been conjectured [1] to be relevant for the fractional quantum Hall effect (see [9, 12, 21] for review) and that is why we study the behavior of their ground state under a magnetic field.

1.1. The model. To describe anyons, the “wave function” has to (formally) behave as

$$\tilde{\Psi}(x_1, \dots, x_j, \dots, x_k, \dots, x_N) = e^{i\alpha\pi} \tilde{\Psi}(x_1, \dots, x_k, \dots, x_j, \dots, x_N)$$

and we can obtain this by writing

$$\tilde{\Psi}(x_1, \dots, x_N) = \prod_{j < k} e^{i\alpha\phi_{jk}} \Psi(x_1, \dots, x_N) \quad \text{where } \phi_{jk} = \arg \frac{x_j - x_k}{|x_j - x_k|}$$

with Ψ a bosonic wave function, symmetric under particle exchange. Note that cases $\alpha = 1$ and $\alpha = 0$ are respectively the fermionic and the bosonic one. Let us now consider an usual Hamiltonian with the magnetic field $\mathbf{B}_e = \text{curl } \mathbf{A}_e$:

$$\mathcal{H}_N = \sum_{j=1}^N (-i\nabla_j + \mathbf{A}_e(x_j))^2 + V(x_j)$$

One can show that acting with it on $\tilde{\Psi}$ is equivalent to acting on Ψ with the Hamiltonian:

$$H_N = \sum_{j=1}^N ((p_j + \mathbf{A}_e(x_j) + \alpha\mathbf{A}(x_j))^2 + V(x_j)) \quad (1.1)$$

where, denoting $(x, y)^\perp = (-y, x)$,

$$p_j = -i\nabla_j \quad \text{and} \quad \mathbf{A}(x_j) = \sum_{j \neq k} \frac{(x_j - x_k)^\perp}{|x_j - x_k|^2}. \quad (1.2)$$

The operator $\mathbf{A}(x_j)$ is the statistical gauge vector potential felt by the particle j due to the influence of all the other particles. The statistics parameter is denoted by α , corresponding to a statistical phase $e^{i\alpha\pi}$ under a continuous simple interchange of two particles. In this so-called “magnetic gauge picture”, 2D anyons are thus described as bosons, each of them carrying an Aharonov-Bohm magnetic flux of strength α .

We henceforth work with the Hamiltonian H_N acting on symmetric wave functions

$$\Psi \in \mathfrak{H} = L^2_{\text{sym}}(\mathbb{R}^{2N}).$$

In this work we will assume that $\alpha \rightarrow 0$ when $N \rightarrow \infty$, a limit that we call ‘‘almost bosonic’’. More precisely we define a fixed constant β such that

$$\alpha = \frac{\beta}{N-1} \xrightarrow{N \rightarrow \infty} 0. \quad (1.3)$$

In this way, the anyon statistics has a leading order effect manifest through *the average field approximation*.

1.2. Extended anyons. The Hamiltonian (1.1) is too singular to be considered as acting on a pure tensor product $u^{\otimes N} \in \bigotimes_{\text{sym}}^N \mathfrak{H}$, and, consequently, to consider a mean field limit. In order to circumvent this problem we will introduce a length R over which the equivalent magnetic charge is smeared. In our approach, $R \rightarrow 0$ will make us recover the point-like anyons point of view.

Let us consider the 2D Coulomb potential generated by a unit charge smeared over the disc of radius R :

$$w_R(x) = (\log |\cdot| * \chi_R)(x), \text{ with the convention } w_0 = \log |\cdot| \quad (1.4)$$

and $\chi_R(x)$ a positive regularizing function with a fixed to one mass and built as follows

$$\chi_R = \frac{1}{R^2} \chi\left(\frac{\cdot}{R}\right) \text{ and } \int_{\mathbb{R}^2} \chi = 1$$

such that

$$\chi(x) = \begin{cases} 1/\pi^2 & |x| \leq 1 \\ 0 & |x| \geq 2 \end{cases} \quad (1.5)$$

being smooth, positive and decreasing between 1 and 2. Since the magnetic field associated to $\mathbf{A}(x_j)$ is:

$$\text{curl} \mathbf{A}(x_j) = 2\pi \sum_{k \neq j} \delta(x_j - x_k)$$

we will recover the same magnetic field in the limit $R \rightarrow 0$ posing:

$$\mathbf{A}^R(x_j) = \sum_{k \neq j} \nabla^\perp w_R(x_j - x_k) \quad (1.6)$$

leading to the regularized Hamiltonian

$$H_N^R = \sum_{j=1}^N \left((p_j + \mathbf{A}_e + \alpha \mathbf{A}^R(x_j))^2 + V(x_j) \right). \quad (1.7)$$

We denote

$$E^R(N) = \inf \sigma(H_N^R) \quad (1.8)$$

with H_N^R acting on $L^2_{\text{sym}}(\mathbb{R}^{2N})$. This regularization method is a generalization of the one used in [17, equation 1.4]. There,

$$\chi = \frac{\mathbb{1}_{B(0,R)}}{\pi R^2}(x)$$

was used but we will need a smoother χ , namely,

$$\int_{\mathbb{R}^2} |\widehat{\chi}(p)| dp < \infty$$

with $\widehat{\chi}$ the Fourier transform of χ .

We expand (1.7) in four terms that we will treat one by one:

$$\begin{aligned} H_R^N &= \sum_{j=1}^N ((p_j^A)^2 + V(x_j)) \quad \text{“Kinetic and potential terms”} \\ &+ \alpha \sum_{j \neq k} \left(p_j^A \cdot \nabla^\perp w_R(x_j - x_k) + \nabla^\perp w_R(x_j - x_k) \cdot p_j^A \right) \quad \text{“Mixed two-body term”} \\ &+ \alpha^2 \sum_{j \neq k \neq l} \nabla^\perp w_R(x_j - x_k) \cdot \nabla^\perp w_R(x_j - x_l) \quad \text{“Three-body term”} \\ &+ \alpha^2 \sum_{j \neq k} \left| \nabla^\perp w_R(x_j - x_k) \right|^2 \quad \text{“Singular two-body term”}. \end{aligned} \quad (1.9)$$

1.3. Average field approximation. The model we have built until now is still hard to study. That is why we will set:

$$\mathbf{A}[\rho] = \nabla^\perp w_0 * \rho \quad \text{and} \quad \mathbf{A}^R[\rho] = \nabla^\perp w_R * \rho \quad (1.10)$$

which makes \mathbf{A} independent of the precise positions x_j . Here ρ is the one-body density normalized in $L^1(\mathbb{R}^2)$ of the bosonic wave function Ψ :

$$\rho(x_1) = \int_{\mathbb{R}^{2(N-1)}} |\Psi(x_1, x_2, \dots, x_N)|^2 dx_2 \dots dx_N$$

We obtain from this the approximate N-body Hamiltonian,

$$H_N^{\text{af}} = \sum_{j=1}^N ((p_j^A + (N-1)\alpha \mathbf{A}^R[\rho])^2 + V(x_j)) \quad (1.11)$$

denoting

$$p_j^A = p_j + \mathbf{A}_e(x_j).$$

This average field approximation Hamiltonian is the usual way to describe almost-bosonic anyons. It has also been studied in [7, 6] with, in these cases, β as a parameter. In this way, the magnetic field becomes

$$\mathbf{B}(x) = \text{curl} \beta \mathbf{A}[\rho](x) = 2\pi\beta\rho(x).$$

We can now consider the usual mean field ansatz taking $\Psi(x_1, \dots, x_N) = u^{\otimes N}$ with the consistency equation $|u|^2 = \rho$ and get the one particle average-field energy functional:

$$\mathcal{E}_R^{\text{af}}[u] = \int_{\mathbb{R}^2} \left(|(-i\nabla + \mathbf{A}_e + \beta \mathbf{A}^R[|u|^2])u|^2 + V|u|^2 \right) \quad (1.12)$$

with

$$E_R^{\text{af}} = \inf \mathcal{E}_R^{\text{af}}[u] \quad (1.13)$$

under the unit mass constraint:

$$\int_{\mathbb{R}^2} |u|^2 = 1.$$

Alternatively, we can write

$$\mathcal{E}_R^{\text{af}}[u] = \int_{\mathbb{R}^2} \left(|p^A u|^2 + V|u|^2 + 2\beta \mathbf{A}^R[|u|^2] \cdot \left(\frac{i}{2}(u\nabla\bar{u} - \bar{u}\nabla u) + \mathbf{A}_e|u|^2 + \frac{\beta}{2} \mathbf{A}^R[|u|^2]|u|^2 \right) \right) dx$$

Our aim is now well defined and consists in proving:

$$\frac{\inf \langle \Psi_N, H_N^R \Psi_N \rangle}{N} \approx \inf \left\{ \mathcal{E}_R^{\text{af}}[u], \int |u|^2 = 1 \right\} \quad (1.14)$$

when $N \rightarrow \infty$. In other words, the average mean field energy is a good approximation for the ground state energy of H_N^R .

1.4. Main results. We state our main theorem, justifying the average field approximation in the almost-bosonic limit at the level of the ground state. For technical reasons we assume that the one-body potential is confining

$$V(x) \geq c|x|^s - C, \quad s > 0, \quad (1.15)$$

and

$$\mathbf{A}_e \in L_{loc}^2(\mathbb{R}^2), \quad \text{curl} \mathbf{A}_e = \mathbf{B}_e = \mathbf{B}_0 + \tilde{\mathbf{B}}, \quad \mathbf{B}_0 = C\text{st}, \quad \tilde{\mathbf{B}} \in W^{1,2+\varepsilon}. \quad (1.16)$$

We also need that the size R of the extended anyons does not go to zero too fast in the limit $N \rightarrow \infty$ i.e $R \sim N^{-\eta}$ for η having to be determined.

Theorem 1.1 (Validity of the average field approximation).

Assume that we have N extended anyons of radius $R \sim N^{-\eta}$ for some

$$0 < \eta < \frac{1}{4} \quad (1.17)$$

and with the statistics parameter

$$\alpha = \frac{\beta}{(N-1)}$$

for fixed $\beta \in \mathbb{R}$. Then, in the limit $N \rightarrow \infty$ we have for the ground-state energy

$$\frac{E_R(N)}{N} \rightarrow E_0^{\text{af}} = E^{\text{af}}. \quad (1.18)$$

Moreover, if Ψ_N is a sequence of ground states for H_N^R , with associated reduced density matrices

$$\gamma_N^{(k)} := \text{Tr}_{k+1 \rightarrow N} [|\Psi_N\rangle\langle\Psi_N|]$$

then along a subsequence we have

$$\gamma_N^{(k)} \rightarrow \int_{\mathcal{M}^{\text{af}}} |u^{\otimes k}\rangle\langle u^{\otimes k}| d\mu(u) \quad (1.19)$$

strongly in the trace-class when $N \rightarrow \infty$, where μ is a Borel probability measure supported on the set of minimizers of \mathcal{E}^{af} ,

$$\mathcal{M}^{\text{af}} := \{u \in L^2(\mathbb{R}^2) : \|u\|_{L^2} = 1, \mathcal{E}^{\text{af}}[u] = E^{\text{af}}\}.$$

There are two main improvements in this theorem compared with [17, Theorem 1.1]. The first is the removal of the constraint

$$\eta < \eta_0 = \frac{1}{4} \left(1 + \frac{1}{s} \right)^{-1}$$

providing us a less restrictive result holding for every $s > 0$. We achieve this using the method of [24] which revisits the quantum de Finetti theorem due to Brandão and Harrow [4]. This implies to write operators of (1.7) in a tensorized form (see (3.1)). This forces us to use a smooth χ complicating the estimate of the 3-body term (third line in (1.9)).

The second improvement is the addition of an external magnetic term \mathbf{A}_e in H_N . Transforming p_j^0 in p_j^A the estimates of the cross product term becomes more difficult. The non commutativity

$$\left[(p_j^A)_1, (p_j^A)_2 \right] \neq 0$$

causes this difficulty.

The needed new bounds will be the aim of the second section. Indeed, we will obtain a control of the Hamiltonian terms as a function of the kinetic energy in the limit $R \rightarrow \infty$. In the third section we combine these results with an a priori bound on the kinetic energy also derived in Section 2 and the de Finetti theorem to get a lower bound on $E^R(N)/N$ as E^{af} plus an error depending on R , N and a kinetic energy cut-off Λ that we finally optimize in the limit $N \rightarrow \infty$. We also prove the convergence of states in this section. We finally derive some auxiliary results such as the convergence $E_R^{\text{af}} \rightarrow E^{\text{af}}$ or the existence of minimizers for \mathcal{E}^{af} in the appendix.

Acknowledgments. I want to thank my thesis advisor Nicolas Rougerie for discussions, explanations and help that made this work possible. I would also like to thank Alessandro Olgiati for inspiring discussions. Funding from the European Research Council (ERC) under the European Union's Horizon 2020 Research and Innovation Programme (Grant agreement CORFRONMAT No 758620) is gratefully acknowledged.

2. THE EXTENDED ANYON HAMILTONIAN

In this section we establish some bounds allowing to control the Hamiltonian (1.7). We can expand it as in (1.9) (and (A.6) for the energy expression) in order to estimate it term by term.

Because of the boundness of the interaction at fixed R , H_N^R is defined uniquely as a self-adjoint operator on $L_{\text{sym}}^2(\mathbb{R}^{2N})$ with the same form domain as the non-interacting bosonic Hamiltonian, (see [16, Proposition 7.20] for a definition of $H_{A_e}^1$):

$$\sum_{j=1}^N \left((p_j^A)^2 + V(x_j) \right), \quad \mathbf{A}_e \in L_{\text{loc}}^2.$$

As we will finally take the limit $R \rightarrow 0$ we need to deduce precise bounds depending on R assuming $R \ll R_0$, $R_0 > 0$ a fixed reference length scale. In the following, the generic constant C may implicitly depends on R_0 . Bounds are here similar to [17] but with p_j^A instead of p_j which complicates the proof of lemma 2.3 concerning the mixed two-body

term. As for the three-body term (lemma 2.4), the new regularizing function χ makes its proof based on the Hardy inequality harder.

2.1. Operator bounds for the interaction terms. Exploiting the regularizing effect of taking $R > 0$ we will estimate the different terms in (A.6).

Lemma 2.1 (The smeared Coulomb potential).

Let w_R be defined as in (1.4). There is a constant $C > 0$ depending only on R_0 such that

$$\sup_{\mathbb{R}^2} |\nabla w_R| \leq \frac{C}{R} \quad \text{and} \quad \|\nabla w_R\|_{L^p(\mathbb{R}^2)} \leq C_p R^{2/p-1} \quad (2.1)$$

for any $2 < p < \infty$.

Proof. We apply Newton's theorem [16, Theorem 9.7] to calculate w_R in each domain:

$$|\nabla w_R(x)| = \begin{cases} 1/|x|, & |x| \in [2R, +\infty[\\ 1/|x| \int_{B(0,|x|)} \chi_R(u) du, & |x| \in]R, 2R[\\ |x|/\pi R^2, & |x| \in [0, R] \end{cases} \quad (2.2)$$

For $|x|$ between 0 and $2R$ the given formula comes from:

$$w_R = \log|x| \int_0^{2\pi} \int_0^{|x|} \chi_R(y) y dy d\theta + \int_0^{2\pi} \int_{|x|}^{2R} \log(y) \chi_R(y) y dy d\theta$$

which gives the first conclusion of the lemma.

To obtain the second we need to compute

$$\|\nabla w_R\|_{L^p(\mathbb{R}^2)}^p \leq 2\pi \int_0^R \frac{r^p}{\pi^p R^{2p}} r dr + 2\pi \int_R^{+\infty} r^{-p} r dr \leq C_p^p R^{2-p} \quad (2.3)$$

with $C_p > 0$ only depending on $p > 2$. □

Lemma 2.2 (Singular two-body term).

For any $\varepsilon > 0$ we have as operators on $L^2(\mathbb{R}^4)$:

$$|\nabla w_R(x-y)|^2 \leq C_\varepsilon R^{-\varepsilon} ((p_x^A)^2 + 1) \quad (2.4)$$

$$|\nabla w_R(x-y)|^2 \leq C_\varepsilon R^{-\varepsilon} (p_x^2 + 1) \quad (2.5)$$

Proof. Taking any $w: \mathbb{R}^2 \mapsto \mathbb{R}$ and $f \in C_c^\infty(\mathbb{R}^4)$ with $u = |f|$, we compute

$$\begin{aligned} \langle f, w(x-y)f \rangle &= \langle u, w(x-y)u \rangle \\ &= \iint_{\mathbb{R} \times \mathbb{R}} w(x-y) |u(x,y)|^2 dx dy \quad \text{because } w \text{ does not see the phase} \\ &\leq \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |w(x-y)|^p dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^2} |u(x,y)|^{2q} dx \right)^{\frac{1}{q}} dy \quad \text{by Hölder's inequality.} \end{aligned}$$

Then, using Sobolev's inequality [16, Theorem 8.5 ii]

$$\begin{aligned}
\langle f, w(x-y)f \rangle &= \|w\|_{L^p} \int_{\mathbb{R}^2} \|u(\cdot, y)\|_{L^{2q}}^2 dy \\
&\leq C \|w\|_{L^p} \int_{\mathbb{R}^2} (\|\nabla u(\cdot, y)\|_{L^2}^2 + \|u(\cdot, y)\|_{L^2}^2) dy \\
&= C \|w\|_{L^p} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (|\nabla u(x, y)|^2 + |u(x, y)|^2) dx dy \\
&\leq C \|w\|_{L^p} \langle f, ((p_x^A)^2 + 1) f \rangle \quad \text{using the diamagnetic inequality (A.2)}
\end{aligned}$$

with $p > 1$ and $q = \frac{p}{p-1}$. We now use (2.1) with $w = |\nabla w_R|^2$ and $p = 1 + \varepsilon'$ to get

$$\|w\|_{L^p} = \|\nabla w_R\|_{L^{2p}}^2 \leq C_{2p}^2 R^{2/p-2} \leq C_\varepsilon R^{-\varepsilon}$$

with a constant $C_\varepsilon > 0$ for given $\varepsilon > 0$. The proof of the second inequality is the same without the diamagnetic inequality step. \square

We next deal with the term mixing the position and the momentum. The first inequality we will get will have a bad R -dependence but a better behavior for large momenta than the other.

Lemma 2.3 (Mixed two-body term).

Consider a magnetic field

$$\mathbf{B}_e = \text{curl} \mathbf{A}_e = \mathbf{B}_0 + \tilde{\mathbf{B}}$$

with $\mathbf{B}_0 = \text{Cst}$ and $\tilde{\mathbf{B}} \in W^{1,2+\varepsilon}$.

For $R < R_0$ and $\varepsilon > 0$ we have, as operator on $L_{sym}^2(\mathbb{R}^4)$:

$$\left| p_x^A \cdot \nabla^\perp w_R(x-y) + \nabla^\perp w_R(x-y) \cdot p_x^A \right| \leq CR^{-1} |p_x^A| \quad (2.6)$$

$$\left| p_x^A \cdot \nabla^\perp w_R(x-y) + \nabla^\perp w_R(x-y) \cdot p_x^A \right| \leq C_\varepsilon R^{-\varepsilon} \left((p_x^A)^2 + 1 \right) \quad (2.7)$$

$$\left| p_x \cdot \nabla^\perp w_R(x-y) + \nabla^\perp w_R(x-y) \cdot p_x \right| \leq C_\varepsilon R^{-\varepsilon} (p_x^2 + 1) \quad (2.8)$$

Proof. Proof of (2.6):

We first note that $\nabla_x \cdot \nabla^\perp w_R(x-y) = 0$, consequently

$$p_x^A \cdot \nabla^\perp w_R(x-y) = \nabla^\perp w_R(x-y) \cdot p_x^A$$

and we square the expression we want to estimate

$$\left(p_x^A \cdot \nabla^\perp w_R(x-y) + \nabla^\perp w_R(x-y) \cdot p_x^A \right)^2 = 4 p_x^A \cdot \nabla^\perp w_R(x-y) \nabla^\perp w_R(x-y) \cdot p_x^A.$$

Then for any $f(x, y) \in C_c^\infty(\mathbb{R}^4)$

$$\begin{aligned}
&\left| \left\langle f, \left(p_x^A \cdot \nabla^\perp w_R(x-y) + \nabla^\perp w_R(x-y) \cdot p_x^A \right)^2 f \right\rangle \right| \\
&\leq 4 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |p_x^A f(x, y)|^2 \left| \nabla^\perp w_R(x-y) \right|^2 dx dy
\end{aligned}$$

using (2.4) we get

$$\begin{aligned} \left| \left\langle f, \left(p_x^A \cdot \nabla^\perp w_R(x-y) + \nabla^\perp w_R(x-y) \cdot p_x^A \right)^2 f \right\rangle \right| &\leq \frac{C}{R^2} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |p_x^A f(x, y)|^2 dx dy^2 \\ &\leq \frac{C}{R^2} (p_x^A)^2 \end{aligned}$$

and we deduce (2.6) because the square root is operator monotone, see [3, Chapter 5].

For the proof of (2.7), writing $x = (x_1, x_2)$ the coordinates of the particle located in x , we start from

$$\begin{aligned} T &= \left| \left\langle f, \left(p_x^A \cdot \nabla^\perp w_R(x-y) + \nabla^\perp w_R(x-y) \cdot p_x^A \right)^2 f \right\rangle \right| \\ &\leq 4 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |p_x^A f(x, y)|^2 \left| \nabla^\perp w_R(x-y) \right|^2 dx dy \\ &= 4 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (|p_{x_1}^A f(x, y)|^2 + |p_{x_2}^A f(x, y)|^2) \left| \nabla^\perp w_R(x-y) \right|^2 dx dy \\ &= 4 \left(\left\langle p_{x_1}^A f, \left| \nabla^\perp w_R(x-y) \right|^2 p_{x_1}^A f \right\rangle + \left\langle p_{x_2}^A f, \left| \nabla^\perp w_R(x-y) \right|^2 p_{x_2}^A f \right\rangle \right) \\ &\leq \frac{C_\varepsilon}{R^\varepsilon} (\langle p_{x_1}^A f, ((p_x^A)^2 + 1) p_{x_1}^A f \rangle + \langle p_{x_2}^A f, ((p_x^A)^2 + 1) p_{x_2}^A f \rangle) \end{aligned}$$

Using the bound (2.4).

In order to recover $(p_{x_i}^A)^2$ we have to calculate commutators involving $p_{x_1}^A$ and $(p_x^A)^2$.

$$\begin{aligned} T &\leq \frac{C_\varepsilon}{R^\varepsilon} \langle f, ((p_x^A)^2 + 1) (p_x^A)^2 f \rangle + \frac{C_\varepsilon}{R^\varepsilon} \langle f, ([p_{x_1}^A, (p_{x_2}^A)^2] p_{x_1}^A + [p_{x_2}^A, (p_{x_1}^A)^2] p_{x_2}^A) f \rangle \\ &\leq \frac{C_\varepsilon}{R^\varepsilon} \langle f, ((p_x^A)^2 + 1)^2 f \rangle + \frac{C_\varepsilon}{R^\varepsilon} \langle f, F f \rangle \end{aligned}$$

To control F we calculate

$$[p_{x_1}^A, p_{x_2}^A] = -i(\nabla \wedge \mathbf{A}_e) = -i\mathbf{B}_e$$

and

$$[p_{x_1}^A, -i\mathbf{B}_e] = -\partial_{x_1} \mathbf{B}_e \quad \text{and} \quad [p_{x_2}^A, -i\mathbf{B}_e] = \partial_{x_2} \mathbf{B}_e.$$

Thus we obtain:

$$\begin{aligned} F &= p_{x_2}^A (-i\mathbf{B}_e) p_{x_1}^A - i\mathbf{B}_e p_{x_2}^A p_{x_1}^A - p_{x_1}^A (-i\mathbf{B}_e) p_{x_2}^A + i\mathbf{B}_e p_{x_1}^A p_{x_2}^A \\ &= (\partial_{x_2} \mathbf{B}_e - i\mathbf{B}_e p_{x_2}^A) p_{x_1}^A - i\mathbf{B}_e p_{x_2}^A p_{x_1}^A - (\partial_{x_1} \mathbf{B}_e - i\mathbf{B}_e p_{x_1}^A) p_{x_2}^A + i\mathbf{B}_e p_{x_1}^A p_{x_2}^A \\ &= \partial_{x_2} \mathbf{B}_e p_{x_1}^A + \partial_{x_1} \mathbf{B}_e p_{x_2}^A - 2i\mathbf{B}_e (p_{x_2}^A p_{x_1}^A - p_{x_1}^A p_{x_2}^A) \\ &= 2\mathbf{B}_e^2 - (\partial_{x_2} \mathbf{B}_e) p_{x_1}^A - (\partial_{x_1} \mathbf{B}_e) p_{x_2}^A \\ &\leq 2\mathbf{B}_e^2 + ((\partial_{x_1} \mathbf{B}_e)^2 + (\partial_{x_2} \mathbf{B}_e)^2 + (p_{x_1}^A)^2 + (p_{x_2}^A)^2) \\ &= 2\mathbf{B}_e^2 + (\nabla \mathbf{B}_e)^2 + (p^A)^2 \end{aligned}$$

Splitting the magnetic field as $\mathbf{B}_e = \mathbf{B}_0 + \tilde{\mathbf{B}}$ with $\mathbf{B}_0 = \text{Cst}$ we get for the $\mathbf{B}_e^2 = \mathbf{B}_0^2 + 2\mathbf{B}_0 \cdot \tilde{\mathbf{B}} + \tilde{\mathbf{B}}^2$ and the $(\nabla \mathbf{B}_e)^2 = (\nabla \tilde{\mathbf{B}})^2$ terms:

$$\begin{aligned} \langle f, \mathbf{B}_e^2 f \rangle &= \mathbf{B}_0^2 + 2\mathbf{B}_0 \langle f, \tilde{\mathbf{B}} f \rangle + \langle f, \tilde{\mathbf{B}}^2 f \rangle \\ &\leq \mathbf{B}_0^2 + \left(2\mathbf{B}_0 \|\tilde{\mathbf{B}}\|_{L^p} + \|\tilde{\mathbf{B}}\|_{L^{2p}}^2 \right) \|f\|_{L^{2q}}^2 \leq C \langle f, \left((p_x^A)^2 + 1 \right) f \rangle \\ \langle f, (\nabla \tilde{\mathbf{B}})^2 f \rangle &\leq \|(\nabla \tilde{\mathbf{B}})^2\|_{L^p} \|f\|_{L^{2q}}^2 \leq C \|(\nabla \tilde{\mathbf{B}})^2\|_{L^p} \langle f, \left((p_x^A)^2 + 1 \right) f \rangle \end{aligned}$$

using Hölder's inequality with $1 = 1/p + 1/q$, that \mathbf{B} and $(\nabla \mathbf{B})^2$ are multiplier operators and the Sobolev' inequality [16, Theorem 8.5 ii]:

$$\|g\|_{L^{2q}}^2 \leq C_q \left(\|\nabla g\|_{L^2}^2 + \|g\|_{L^2}^2 \right)$$

for $1 \leq q \leq \infty$.

So because we have $\tilde{\mathbf{B}} \in W^{1,2+\varepsilon}$:

$$F \leq C \left((p_x^A)^2 + 1 \right)$$

and we get

$$\begin{aligned} T &\leq \frac{C_\varepsilon}{R^\varepsilon} \langle f, \left((p_x^A)^2 + 1 \right)^2 f \rangle \\ &\left(p_x^A \cdot \nabla^\perp w_R(x-y) + \nabla^\perp w_R(x-y) \cdot p_x^A \right)^2 \leq \frac{C_\varepsilon}{R^\varepsilon} \left((p_x^A)^2 + 1 \right)^2 \end{aligned}$$

and the result follows by taking the square root. The proof of the third inequality (2.8) is a simpler version of what we just did. \square

Lemma 2.4 (Three-body term).

We have that, as operators on $L_{sym}^2(\mathbb{R}^6)$:

$$\left| \nabla^\perp w_R(x-y) \cdot \nabla^\perp w_R(x-z) \right| \leq C \left((p_x^A)^2 + 1 \right) \quad (2.9)$$

$$\left| \nabla^\perp w_R(x-y) \cdot \nabla^\perp w_R(x-z) \right| \leq C (p_x^2 + 1) \quad (2.10)$$

To prove this lemma we will use the three-particle Hardy inequality [10, Lemma 3.6] as done in [17, Lemma 2.4] but with our new $\chi \neq \mathbb{1}_{B(0,R)}$ which implies to take into account the sign of each considered term. The second difference is that we do not use Clifford algebra.

Lemma 2.5 (Three-body Hardy inequality, [10]).

Let $d = 2$ and $u : \mathbb{R}^6 \rightarrow \mathbb{C}$. Let $\mathcal{R}(x, y, z)$ be the circumradius of the triangle with vertices $x, y, z \in \mathbb{R}^2$, and $\rho(x, y, z) := \sqrt{|x-y|^2 + |y-z|^2 + |z-x|^2}$. Then $\mathcal{R}^{-2} \leq 9\rho^{-2}$ pointwise, and, for all $u \in H^1$

$$3 \int_{\mathbb{R}^6} \frac{|u(x, y, z)|^2}{\rho(x, y, z)^2} dx dy dz \leq \int_{\mathbb{R}^6} \left(|\nabla_x u|^2 + |\nabla_y u|^2 + |\nabla_z u|^2 \right) dx dy dz \quad (2.11)$$

Proof of Lemma 2.4. Since we consider the operator as acting on symmetric wave functions it is equivalent to estimate the sum:

$$S = \sum_{x, y, z} \nabla w_R(x-y) \cdot \nabla w_R(x-z) = \sum_{x, y, z} \frac{x-y}{|x-y|} \cdot \frac{x-z}{|x-z|} v(x-y) v(x-z) \quad (2.12)$$

where x, y, z under the sum means cyclic in x, y, z and $v(u) = \partial_u w_R(u)$ is a radial function whose expression has been calculated in (2.2) such that

$$v(|x|) = |\nabla w_R(x)| = \begin{cases} 1/|x|, & |x| \in [2R, +\infty[\\ 1/|x| \int_{B(0,|x|)} \chi_R(u) du, & |x| \in]R, 2R[\\ |x|/\pi R^2, & |x| \in [0, R] \end{cases} \quad (2.13)$$

In general, let each $x, y, z \in \mathbb{R}^2$ denote the vertices of a triangle. Then we claim the following geometric fact:

$$|S| = \left| \sum_{x,y,z} \frac{x-y}{|x-y|} \cdot \frac{x-z}{|x-z|} v(x-y)v(x-z) \right| \leq \frac{C}{\rho(x,y,z)^2}. \quad (2.14)$$

where C is a positive constant independent of R . Observe that for $R \leq |x| \leq 2R$, $v(x) \geq 0$ and that

$$\frac{C_1}{R} \leq v(x) \leq \frac{C_2}{R}$$

where $C_1 \leq C_2$ are two non-negative constants independent of R . We will proceed in several steps respectively considering the cases:

- (1) All lengths of the triangle are larger than $2R$
- (2) All lengths are smaller than $2R$
- (3) Two of the lengths are smaller than $2R$ and one greater.
- (4) Two of the lengths are larger than $2R$ and one smaller.

and establish for each the inequality $S \leq C\rho^{-2}$. The second, $S \geq -C\rho^{-2}$ being easy or similar, depending on the case, will not be treated.

Step 1. All lengths of the triangle are greater than $2R$ i.e $|x| \geq 2R$. Then

$$S = \sum_{x,y,z} \frac{x-y}{|x-y|^2} \cdot \frac{x-z}{|x-z|^2} \quad (2.15)$$

We proceed as in [10, Lemma 3.2]: $a = x - y$, $b = x - z$ and ϕ the angle between a and b . A simple computation of S leads to:

$$\begin{aligned} S &= \frac{a \cdot b}{|a|^2 |b|^2} - \frac{a}{|a|^2} \cdot \frac{(b-a)}{|b-a|^2} - \frac{b}{|b|^2} \cdot \frac{(a-b)}{|b-a|^2} \\ \frac{2(|a|^2 |b|^2 - (a \cdot b)^2)}{|a|^2 |b|^2 |b-a|^2} &= \frac{2 \sin(\phi)^2}{|y-z|^2} = \frac{1}{2} R^{-2}(x, y, z) \leq \frac{9}{2} \rho^{-2} \end{aligned}$$

by Lemma 2.5.

Step 2. Before initiating the proof we need to enlight some general facts about this geometric configuration. Indeed we will need to know which term of the sum is positive or negative in each case we will study. First, as the sum of the triangle's angles is π we know that if one of them is larger than $\pi/2$ the others have to be smaller. That implies that only one of the scalar products of the sum can be negative in each configuration and we are able

to identify it. For example if $\widehat{xyz} \geq \pi/2$ then the first term of the sum has to be negative and the two others positive. If all of them are under $\pi/2$, then the sum only consists of positive terms. This case being easier and involving the same reasoning than the others we will systematically ignore it.

Here all lengths are smaller than $2R$ and the negative term is the second. This term can be bounded in three different ways. First:

$$\frac{(y-z).(y-x)}{|y-z||y-x|}v(y-z)v(y-x) \leq \frac{(y-z).(y-x)}{\pi^2 R^4}$$

if $|y-z|$ and $|y-x|$ are smaller than R (here the inequality is actually an equality). Or it can be bounded as

$$\frac{(y-z).(y-x)}{|y-z||y-x|}v(y-z)v(y-x) \leq C_1 \frac{(y-z).(y-x)}{2R^4}$$

if only $|y-z|$ or $|y-x|$ is smaller than R and by

$$\frac{(y-z).(y-x)}{|y-z||y-x|}v(y-z)v(y-x) \leq \frac{C_1^2}{4} \frac{(y-z).(y-x)}{R^4}$$

if $|y-z|$ and $|y-x|$ are between R and $2R$. Arguing similarly for the positive terms we get

$$\begin{aligned} R^4 S &\leq \max(1, C_2, C_2^2) (x-y).(x-z) + \min(1, C_1/2, C_1^2/4) (y-z).(y-x) \\ &\quad + \max(1, C_2, C_2^2) (z-x).(z-y) \\ &= \max(1, C_2, C_2^2) ((x-y).(x-z) + (y-z).(y-x) + (z-x).(z-y)) \\ &\quad + (\min(1, C_1/2, C_1^2/4) - \max(1, C_2, C_2^2)) (y-z).(y-x) \end{aligned}$$

and we obtain two positive constants C_3 and C_4 such that

$$|S| \leq \frac{C_3}{R^4} \rho^2 \leq C_4 \rho^{-2}. \quad (2.16)$$

Step 3. We consider the case where two of the edges are short and one long. Let's take $|x-y|, |y-z| \leq 2R$ and $|x-z| \geq 2R$. In this configuration the negative term is the second. As we did in Step 2 we start by writing

$$\begin{aligned} S_3 &\leq \max(1, C_2) \frac{(x-y).(x-z)}{R^2|x-z|^2} + \min(1, C_1/2, C_1^2) \frac{(y-z).(y-x)}{R^4} \\ &\quad + \max(1, C_2) \frac{(y-z).(y-x)}{R^2|x-z|^2} \\ &= \max(1, C_2) \left[\frac{(x-y).(x-z)}{R^2|x-z|^2} + \frac{(y-z).(y-x)}{R^4} + \frac{(y-z).(y-x)}{R^2|x-z|^2} \right] \\ &\quad + \frac{(y-z).(y-x)}{R^4} [\min(1, C_1/2, C_1^2) - \max(1, C_2)] \end{aligned}$$

From there, we proceed as in [17, Lemma 2.4]. For the first term of the above

$$\begin{aligned} & (x-y).(x-z)R^2 + (y-z).(y-x)|x-z|^2 + R^2(z-x).(z-y) \\ &= |x-z|^2(y-z).(y-x) + R^2(x-z)(x-y-z+y) \\ &\leq 5R^2|x-z|^2 \end{aligned}$$

and for the second

$$\left| \frac{(y-z).(y-x)}{R^4} \right| |x-z|^2 \leq \frac{4}{R^2} |x-z|^2.$$

Comparing to

$$\begin{aligned} & |x-z|^2 R^4 \rho^{-2} \frac{|x-z|^2 R^4}{|x-z|^2 + |x-y|^2 + |y-z|^2} \\ &\geq \frac{R^4 |x-z|^2}{4R^2 + |x-z|^2} \\ &\geq \frac{R^2}{20} |x-z|^2 \end{aligned}$$

because $|x-z| \leq |x-y| + |y-z| \leq 4R$, we conclude that

$$|S| \leq C_5 \rho^{-2}$$

Step 4. We now study the last case when only one edge is short and the two others long. Let's take $|x-y| \leq 2R$ and $|y-z|, |z-x| \geq 2R$. We first notice that the two longer edges have to be of the same order:

$$\begin{aligned} |y-z| &\leq |y-x| + |x-z| \leq 2|x-z| \\ |x-z| &\leq |x-y| + |y-z| \leq 2|y-z| \\ \frac{1}{2}|x-z| &\leq |y-z| \leq 2|x-z|. \end{aligned} \tag{2.17}$$

We write

$$|x-y|^2 |x-z|^2 |y-z|^2 S = \sum_{x,y,z} (x-y).(x-z)|y-z|^2 |x-y| |x-z| v(x-y)v(x-z)$$

where $|x-z|v(x-z) = 1$. We can now evaluate the right side of the inequality (2.14) multiplied by $|x-y|^2 |x-z|^2 |y-z|^2$.

$$\rho^{-2} |x-y|^2 |x-z|^2 |z-y|^2 \geq C_6 R^2 \frac{|x-z|^4}{|x-z|^2 + |z-y|^2} \geq C_6 R^2 |x-z|^2$$

because $|x-y|^2 \leq |x-z|^2$. All that is left is to show that

$$\begin{aligned} & (x-y).(x-z)|z-y|^2 + (y-z).(y-x)|x-z|^2 \\ &= |z-y|^2 |x-y|^2 + |z-y|^2 (x-y).(y-z) + |x-z|^2 |x-y|^2 + |x-z|^2 (x-y).(z-x) \\ &= |x-z|^2 |x-y|^2 + (x-y).(z-x) [|x-z|^2 - |z-y|^2] \\ &= |x-z|^2 |x-y|^2 + (x-y).(z-x) [(|z-x| + |z-y|)(|z-x| - |z-y|)] \\ &\leq C_7 R^2 |x-z|^2 \end{aligned}$$

Using the triangular inequality

$$||z - x| - |z - y|| \leq |z - x - (z - y)|$$

so we deduce

$$\begin{aligned} |x - y|^2 |x - z|^2 |y - z|^2 S &= |x - y| v(x - y) [(x - y) \cdot (x - z) |z - y|^2 + (y - z) \cdot (y - x) |x - z|^2] \\ &\quad + (z - x) \cdot (z - y) |x - y|^2 \leq C_8 R^2 |x - z|^2 \end{aligned}$$

and conclude that

$$|S| \leq C \rho^{-2}.$$

Now we have

$$\left| \nabla^\perp w_R(x - y) \cdot \nabla^\perp w_R(x - z) \right| \leq C \left((p_x)^2 + 1 \right)$$

using Lemma 2.5. Then, as a multiplication operator $\nabla^\perp w_R$ does not see the phase so

$$\begin{aligned} \left\langle u \left| \nabla^\perp w_R(x - y) \cdot \nabla^\perp w_R(x - z) \right| u \right\rangle &= \left\langle |u| \left| \nabla^\perp w_R(x - y) \cdot \nabla^\perp w_R(x - z) \right| |u| \right\rangle \\ &\leq C \left(|\nabla |u||^2 + |u|^2 \right) \\ &\leq C \left\langle u, \left((p_x^A)^2 + 1 \right) u \right\rangle \end{aligned}$$

which gives the expected result using the diamagnetic inequality. \square

2.2. A priori bound for the ground state. For the estimates of the previous subsection to apply efficiently, we need an a priori bound on ground states (or approximate ground states) of the N -body Hamiltonian (1.7), provided in the following:

Proposition 2.6 (A priori bound for many-body ground states).

Let $\Psi_N \in L_{sym}^2(\mathbb{R}^{2N})$ be a sequence of approximate ground state for H_N^R that is:

$$\langle \Psi_N, H_N^R \Psi_N \rangle \leq E^R(N)(1 + o(1)) \quad \text{when } N \rightarrow \infty$$

Denote by $\gamma_N^{(1)}$ the associated sequence of one-body density matrices. In the regime $\alpha = \beta/(N - 1) \rightarrow 0$, assuming a bound $R \geq N^{-\eta}$ for some $\eta > 0$ independent of N , we have:

$$\text{Tr} \left[\left((p^A)^2 + V \right) \gamma_N^{(1)} \right] \leq C(1 + \beta) E_R^{\text{af}} \quad (2.18)$$

Proof. We proceed in two steps:

Step 1. Using a trial state $u^{\otimes N}$ such that $\mathcal{E}_R^{\text{af}}[u] = E_R^{\text{af}}$ we can obtain from (A.6), (A.7), (A.8) and from the bounds (2.5), (2.8) and (2.10) plus the diamagnetic inequality (A.3):

$$\begin{aligned} \frac{E^R(N)}{N} &\leq E_R^{\text{af}} - \frac{\beta^2}{N - 1} \text{Tr} \left[\left(\nabla^\perp w_R(x_1 - x_2) \cdot \nabla^\perp w_R(x_1 - x_3) \right) \gamma_N^{\text{af}(3)} \right] \\ &\quad + \frac{\beta^2}{N - 1} \text{Tr} \left[\left| \nabla^\perp w_R(x_1 - x_2) \right|^2 \gamma_N^{\text{af}(2)} \right] \\ &\leq E_R^{\text{af}} + \left(E_R^{\text{af}} + 1 \right) \left(\frac{C\beta^2}{N - 1} + \frac{C_\varepsilon \beta^2 R^{-\varepsilon}}{N - 1} \right) \\ &= E_R^{\text{af}} + o_N(1) \rightarrow E^{\text{af}} \end{aligned} \quad (2.19)$$

since $R \sim N^{-\eta}$ with $N \rightarrow \infty$, and we used Proposition (A.5).

We now use the diamagnetic inequality in each variable to obtain:

$$\begin{aligned} \langle \Psi_N, H_N^R \Psi_N \rangle &= \sum_{j=1}^N \int_{\mathbb{R}^{2N}} \left(|(-i\nabla_j + A_e + \alpha A_j^R) \Psi_N|^2 + V(x_j) |\Psi_N|^2 \right) dx_1 \dots dx_N \\ &\geq \sum_{j=1}^N \int_{\mathbb{R}^{2N}} \left(|\nabla_j |\Psi_N||^2 + V(x_j) |\Psi_N|^2 \right) dx_1 \dots dx_N \end{aligned}$$

We deduce

$$\mathrm{Tr} \left[(p^2 + V) \gamma_{N,+}^{(1)} \right] \leq C E_R^{\mathrm{af}} \quad (2.20)$$

where we denote the reduced k-body density matrix of $|\Psi_N\rangle$

$$\gamma_{N,+}^{(k)} = \mathrm{Tr}_{k+1 \rightarrow N} [|\Psi_N\rangle \langle \Psi_N|]$$

Step 2. We expand the Hamiltonian and use the Cauchy-Schwarz inequality for operators to get:

$$\begin{aligned} H_N^R &= \sum_{j=1}^N \left((p_j^A)^2 + \alpha p_j^A \cdot A_j^R + \alpha A_j^R \cdot p_j^A + \alpha^2 |A_j^R|^2 + V(x_j) \right) \\ &\geq \sum_{j=1}^N \left((1 - 2\delta^{-1}) (p_j^A)^2 + (1 - 2\delta) \alpha^2 |A_j^R|^2 + V(x_j) \right) \\ &= \sum_{j=1}^N \left(\frac{1}{2} ((p_j^A)^2 + V(x_j)) - 7 \frac{\beta^2}{(N-1)^2} |A_j^R|^2 \right) \end{aligned}$$

choosing $\delta = 4$. Thus using (2.19) we have:

$$\mathrm{Tr} \left[\left((p^A)^2 + V \right) \gamma_N^{(1)} \right] \leq C E_R^{\mathrm{af}} + \frac{C\beta}{N(N-1)^2} \left\langle \Psi_N, \sum_{j=1}^N |A_j^R|^2 \Psi_N \right\rangle \quad (2.21)$$

and since the second term is a purely potential term which does not see the phase:

$$\left\langle \Psi_N, \sum_{j=1}^N |A_j^R|^2 \Psi_N \right\rangle = \left\langle |\Psi_N|, \sum_{j=1}^N |A_j^R|^2 |\Psi_N| \right\rangle.$$

We can now expand the square and use (2.5) and (2.8) to obtain:

$$\begin{aligned} \frac{1}{N(N-1)^2} \left\langle |\Psi_N|, \sum_{j=1}^N |A_j^R|^2 |\Psi_N| \right\rangle &\leq C \mathrm{Tr} \left[\left(\nabla^\perp w_R(x_1 - x_2) \cdot \nabla^\perp w_R(x_1 - x_3) \right) \gamma_{N,+}^{(3)} \right] \\ &+ C N^{-1} \mathrm{Tr} \left[\left| \nabla^\perp w_R(x_1 - x_2) \right|^2 \gamma_{N,+}^{(2)} \right] \\ &\leq C \mathrm{Tr} \left[(p_1^2 + 1) \otimes \mathbb{1} \otimes \mathbb{1} \gamma_{N,+}^{(3)} \right] + C_\varepsilon R^{-\varepsilon} N^{-1} \mathrm{Tr} \left[(p_1^2 + 1) \otimes \mathbb{1} \gamma_{N,+}^{(2)} \right] \\ &\leq C(1 + C_\varepsilon R^{-\varepsilon} N^{-1}) \mathrm{Tr} \left[(p_1^2 + 1) \gamma_{N,+}^{(1)} \right] \end{aligned}$$

□

3. MEAN-FIELD LIMIT

We start the study of the mean-field limit. We use the same strategy as in [17] but another version of the quantum de Finetti theorem. This version comes from quantum information theory (see [15, 4]) and provides us with less restrictive constraint for the domain of validity of our main theorem but forces us to put all operators under a tensorized form. We mean: each N -particles operator Θ will be written under the form:

$$\Theta = \int \Theta_1(p) \otimes \Theta_2(p) \dots \otimes \Theta_N(p) dP(p) \quad (3.1)$$

where each Θ_i is a 1-particle operator.

The rest is similar to what is done in [17] combining with the methods of [24, Results and discussion].

3.1. Energy cut-off. We introduce the spectral projector P of the operator $h = (p^A)^2 + V$ below an energy cut-off Λ to be eventually optimized later.

$$P = \mathbb{1}_{h \leq \Lambda}, \quad Q = \mathbb{1} - P \quad (3.2)$$

We denote

$$N_\Lambda = \dim(PL^2(\mathbb{R}^2))$$

the number of energy levels obtained in this way and recall the following Cwikel-Lieb-Rozenblum type inequality proved in [13, Lemma 3.3](see [5, 26] for more details).

Lemma 3.1 (Number of energy levels below the cut-off).

For Λ large enough we have

$$N_\Lambda \leq C\Lambda^{1+2/s} \quad (3.3)$$

3.2. Localisation in Fock space. Let γ_N be an arbitrary N -body mixed state. Associated with the projector P , there is a localised state G_N^P on the Fock space

$$\mathcal{F}(\mathfrak{H}) = \mathbb{C} \oplus \mathfrak{H} \oplus \mathfrak{H}^2 \oplus \dots$$

with the property that its reduced density matrices satisfy

$$P^{\otimes n} \gamma_N^{(n)} P^{\otimes n} = (G_N^P)^{(n)} = \binom{N}{n}^{-1} \sum_{k=n}^N \binom{k}{n} \text{Tr} [G_{N,k}^P] \quad (3.4)$$

for any $0 \leq n \leq N$.

We use the convention that

$$\gamma_N^{(n)} = \text{Tr}_{n+1 \rightarrow N} [\gamma_N]$$

The localized state G_N^Q corresponding to the projector Q is defined similarly. The relations (3.4) determine the localized states G_N^P and G_N^Q uniquely ensuring they are mixed states on the projected Fock spaces $\mathcal{F}(P\mathfrak{H})$ and $\mathcal{F}(Q\mathfrak{H})$, respectively:

$$\sum_{k=0}^N \text{Tr} [G_{N,k}^{P/Q}] = 1 \quad (3.5)$$

The precise construction can be found in [23, Chapter 5]. We will now apply the de Finetti representation of projected density matrices of [24, Proposition 3.2] in order to approximate its three-body density matrix. This inequality combine Fock-space localisation and the

information-theoretic quantum de Finetti theorem ([24, Appendix A]) as we can find in the proof of [24, Proposition 3.2].

3.3. Quantum de Finetti theorem. We call \mathfrak{S}^1 the set of trace-class operators.

Theorem 3.2 (de Finetti representation of projected density matrices).

Let \mathfrak{H} be a complex separable Hilbert space with the corresponding bosonic space $\mathfrak{H}_N = \mathfrak{H}^{\otimes_{\text{sym}} N}$. Let $\gamma_N^{(3)}$ be the 3-body reduced density matrix of a N -body state vector $\Psi_N \in \mathfrak{H}_N$ or a general mixed state in:

$$\mathcal{S}(L_{\text{sym}}^2(\mathbb{R}^{2N})) = \{\Gamma \in \mathfrak{S}^1(L_{\text{sym}}^2(\mathbb{R}^{2N})), \Gamma = \Gamma^*, \Gamma \geq 0, \text{Tr } \Gamma = 1\}$$

Let P be a finite dimensional orthogonal projector.

There exists a positive Borel measure $\mu_N^{(3)}$ on the set of one-body mixed states $S_P = \mathcal{S}(PL_{\text{sym}}^2(\mathbb{R}^{2N}))$ such that for all A, B and C hermitian operators:

$$\sup_{A,B,C} \text{Tr} \left| A \otimes B \otimes C \left(P^{\otimes 3} \gamma_N^{(3)} P^{\otimes 3} - \int_{S_P} \gamma^{\otimes 3} d\mu_N^{(3)}(\gamma) \right) \right| \leq C \sqrt{\frac{\log(\dim(P))}{N}} \|A\| \|B\| \|C\| \quad (3.6)$$

where the norm $\|A\|_{\text{op}}$ is the operator norm defined as usual by: $\sup_{\|u\|=1} \|Au\|_{L^2(\mathbb{R}^2)}$.

The theorem gives us a better dependency in $\dim(P)$. This comes at the cost of the measure charging general mixed states $\gamma^{\otimes 3}$ instead of $(|u\rangle\langle u|)^{\otimes 3}$. Comparing to [24, Proposition 3.2] we extend the theorem to all hermitian operators.

Proof. To begin, we call

$$\Delta\gamma^{(3)} = \left(P^{\otimes 3} \gamma_N^{(3)} P^{\otimes 3} - \int \gamma^{\otimes 3} d\mu_N^{(3)}(\gamma) \right)$$

where the measure μ_N is defined as in [24] and $A = A^+ + A^- = \Pi_{A \geq 0} A + \Pi_{A \leq 0} A$ with Π an orthogonal projector acting on $P\mathfrak{H}$ defined by the projection on the positive and negative part of A (recall that this operator describes an observables and is thus hermitian). The first term is then

$$\begin{aligned} & \sup_{A,B,C} \text{Tr} \left| A \otimes B \otimes C \left(\Delta\gamma^{(3)} \right) \right| \\ &= \sup_{A,B,C} \text{Tr} \left| \sum_{(i,j,k) \in \{+,-\}} \|A^i\| \|B^j\| \|C^k\| \frac{A^i \otimes B^j \otimes C^k (\Delta\gamma^{(3)})}{\|A^i\| \|B^j\| \|C^k\|} \right| \\ &\leq C \sqrt{\frac{\log(\dim(P))}{N}} \sum_{(i,j,k) \in \{+,-\}} \|A^i\| \|B^j\| \|C^k\| \end{aligned}$$

using the fact that $\frac{A^i}{\|A^i\|}$ is a bounded by 1 operator which allows us to use [24, Theorem A.3] with the method of [24, Proposition 3.2] but considering three operators instead and using that $\|A^+\| \leq \|A\|$, $\|A^-\| \leq \|A\|$. \square

3.4. Truncated Hamiltonian. Our aim is now to find an energy lower bound to H_N^R . We introduce the effective three-body Hamiltonian \tilde{H}_3^R such that for $\|u\| = 1$:

$$\langle u^{\otimes 3}, \tilde{H}_3^R u^{\otimes 3} \rangle = \mathcal{E}_R^{\text{af}}[u] \geq E_R^{\text{af}}$$

and denote

$$\text{Tr} \left[\tilde{H}_3^R \gamma^{\otimes 3} \right] = \mathcal{E}_R^{\text{af}}[\gamma].$$

Posing

$$h = (p^A)^2 + V \tag{3.7}$$

we note:

$$\begin{aligned} \tilde{H}_3^R &= \frac{1}{3}(h_1 + h_2 + h_3) \\ &+ \frac{\beta}{6} \sum_{1 \leq j \neq k \leq 3} \left(p_j^A \cdot \nabla^\perp w_R(x_j - x_k) + \nabla^\perp w_R(x_j - x_k) \cdot p_j^A \right) \\ &+ \beta^2 \nabla^\perp w_R(x_1 - x_2) \cdot \nabla^\perp w_R(x_1 - x_3) \end{aligned} \tag{3.8}$$

We denote for shortness

$$W_2 = p_1 \cdot \nabla^\perp w_R(x_1 - x_2) + \nabla^\perp w_R(x_1 - x_2) \cdot p_1 \tag{3.9}$$

the two-body part and

$$W_3 = \nabla^\perp w_R(x_1 - x_2) \cdot \nabla^\perp w_R(x_1 - x_3) \tag{3.10}$$

the three-body part. Thus

$$\tilde{H}_3^R = \frac{1}{3}(h_1 + h_2 + h_3) + \frac{\beta}{6} \sum_{1 \leq j \neq k \leq 3} W_2(j, k) + \beta^2 W_3$$

We now bound the full energy from below in terms of a projected version of \tilde{H}_3^R .

Lemma 3.3 (Truncated three-body Hamiltonian).

Let Ψ_N be a sequence of approximate ground states for H_N^R with associated reduced matrices $\gamma_N^{(k)}$. Then for $\varepsilon > 0$ and R small enough:

$$N^{-1} \langle \Psi_N, H_N^R \Psi_N \rangle \geq \text{Tr} \left[\tilde{H}_3^R P^{\otimes 3} \gamma_N^{(3)} P^{\otimes 3} \right] + C \text{Tr} \left[h Q \gamma_N^{(1)} Q \right] - C_\beta \left(\frac{C_\varepsilon R^{-\varepsilon}}{\sqrt{\Lambda} R} + \frac{C}{\Lambda R^2} \right) - C_\beta N^{-1} \tag{3.11}$$

Proof. The proof is quite the same as in [17, Proposition 3.3]. We just have to use our new bounds (2.4), (2.7), (2.9) and Proposition 2.6 instead of those of the original paper. \square

3.5. Application of the de Finetti theorem. We need to apply the de Finetti Theorem 3.2 to the first term of (3.11) in order to get our energy

$$\int_{S_P} \text{Tr}[\tilde{H}_3^R \gamma^{\otimes 3}] d\mu_N^{(3)}(\gamma)$$

with a controlled error.

Lemma 3.4 (Error for the truncated Hamiltonian).

$$\mathrm{Tr} \left[\tilde{H}_3^R P^{\otimes 3} \gamma_N^{(3)} P^{\otimes 3} \right] \geq \int_{S_P} \mathcal{E}_R^{\mathrm{af}}[\gamma] d\mu_N^{(3)}(\gamma) - \frac{C_\beta}{R^2} \left(R^2 \Lambda + R\sqrt{\Lambda} + 1 \right) \sqrt{\frac{\log N_\Lambda}{N}} \quad (3.12)$$

Proof. The only thing we have to do to apply Theorem 3.2 is to put our operators under a tensorized form (3.1).

Step 1. Our first operator is h_i defined in (3.7). As it is a one particle observable its tensorized form is simply

$$h_1 \otimes \mathbb{1} \otimes \mathbb{1}$$

if $i = 1$.

Step 2. To put W_2 defined in (3.9) in the form (3.1) we will use the Fourier transform (FT[.]) of $\nabla^\perp w_R(x)$. We can then write

$$W_2 = -2\pi i \int \hat{\chi}_R \frac{\mathbf{e}_{p^\perp}}{|p|} e^{-ip \cdot x_2} \mathbf{p}_1^A e^{ip \cdot x_1} dp - 2\pi i \int \hat{\chi}_R \frac{\mathbf{e}_{p^\perp}}{|p|} e^{-ip \cdot x_2} e^{ip \cdot x_1} \mathbf{p}_1^A dp$$

where $\hat{\chi}_R = \mathrm{FT}[\chi_R]$ and with \mathbf{e}_{p^\perp} being the direction of the vector p^\perp such that $\|\mathbf{e}_{p^\perp}\| = 1$. In order to bring together the two parts of W_2 we note that

$$\mathbf{p}_1^A e^{ip \cdot x_1} \Psi = (\mathbf{p}_1 + \mathbf{A}_e) e^{ip \cdot x_1} \Psi = (i\mathbf{p} e^{ip \cdot x_1} + e^{ip \cdot x_1} \mathbf{p}_1^A) \Psi$$

and obtain

$$W_2 = -2i\pi \int \hat{\chi}_R(p) \frac{\mathbf{e}_{p^\perp}}{|p|} e^{-ip \cdot x_2} e^{ip \cdot x_1} \cdot [i\mathbf{p} + 2\mathbf{p}_1^A] dp \quad (3.13)$$

$$= -4i\pi \int \frac{\hat{\chi}_R(p)}{|p|} e^{-ip \cdot x_2} \otimes e^{ip \cdot x_1} \mathbf{e}_{p^\perp} \cdot \mathbf{p}_1^A \otimes \mathbb{1} dp \quad (3.14)$$

because $\mathbf{e}_{p^\perp} \cdot \mathbf{p} = 0$. The last problem is that e^{ipx} is a complex number and we need hermitian operators. We rewrite

$$ie^{-ip \cdot x_2} e^{ip \cdot x_1} = i[\cos(p \cdot x_1) + i \sin(p \cdot x_1)][\cos(p \cdot x_2) - i \sin(p \cdot x_2)] \quad (3.15)$$

$$= i[\cos(p \cdot x_1) \cos(p \cdot x_2) + \sin(p \cdot x_1) \sin(p \cdot x_2)] \quad (3.16)$$

$$+ [\sin(p \cdot x_2) \cos(p \cdot x_1) - \sin(p \cdot x_1) \cos(p \cdot x_2)] \quad (3.17)$$

Here the imaginary part is an even function whereas $\hat{\chi}_R(p) \frac{\mathbf{e}_{p^\perp}}{|p|}$ is odd. Consequently this part of the integral gives zero and we get:

$$W_2 = 4\pi \int \frac{\hat{\chi}_R}{|p|} (\cos(p \cdot x_2) \otimes \sin(p \cdot x_1) - \sin(p \cdot x_2) \otimes \cos(p \cdot x_1)) \mathbf{e}_{p^\perp} \cdot \mathbf{p}_1^A \otimes \mathbb{1} dp. \quad (3.18)$$

Step 3. We now consider the three-body operator W_3 defined in (3.10).

$$W_3 = \int \hat{\chi}_R(p) \frac{\mathbf{e}_{p^\perp}}{|p|} ie^{ip(x_1-x_2)} dp \cdot \int \hat{\chi}_R(p') \frac{\mathbf{e}_{p'^\perp}}{|p'|} ie^{ip'(x_1-x_3)} dp'$$

Then, rewriting the exponential term as in (3.17) we obtain the tensorized form:

$$\begin{aligned}
W_3 = & \int \widehat{\chi}_R(p) \widehat{\chi}_R(p') \frac{\mathbf{e}_{p^\perp}}{|p|} \cdot \frac{\mathbf{e}_{p'^\perp}}{|p'|} \\
& \left[\cos(p.x_1) \cos(p'.x_1) \otimes \sin(p.x_2) \otimes \sin(p'.x_3) \right. \\
& + \sin(p.x_1) \sin(p'.x_1) \otimes \cos(p.x_2) \otimes \cos(p'.x_3) \\
& - \cos(p.x_1) \sin(p'.x_1) \otimes \sin(p.x_2) \otimes \cos(p'.x_3) \\
& \left. - \sin(p.x_1) \sin(p'.x_1) \otimes \cos(p.x_2) \otimes \cos(p'.x_3) \right] dp dp'
\end{aligned}$$

Step 4. All that is left is to compute the entire Hamiltonian and to apply (p being fixed) Theorem 3.2. Thus we get

$$\left| \text{Tr} \left[\frac{1}{3} \sum_{i=1}^3 P^{\otimes 3} \mathbb{1} \otimes h_i \otimes \mathbb{1} P^{\otimes 3} \left(\Delta \gamma^{(3)} \right) \right] \right| \leq C \sqrt{\frac{\log N_\Lambda}{N}} \|PhP\| \leq C\Lambda \sqrt{\frac{\log N_\Lambda}{N}} \quad (3.19)$$

Recalling that

$$\Delta \gamma^{(3)} = \left(P^{\otimes 3} \gamma_N^{(3)} P^{\otimes 3} - \int \gamma^{\otimes 3} d\mu_N^{(3)}(\gamma) \right)$$

We do the same for W_2 and W_3 using that $\|\cos(x.p)\|_{\text{op}} \leq 1$ as for the sine, and the triangle inequality

$$\begin{aligned}
\left| \text{Tr} \left[\frac{\beta}{6} \sum_{1 \leq j \neq k \leq 3} P^{\otimes 3} W_2(i,j) \otimes \mathbb{1} P^{\otimes 3} \left(\Delta \gamma^{(3)} \right) \right] \right| & \leq C_\beta \sqrt{\frac{\log N_\Lambda}{N}} \|P \mathbf{e}_{p^\perp} \cdot \mathbf{p}^A P\| \int \left| \frac{\widehat{\chi}_R(p)}{p} \right| dp \\
& \leq \frac{C_\beta \sqrt{\Lambda}}{R} \sqrt{\frac{\log N_\Lambda}{N}} \quad (3.20)
\end{aligned}$$

where we used

$$\widehat{\chi}_R(p) = \widehat{\chi}(Rp)$$

and for the third term

$$\left| \text{Tr} \left[\beta^2 P^{\otimes 3} W_3 P^{\otimes 3} \left(\Delta \gamma^{(3)} \right) \right] \right| \leq C_\beta \sqrt{\frac{\log N_\Lambda}{N}} \int \left| \frac{\widehat{\chi}_R(p)}{p} \frac{\widehat{\chi}_R(p')}{p'} \right| dp dp' \leq \frac{C_\beta}{R^2} \sqrt{\frac{\log N_\Lambda}{N}} \quad (3.21)$$

We now combine (3.19), (3.20) and (3.21) to get (3.12) by the triangle inequality. \square

3.6. Energy bound. As we did in (2.19) we test (A.6) against a factorized trial state $\Psi_N = (u_R^{\text{af}})^{\otimes N}$ such as $\mathcal{E}_R^{\text{af}}[\Psi_N] = E_R^{\text{af}}$ to get an energy upper bound.

$$\frac{E^R(N)}{N} \leq \mathcal{E}_R^{\text{af}}[u_R^{\text{af}}] + o(1) \rightarrow E^{\text{af}} \quad (3.22)$$

as $R \sim N^{-\eta}$ with $N \rightarrow \infty$ we also used (A.5) to get $E_R^{\text{af}} \rightarrow E^{\text{af}}$ when $R \rightarrow 0$.

We derive the lower bound by using the quantitative quantum de Finetti for localized states Theorem 3.2 on the first term of the estimate (3.11) and we get for any sequence of ground states Ψ_N that

$$\begin{aligned} N^{-1} \langle \Psi_N, H_N^R \Psi_N \rangle &\geq \int_{S_P} \mathcal{E}_R^{\text{af}}[\gamma] d\mu_N^{(3)}(\gamma) + C\Lambda \text{Tr} [Q\gamma_N^1] \\ &\quad - \frac{C_\beta}{R^2} \left(R^2\Lambda + R\sqrt{\Lambda} + 1 \right) \sqrt{\frac{\log N_\Lambda}{N}} \\ &\quad - C_\beta \left(\frac{1}{N} + \frac{C_\varepsilon R^{-\varepsilon}}{\sqrt{\Lambda R}} + \frac{1}{\Lambda R^2} \right) \end{aligned} \tag{3.23}$$

To get this result we used (3.12) and will use (3.3) to bound N_Λ .

We now choose Λ to minimize the error in (3.11) with R behaving at worst as

$$R \sim N^{-\eta}$$

Changing a little bit η to $\eta - \varepsilon$ we can ignore the R^ε factor. Our choice of Λ has to minimize

$$\left(\frac{C_\varepsilon}{\sqrt{\Lambda R}} + \frac{1}{\Lambda R^2} \right) + \frac{1}{\sqrt{N}} \left(\Lambda + \frac{\sqrt{\Lambda}}{R} + \frac{1}{R^2} \right)$$

We see that the limiting terms are $\frac{\Lambda}{\sqrt{N}}$ and $\frac{1}{\Lambda R^2}$. We pick $\Lambda = N^{2\eta}$ to minimize their sum, with

$$\eta < \frac{1}{4}$$

Under this assumption we can get rid of the $C\text{Tr} [Q\gamma_N^1]$ when $\Lambda \rightarrow \infty$. Indeed, we claim that:

$$\mathcal{E}_R^{\text{af}}[\gamma] \geq 0 \tag{3.24}$$

Then it follows from (3.23) that for a constant C

$$C \geq \Lambda \text{Tr} [Q\gamma_N^{(1)}] - C_\eta o_N(1)$$

and the only way the last inequality remains true when $\Lambda \rightarrow \infty$ is

$$\text{Tr} [Q\gamma_N^{(1)}] \xrightarrow{N \rightarrow \infty} 0 \tag{3.25}$$

To see that (3.24) holds and because γ can be a mixed state it is convenient to write its kernel

$$\gamma(x; y) = \sum_j \lambda_j u_j(x) \overline{u_j(y)}, \quad \lambda_i \geq 0$$

Then

$$\begin{aligned}
\mathrm{Tr} \left[\tilde{H}_3^R \gamma^{\otimes 3} \right] &= \sum_i \lambda_i \int_{\mathbb{R}^2} h |u_i(x)|^2 dx + \beta \sum_{i,j} \lambda_i \lambda_j \iint_{\mathbb{R}^2 \times \mathbb{R}^2} W_2 |u_i(x)|^2 |u_j(y)|^2 dx dy \\
&\quad + \beta^2 \sum_{i,j,k} \lambda_i \lambda_j \lambda_k \iiint_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} W_3 |u_i(x)|^2 |u_j(y)|^2 |u_k(z)|^2 \\
&= \sum_i \lambda_i \int_{\mathbb{R}^2} \left(\left| \left(\nabla + i\mathbf{A}_e + i\beta \mathbf{A}^R \left[\sum_i \lambda_j |u_j|^2 \right] \right) u_i \right|^2 + V |u_i|^2 \right) \\
&\geq \sum_i \lambda_i \int |\nabla |u_i||^2 \geq 0
\end{aligned}$$

by (A.3). We conclude that:

$$N^{-1} \langle \Psi_N, H_N^R \Psi_N \rangle \geq \int_{S_P} \mathcal{E}_R^{\mathrm{af}}[\gamma] d\mu_N^{(3)}(\gamma) + o(1).$$

3.7. Convergence of states. We first prove that the de Finetti measures converge. Returning to the proof of [24, Proposition 3.2] we have that

$$\begin{aligned}
\int_{S_P} d\mu_N^{(3)}(\gamma) &= \mathrm{Tr} \left(P^{\otimes 3} \gamma_N^{(3)} P^{\otimes 3} \right) = \sum_{\ell=3}^N \mathrm{Tr} [G_{N,\ell}^P] \frac{\ell(\ell-1)(\ell-2)}{N(N-1)(N-2)} \\
&\geq \sum_{\ell=0}^N \mathrm{Tr} [G_{N,\ell}^P] \frac{\ell^3}{N^3} - \frac{C}{N} \geq \left(\sum_{\ell=0}^N \mathrm{Tr} [G_{N,\ell}^P] \frac{\ell}{N} \right)^3 - \frac{C}{N} \\
&= \left(\mathrm{Tr} \left(P \gamma_N^{(1)} P \right) \right)^3 - \frac{C}{N}
\end{aligned} \tag{3.26}$$

where we used Jensen's inequality.

But (3.25) implies

$$\mathrm{Tr} \left(P \gamma_N^{(1)} P \right) \xrightarrow{N \rightarrow \infty} 1.$$

so

$$\int_{S_P} d\mu_N^{(3)}(\gamma) \xrightarrow{N \rightarrow \infty} 1.$$

Thus the sequence $(\mu_N^{(3)})_N$ of measures given by Theorem (3.2) is tight on the set of one-body mixed states

$$S_P = S(PL^2(\mathbb{R}^2))$$

and modulo a (not-reabeled) subsequence $(\mu_N^{(3)})_N$ converges to a probability measure μ by a tightness argument

$$\mu_N^{(3)} \rightarrow \mu. \tag{3.27}$$

Convergence of reduced density matrices.

Given the previous constructions and energy estimates, the proof of (1.19) follows almost exactly the one used in [24].

From Proposition 2.6 we know that $\left((p^A)^2 + V\right) \gamma_N^{(1)}$ is uniformly bounded in trace-class.

Under our assumptions, $\left((p^A)^2 + V\right)^{-1}$ is compact (see [2, Theorem 2.7]) and we may thus, modulo a further extraction, assume that

$$\gamma_N^{(1)} \xrightarrow{N \rightarrow \infty} \gamma^{(1)}$$

strongly in trace-class norm. Modulo extraction of subsequences we have

$$\gamma_N^{(k)} \rightharpoonup_* \gamma^{(k)}$$

weakly-* in the trace-class as $N \rightarrow \infty$. So by [25, Corollary 2.4] we have

$$\gamma_N^{(k)} \rightarrow \gamma^{(k)}$$

strongly in class-trace norm for $k \geq 0$.

Applying the weak quantum de Finetti theorem [14, Theorem 2.2] we deduce that there exists a measure ν on the unit ball of $L^2(\mathbb{R}^2)$ such as

$$\gamma^{(k)} = \int |u^{\otimes k}\rangle\langle u^{\otimes k}| d\nu(u).$$

But since $\gamma^{(1)}$ must have trace 1, the measure ν must actually live on

$$SL^2(\mathbb{R}^d) = \left\{ u \in L^2(\mathbb{R}^2), \int_{\mathbb{R}^2} |u|^2 = 1 \right\},$$

the unit sphere of $L^2(\mathbb{R}^2)$.

Next we claim that the two measures μ (of (3.27)) and ν just found are related by

$$\int |u^{\otimes 3}\rangle\langle u^{\otimes 3}| d\nu(u) = \int |u^{\otimes 3}\rangle\langle u^{\otimes 3}| d\mu(u). \quad (3.28)$$

Indeed, let

$$\tilde{P} = \mathbb{1}_{h \leq \tilde{\Lambda}}$$

where $\tilde{\Lambda}$ is a fixed cut-off (different from Λ above). Testing (3.6) with A_1, A_2, A_3 finite rank operators whose ranges lie within that of \tilde{P} we get

$$\mathrm{Tr} \left(A_1 \otimes A_2 \otimes A_3 \gamma_N^{(3)} \right) \xrightarrow{N \rightarrow \infty} \mathrm{Tr} \left(A_1 \otimes A_2 \otimes A_3 \int_{S_{\tilde{P}}} \gamma^{\otimes 3} d\mu(\gamma) \right)$$

using the convergence of $\mu_N^{(3)}$ to μ . On the other hand, by the convergence of $\gamma_N^{(3)}$ to $\gamma^{(3)}$ we also have

$$\begin{aligned} & \mathrm{Tr} \left(A_1 \otimes A_2 \otimes A_3 \gamma_N^{(3)} \right) \xrightarrow{N \rightarrow \infty} \mathrm{Tr} \left(A_1 \otimes A_2 \otimes A_3 \gamma^{(3)} \right) \\ & = \mathrm{Tr} \left(A_1 \otimes A_2 \otimes A_3 \int_{SL^2(\mathbb{R}^2)} |u^{\otimes 3}\rangle\langle u^{\otimes 3}| d\nu(u) \right) \end{aligned}$$

Thus

$$\mathrm{Tr} \left(A_1 \otimes A_2 \otimes A_3 \int_{\mathcal{S}_{\tilde{P}}} \gamma^{\otimes 3} d\mu(\gamma) \right) = \mathrm{Tr} \left(A_1 \otimes A_2 \otimes A_3 \int_{SL^2(\mathbb{R}^2)} |u^{\otimes 3}\rangle\langle u^{\otimes 3}| d\nu(u) \right) \quad (3.29)$$

for any A_1, A_2, A_3 with range within that of \tilde{P} . Letting finally $\tilde{\Lambda} \rightarrow \infty$ yields $\tilde{P} \rightarrow \mathbb{1}$ and thus (3.29) holds for any compact operators A_1, A_2, A_3 . This implies (3.28). In particular, since the left-hand side of (3.28) is $\gamma^{(3)}$, a bosonic operator, μ must be supported on pure states $\gamma = |u\rangle\langle u|$, see [11].

Final passage to the liminf. Let us return to (3.23). We split the integral over one-body states γ between low and high kinetic energy states:

$$\mathcal{S}^- = \{\gamma \in \mathcal{S}, \mathrm{Tr}(h\gamma) \leq L\}, \quad \mathcal{S}^+ = \mathcal{S} \setminus \mathcal{S}^-.$$

Using Lemma 2.6 we obtain

$$\begin{aligned} \int_{\mathcal{S}_P} \mathcal{E}_R^{\mathrm{af}}[\gamma] d\mu_N^{(3)}(\gamma) &\geq L \int_{\mathcal{S}^+} d\mu_N^{(3)}(\gamma) + \int_{\mathcal{S}^-} \mathcal{E}_R^{\mathrm{af}}[\gamma] d\mu_N^{(3)}(\gamma) - o_N(1) \\ &\geq \int_{\mathcal{S}_P} \min(L, \mathcal{E}_R^{\mathrm{af}}[\gamma]) d\mu_N^{(3)}(\gamma) - o_N(1) \end{aligned}$$

Inserting in (3.23) and passing to the liminf in $N \rightarrow \infty$ this implies

$$E^{\mathrm{af}} \geq \liminf_{N \rightarrow \infty} \frac{E(N)}{N} \geq \int_{\mathcal{S}} \min(L, \mathcal{E}^{\mathrm{af}}[\gamma]) d\mu(\gamma).$$

Finally, we pass to the limit $L \rightarrow \infty$ to deduce

$$E^{\mathrm{af}} \geq \liminf_{N \rightarrow \infty} \frac{E(N)}{N} \geq \int_{\mathcal{S}} \mathcal{E}^{\mathrm{af}}[\gamma] d\mu(\gamma). \quad (3.30)$$

But as we saw above μ must be supported on pure states $\gamma = |u\rangle\langle u|$ which yields both the energy lower bound concluding the proof of (1.18) and the fact that μ must be supported on $\mathcal{M}^{\mathrm{af}}$. Because $\mathcal{E}^{\mathrm{af}}[\gamma]$ is a linear function of $\gamma^{\otimes 3}$ we can also combine (3.30) with (3.28) to deduce that also ν must be supported on $\mathcal{M}^{\mathrm{af}}$, which proves (1.19) and

$$E^{\mathrm{af}} \geq \liminf_{N \rightarrow \infty} \frac{E(N)}{N} \geq E^{\mathrm{af}}.$$

APPENDIX A. PROPERTIES OF THE AVERAGE-FIELD FUNCTIONAL

In this appendix we establish some of the fundamental properties of the functional $\mathcal{E}_R^{\mathrm{af}}$ and its limit $R \rightarrow 0$. We call $\nabla_{A_e} = \nabla + i\mathbf{A}_e$ the covariant derivative. For $\beta \in \mathbb{R}$, $\mathbf{A}_e \in L_{loc}^2(\mathbb{R}^2)$ and $V : \mathbb{R}^2 \rightarrow \mathbb{R}^+$ we define the average-field energy functional

$$\mathcal{E}^{\mathrm{af}}[u] := \int_{\mathbb{R}^2} \left(|(\nabla_{A_e} + i\beta\mathbf{A}[|u|^2])u|^2 + V|u|^2 \right), \quad (\text{A.1})$$

with the self-generated magnetic potential

$$\mathbf{A}[\rho] := \nabla^\perp w_0 * \rho = \int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} \rho(y) dy, \quad \mathrm{curl} \mathbf{A}[\rho] = 2\pi\rho.$$

Lemma A.1 (Bound on the magnetic term).

We have for any $u \in L^2(\mathbb{R}^2)$ that

$$\int_{\mathbb{R}^2} |\mathbf{A}[|u|^2]|^2 |u|^2 \leq \frac{3}{2} \|u\|_{L^2(\mathbb{R}^2)}^4 \int_{\mathbb{R}^2} |\nabla|u||^2 \leq \frac{3}{2} \|u\|_{L^2(\mathbb{R}^2)}^4 \int_{\mathbb{R}^2} |\nabla_{A_e}|u||^2$$

Proof. See [17, Lemma A.1]; □

Using the same reasoning than in [17, Equation A2] we can, using (A.1) define the domain of \mathcal{E}^{af} to be (see [16][Proposition 7.20]) and otherwise let $\mathcal{E}^{\text{af}}[u] := +\infty$

$$\mathcal{D}^{\text{af}} := \left\{ u \in H_{A_e}^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V|u|^2 < \infty \right\},$$

We find using Cauchy-Schwarz, Lemma A.1, and $|\nabla_{A_e}|u|| \leq |\nabla_{A_e} u|$ (as multiplier operator A_e does not see the phase) that for $u \in \mathcal{D}^{\text{af}}$

$$\begin{aligned} 0 \leq \mathcal{E}^{\text{af}}[u] &\leq 2\|\nabla_{A_e} u\|^2 + 2\|A_e u\|^2 + 3\beta^2 \|\mathbf{A}[|u|^2]u\|^2 + \int V|u|^2 \\ &\leq (2 + 3\beta^2 \|u\|^4) \|\nabla_{A_e} u\|^2 + C \int V|u|^2 < \infty. \end{aligned}$$

The ground-state energy of the average-field functional is then given by

$$E^{\text{af}} := \inf \left\{ \mathcal{E}^{\text{af}}[u] : u \in \mathcal{D}^{\text{af}}, \int_{\mathbb{R}^2} |u|^2 = 1 \right\}.$$

For convenience we also make the assumption on V that $V(x) \rightarrow +\infty$ as $|x| \rightarrow \infty$ and that $C_c^\infty(\mathbb{R}^2) \subseteq \mathcal{D}^{\text{af}}$ is a form core (algebraic interior) for $\|u\|_{L_V^2}^2 := \int_{\mathbb{R}^2} V|u|^2$, with $(p^A)^2 + V$ essentially self-adjoint (symmetric, densely-defined with an unique self-adjoint extension) and with purely discrete spectrum (see, e.g., [2, Theorem 2.7]). This is then also a core for \mathcal{E}^{af} .

Lemma A.2 (Basic magnetic inequalities).

We have for $u \in \mathcal{D}^{\text{af}}$ that (diamagnetic inequality)

$$\int_{\mathbb{R}^2} |(\nabla + i\mathbf{A}_e)u|^2 \geq \int_{\mathbb{R}^2} |\nabla|u||^2 \tag{A.2}$$

$$\int_{\mathbb{R}^2} |(\nabla + i\mathbf{A}_e + i\beta\mathbf{A}[|u|^2])u|^2 \geq \int_{\mathbb{R}^2} |\nabla|u||^2, \tag{A.3}$$

and

$$\int_{\mathbb{R}^2} |(\nabla + i\mathbf{A}_e + i\beta\mathbf{A}[|u|^2])u|^2 \geq \int_{\mathbb{R}^2} \mathbf{B}|u|^2 + 2\pi|\beta| \int_{\mathbb{R}^2} |u|^4. \tag{A.4}$$

Proof. The first is the usual diamagnetic inequality (see e.g. Theorem 2.1.1 in [8]). By density we can w.l.o.g. assume $u \in C_c^\infty(\mathbb{R}^2)$. We then have $\mathbf{A}[|u|^2] \in C^\infty(\mathbb{R}^2) \subseteq L_{\text{loc}}^2(\mathbb{R}^2)$ and hence the second inequality follows by the usual diamagnetic inequality. Furthermore, by e.g. Lemma 1.4.1 in [8],

$$\int_{\mathbb{R}^2} |(\nabla_{A_e} + i\beta\mathbf{A}[|u|^2])u|^2 \geq \pm \int_{\mathbb{R}^2} \text{curl}(\mathbf{A}_e + \beta\mathbf{A}[|u|^2])|u|^2,$$

which proves the second inequality since $\text{curl}\mathbf{A}[|u|^2] = 2\pi|u|^2$ and $\text{curl}\mathbf{A}_e = \mathbf{B}$. □

Proposition A.3 (Existence of minimizers).

For any value of $\beta \in \mathbb{R}$ there exists $u^{\text{af}} \in \mathcal{D}^{\text{af}}$ with $\int_{\mathbb{R}^2} |u^{\text{af}}|^2 = 1$ and $\mathcal{E}^{\text{af}}[u^{\text{af}}] = E^{\text{af}}$.

Proof. First note that for $u \in \mathcal{D}^{\text{af}}$, by Lemma A.1 and Lemma A.2,

$$\begin{aligned} \|\nabla_{A_e} u\|_2 &= \|\nabla_{A_e} u + i\beta \mathbf{A}[|u|^2]u - i\beta \mathbf{A}[|u|^2]u\|_2 \leq \mathcal{E}^{\text{af}}[u]^{1/2} + |\beta| \|\mathbf{A}[|u|^2]u\|_2 \\ &\leq \mathcal{E}^{\text{af}}[u]^{1/2} + |\beta| \sqrt{\frac{3}{2}} \|u\|_2^2 \|\nabla|u|\|_2 \leq \left(1 + |\beta| \sqrt{\frac{3}{2}} \|u\|_2^2\right) \mathcal{E}^{\text{af}}[u]^{1/2}. \end{aligned}$$

Now take a minimizing sequence

$$(u_n)_{n \rightarrow \infty} \subset \mathcal{D}^{\text{af}}, \quad \|u_n\|_2 = 1, \quad \lim_{n \rightarrow \infty} \mathcal{E}^{\text{af}}[u_n] = E^{\text{af}}.$$

Then clearly (u_n) is uniformly bounded in both $L^2(\mathbb{R}^2)$, L_V^2 , and $H_{A_e}^1(\mathbb{R}^2)$ and therefore there exists $u^{\text{af}} \in \mathcal{D}^{\text{af}}$ and a weakly convergent subsequence (still denoted u_n) such that

$$u_n \rightharpoonup u^{\text{af}} \text{ in } L^2(\mathbb{R}^2) \cap L_V^2 \cap H_{A_e}^1(\mathbb{R}^2), \quad \nabla_{A_e} u_n \rightharpoonup \nabla_{A_e} u^{\text{af}} \text{ in } L^2(\mathbb{R}^2).$$

Moreover, since $((p^A)^2 + V + 1)^{-1/2}$ is compact we have that

$$u_n = \left((p^A)^2 + V + 1\right)^{-1/2} \left((p^A)^2 + V + 1\right)^{1/2} u_n$$

is actually strongly convergent (again, extracting a subsequence), hence

$$u_n \rightarrow u^{\text{af}} \text{ in } L^2(\mathbb{R}^2).$$

It then follows that

$$\|(\nabla_{A_e} + i\beta \mathbf{A}[|u|^2])u\|_2^2 = \langle (\nabla_{A_e} + i\beta \mathbf{A}[|u|^2])u, (\nabla_{A_e} + i\beta \mathbf{A}[|u|^2])u \rangle$$

with

$$\begin{aligned} \liminf_{n \rightarrow \infty} \langle \nabla_{A_e} u_n, \nabla_{A_e} u_n \rangle &\geq \int |\nabla_{A_e} u|^2 \\ \lim_{n \rightarrow \infty} \langle \nabla_{A_e} u_n, \mathbf{A}[|u_n|^2]u_n \rangle &= \langle \nabla_{A_e} u, \mathbf{A}[|u|^2]u \rangle \\ \lim_{n \rightarrow \infty} \langle \mathbf{A}[|u_n|^2]u_n, \mathbf{A}[|u_n|^2]u_n \rangle &= \int |\mathbf{A}[|u|^2]u|^2 \end{aligned}$$

by the weak lower semicontinuity of the norm, the weak-strong convergence and the fact that the functions $\mathbf{A}[|u_n|^2]u_n$ are strongly converging to $\mathbf{A}[|u|^2]u$ in $L^2(\mathbb{R}^2)$, see [17, Proposition A.3]. And since $\|\cdot\|_{L_V^2}$ is also weakly lower semicontinuous (see, e.g., [22, Supplement to IV.5]), we have $\liminf_{n \rightarrow \infty} \mathcal{E}^{\text{af}}[u_n] \geq \mathcal{E}^{\text{af}}[u^{\text{af}}]$. Thus, with $\|u^{\text{af}}\| = \lim_{n \rightarrow \infty} \|u_n\| = 1$, we also have $\mathcal{E}^{\text{af}}[u^{\text{af}}] = E^{\text{af}}$. \square

Proposition A.4 (Regularized functional).

Let us now consider the corresponding situation for the regularized functional (extended anyons)

$$\mathcal{E}_R^{\text{af}}[u] := \int_{\mathbb{R}^2} \left(|(\nabla + i\mathbf{A}_e + i\beta \mathbf{A}^R[|u|^2])u|^2 + V|u|^2 \right), \quad \mathbf{A}^R[\rho] := \nabla^\perp w_R * \rho, \quad R > 0.$$

Since $\nabla w_R \in L^\infty(\mathbb{R}^2)$ we have $\mathbf{A}^R[|u|^2] \in L^\infty(\mathbb{R}^2)$ with

$$\|\mathbf{A}^R[|u|^2]\|_\infty \leq \frac{C}{R} \|u\|_2^2$$

and instead of Lemma A.1 we have

$$\|\mathbf{A}^R[|u|^2]u\|_2 \leq C \|u\|_2^2 \|u\|_{H_{A_e}^1}$$

using Lemma 2.4.

Hence the natural domain is again \mathcal{D}^{af} and all properties established above for \mathcal{E}^{af} are also found to be valid for $\mathcal{E}_R^{\text{af}}$ (except (A.4) which now has regularized versions). Denoting

$$E_R^{\text{af}} := \min\{\mathcal{E}_R^{\text{af}}[u] : u \in \mathcal{D}^{\text{af}}, \|u\|_2 = 1\},$$

we furthermore have the following relationship:

Proposition A.5 (Convergence to point-like anyons).

The functional $\mathcal{E}_R^{\text{af}}$ converges pointwise to \mathcal{E}^{af} as $R \rightarrow 0$. More precisely, for any $u \in \mathcal{D}^{\text{af}}$

$$\left| \mathcal{E}_R^{\text{af}}[u] - \mathcal{E}^{\text{af}}[u] \right| \leq C_u \beta^2 \mathcal{E}^{\text{af}}[u]^{1/2} (1 + \mathcal{E}^{\text{af}}[u])^{1/2} R, \quad (\text{A.5})$$

where C_u depends only on $\|u\|_2$. Hence,

$$E_R^{\text{af}} \xrightarrow{R \rightarrow 0} E^{\text{af}},$$

and if $(u_R)_{R \rightarrow 0} \subset \mathcal{D}^{\text{af}}$ denotes a sequence of minimizers of $\mathcal{E}_R^{\text{af}}$, then there exists a subsequence $(u_{R'})_{R' \rightarrow 0}$ s.t. $u_{R'} \rightarrow u^{\text{af}}$ as $R' \rightarrow 0$, where u^{af} is some minimizer of \mathcal{E}^{af} .

Proof. This is an easy variant of the proof of [17, Proposition A.5]. \square

Proposition A.6 (Link with the regularized functional).

The Hamiltonian (1.7) can be expanded (with the notation $p_j + \mathbf{A}_e = p_j^A$) and corresponds to the energy:

$$\begin{aligned} N^{-1} \langle \Psi_N, H_R^N \Psi_N \rangle &= \text{Tr} \left[((p_j^A)^2 + V(x_j)) \gamma_N^{(1)} \right] \\ &+ \beta \text{Tr} \left[\left(p_1^A \cdot \nabla^\perp w_R(x_1 - x_2) + \nabla^\perp w_R(x_1 - x_2) \cdot p_1^A \right) \gamma_N^{(2)} \right] \\ &+ \beta^2 \frac{N-2}{N-1} \text{Tr} \left[\left(\nabla^\perp w_R(x_1 - x_2) \cdot \nabla^\perp w_R(x_1 - x_3) \right) \gamma_N^{(3)} \right] \\ &+ \beta^2 \frac{1}{N-1} \text{Tr} \left[\left| \nabla^\perp w_R(x_1 - x_2) \right|^2 \gamma_N^{(2)} \right] \end{aligned} \quad (\text{A.6})$$

In particular, taking a trial state $\psi_N = u^{\otimes N}$ we observe that:

$$\text{Tr} \left[\left(p_x^A \cdot \nabla^\perp w_R(x - y) + \nabla^\perp w_R(x - y) \cdot p_x^A \right) |u^{\otimes 2}\rangle \langle u^{\otimes 2}| \right]$$

$$\begin{aligned}
&= i \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \nabla \bar{u}(x) \bar{u}(y) \cdot \nabla^\perp w_R(x-y) u(x) u(y) dx dy + \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \mathbf{A}_e(x) \cdot \nabla^\perp w_R(x-y) |u(x)|^2 |u(y)|^2 dx dy \\
&- i \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \bar{u}(x) \bar{u}(y) \cdot \nabla^\perp w_R(x-y) \nabla u(x) u(y) dx dy + \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \mathbf{A}_e(x) \cdot \nabla^\perp w_R(x-y) |u(x)|^2 |u(y)|^2 dx dy \\
&= \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |u(y)|^2 \nabla^\perp w_R(x-y) \cdot (i [u(x) \nabla \bar{u}(x) - \bar{u}(x) \nabla u(x)] + 2\mathbf{A}_e(x) |u(x)|^2) dx dy \\
&= 2 \int_{\mathbb{R}^2} |\mathbf{A}^R[|u|^2] \cdot (\mathbf{J}[u(x)] + \mathbf{A}_e(x) |u(x)|^2) dx \tag{A.7}
\end{aligned}$$

where

$$\mathbf{J}[u] = \frac{i}{2} (u \nabla \bar{u} - \bar{u} \nabla u)$$

$$\mathbf{J}^{tot}[u] = \mathbf{J}[u] + 2\mathbf{A}_e |u|^2$$

with

$$|\mathbf{A}^R[|u|^2]| = \int_{\mathbb{R}^2} |u(y)|^2 \nabla^\perp w_R(x-y) dy$$

for the second term and:

$$\begin{aligned}
&\text{Tr} \left[\left(\nabla^\perp w_R(x-y) \cdot \nabla^\perp w_R(x-z) \right) \gamma_N^{(3)} \right] \\
&= \iiint_{\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2} |u(x)|^2 |u(y)|^2 |u(z)|^2 \cdot \nabla^\perp w_R(x-y) \cdot \nabla^\perp w_R(x-z) dx dy dz \\
&= \int_{\mathbb{R}^2} |u|^2 |\mathbf{A}^R[|u|^2]|^2 dx \tag{A.8}
\end{aligned}$$

for the third.

REFERENCES

- [1] S. AROVAS, J. SCHRIEFFER, AND F. WILCZEK, *Fractional statistics and the quantum Hall effect*, Phys. Rev. Lett., 53 (1984), pp. 722–723.
- [2] I. AVRON, J. HERBST AND B. SIMON, *Schrödinger operators with magnetic fields. 1. general interactions*, Duke Mathematical Journal, 45 N 4 (1978).
- [3] R. BHATIA, *Matrix analysis*, vol. 169, Springer, 1997.
- [4] F. BRANDÃO AND A. HARROW, *Quantum de Finetti Theorems under Local Measurements with Applications*, Commun. Math. Phys., 353 (2017), pp. 469–506.
- [5] J. COMBES, R. SCHRADER, AND R. SEILER, *Classical bounds and limits for energy distributions of hamilton operators in electromagnetic fields*, Annals of Physics, 111 (1978), pp. 1 – 18.
- [6] M. CORREGGI, R. DUBOSCQ, N. ROUGERIE, AND D. LUNDHOLM, *Vortex patterns in the almost-bosonic anyon gas*. to appear, 2019.
- [7] M. CORREGGI, D. LUNDHOLM, AND N. ROUGERIE, *Local density approximation for the almost-bosonic anyon gas*, Analysis and PDEs, 10 (2017), pp. 1169–1200.
- [8] S. FOURNAIS AND B. HELFFER, *Spectral methods in surface superconductivity*, Progress in Nonlinear Differential Equations and their Applications, 77, Birkhäuser Boston Inc., Boston, MA, 2010.
- [9] M. O. GOERBIG, *Quantum Hall effects*. arXiv:0909.1998, 2009.
- [10] M. HOFFMANN-OSTENHOF, T. HOFFMANN-OSTENHOF, A. LAPTEV, AND J. TIDBLM, *Many-particle Hardy Inequalities*, J. London Math. Soc., 77 (2008), pp. 99–114.
- [11] R. L. HUDSON AND G. R. MOODY, *Locally normal symmetric states and an analogue of de Finetti’s theorem*, Z. Wahrscheinlichkeitstheor. und Verw. Gebiete, 33 (1975/76), pp. 343–351.
- [12] R. B. LAUGHLIN, *Nobel lecture: Fractional quantization*, Rev. Mod. Phys., 71 (1999), pp. 863–874.

- [13] M. LEWIN, P. NAM, AND N. ROUGERIE, *Derivation of Hartree's theory for generic mean-field Bose systems*, Adv. Math., 254 (2014), pp. 570–621.
- [14] ———, *A note on 2D focusing many-boson systems*, Proc. Ame. Math. Soc., 145 (2017), pp. 2441–2454.
- [15] K. LI AND G. SMITH, *Quantum de Finetti Theorems under fully-one-way adaptative measurements*, Phys. Rev. Lett. 114, 114 (2015), p. 160503.
- [16] E. H. LIEB AND M. LOSS, *Analysis*, vol. 14 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2nd ed., 2001.
- [17] D. LUNDHOLM AND N. ROUGERIE, *The average field approximation for almost bosonic extended anyons*, J. Stat. Phys., 161 (2015), pp. 1236–1267.
- [18] D. LUNDHOLM AND J. P. SOLOVEJ, *Hardy and Lieb-Thirring inequalities for anyons*, Comm. Math. Phys., 322 (2013), pp. 883–908.
- [19] ———, *Local exclusion principle for identical particles obeying intermediate and fractional statistics*, Phys. Rev. A, 88 (2013), p. 062106.
- [20] ———, *Local exclusion and Lieb-Thirring inequalities for intermediate and fractional statistics*, Ann. Henri Poincaré, 15 (2014), pp. 1061–1107.
- [21] D. LUNDHOLM, N. ROUGERIE, *Emergence of fractional statistics for tracer particles in a Laughlin liquid*, Phys. Rev. Lett., (2016).
- [22] M. REED AND B. SIMON, *Methods of Modern Mathematical Physics. I. Functional analysis*, Academic Press, 1972.
- [23] N. ROUGERIE, *Théorèmes de De Finetti, limites de champ moyen et condensation de Bose-Einstein*, Les cours Peccot, Spartacus IDH, Paris, 2016. Cours Peccot, Collège de France : février-mars 2014.
- [24] N. ROUGERIE, *Non linear schrödinger limit of bosonic ground states, again.*, arXiv:1901.09561, (2019).
- [25] N. P. T. LEWIN, M. AND N. ROUGERIE, *Derivation of Hartree's theory for generic mean-field Bose systems.*, Advances in Mathematics 254, (2014).
- [26] B. SIMON, *Functional integration and quantum physics*, AMS Chelsea Publishing, Providence, RI, second ed., 2005.

UNIVERSITÉ GRENOBLE ALPES & CNRS, LPMMC (UMR 5493), B.P. 166, F-38042 GRENOBLE, FRANCE

E-mail address: `theoime.girardot@lpmmc.cnrs.fr`