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# Cassini states for black hole binaries

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## ABSTRACT

Cassini states correspond to the equilibria of the spin axis of a body when its orbit is perturbed. They were initially described for planetary satellites, but the spin axes of black hole binaries also present this kind of equilibria. In previous works, Cassini states were reported as spin-orbit resonances, but actually the spin of black hole binaries is in circulation and there is no resonant motion. Here we provide a general description of the spin dynamics of black hole binary systems based on a Hamiltonian formalism. In absence of dissipation, the problem is integrable and it is easy to identify all possible trajectories for the spin for a given value of the total angular momentum. As the system collapses due to radiation reaction, the Cassini states are shifted to different positions, which modifies the dynamics around them. This is why the final spin distribution may differ from the initial one. Our method provides a simple way of predicting the distribution of the spin of black hole binaries at the end of the inspiral phase.

**Key words:** black hole physics – gravitation – binaries: close.

## 1 INTRODUCTION

Following observations of the Moon, Cassini (1693) established three empirical laws on its rotational motion. The first stated that the rotation rate and the orbital mean motion are synchronous, the second that the angle between Moon’s equator and the ecliptic is constant, and the third that the Moon’s spin axis and the normals to its orbital plane and ecliptic remain coplanar. The observed physical librations are described as departures of the rotational motion from these three equilibrium laws. Colombo (1966) has shown that the second and third laws are independent of the first one, and generalized these laws to any satellite or planet whose nodal line on the invariant plane shifts because of perturbations. In his approach, the Hamiltonian of a slightly aspherical body is developed in a reference frame that precesses with the orbit. If the angular momentum and the energy are approximately conserved, the precession of the spin axis relative to the coordinate system fixed in the orbital plane is determined by the intersection of a sphere and a parabolic cylinder. The spin axis is fixed relative to the precessing orbit when the energy has an extreme value. Thus, these equilibria states for the spin axis can be the end point of dissipation, and they received the name of Cassini states (Colombo 1966; Peale 1969; Ward 1975; Correia 2015).

Schnittman (2004) has found that spinning black hole binaries can also present stable configurations where the two spin axes and the orbital angular momentum vector remain coplanar. Because of the loss of energy and orbital angular momentum through gravitational radiation reaction, the spins may end in libration around these equilibria. The final spin evolution of black holes

is particularly interesting, since the spin alignment during the inspiral phase can change significantly the distribution of black hole recoil velocities (Bogdanović, Reynolds & Miller 2007; Kesden, Sperhake & Berti 2010a,b; Berti, Kesden & Sperhake 2012; Gerosa et al. 2015a; Gupta & Gopakumar 2014; Gerosa et al. 2013). One of the most difficult aspects of studying the spinning black hole binary system is the problem of visualizing and analysing the orientation of the two spins and the angular momentum in an informative way. Previous studies always describe the libration around the coplanar configurations as spin-orbit resonances (Schnittman 2004; Kesden et al. 2010a; Berti et al. 2012; Gerosa et al. 2013, 2014, 2015a,b; Kesden et al. 2015). However, this description is not consistent with the expectation that the two normal modes of this problem become resonant, since for black hole binaries, one frequency is usually smaller than the other (Racine 2008). Indeed, by a suitable variable transformation, one can change the critical argument from libration to circulation. More generally, one can say there is resonant motion only if there is a clear change in the topology of the phase space, with a separatrix between the circulation and libration regions (see Henrard & Lemaître 1983).

In this Letter, we revisit the spin dynamics of black hole binaries adopting an Hamiltonian formalism. In absence of dissipation, the problem is integrable and we provide a simple analytical method to find its solutions. We show that the equilibria found by Schnittman (2004) is similar to the one observed by Cassini (1693) for the Moon.

## 2 SECULAR DYNAMICS

We consider a binary composed of two black holes with relative position  $\mathbf{r}$ , masses  $m_1$  and  $m_2$ , and spins  $\mathbf{S}_1$  and  $\mathbf{S}_2$ , respectively. The

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inspiral of the system is governed by radiation reaction at binary separations  $r = ||\mathbf{r}|| < 10^4 r_g$ , where  $r_g = GM/c^2$  is the gravitational radius. Numerical relativity simulations with initial separations  $r/r_g > 10$  are still too computationally expensive. However, in the range  $10 < r/r_g < 10^4$ , the binary dynamics can be described using the spinning Taylor-expanded PN Hamiltonian. As in previous studies (Schnittman 2004; Kesden et al. 2010a; Berti et al. 2012; Gerosa et al. 2013, 2014, 2015a,b; Kesden et al. 2015), we focus our analysis in this range. In the barycenter frame, the Hamiltonian depends on the canonical variables  $(\mathbf{r}, \mathbf{p})$  and on the spins vectors. For the purposes of our analysis, it is sufficient to restrict the discussion to the Newtonian contribution,  $H_N$ , and include only the leading 1.5PN spin-orbit interaction,  $H_{SO}$ , and the leading 2PN interaction,  $H_{SS}$ , which includes spin-induced monopole–quadrupole terms (Barker & O’Connell 1975; Damour 2001; Buonanno et al. 2011). The full Hamiltonian then reads  $H = H_N + H_{SO} + H_{SS}$ , where

$$H_N = \frac{\mathbf{p}^2}{2\mu} - \frac{GM\mu}{r}, \quad (1)$$

$$H_{SO} = \frac{2G}{c^2 r^3} \mathbf{S}_e \cdot \mathbf{L}, \quad (2)$$

$$H_{SS} = \frac{G\mu}{2Mc^2 r^3} [3(\mathbf{S}_0 \cdot \hat{\mathbf{r}})^2 - S_0^2], \quad (3)$$

with  $\mu = m_1 m_2 / M$ ,  $M = m_1 + m_2$ ,  $q = m_2 / m_1$ ,  $\hat{\mathbf{r}} = \mathbf{r} / r$ ,  $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ ,

$$\mathbf{S}_0 = (1 + q) \mathbf{S}_1 + (1 + 1/q) \mathbf{S}_2, \quad (4)$$

$$\mathbf{S}_e = \left(1 + \frac{3q}{4}\right) \mathbf{S}_1 + \left(1 + \frac{3}{4q}\right) \mathbf{S}_2. \quad (5)$$

The evolution in the PN limit occurs on three distinct time-scales: the orbital time  $t_o \sim (r^3/GM)^{1/2}$ , the spin precession time  $t_p \sim t_o(M/\mu)(r/r_g)$ , and the radiation–reaction time on which the orbital angular momentum decreases  $t_{rr} \sim t_p(r/r_g)^{3/2}$ . Therefore, for  $r/r_g > 10$  we have  $t_o \ll t_p \ll t_{rr}$ , which means that (1) we can neglect the effect of radiation–reaction over one precession cycle; (2) we can average the Hamiltonian over one orbital period to obtain the secular motion of the spin (Correia et al. 2011):

$$\overline{H} = -\frac{GM\mu}{2a} + 4\alpha \mathbf{S}_e \cdot \mathbf{L} - \frac{\alpha\mu}{2M} \left[ \frac{3}{L^2} (\mathbf{S}_0 \cdot \mathbf{L})^2 - S_0^2 \right], \quad (6)$$

where  $\alpha = G/(2c^2 r_0^3)$ ,  $r_0 = a\sqrt{1-e^2}$ ,  $L = ||\mathbf{L}|| = \mu\sqrt{GMa(1-e^2)}$ ,  $a$  is the semimajor axis, and  $e$  is the eccentricity. In the secular conservative problem, all quantities appearing in the Hamiltonian (6) are constant, except for the angular momentum components. The evolution of the system can therefore be obtained from the Hamiltonian through Poisson brackets (Dullin 2004; Tremaine, Touma & Namouni 2009)

$$\dot{\mathbf{S}}_i = \{\mathbf{S}_i, \overline{H}\} = \frac{\partial \overline{H}}{\partial \mathbf{S}_i} \times \mathbf{S}_i, \quad \dot{\mathbf{L}} = \{\mathbf{L}, \overline{H}\} = \frac{\partial \overline{H}}{\partial \mathbf{L}} \times \mathbf{L}, \quad (7)$$

which gives

$$\dot{\mathbf{S}}_i = \alpha \left[ \left( 4 + \frac{3m_j}{m_i} - \frac{3\mu}{m_i} \lambda \right) \mathbf{L} + \mathbf{S}_j \right] \times \mathbf{S}_i, \quad (8)$$

$$\dot{\mathbf{L}} = \alpha \left[ 4\mathbf{S}_e - \frac{3\mu}{M} \lambda \mathbf{S}_0 \right] \times \mathbf{L}, \quad (9)$$

with  $i \neq j = 1, 2$ , and  $\lambda = \mathbf{S}_0 \cdot \mathbf{L} / L^2$ . From previous expressions, we also see that in absence of radiation reaction, (1) the norms of the individual angular momentum vectors are conserved; (2) the total angular momentum vector is also conserved:

$$\mathbf{J} = \mathbf{L} + \mathbf{S}_1 + \mathbf{S}_2 = \text{const}. \quad (10)$$

### 3 REDUCED PROBLEM

The equations of motion can be simplified if we consider only the relative position in space of the unit vectors  $\mathbf{s}_i = \mathbf{S}_i / S_i$  and  $\mathbf{k} = \mathbf{L} / L$ , given by the direction cosines (Goldreich 1966; Schnittman 2004; Boué & Laskar 2006)

$$x_i = \cos \theta_i = \mathbf{s}_i \cdot \mathbf{k}, \quad z = \cos \theta_{12} = \mathbf{s}_1 \cdot \mathbf{s}_2, \quad (11)$$

together with the ‘berlingot’-shaped volume

$$w = \mathbf{k} \cdot (\mathbf{s}_1 \times \mathbf{s}_2) = \pm \sqrt{1 - x_1^2 - x_2^2 - z^2 + 2x_1 x_2 z}. \quad (12)$$

The equations of motion (8) and (9) become

$$\dot{x}_i = (-1)^i \beta S_j \frac{\mu \lambda - M}{m_j}, \quad \dot{z} = \beta L \left( \frac{\Delta m}{M} \lambda - \frac{\Delta m}{\mu} \right), \quad (13)$$

with  $\beta = 3\alpha w$ ,  $\Delta m = m_1 - m_2$ , and

$$\dot{w} = (x_2 z - x_1) \frac{\dot{x}_1}{w} + (x_1 z - x_2) \frac{\dot{x}_2}{w} + (x_1 x_2 - z) \frac{\dot{z}}{w}. \quad (14)$$

We can get  $w$  directly from expression (12), although this last equation can be useful for determining whether  $w$  is positive or negative. In addition, we still have two remaining integrals,<sup>1</sup> one from the total angular momentum (10)

$$J_0 = L S_1 x_1 + L S_2 x_2 + S_1 S_2 z = \text{const}, \quad (15)$$

and another from the Hamiltonian (6). Thus, equations (13) reduce to an integrable problem (Boué & Laskar 2006). From expression (15), we can write

$$\begin{aligned} S_0^2 &= 2(2 + q + 1/q) S_1 S_2 z + (1 + q)^2 S_1^2 + (1 + 1/q)^2 S_2^2 \\ &= -2(2 + q + 1/q) L (S_1 x_1 + S_2 x_2) + \text{const}. \end{aligned} \quad (16)$$

Replacing in expression (6) gives for the Hamiltonian

$$\overline{H}_0 = 3\alpha L^2 \lambda - \frac{3\alpha L^2 \mu}{2M} \lambda^2 = \text{const}, \quad (17)$$

hence we conclude that

$$\lambda(x_1, x_2) = \frac{M}{L} \left( \frac{S_1}{m_1} x_1 + \frac{S_2}{m_2} x_2 \right), \quad (18)$$

is also a conserved quantity (see also Racine 2008).

### 4 PRECESSIONAL MOTION

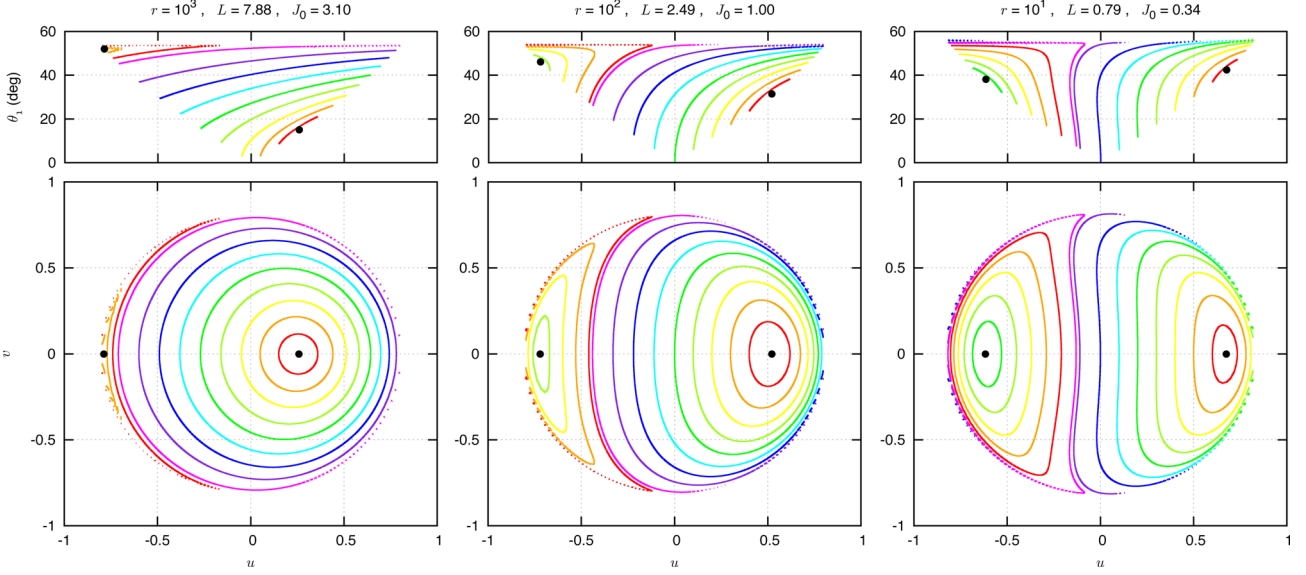
The spin evolution is better described by the projection of the spin axis in the orbital plane  $(u, v)$ , obtained as

$$u = \frac{x_2 - x_1 z}{\sqrt{1 - z^2}} = \sin \theta_1 \cos \Delta \phi, \quad (19)$$

and

$$v = \frac{w}{\sqrt{1 - z^2}} = \sin \theta_1 \sin \Delta \phi, \quad (20)$$

<sup>1</sup> For equal masses ( $\Delta m = 0$ ), the angle  $\theta_{12}$  between the spin axes is also conserved (equation 13).



**Figure 1.** Secular trajectories for the spin of the more massive object in a black hole binary with  $q = 0.9$  and  $\chi_1 = \chi_2 = 1$  (the units are  $G = M = c = 1$ ). Each panel shows the spin evolution at different binary separations,  $r$ , for which all trajectories have the same total angular momentum  $J_0$  (equation 15), they only differ by the value of  $\lambda$  (equation 18). We show the spin projected on the orbit normal (top), and its projection on the orbital plane (bottom). Cassini states are marked with a dot.

where  $\Delta\phi$  is the angle measured along the orbital plane from the projection of  $\mathbf{s}_1$  to the projection of  $\mathbf{s}_2$  (see fig. 1 in Gerosa et al. 2015a). Thus, when  $\sin \Delta\phi = 0$ , the unit vectors  $(\mathbf{s}_1, \mathbf{s}_2, \mathbf{k})$  lie in the same plane. With this choice,  $x_1$  only depends on the new variables<sup>2</sup>

$$x_1 = \sqrt{1 - u^2 - v^2}. \quad (21)$$

We can get  $z$  from spherical trigonometry

$$z = x_1 x_2 + u \sqrt{1 - x_2^2}, \quad (22)$$

while  $x_2$  can be obtained by eliminating  $z$  in expression (15) using the previous identity:

$$(L + S_1 x_1) x_2 + S_1 u \sqrt{1 - x_2^2} = (J_0 - L S_1 x_1) / S_2, \quad (23)$$

which can be explicitly solved for  $x_2$  as

$$x_2 = \frac{(L + S_1 x_1) X(x_1, u) - S_1 u \sqrt{1 - X^2(x_1, u)}}{S(x_1, u)}, \quad (24)$$

with

$$X(x_1, u) = \frac{J_0 - L S_1 x_1}{S_2 S(x_1, u)}, \quad (25)$$

$$S(x_1, u) = \sqrt{(L + S_1 x_1)^2 + (S_1 u)^2}. \quad (26)$$

Therefore,  $x_2$  depends only on  $(x_1, u)$ , hence on the new variables  $(u, v)$ , as well as  $\lambda$  (equation 18)

$$\lambda(x_1, x_2) = \lambda(x_1, u, J_0) = \lambda(u, v, J_0). \quad (27)$$

In Fig. 1, we show the secular trajectories for the spin of the more massive object in a black hole binary with  $q = 0.9$  at different binary

separations (we adopt  $S_i = \chi_i m_i^2$ , with  $\chi_i = 1$ ). In each panel, all trajectories have the same total angular momentum, obtained with initial  $x_1 = z = 1$  and  $x_2 = 0.5$  (equation 15). They only differ by the value of  $\lambda$ , that corresponds to different initial orientations of the spin vectors (equation 18). The level curves, obtained directly from expression (27) without integrating the equations of motion, fully characterize the spin dynamics for a given separation and total angular momentum.

We observe that the spin is always in circulation around a fix point (Cassini state). Previous studies (Schnittman 2004; Kesden et al. 2010a; Gerosa et al. 2013, 2015a) report that when the angle  $\Delta\phi$  librates there is resonant motion. However, we clearly see there is no resonance in this problem, since there is no separatrix emerging from a fix point. Because the fix points are displaced from the origin ( $u = 0$ ), for trajectories where we always have  $u > 0$  (or  $u < 0$ ), it appears that  $\Delta\phi$  librates around 0 (or  $\pi$ ).

## 5 CASSINI STATES

Cassini states correspond to equilibria of the spin axis. Thus, they are given by the extrema of the Hamiltonian (17):

$$\frac{\partial \lambda}{\partial u} = 0 \quad \wedge \quad \frac{\partial \lambda}{\partial v} = 0. \quad (28)$$

Since  $\lambda = \lambda(x_1, u)$ , we have

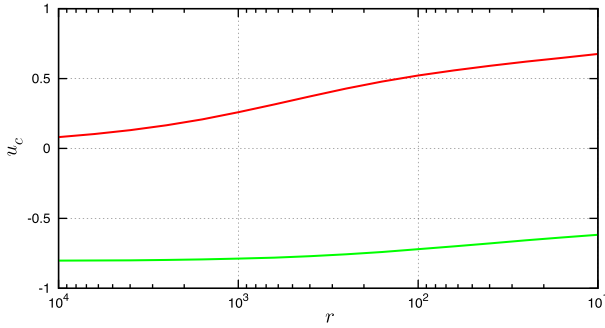
$$\frac{\partial \lambda}{\partial v} = \frac{\partial \lambda}{\partial x_1} \frac{\partial x_1}{\partial v} = -\frac{\partial \lambda}{\partial x_1} \frac{v}{x_1} = 0. \quad (29)$$

We then conclude that  $v = 0$  is always a possible equilibrium solution (equivalent to  $\Delta\phi = 0$  or  $\pi$ ), where the unit vectors  $\mathbf{s}$ ,  $\mathbf{k}_1$ , and  $\mathbf{k}_2$  remain coplanar. Using  $v = 0$  in (28) gives an implicit condition for the coplanar states,  $u_c = \sin \theta_c$ :

$$\tan \theta_c = \frac{u_c}{x_c} = \frac{m_1}{m_2} \frac{S_2}{S_1} \frac{\partial x_2}{\partial u_c} \Big|_{v=0}, \quad (30)$$

where the derivative is computed using expression (24) with  $u = u_c$  and  $x_1 = x_c = \sqrt{1 - u_c^2}$ . The roots of (30) can be found in the

<sup>2</sup> We considered that  $x_1 > 0$ , but this method is still valid for  $x_1 < 0$  adopting  $x_1 = -\sqrt{1 - u^2 - v^2}$ . It also stands for  $x_2$  by switching the indexes 1 and 2. However, if  $x_1$  and  $x_2$  simultaneously oscillate around 0, the validity of this method is not assured.



**Figure 2.** Cassini states as a function of the binary separation  $r$ . These equilibria are obtained by solving equation (30). The vertical lines correspond to the configurations shown in Fig. 1.

interval  $u_c \in [-1, 1]$  using numerical methods or simply by plotting its graph.

Alternatively, coplanar states can be obtained as stationary solutions for the equations of motion for which  $v = 0$  (Schnittman 2004). Therefore, they can also be obtained by setting  $w = 0$  and  $\dot{w} = 0$  (equation 14), that is

$$u_c = \frac{(\mu\lambda - M) \left[ (x_c z - x_2) \frac{S_1}{m_1} - (x_2 z - x_c) \frac{S_2}{m_2} \right]}{L \left( \frac{\Delta m}{M} \lambda - \frac{\Delta m}{\mu} \right) \sqrt{1 - x_2^2}}, \quad (31)$$

where  $z$  and  $x_2$  are obtained from (22) and (24) with  $u = u_c$  and  $x_1 = x_c$ .

In Fig. 2, we plot the Cassini states as a function of the binary separation for the same system shown in Fig. 1. We observe that there are always two equilibrium points  $u_c = (u_0, u_\pi)$ , one corresponding to an aligned configuration for the spins ( $u_0 > 0$  and  $\Delta\phi = 0$ ), and another corresponding to an anti-aligned configuration ( $u_\pi < 0$  and  $\Delta\phi = \pi$ ). For  $r \gg 10^4$ , we have  $u_0 \rightarrow 0$ , hence the angle  $\Delta\phi$  appears to always circulate around zero. As the binary separation decreases, both  $u_c$  increase. Therefore, for trajectories initially circulating around  $u_0$  (or  $u_\pi$ ), the average obliquity of the spin increases (or decreases). Moreover, for the trajectories around  $u_0$ , the angle  $\Delta\phi$  appears to switch from circulation to libration as  $u_0$  moves away from the origin.

The secular spin-orbit problem for black hole binaries has two independent frequencies (normal modes),  $v_0$  and  $v_\pi$ . In the case of similar masses ( $q \approx 1$ ), we have (Racine 2008)

$$v_0 \approx \alpha ||\mathbf{J}|| \left( 7 - \frac{3}{2}\lambda \right), \quad \frac{v_\pi}{v_0} \approx 1 - \frac{12 - 3\lambda ||\mathbf{S}||}{14 - 3\lambda ||\mathbf{J}||}, \quad (32)$$

where  $v_0$  can be seen as the precession rate of the total spin  $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$  about the total angular momentum  $\mathbf{J}$  (or the precession of  $\mathbf{L}$  about  $\mathbf{J}$ ), and  $v_\pi$  as a nutation frequency.

Any projection of the spin  $S_p(t)$  can thus be written as

$$S_p(t) = \sum_{j,k} A_{jk} e^{i(jv_0 + kv_\pi)t} \approx A_{00} + A_{10} e^{iv_0 t} + A_{01} e^{iv_\pi t}, \quad (33)$$

where the  $A_{jk}$  are determined by the initial conditions. We have  $|A_{10}| \gtrsim |A_{01}|$  for trajectories circulating around the Cassini state  $u_0$  (or  $|A_{10}| \lesssim |A_{01}|$  for those around  $u_\pi$ ), with  $|A_{01}| = 0$  at the equilibrium point  $u_0$  (or  $|A_{10}| = 0$  at  $u_\pi$ ). Spin-orbit resonances can occur whenever  $v_0 = v_\pi$ . For black hole binaries, resonant motion is unlikely since we usually have  $v_\pi < v_0$ . For lower mass ratios ( $q \ll 1$ ), the dynamics is dominated by state  $u_0$  and there is also no resonant motion. However, we cannot rule out any extreme

example where the system would actually exhibit formal resonant behaviour. In that case, the analytical model discussed in previous sections is still valid. We would see appearing a separatrix and additional Cassini states, as it happens in the classic planetary case (see Correia 2015).

## 6 LONG-TERM EVOLUTION

In the previous analysis, we neglected the radiation reaction damping of the system through gravitational wave emission. Of course, in all physically realistic black hole binaries, gravitational radiation plays a major role in the secular evolution of the orbit. With the inclusion of radiation reaction, the orbital angular momentum and the eccentricity evolve according to (e.g. Peters 1964; Gergely, Perjés & Vasúth 1998)

$$\dot{L} = -\frac{32}{5} \frac{G^{7/2} M^{5/2} \mu^2}{c^5 a^{7/2} (1 - e^2)^2} \left( 1 + \frac{7}{8} e^2 \right) \mathbf{k}, \quad (34)$$

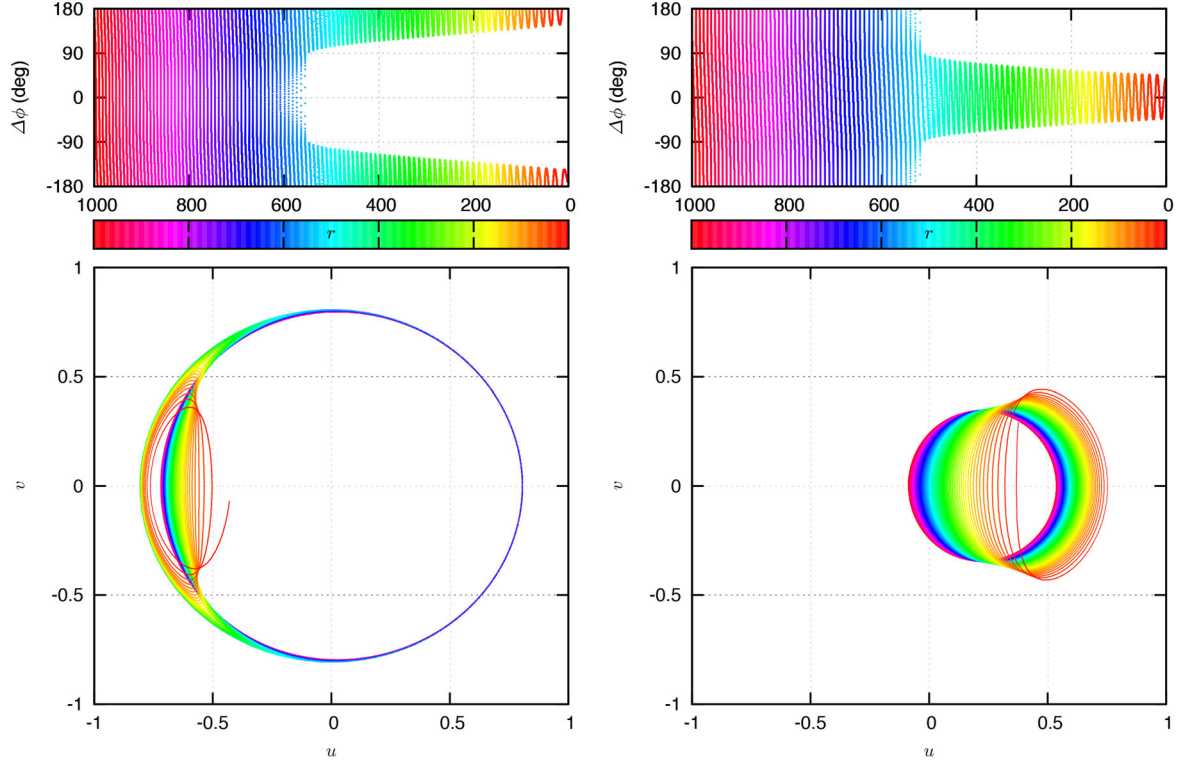
$$\frac{\dot{e}}{e} = -\frac{304}{15} \frac{G^3 M^2 \mu}{c^5 a^4 (1 - e^2)^{5/2}} \left( 1 + \frac{121}{304} e^2 \right). \quad (35)$$

The full problem is no longer integrable, so we need to perform numerical simulations of the previous equations together with (8) and (9) to track the spin evolution of the system. In Fig. 3, we plot the evolution of the spin of the same system shown in Fig. 1 starting with initial  $a = 10^3$ ,  $e = 0$ ,  $v = 0$ , and  $u = -0.73$  (left) or  $u = -0.05$  (right). For initial  $u = -0.73$ , the spin is initially dominated by the Cassini state  $u_0$ . Around  $r \approx 600$ , it switches to state  $u_\pi$ , since for  $q = 0.9$ , this state dominates the majority of the trajectories with  $u < 0$  at small separations (Fig. 1). For initial  $u = -0.05$ , the spin is always dominated by the Cassini state  $u_0$ . The equilibrium value for this state increases for small  $r$  (Fig. 2), so the trajectories moving around it also increase its average obliquity. As a result, the angle  $\Delta\phi$  appears to switch from circulation to libration at  $r \approx 500$  because  $u$  becomes always positive for smaller separations.

## 7 DISCUSSION

In this Letter, we presented a simple method for determining the equilibrium states and the secular trajectories for the spins of black hole binaries with separations  $r/r_g > 10$ . For a given value of the total angular momentum, we are able to plot the global spin dynamics, including the equilibria points (Cassini states). Our method only depends on the geometry of the Hamiltonian and thus does not require an integration of the equations of motion. In addition, it allows us to correctly understand the dynamical regimes and to exclude the occurrence of spin-orbit resonances in this problem. For black hole binaries, all trajectories actually circulate around the Cassini equilibria. As the binary separation decreases due to radiation reaction, the spin dynamics is also modified. However, since dissipation only occurs in the orbital angular momentum, the changes in the spin are a result of the modification in the phase space of the system. Therefore, by looking at the initial position of the spin in the phase space and how the phase space is modified for small separations, it becomes possible to predict the final spin geometries at the end of the inspiral phase. This work is important to understand the distribution of black hole recoil velocities.





**Figure 3.** Long-term spin evolution of the more massive object in a black hole binary with  $q = 0.9$  and  $\chi_1 = \chi_2 = 1$ . The initial conditions are  $a = 10^3$ ,  $e = 0$ ,  $v = 0$ , and  $u = -0.73$  (left) or  $u = -0.05$  (right). We show the angle  $\Delta\phi$  (top) and the spin projected on the orbital plane (bottom). The colour scale is related to the binary separation.

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