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# IMPROVED BOUNDS FOR HERMITE-HADAMARD INEQUALITIES IN HIGHER DIMENSIONS

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ABSTRACT. Let  $\Omega \subset \mathbb{R}^n$  be a convex domain and let  $f : \Omega \rightarrow \mathbb{R}$  be a positive, subharmonic function (i.e.  $\Delta f \geq 0$ ). Then

$$\frac{1}{|\Omega|} \int_{\Omega} f dx \leq \frac{c_n}{|\partial\Omega|} \int_{\partial\Omega} f d\sigma,$$

where  $c_n \leq 2n^{3/2}$ . This inequality was previously only known for convex functions with a much larger constant. We also show that the optimal constant satisfies  $c_n \geq n - 1$ . As a byproduct, we establish a sharp geometric inequality for two convex domains where one contains the other  $\Omega_2 \subset \Omega_1 \subset \mathbb{R}^n$ :

$$\frac{|\partial\Omega_1|}{|\Omega_1|} \frac{|\Omega_2|}{|\partial\Omega_2|} \leq n.$$

## 1. INTRODUCTION

**1.1. Convex functions.** The Hermite-Hadamard inequality dates back to an 1883 observation of Hermite [10] with an independent use by Hadamard [9] in 1893: it says that for convex functions  $f : [a, b] \rightarrow \mathbb{R}$

$$\frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.$$

This inequality is rather elementary and has been refined in many ways – we refer to the monograph of Dragomir & Pearce [7]. However, there is relatively little work outside of the one-dimensional case; we refer to [3, 4, 14, 15, 16, 17, 18, 21]. The strongest possible statement that one could hope for is, for convex functions  $f : \Omega \rightarrow \mathbb{R}$  defined on convex domains  $\Omega \subset \mathbb{R}^n$ ,

$$\frac{1}{|\Omega|} \int_{\Omega} f dx \leq \frac{1}{|\partial\Omega|} \int_{\partial\Omega} f d\sigma.$$

This inequality has been shown to be true for many special cases: it is known for  $\Omega = \mathbb{B}_3$  the 3-dimensional ball by Dragomir & Pearce [7] and  $\Omega = \mathbb{B}_n$  by de la Cal & Carcamo [3] (other proofs are given by de la Cal, Carcamo & Escarriaza [4] and Pasteczka [18]), the simplex [2], the square [6], triangles [5] and Platonic solids [18]. It was pointed out by Pasteczka [18] that if the inequality holds for a domain  $\Omega$  with constant 1, then plugging in affine functions shows that the center of mass of  $\Omega$  and the center of mass of  $\partial\Omega$  coincide, which is not generally true for convex bodies; therefore the inequality cannot hold with constant 1 in higher dimensions uniformly over all convex bodies. The first uniform estimate was shown in [21]: if

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$f : \Omega \rightarrow \mathbb{R}$  is a convex, positive function on the convex domain  $\Omega \subset \mathbb{R}^n$ , then we have

$$(1) \quad \frac{1}{|\Omega|} \int_{\Omega} f \, dx \leq \frac{c_n}{|\partial\Omega|} \int_{\partial\Omega} f \, d\sigma$$

with  $c_n \leq 2n^{n+1}$ . In this paper, we will improve this uniform estimate and show that the optimal constant satisfies  $n-1 \leq c_n \leq 2n^{3/2}$ . We do not have a characterization of the extremal convex functions  $f$  on a given domain  $\Omega$  (however, see below, we have such a characterization in the larger family of subharmonic functions).

**1.2. Subharmonic functions.** Niculescu & Persson [17] (see also [4, 15]) have pointed out that one could also seek such inequalities for subharmonic functions, i.e. functions satisfying  $\Delta f \geq 0$ . We note that all convex functions are subharmonic. Jianfeng Lu and the last author [21] showed that for all positive, subharmonic functions  $f : \Omega \rightarrow \mathbb{R}$  on convex domains  $\Omega \subset \mathbb{R}^n$

$$(2) \quad \int_{\Omega} f \, dx \leq |\Omega|^{1/n} \int_{\partial\Omega} f \, d\sigma.$$

Estimates relating the integral of a positive subharmonic function  $f$  over  $\Omega$  to the integral over the boundary  $\partial\Omega$  are linked to the torsion function on  $\Omega$  given by

$$\begin{aligned} -\Delta u &= 1 && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Integration by parts and the inequalities  $u \geq 0, \Delta f \geq 0$  show that

$$\begin{aligned} \int_{\Omega} f \, dx &= \int_{\Omega} f(-\Delta u) \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} f \, d\sigma - \int_{\Omega} (\Delta f) u \, dx \\ &\leq \int_{\partial\Omega} \frac{\partial u}{\partial \nu} f \, d\sigma \\ &\leq \max_{x \in \partial\Omega} \frac{\partial u}{\partial \nu}(x) \int_{\partial\Omega} f \, d\sigma, \end{aligned}$$

where  $\nu$  is the inward pointing normal vector. This computation suggests that we may have the following characterization of the optimal constant for a given convex domain  $\Omega$ .

**Proposition** (see e.g. [8, 17]). *The optimal constant  $c(\Omega)$  in the inequality*

$$\int_{\Omega} f \, dx \leq c(\Omega) \int_{\partial\Omega} f \, d\sigma$$

*for positive subharmonic functions is given by*

$$c(\Omega) = \max_{x \in \partial\Omega} \frac{\partial u}{\partial \nu}(x).$$

The lower bound on  $c(\Omega)$  follows from setting  $f$  to be the Poisson extension of a Dirac measure located at the point at which the normal derivative assumes its maximum. The derivation also shows that it suffices to consider the case of harmonic functions  $f$ . Implicitly, this also gives a characterization of extremizing functions (via the Green's function). Jianfeng Lu and the last author [14] used this proposition in combination with a gradient estimate for the torsion function to show that the best constant in (2) is uniformly bounded in the dimension. We will follow a similar strategy to obtain an improved bound for the optimal constant in (2).

## 2. THE RESULTS

Our first result improves the constant  $c_n$  from (1) in all dimensions for subharmonic functions and shows that the growth is at most polynomial.

**Theorem 1.** *Let  $\Omega \subset \mathbb{R}^n$  be convex and let  $f : \Omega \rightarrow \mathbb{R}$  be a positive, subharmonic function. Then*

$$(3) \quad \frac{1}{|\Omega|} \int_{\Omega} f dx \leq \frac{c_n}{|\partial\Omega|} \int_{\partial\Omega} f d\sigma,$$

where the optimal constant  $c_n$  satisfies

$$c_n \leq \begin{cases} n^{3/2} & \text{if } n \text{ is odd,} \\ \frac{n^2+n}{\sqrt{n+2}} & \text{if } n \text{ is even.} \end{cases}$$

In particular, for  $n = 2$  dimensions, our proof shows the inequality

$$\frac{1}{|\Omega|} \int_{\Omega} f dx \leq \frac{3}{|\partial\Omega|} \int_{\partial\Omega} f d\sigma,$$

where the constant 3 improves on constant 8 obtained earlier for convex functions in [21]. To complement the result in Theorem 1 we prove that any constant for which (3) is valid must grow at least linearly with the dimension.

**Theorem 2.** *The optimal constant  $c_n$  in (3) is non-decreasing in  $n$  and satisfies*

$$(4) \quad c_n \geq \max\{n - 1, 1\}.$$

In order to prove Theorem 2 we establish a connection to an isoperimetric problem that is of interest in its own right. Specifically, we prove the following Lemma.

**Lemma.** *In any dimension  $n \geq 1$ ,*

$$(5) \quad \sup \left\{ \frac{|\partial\Omega_1|}{|\Omega_1|} \frac{|\Omega_2|}{|\partial\Omega_2|} : \Omega_2 \subset \Omega_1 \text{ both convex domains in } \mathbb{R}^n \right\} = n.$$

We are not aware of any prior treatment of this shape optimization problem in the literature. Problem (5) can be equivalently written as

$$\sup \left\{ \frac{|\partial\Omega|}{|\Omega|} \frac{1}{h(\Omega)} : \Omega \text{ a convex set in } \mathbb{R}^{n-1} \right\}, \text{ where } h(\Omega) = \inf_{X \subset \bar{\Omega}} \frac{|\partial X|}{|X|}$$

denotes the Cheeger constant. We refer to Alter & Caselles [1] and Kawohl & Lachand-Robert [11]. The result of Kawohl & Lachand-Robert [11] will be a crucial ingredient in the proof of Theorem 2 (we note that the infimum runs over all subsets, it is known that the Cheeger set is unique and convex).

We also obtain a slight improvement of the constant in (2).

**Theorem 3.** *Let  $\Omega \subset \mathbb{R}^n$  be convex and let  $f : \Omega \rightarrow \mathbb{R}$  be a positive, subharmonic function. Then*

$$\int_{\Omega} f dx \leq \frac{|\Omega|^{1/n}}{\omega_n^{1/n} \sqrt{n}} \int_{\partial\Omega} f d\sigma,$$

where  $\omega_n$  is the volume of the unit ball in  $n$ -dimensions.

We observe that, as  $n$  tends to infinity,  $\omega_n^{1/n} \sqrt{n} \rightarrow \sqrt{2\pi e}$ . We also note a construction from [14] which shows that the constant in Theorem 3 is at most a factor  $\sqrt{2}$  from optimal in high dimensions.

## 3. PROOF OF THEOREM 1

**3.1. Convex functions.** We first give a proof of Theorem 1 under the assumption that  $f$  is convex; this argument is fairly elementary and is perhaps useful in other settings. A full proof of Theorem 1 is given in §3.2.

*Proof.* This proof combines three different arguments. The first argument is that

$$(6) \quad \int_{\Omega} f dx \leq \frac{w(\Omega)}{2} \int_{\partial\Omega} f d\sigma$$

from the one-dimensional Hermite-Hadamard inequality applied along fibers that are orthogonal to the hyperplanes realizing the width  $w(\Omega)$ .

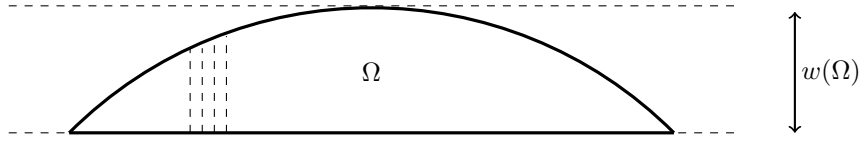


FIGURE 1. Application of the one-dimensional inequality on a one-dimensional fiber. This step is lossy if the boundary is curved.

Steinhagen [22] showed that width can be bounded in terms of the inradius

$$(7) \quad w(\Omega) \leq \begin{cases} 2\sqrt{n} \cdot \text{inrad}(\Omega) & \text{if } n \text{ is odd,} \\ 2\frac{n+1}{\sqrt{n+2}} \cdot \text{inrad}(\Omega) & \text{if } n \text{ is even.} \end{cases}$$

The last inequality follows from [12]: if  $\Omega \subset \mathbb{R}^n$  is a convex body and

$$\Omega_t = \{x \in \Omega : d(x, \partial\Omega) > t\},$$

where  $d(x, \partial\Omega)$  denotes the distance to the boundary

$$d(x, \partial\Omega) = \inf_{y \in \partial\Omega} \|x - y\|,$$

then

$$|\partial\Omega_t| \geq |\partial\Omega| \left(1 - \frac{t}{\text{inrad}(\Omega)}\right)_+^{n-1}.$$

Since  $|\nabla d(x, \partial\Omega)| = 1$  almost everywhere, the coarea formula implies

$$\begin{aligned} |\Omega| &= \int_0^{\text{inrad}(\Omega)} |\partial\Omega_t| dt \\ &\geq |\partial\Omega| \int_0^{\text{inrad}(\Omega)} \left(1 - \frac{t}{\text{inrad}(\Omega)}\right)^{n-1} dt = |\partial\Omega| \frac{\text{inrad}(\Omega)}{n} \end{aligned}$$

and thus we obtain (also stated in [13, Eq. 13])

$$(8) \quad \text{inrad}(\Omega) \leq n \frac{|\Omega|}{|\partial\Omega|}.$$

Combining inequalities (6), (7) and (8) implies the result.  $\square$

Both Steinhagen's inequality as well as inequality (8) are sharp for the regular simplex. However, see Fig. 1, an application of the one-dimensional Hermite-Hadamard inequality can only be sharp if the fibers are hitting the boundary at a point at which they are normal, otherwise there is a Jacobian determinant determined by the slope of the boundary and better results are expected. It is not clear to us how to reconcile these two competing factors.

### 3.2. A Proof of Theorem 1.

*Proof.* We have, for all positive, subharmonic functions  $f : \Omega \rightarrow \mathbb{R}$ ,

$$\int_{\Omega} f dx \leq \max_{x \in \partial\Omega} \frac{\partial u}{\partial \nu}(x) \int_{\partial\Omega} f d\sigma,$$

where  $u$  is the torsion function. A classic bound on the torsion functions is given in Sperb [20, Eq. (6.12)],

$$\max_{x \in \partial\Omega} \frac{\partial u}{\partial \nu}(x) \leq \sqrt{2} \|u\|_{L^\infty}^{\frac{1}{2}}.$$

Moreover, using Steinhagen's inequality in combination with (8), we know that  $\Omega$  is contained within a strip of width

$$w(\Omega) \leq \frac{|\Omega|}{|\partial\Omega|} \begin{cases} 2n^{3/2} & \text{if } n \text{ is odd,} \\ 2 \frac{n^2+n}{\sqrt{n+2}} & \text{if } n \text{ is even.} \end{cases}$$

We can now use the maximum principle to argue that the torsion function in  $\Omega$  is bounded from above by the torsion function in the strip of width  $w(\Omega)$  (see Fig. 2). That torsion function, however, is easy to compute since the problem becomes one-dimensional. Orienting the strip to be given by

$$S = \left\{ (x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : |y| \leq \frac{w(\Omega)}{2} \right\},$$

we see that the torsion function on the strip is given by

$$v(x, y) = \frac{w(\Omega)^2}{8} - \frac{y^2}{2}.$$

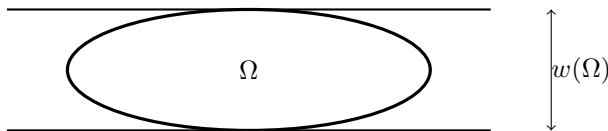


FIGURE 2. The torsion function in  $\Omega$  is bounded from above by the torsion function of the strip.

This shows

$$\|u\|_{L^\infty} \leq \frac{w(\Omega)^2}{8}$$

and thus

$$\begin{aligned} \max_{x \in \partial\Omega} \frac{\partial u}{\partial \nu}(x) &\leq \sqrt{2} \|u\|_{L^\infty}^{1/2} \\ &\leq \frac{w(\Omega)}{2} \leq \frac{|\Omega|}{|\partial\Omega|} \begin{cases} n^{3/2} & \text{if } n \text{ is odd,} \\ \frac{n^2+n}{\sqrt{n+2}} & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

□

#### 4. PROOF OF THEOREM 2

The purpose of this section is to prove  $c_{n+1} \geq c_n$  as well as the inequality

$$c_n \geq \sup \left\{ \frac{|\partial\Omega_1|}{|\Omega_1|} \frac{|\Omega_2|}{|\partial\Omega_2|} : \Omega_2 \subset \Omega_1 \text{ both convex domains in } \mathbb{R}^{n-1} \right\}.$$

Theorem 2 is then implied by this statement together the proof of the Geometric Lemma in Section §5.

*Proof.* The proof is based on explicit constructions. We first show that  $c_{n+1} \geq c_n$ . This is straightforward and based on an extension in the  $(n+1)$ -first coordinate: for any  $\varepsilon > 0$ , we can find a convex domain  $\Omega_\varepsilon \subset \mathbb{R}^n$  and a positive, convex function  $f_\varepsilon : \Omega_\varepsilon \rightarrow \mathbb{R}$  such that

$$\frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} f_\varepsilon dx \geq \frac{c_n - \varepsilon}{|\partial\Omega_\varepsilon|} \int_{\partial\Omega_\varepsilon} f_\varepsilon d\sigma.$$

We define, for any  $z > 0$ ,

$$\Omega_{z,\varepsilon} = \{(x, y) : x \in \Omega_\varepsilon \text{ and } 0 \leq y \leq z\} \subset \mathbb{R}^{n+1}$$

and  $f_{z,\varepsilon} : \Omega_{z,\varepsilon} \rightarrow \mathbb{R}$  via

$$f_{z,\varepsilon}(x, y) = f_\varepsilon(x).$$

Then

$$\frac{1}{|\Omega_{z,\varepsilon}|} \int_{\Omega_{z,\varepsilon}} f_{z,\varepsilon} dx dy = \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon} f_\varepsilon dx.$$

This integral simplifies for  $z$  large since

$$\lim_{z \rightarrow \infty} \frac{1}{|\partial\Omega_{z,\varepsilon}|} \int_{\partial\Omega_{z,\varepsilon}} f_{z,\varepsilon} d\sigma = \frac{1}{|\partial\Omega_\varepsilon|} \int_{\partial\Omega_\varepsilon} f_\varepsilon d\sigma.$$

Picking  $\varepsilon$  sufficiently small and  $z$  sufficiently large shows that  $c_{n+1} < c_n$  leads to a contradiction.

We now establish the inequality

$$c_n \geq \sup \left\{ \frac{|\partial\Omega_1|}{|\Omega_1|} \frac{|\Omega_2|}{|\partial\Omega_2|} : \Omega_2 \subset \Omega_1 \text{ both convex domains in } \mathbb{R}^{n-1} \right\}.$$

To this end pick  $0 \in \Omega_2 \subset \Omega_1 \subset \mathbb{R}^{n-1}$  in such a way that both domains are convex. We will now define a domain  $\Omega_N \subset \mathbb{R}^n$  and a convex function  $f_N : \Omega_N \rightarrow \mathbb{R}$  where  $N \gg 1$  will be a large parameter. We first define the convex sets

$$C_1 = \{(x, y) : x \in \Omega_1 \text{ and } y \geq -N^3\}$$

and

$$C_2 = \left\{ (x, y) : x \in \left(1 - \frac{y}{N^2}\right) \Omega_2 \text{ and } y \leq N \right\}.$$

The set  $\Omega_N$  is then given as the intersection  $\Omega_N = C_1 \cap C_2$  (see Fig. 3). We observe that  $\Omega_N$  is the intersection of two convex sets and is therefore convex. Also, looking at the scaling, we see that  $C_1$  dominates: looking at  $\Omega_N$  from ‘far away’, it looks essentially like  $C_1$  truncated. We now make this precise: note that there exists a constant  $\lambda \geq 1$  such that  $\Omega_1 \subseteq \lambda\Omega_2$  and then

$$\Omega_N \cap \{(x, y) \in \mathbb{R}^n : y \leq -(\lambda - 1)N^2\} = C_1 \cap \{(x, y) \in \mathbb{R}^n : y \leq -(\lambda - 1)N^2\}.$$

This means, that for  $N \gg \lambda$ , the ‘left’ part of the convex domain dominates area and volume. We also observe that

$$\begin{aligned} |\Omega_N| &= N^3|\Omega_1| + \mathcal{O}(N^2) \\ |\partial\Omega_N| &= N^3|\partial\Omega_1| + \mathcal{O}(N^2), \end{aligned}$$

where the implicit constants depend on  $\Omega_1$  and  $\Omega_2$ .

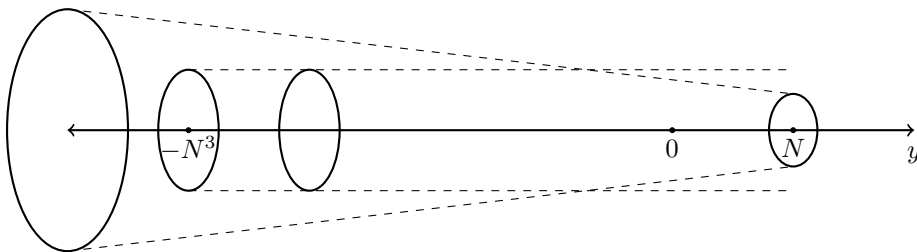


FIGURE 3. The construction of  $C_1$  and  $C_2$ .

Since  $\Omega_2 \subset \Omega_1$ , we have that

$$\Omega_N \cap \{(x, y) \in \mathbb{R}^n : y \geq 0\} = C_2 \cap \{(x, y) \in \mathbb{R}^n : y \geq 0\}.$$

We now define a convex function on  $\mathbb{R}^n$  via

$$f(x, y) = \begin{cases} y & \text{if } y \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We obtain

$$\begin{aligned} \int_{\Omega_N} f \, dx dy &= \int_{\Omega_N \cap \{y > 0\}} f \, dx dy = \int_{C_2 \cap \{y > 0\}} f \, dx dy = (1 + o(1)) \frac{N^2}{2} |\Omega_2| \\ \int_{\partial\Omega_N} f \, d\sigma &= \int_{\partial\Omega_N \cap \{y > 0\}} f \, d\sigma = \int_{\partial C_2 \cap \{y > 0\}} f \, d\sigma = (1 + o(1)) \frac{N^2}{2} |\partial\Omega_2|. \end{aligned}$$

This shows that

$$\frac{1}{|\Omega_N|} \int_{\Omega_N} f \, dx dy = \frac{(1 + o(1))}{2N} \frac{|\Omega_2|}{|\Omega_1|}$$

and

$$\frac{1}{|\partial\Omega_N|} \int_{\partial\Omega_N} f \, d\sigma = \frac{(1 + o(1))}{2N} \frac{|\partial\Omega_2|}{|\partial\Omega_1|}$$

which implies the desired result for  $N \rightarrow \infty$ .  $\square$



## 5. PROOF OF THE GEOMETRIC LEMMA

*Proof.* By the inequality

$$\frac{|\Omega|}{|\partial\Omega|} \leq \text{inrad}(\Omega) \leq n \frac{|\Omega|}{|\partial\Omega|},$$

a proof of which can be found in [13], the supremum is no larger than  $n$  since

$$(9) \quad \frac{|\partial\Omega|}{|\Omega|} \frac{|\Omega'|}{|\partial\Omega'|} \leq \frac{n \cdot \text{inrad}(\Omega')}{\text{inrad}(\Omega)} \leq n.$$

What remains is to prove that this upper bound is saturated. The underlying idea of our proof is a theorem of Kawohl and Lachand-Robert [11] characterizing the Cheeger set of a convex set  $\Omega \subset \mathbb{R}^2$ . Specifically, their theorem states that for a convex  $\Omega \subset \mathbb{R}^2$  the Cheeger problem

$$h(\Omega) = \inf \left\{ \frac{|\partial\Omega'|}{|\Omega'|} : \Omega' \subset \overline{\Omega} \right\}$$

is solved by the set

$$\Omega' = \{x \in \Omega : \exists y \in \Omega \text{ such that } x \in B_{1/h(\Omega)}(y) \subset \Omega\},$$

where  $B_r(x_0)$  is a ball of radius  $r$  centered at  $x_0$ . We recall our use of the notation

$$\Omega_t = \{x \in \Omega : d(x, \partial\Omega) > t\},$$

where  $d(x, \partial\Omega)$  denotes the distance to the boundary

$$d(x, \partial\Omega) = \inf_{y \in \partial\Omega} \|x - y\|,$$

and we can equivalently write the Cheeger set of  $\Omega$  as  $\Omega' = \Omega_{1/h(\Omega)} + B_{1/h(\Omega)}$ , where  $B_r$  denotes a ball of radius  $r$  centered in 0. Here and in what follows the sum of two sets is to be interpreted in the sense of the Minkowski sum:

$$A + B = \{x : x = a + b, a \in A, b \in B\}.$$

The situation when  $n \geq 2$  is more complicated, and as far as we know a precise solution of the Cheeger problem is not available [1]. Nonetheless, our aim in what follows is to prove that by taking  $\Omega$  as a very thin  $n$ -simplex we can find a good enough candidate for  $\Omega'$  among the one-parameter family of sets

$$(10) \quad \Omega_t + B_t, \quad 0 \leq t \leq \text{inrad}(\Omega).$$

We construct our candidate for  $\Omega$  as follows. Let  $\Omega(\eta) \subset \mathbb{R}^n$  be the  $n$ -simplex obtained by taking a regular  $(n-1)$ -simplex of sidelength  $\eta \gg 1$  in the hyperplane  $\{x \in \mathbb{R}^n : x_1 = -1\}$  with  $(-1, 0, \dots, 0)$  as center of mass and adding the last vertex at  $(h(\eta), 0, \dots, 0)$ , where  $h(\eta)$  is chosen so that  $\text{inrad}(\Omega(\eta)) = 1$ . Note that, as  $\eta$  becomes large,  $h(\eta)$  is approximately 1 and  $|\Omega(\eta)| \sim \eta^{n-1}$ .

By construction  $B_1 \subset \Omega(\eta)$ , and it is the unique unit ball of maximal radius contained in  $\Omega(\eta)$ . Moreover, the set  $\Omega(\eta)$  is a tangential body to this ball (that is, a convex body all of whose supporting hyperplanes are tangential to the same ball). Since every tangential body to a ball is homothetic to its form body [19] (in our case  $\Omega(\eta)$  is in fact equal to its form body), the main result in [12] implies

$$|(\partial\Omega(\eta))_t| = (1-t)^{n-1} |\partial\Omega(\eta)|, \quad \text{for all } t \in [0, 1].$$

An application of the coarea formula now yields the identity

$$(11) \quad |\Omega(\eta)| = \int_0^1 |\partial(\Omega(\eta))_t| dt = \frac{|\partial\Omega(\eta)|}{n}.$$

We also note that  $\Omega(\eta) \subset B_{2\eta}$ . To see why this is true, we first note that the inradius of the regular  $n$ -simplex (by which we mean  $n+1$  points all at distance 1 from each other embedded in  $\mathbb{R}^n$ ) is given by

$$r_n = \frac{1}{\sqrt{2n(n+1)}}.$$

The regular simplex is the convex body for which John's theorem is sharp, the circumradius is thus given by

$$R_n = n \cdot r_n = \frac{\sqrt{n}}{\sqrt{2(n+1)}} \leq \frac{1}{\sqrt{2}}.$$

This shows that  $\Omega(\eta) \subset B_{2\eta}$  (for the purpose of the proof, the constant 2 is not important and could be replaced by a much larger (absolute) constant). Since it makes the computations somewhat simpler we consider, for a suitably chosen number  $t$ , the set  $(1+t)\Omega(\eta)$ . By construction  $B_1 \subset \Omega(\eta)$  which implies the inclusion

$$\Omega(\eta) + B_t \subset \Omega(\eta) + t\Omega(\eta) = (1+t)\Omega(\eta).$$

In particular, we can test (5) with  $\Omega = (1+t)\Omega(\eta)$  and  $\Omega' = \Omega(\eta) + B_t$  for any values of  $t, \eta \gg 1$ . We note that up to rescaling by  $(1+t)^{-1}$  this is exactly the family of sets in (10). Indeed, for each  $t$  the set  $((1+t)\Omega(\eta))_t = \Omega(\eta)$ .

The final step of the proof is to show that by letting  $t, \eta \rightarrow \infty$  appropriately

$$(12) \quad \frac{|\partial((1+t)\Omega(\eta))|}{|(1+t)\Omega(\eta)|} \frac{|\Omega(\eta) + B_t|}{|\partial(\Omega(\eta) + B_t)|} \rightarrow n.$$

To prove (12) we recall the definition and some basic properties of mixed volumes [19, p. 275ff]. Let  $\mathcal{K}$  denote the set of convex bodies in  $\mathbb{R}^n$  with nonempty interior. The mixed volume is defined as the unique symmetric function  $W : \mathcal{K}^n \rightarrow \mathbb{R}_+$  satisfying

$$|\eta_1\Omega_1 + \dots + \eta_m\Omega_m| = \sum_{j_1=1}^m \dots \sum_{j_n=1}^m \eta_{j_1} \dots \eta_{j_n} W(\Omega_{j_1}, \dots, \Omega_{j_n}),$$

for any  $\Omega_1, \dots, \Omega_m \in \mathcal{K}$  and  $\eta_1, \dots, \eta_m \geq 0$  [19]. Then  $W$  satisfies the following properties:

- (1)  $W(\Omega_1, \dots, \Omega_n) > 0$  for  $\Omega_1, \dots, \Omega_n \in \mathcal{K}$ .
- (2)  $W$  is a multilinear function with respect to Minkowski addition.
- (3)  $W$  is increasing with respect to inclusions in each of its arguments.
- (4) The volume and perimeter of  $\Omega \in \mathcal{K}$  can be written in terms of  $W$ :

$$|\Omega| = W(\underbrace{\Omega, \dots, \Omega}_{n \text{ times}}) \quad \text{and} \quad |\partial\Omega| = nW(\underbrace{\Omega, \dots, \Omega}_{n-1 \text{ times}}, B_1).$$

Since only mixed volumes of two distinct sets appear in our proof we introduce the shorthand notation

$$W_j(K, L) = W(\underbrace{K, \dots, K}_{n-j}, \underbrace{L, \dots, L}_j).$$

By (11), we have

$$\frac{|\partial((1+t)\Omega(\eta))|}{|(1+t)\Omega(\eta)|} = \frac{n}{1+t}.$$

The definition of  $W$  implies

$$|\Omega(\eta) + B_t| = W(\Omega(\eta) + B_t, \dots, \Omega(\eta) + B_t).$$

Multilinearity allows us to expand this term as

$$W(\Omega(\eta) + B_t, \dots, \Omega(\eta) + B_t) = \sum_{j=0}^n \binom{n}{j} t^j W_j(\Omega(\eta), B_1)$$

and the same argument shows

$$|\partial(\Omega(\eta) + B_t)| = n \sum_{j=0}^{n-1} \binom{n-1}{j} t^j W_{j+1}(\Omega(\eta), B_1).$$

Altogether, we can write the expression of interest as

$$(13) \quad \frac{|\partial((1+t)\Omega(\eta))|}{|(1+t)\Omega(\eta)|} \frac{|\Omega(\eta) + B_t|}{|\partial(\Omega(\eta) + B_t)|} = \frac{n}{(1+t)} \frac{\sum_{j=0}^n \binom{n}{j} t^j W_j(\Omega(\eta), B_1)}{n \sum_{j=0}^{n-1} \binom{n-1}{j} t^j W_{j+1}(\Omega(\eta), B_1)}.$$

In order to prove (12) we need a bound from below. For the sum in the numerator it suffices to keep the first two terms in the expansion and to use Property (1) of  $W$  resulting in

$$\begin{aligned} \sum_{j=0}^n \binom{n}{j} t^j W_j(\Omega(\eta), B_1) &\geq W_0(\Omega(\eta), B_1) + ntW_1(\Omega(\eta), B_1) \\ &= |\Omega(\eta)| + t|\partial\Omega(\eta)|. \end{aligned}$$

To bound the sum in the denominator we wish to keep the term with  $j = 0$  as is. For  $j \geq 1$ , we now use that  $\Omega(\eta) \subset B_{2\eta}$  together with Property (3) to bound

$$W_{j+1}(\Omega(\eta), B_1) \leq W_{j+1}(B_{2\eta}, B_1) = (2\eta)^{n-j-1} |B_1|.$$

Inserting the two bounds above into (13) yields

$$\begin{aligned} \frac{|\partial((1+t)\Omega(\eta))|}{|(1+t)\Omega(\eta)|} \frac{|\Omega(\eta) + B_t|}{|\partial(\Omega(\eta) + B_t)|} &\geq \frac{n}{(1+t)} \frac{|\Omega(\eta)| + t|\partial\Omega(\eta)|}{|\partial\Omega(\eta)| + n \sum_{j=1}^{n-1} \binom{n-1}{j} 2^{n-j-1} t^j \eta^{n-j-1} |B_1|}. \end{aligned}$$

We recall that  $|\Omega(\eta)| = n^{-1}|\partial\Omega(\eta)|$  and therefore

$$\begin{aligned} \frac{|\partial((1+t)\Omega(\eta))|}{|(1+t)\Omega(\eta)|} \frac{|\Omega(\eta) + B_t|}{|\partial(\Omega(\eta) + B_t)|} &\geq \frac{nt}{(1+t)} \frac{1 + 1/(nt)}{1 + n|B_1| \sum_{j=1}^{n-1} \binom{n-1}{j} 2^{n-j-1} t^j \eta^{n-j-1} / |\partial\Omega(\eta)|}. \end{aligned}$$

By construction  $|\partial\Omega(\eta)| \sim \eta^{n-1}$  as  $\eta \rightarrow \infty$ . Consequently, by choosing  $t = \sqrt{\eta}$  (though the more general choice  $t = \eta^\alpha$  for  $0 < \alpha < 1$  would also work) we find  $t^j \eta^{n-j-1} / |\partial\Omega(\eta)| \sim \eta^{-j/2}$ . Therefore, taking  $\eta$  (and thus  $t$ ) to infinity, we obtain

$$\liminf_{\eta \rightarrow \infty} \frac{|\partial((1+\sqrt{\eta})\Omega(\eta))|}{|(1+\sqrt{\eta})\Omega(\eta)|} \frac{|\Omega(\eta) + B_{\sqrt{\eta}}|}{|\partial(\Omega(\eta) + B_{\sqrt{\eta}})|} \geq n,$$

which when combined with the matching upper bound (9) completes the proof.  $\square$

## 6. PROOF OF THEOREM 3

*Proof.* We use the estimate

$$\int_{\Omega} f dx \leq \max_{x \in \partial\Omega} \frac{\partial u}{\partial \nu}(x) \int_{\partial\Omega} f d\sigma$$

introduced in Proposition 1 above and use, inspired by the argument in [14], estimates for the torsion function. One such estimate for the torsion function comes from  $P$ -functions, we refer to the classic book of Sperb [20, Eq. (6.12)],

$$\max_{x \in \partial\Omega} \frac{\partial u}{\partial \nu}(x) \leq \sqrt{2} \|u\|_{L^\infty}^{\frac{1}{2}}.$$

It remains to estimate the largest value of the torsion function. There are two different approaches: we can interpret it as the maximum lifetime of Brownian motion inside a domain of given measure or we can interpret it as the solution of a partial differential equation to which Talenti's theorem [23] can be applied. In both cases, we end up with a standard isoperimetric estimate [23] (that was also used in [14])

$$\|u\|_{L^\infty} \leq \frac{1}{2n} \left( \frac{|\Omega|}{\omega_n} \right)^{\frac{2}{n}}$$

to obtain

$$\max_{x \in \partial\Omega} \frac{\partial u}{\partial \nu}(x) \leq \frac{|\Omega|^{1/n}}{\omega_n^{1/n} \sqrt{n}}.$$

$\square$

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