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## ► To cite this version:

T Maquart, T. Elguedj, A. Gravouil, M. Rochette. A note on topological properties of volumes constructed from surfaces. [Research Report] INSA Lyon; LaMCoS - Laboratoire de Mécanique des Contacts et des Structures, UMR 5259 (Lyon, INSA). 2019. hal-02318094v2

**HAL Id: hal-02318094**

**<https://hal.science/hal-02318094v2>**

Submitted on 18 Oct 2019

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# A note on topological properties of volumes constructed from surfaces

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## Abstract

Converting surfaces into a volume has a long interest in several communities, e.g. the computational mechanics community. This process involves having specific surfaces which can be converted into a solid, i.e., a volume. This paper presents in a clear and brief manner the topological properties conserved during surface to volume transformation. We state the limits of this approach if a specific volume structure is required. Volume structures can be a coarse volume organization or meshes. For that purpose, surface manifolds are mathematically turned into a volume manifold. Topological tools are presented to understand which properties are transmitted to the volume and which ones are unset. Developments are submitted both for continuous and discrete manifolds using CW-complexes.

## 1 Introduction

Problematic of surface to volume transformation finds its interest in various fields of application. For many years, different approaches have been used to solve this conversion. Despite this particular interest, it currently not exists generic methods to generate a volume with a prescribed simplicial complex connectivity from arbitrary surfaces. Such notes are given to better understand topological aspects relative to this issue. This investigation will serves application fields who apply physics on solid domains [Maq19].

Current challenges in computational mechanics and more globally in the physics field are the generation of volume domains which are structured. For instance, pure hexahedral meshes have always been greatly appreciated. They stand for a more homogeneous solution field with less local effects and a more stable computation during large deformations. Structured mesh generation goes with topology. Isogeometric analysis [HCB05, CHB09, Occ18] is a new method that represents the geometry better than standard meshes. Despite this benefit, isogeometric meshes also need structured domains. Attempts to compute fields with a specific symmetry embedded in volumes are on current research [CHRVS18] to produce hexahedral meshes [RS15, VCD<sup>+</sup>16, SRUL16, SVB17].

Furthermore, today's artificial intelligence applications need relevant ordered data. In other words, the discretization of the geometry should be the same in order to learn important information from identical structures [Maq19, LBG18].

We present mathematical formulations on topological properties of 2, 3-dimensional manifolds embedded in  $\mathbb{R}^3$ . In particular with a goal to convert many 2-dimensional manifolds into a 3-dimensional manifold. Using the well known generalized Euler characteristic and references in the literature, our developments will be securely anchored. Close links between surfaces and its associated volume are then determined.

Presented concepts are given without prerequisites, references in the topology field are recommended [HLS00, Hat01, Jub09, SOA07]. In the following we consider only finite and connected volumes constructed from one or more surfaces.

## 2 Surfaces to volume conversion

We strive to find topological properties of 3-dimensional manifolds  $V$  that respect some constraints inherited from topological and geometrical properties of a surface  $M$ , and in the following from many surfaces  $M$ . We seek for constraints provided from the surface to be mapped into the volume manifold. Into a volume manifold  $V$ , the number of volumes  $k_3$  is defined by equation (1a). Naturally, the considered surface  $M$  becomes the boundary of  $V$  as formulated in equation (1b).

$$k_3 = -\chi(V) + k_0 - k_1 + k_2, \quad (1a)$$

$$\text{s.t. } \{k_{0_M} - k_{1_M} + k_{2_M} = \chi(\partial V) = \chi(M_c) \ \& \ |\partial V| = 1\}. \quad (1b)$$

$\chi(V)$  is the Euler characteristic of the volume  $V$  and  $\partial V$  its boundary.  $k_{i_M}$  are the CW-complex or simplicial complex entities of the closed surface  $M_c$ . Likewise,  $k_i$  are related to  $V$ .  $M_c$  is given with no boundaries, i.e., without 1-dimensional manifolds and is provided from a surface  $M$  possibly with boundaries. Since equation (1a) has overabundant unknown terms, and in general not enough constraints in equation (1b), closed surface  $M_c$  can refer to many associated volumes  $V$  with different structures. The constraint  $|\partial V| = 1$  if the volume manifold is built from only one compact, connected and oriented 2-dimensional manifold. Hence, for one given closed surface  $M_c$ , there exists many different volume structures composed by more or less  $k_3$  entities that satisfying above constraints. Depending on the surface features, choosing the best volume structure is done by analyzing geometry. It leads us to the following statement in proposition 1.

**Proposition 1 *Non-isomorphic associated volumes.*** *Let  $M_c$  be a closed compact oriented 2-dimensional manifold  $M$ . If we want to turn  $M_c$  into a volume manifold  $V$ , there exists a set  $U_V$  with an infinite size of non-isomorphic volume structures that satisfies constraints provided in equation (1b):*

$$M_c \Rightarrow V^i \not\equiv_{iso} V^j, \forall i, j < |U_V| = \infty. \quad (2)$$

*Where  $\equiv_{iso}$  denotes the isomorphic equivalence in terms of simplicial complex entities, i.e., related to the number of entities of each dimension. Transformations with a discontinuous function are permitted.*

Non-isomorphism between different volumes  $V$  in equation (2) differ just by number of volumes  $k_3$ , number of surfaces  $k_2$  and number of edges  $k_1$  for a same  $\chi(V)$ . In other words, two volumes are isomorphic if they have the same number of simplices of each dimension  $n$ ,  $n \in \{0, 1, 2, 3\}$ , if a mapping with a discontinuous function is allowed. Homeomorphic volumes are isomorphic volumes holding exactly the same simplicial complex connectivity. The set  $U_V$  has an infinite size as explained in proof 1.

**Proof 1 *Infinite size of the set  $U_V$ .*** *The set of non-isomorphic volumes  $U_V$  of an associated closed surface  $M_c$  has an infinite size because of the properties of the 3-simplex subdivision. It is always possible to subdivide a  $n$ -simplex in the way that its  $(n - 1)$ -simplex entities remains topologically and geometrically the same; and the created manifold is also embedded in the same  $n$ -dimensional topological space, so:*

$$|U_V| = \infty, \forall V. \quad (3)$$

Taking into account previous developments and constraints, we now seek to find which volume Euler characteristic  $\chi(V)$  is interesting. In other words, we expect a 3-dimensional manifold embedded in  $\mathbb{R}^3$  with specific properties inherited from a surface  $M$  or several surfaces  $M$ . Furthermore, a volume which lies in  $\mathbb{R}^3$  with one or more  $(n-1)$ -dimensional compact, connected and orientable entities is desired. However, we can not accept any 1-dimensional entity, i.e., boundaries of 2-dimensional manifolds. Finally, equation (4) and equation (5) in proposition 2 state our interesting volume Euler characteristics.

**Proposition 2 Equation**  $2\chi(V) = \chi(M_c)$ . In  $\mathbb{R}^3$ , a 3-dimensional manifold  $V$  constructed from a compact, connected and oriented 2-dimensional manifold  $M_c$  is restricted to the following volume Euler characteristic  $\chi(V)$ :

$$\chi(V) = \frac{\chi(M_c)}{2} = 1 - g. \quad (4)$$

Using  $g$  as the genus of the surface  $M$  or the closed surface  $M_c$ , i.e.,  $g(M)$  or  $g(M_c)$ . Indeed, adding 1-dimensional manifold entities to a 2-dimensional one does not change its genus  $g$ . If several compact, connected and oriented 2-dimensional manifolds are used to create the volume, the disjoint union of Euler characteristics implies:

$$\chi(V) = \frac{\chi(M_c^1 \sqcup M_c^2 \sqcup \dots \sqcup M_c^{n_s})}{2} = \sum_{i=1}^{n_s} \frac{\chi(M_c^i)}{2} = \sum_{i=1}^{n_s} 1 - g(M_c^i). \quad (5)$$

With  $n_s$  the number of given surface manifolds which serve to generate the volume  $V$ . An example is shown in figure 1.

This result can be demonstrated using CW-complex properties and Betti numbers in a topological space  $\mathbb{R}^3$  [Dam10], see equation (9) in proof 2. Notice that meshes are simplicial complexes due to their combinatorial form. Remember that CW-complexes are themselves both simplicial complexes and continuous entities on which homotopy theory is applicable.

**Proof 2 Equation**  $2\chi(V) = \chi(M_c)$ . To turn any 3-dimensional volume manifold  $V$  embedded in  $\mathbb{R}^3$  into a topological object homeomorphic to the volume ball  $\mathbf{B}^3$ , we have to add CW-complex objects to  $V$ . We first create  $g$  surfaces  $k_2$ , i.e., for each 1-dimensional tunnel, see e.g. figure 2 (A). Secondly, we create two edges  $k_1$  and one surface  $k_2$  for each 2-dimensional cavity, see e.g. figure 2 (B):

$$\chi(V) = k_0 - (k_1 + 2b_2) + (k_2 + b_1 + b_2) - k_3, \quad (6)$$

$$= k_{0_S} - k_{1_S} + k_{2_S} - 2b_2 + b_1 + b_2 - k_3, \quad (7)$$

$$= \chi(S) + b_1 - b_2 - 1, \quad (8)$$

$$= \chi(S) - \chi(V) \Rightarrow 2\chi(V) = \chi(S). \quad (9)$$

The Betti number  $b_3 = 0$ , because the topological space  $\mathbb{R}^3$  is considered.  $S$  is the set of closed surfaces,  $S = \sqcup M_c^i$ .  $k_0 = k_{0_S}$ ,  $k_1 = k_{1_S}$  and  $k_2 = k_{2_S}$ .  $S$  remains unchanged.  $k_3 = 1$  implies only one connected volume.

Developments leading to equation (9) establish the topological relation between the associated volume  $V$  of many closed surfaces  $M_c^i$ . We can now express in equation (10) the Betti numbers of  $V$  using the previous results.

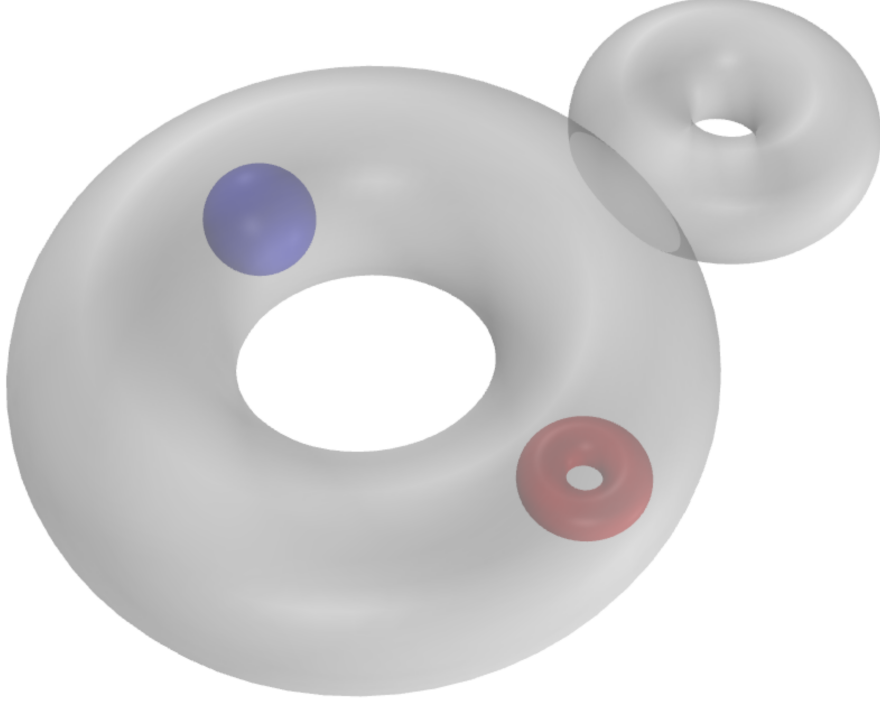


Figure 1: A 3-dimensional manifold  $V$ . Three compact, connected and oriented 2-dimensional manifolds  $M_c^1$ ,  $M_c^2$  and  $M_c^3$  are used to determine  $V$ . According to Betti numbers  $b_i$ ,  $\chi(V) = 1 - 3 + 2 = 0$ . Using equation (5) the same result is found.

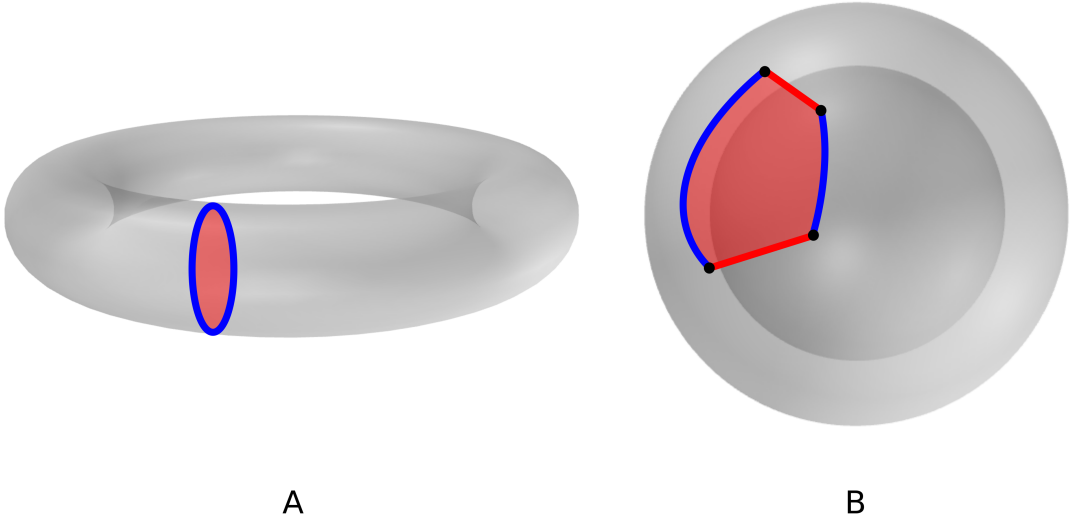


Figure 2: 3-dimensional manifold  $V$  to volume ball  $\mathbf{B}^3$  conversion using CW-complexes. (A)  $g$  surfaces  $k_2$  are created for a genus- $g$  surface (red surface), i.e., for each tunnel. (B) Two edges  $k_1$  and one surface  $k_2$  are constructed (red curves and surface) for a cavity. Black and blue objects are not created because they are lying on a lower dimensional object than a volume.

$$\begin{cases} b_0 = 1, \\ b_1 = b_0 + b_2 - \frac{\chi(M_c)}{2} \text{ or } b_1 = b_0 + b_2 - \sum_{i=1}^{n_s} \frac{\chi(M_c^i)}{2}, \\ b_2 = |M|. \end{cases} \quad (10)$$

With  $|M|$  the number of connected 2-dimensional manifolds used to define  $V$  minus one. If only one surface manifold is used to generate the volume,  $|M| = 0$ . We can now re-write equation (1) to include interesting volume Euler characteristics and needed topological space constraints inherited from several surfaces. This idea is formulated in equation (11).

$$k_3 = \left[ \sum_{i=1}^{n_s} 1 - g(M_c^i) \right] + k_0 - k_1 + k_2, \quad (11a)$$

$$\text{s.t. } \left\{ \sum_{i=1}^{n_s} (k_{0_M}^i - k_{1_M}^i + k_{2_M}^i) = \chi(\partial V) = \sum_{i=1}^{n_s} \chi(M_c^i) \right\} \text{ and} \quad (11b)$$

$$\{|\partial V| = n_s \text{ \& } V \in \mathbb{R}^3 \text{ \& } \partial V \equiv \sqcup M_c^i\}. \quad (11c)$$

Where  $\equiv$  denotes the classic geometric equivalence: if the  $n_s$  surfaces  $M_c^i$  are meshes, boundaries of  $V$  are.  $\partial V$  and surfaces  $M_c^i$  are topologically and geometrically the same.  $\partial V$  replicates perfectly the closed surfaces  $M_c^i$ . Notice that, in equation (11b) and equation (11c), constraints are not independent. Since  $\partial V$  is provided from the surfaces  $M_c^i$ ,  $|\partial V|$  and  $\chi(\partial V)$  are already established.

### 3 Conclusion

We have presented topological aspects in order to determine properties of volumes constructed from surfaces. We have shown that topology concepts are fundamental features in the transformation of boundary surfaces to a volume.

In spite of mathematical tools given above, the problem of converting surfaces into a structured volume is still hard. Indeed, there is many possibilities of volume structures satisfying input information during this kind of conversion. It is easy to see that the interior volume is not defined. A lack of constraints and unset properties are clearly observed. Additional constraints should be added to tackle this uncertainty on the solid structure.

Within the computational mechanics community, researchers are trying to turn scanned meshes into volume meshes. For instance, several techniques exist to convert boundary surfaces into solids, in order to apply physics on them. Presented mathematics state the limits of this kind of approaches.

### 4 Acknowledgments

T. Maquart was partially supported by a CIFRE fellowship of the french Association Nationale de la Recherche et de la Technologie and ANSYS research. T. Maquart, T. Elguedj and A. Gravouil were also partially supported by a research contract with ANSYS. These supports are gratefully acknowledged.

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