



HAL
open science

Lines, conics, and all that

Ciro Ciliberto, M Zaidenberg

► **To cite this version:**

| Ciro Ciliberto, M Zaidenberg. Lines, conics, and all that. 2019. hal-02318018v2

HAL Id: hal-02318018

<https://hal.science/hal-02318018v2>

Preprint submitted on 14 Feb 2020 (v2), last revised 5 Jul 2020 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

LINES, CONICS, AND ALL THAT

C. CILIBERTO, M. ZAIDENBERG

To Bernard Shiffman on occasion of his seventy fifths birthday

ABSTRACT. This is a survey on the Fano schemes of linear spaces, conics, rational curves, and curves of higher genera in smooth projective hypersurfaces, complete intersections, Fano threefolds, on the related Abel-Jacobi mappings, etc.

CONTENTS

Introduction	1
1. Counting lines on surfaces	2
2. The numerology of Fano schemes	3
3. Geometry of the Fano scheme	6
4. Counting conics in complete intersection	8
5. Lines and conics on Fano threefolds and the Abel-Jacobi mapping	11
5.1. The Fano-Iskovskikh classification	11
5.2. Lines and conics on Fano threefolds	12
5.3. The Abel-Jacobi mapping	14
5.4. The cylinder homomorphism	15
6. Counting rational curves	16
6.1. Varieties of rational curves in hypersurfaces	16
6.2. Twisted cubics in complete intersections	17
7. Hypersurfaces with few rational curves	17
7.1. Rational curves on K3 surfaces	17
7.2. Rational curves in hypersurfaces	18
7.3. Clemens' Conjecture	19
8. Counting curves of higher genera and hyperbolicity	20
References	22

INTRODUCTION

A classical problem of enumerative geometry asks to count curves with given numerical invariants in a smooth complete intersection variety X in \mathbb{P}^n . This includes the study of various Fano schemes of X , that is, the components of the Hilbert schemes of curves with given numerical invariants, in particular, the Fano schemes of lines and conics. The present paper is a survey on this problem. We concentrate mainly on concrete numerical results. A special attention is paid to the case of surfaces and threefolds. We discuss the lines and conics in Fano threefolds, which

Acknowledgements: This research started during a visit of the second author at the Department of Mathematics of the University Roma "Tor Vergata" in October, 2018. The authors acknowledge the MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006. *Ciro Ciliberto* is a member of GNSAGA of INdAM.

2010 Mathematics Subject Classification: 14H45, 14N10, 14N25, 14J26, 14J28, 14Q05, 14Q10.

Key words: line, conic, rational curve, projective hypersurface, Fano threefold, Abel-Jacobi mapping.

are not necessarily complete intersections. The latter involves the Abel-Jacobi mapping, the related cylinder homomorphism, and Torelli type theorems. To keep a reasonable volume, we do not outline proofs, and restrict the exposition to smooth projective varieties. We are working over the complex number field unless otherwise is stated, while many results remain valid over any algebraically closed field of characteristic zero. In the majority of cases, the names of the authors of cited results appear only in the list of references. Together with the best known result, we often mention its predecessors; we apologize for possible confusions. We hope to provide the reader with a brief introduction to several chapters of this beautiful subject and a practical guide on the extended bibliography.

Several important topics remain outside our survey, such as, e.g.: rational curves in Kähler varieties [87], in Moishezon varieties, see, e.g., [392], in hyperkähler varieties, see, e.g., [4], in fibered varieties, see, e.g., [5, 6], pseudo-holomorphic curves in symplectic manifolds, see, e.g., [13, 216, 242, 231, 357, 358, 400], minimal rational curves and the varieties of minimal rational tangents, see, e.g., [7, 8, 246, 248, 249, 250, 251, 524], (minimal) rational and elliptic curves in flag varieties, Schubert varieties, Bott-Samuelson varieties, etc., see, e.g., [74, 126, 252, 317, 393, 402, 404, 405, 522, 523], varieties covered by lines and conic-connected varieties, see, e.g., [271, 272, 273, 350, 351], Prym varieties, see, e.g., [32, 33, 148, 181, 190, 198, 199, 281, 384, 421, 491, 527, 528], etc. We avoid the vast domain of Mirror Symmetry and the Gromov-Witten invariants, see, e.g., [12, 48, 145, 234, 237, 306, 307, 336, 356, 418]. Among the topics missing in this survey, one should mention as well the results inspired by the Manin conjectures on the asymptotic of rational points and rational curves on Fano varieties [346, 347], see, e.g., [25, 26, 69, 70, 71, 296, 328, 402, 404, 405, 406, 407, 408, 447, 448, 481, 482, 483] and the literature therein.

1. COUNTING LINES ON SURFACES

Cayley and Salmon [81, 446], and also Clebsch [115], discovered that any smooth complex cubic surface in \mathbb{P}^3 contains exactly 27 lines. See, e.g., [177] for the modern treatment and [30, 31, 82, 195, 291, 345, 394, 452, 471] for the count of lines on cubic surfaces over \mathbb{R} and in characteristic $p > 0$, including [228] for the monodromy of lines in families of cubic surfaces, and [104] for lines on singular cubic surfaces.

In the case of quartic surfaces in \mathbb{P}^3 the following is known.

- Theorem 1.1.** (a) *The maximal number of lines on a smooth quartic in \mathbb{P}^3 is 64. This maximum is achieved by the F. Schur quartic. Any smooth quartic with 64 lines is isomorphic to the F. Schur quartic [164, 425, 451, 453].*
- (b) *In the projective space $|\mathcal{O}_{\mathbb{P}^3}(4)|$ parameterizing the quartics in \mathbb{P}^3 , the subvariety of quartics containing a line has codimension one and degree 320. The general point of this subvariety represents a quartic surface with a unique line [348, 356].*

The first claim in (a) is due to B. Segre [453], but his proof contains a mistake. In [164] and [425] there are two different proofs based on the ideas of B. Segre.

In the case of quintics we have the following weaker results. We let $\Sigma(d, n)$ stand for the projective space $|\mathcal{O}_{\mathbb{P}^n}(d)| = \mathbb{P}^{\binom{n+d}{n}-1}$ parameterizing the degree d effective divisors on \mathbb{P}^n . The general point of $\Sigma(d, n)$ corresponds to a smooth hypersurface of degree d in \mathbb{P}^n .

- Theorem 1.2.** (a) *A smooth quintic surface in \mathbb{P}^3 contains at most 127 lines [427].*
- (b) *The variety of quintic surfaces in \mathbb{P}^3 containing a line is irreducible of degree 1990 and of codimension 2 in $\Sigma(5, 3)$. The general point of this variety corresponds to a quintic surface with a unique line [344, 348].*

The exact upper bound for the number of lines in a smooth quintic is unknown. The Fermat quintic and the Barth quintic contain exactly 75 lines each.

For higher degree surfaces in \mathbb{P}^3 the following is known (cf. Theorem 1.1.b).

Theorem 1.3. (a) *A smooth surface of degree $d \geq 3$ in \mathbb{P}^3 contains at most $11d^2 - 32d + 24$ lines [29].¹*

(b) *The surfaces of degree d in \mathbb{P}^3 containing a line are parameterized by an irreducible subvariety in $\Sigma(d, 3)$ of codimension $d - 3$ and degree*

$$(1) \quad \frac{1}{24} \binom{d+1}{4} (3d^4 + 6d^3 + 17d^2 + 22d + 24).$$

The general such surface contains a unique line [344, 348].

Once again, for $d \geq 5$ the exact upper bound in (a) is unknown. The Fermat surface of degree d contains exactly $3d^2$ lines. As for (b), there are analogous formulas for the degrees of the loci of degree d surfaces in \mathbb{P}^3 passing through a given line and containing an extra line, and, respectively, containing three lines in general position; see [344, Prop. 3.3 and 6.4.1]. The latter locus is of codimension $3d + 3 - 12 = 3d - 9$ in $\Sigma(d, 3)$. For $d = 3$ its degree equals 720, in agreement with the combinatorics of triplets of skew lines contained in a smooth cubic surface.

2. THE NUMEROLOGY OF FANO SCHEMES

Given a variety $X \subset \mathbb{P}^n$, the Fano scheme $F_h(X)$ is the scheme of h -planes contained in X . Recall that a hypersurface is *very general* if it belongs to the complement of countably many proper subvarieties in the space of hypersurfaces of a given degree. For the Fano schemes of lines on hypersurfaces one has the following results.

Theorem 2.1. *For a smooth hypersurface X of degree d in \mathbb{P}^n , where $n \geq 3$, the following holds.*

- (a) ([22, 41, 43, 46, 318, 319, 330], [300, Ch. V, 2.9, 4.3, 4.5])
 - *If $d \leq 2n - 3$ then the Fano scheme of lines $F_1(X)$ is nonempty;*
 - *if $d < n$ then X is covered by lines. The latter holds also for non-smooth hypersurfaces;*
 - *$F_1(X)$ is smooth of the expected dimension $\delta = 2n - 3 - d$ if either X is general, or $d \leq \min\{8, n\}$ and $\delta \geq 0$;*
 - *$F_1(X)$ is irreducible if $d \leq (n + 1)/2$ and X is not a smooth quadric in \mathbb{P}^3 ; it is connected if $d \leq 2n - 5$.*
- (b)
 - *If $\binom{d+1}{2} \leq n$ then $F_1(X)$ is rationally connected and is a Fano variety for X general [300, Exerc. V.4.7].*
 - *If $\binom{d+2}{2} \geq 3n$ and X is very general then $F_1(X)$ contains no rational curve [438, Thm. 3.3].*
- (c) *Any hypersurface of degree $d = 2n - 3$ in \mathbb{P}^n contains a line. The number of lines in a general such hypersurface is ²*

$$d \cdot d! \sum_{k=0}^{n-3} \frac{(2k)!}{k!(k+1)!} \sum_{I \subset \{1, \dots, n-2\}, |I|=n-2-k} \prod_{i \in I} \frac{(d-2i)^2}{i(d-i)}.$$

For instance, a general quintic threefold in \mathbb{P}^4 contains exactly 2875 lines [222].

The De Jong–Debarre Conjecture [152] states that $\dim(F_1(X)) = \delta$ for any smooth hypersurface in \mathbb{P}^n of degree $d \leq n$. The upper bound n is optimal; indeed, for any $d > n$, there exists a smooth hypersurface X of degree d in \mathbb{P}^n with $\dim(F_1(X)) > \delta$ [46, Cor. 3.2]. For instance, the Fano variety of lines on a Fermat hypersurface of degree $d \geq n$ in \mathbb{P}^n is of dimension $n - 3$ [151,

¹This improves the former upper bound $11d^2 - 28d + 12$ [453].

²See also [504] for another expression of this number.

Ch. 2, p. 50] (see also [480]), which is larger than δ when $d > n$. On the Fermat quintic threefold X_5 in \mathbb{P}^4 there are 50 one-parameter families of lines [1], cf. (c). Whereas for $n = 4$, $d = 3$, the Fano variety of lines $F_1(X_3)$ on the cubic Fermat threefold $X_3 = \{x_1^3 + x_2^3 + x_3^3 + x_4^3 + x_5^3 = 0\}$ in \mathbb{P}^4 is a smooth surface (and so, $\dim(F_1(X_3)) = \delta = 2$) carrying 30 smooth elliptic Fermat cubic curves [442].

For varieties with too many lines we have the following classification results, see [84, Thm. 4.2].

Theorem 2.2. ([441, 454]) *Let $X \subset \mathbb{P}^n$ be a k -dimensional subvariety of codimension at least 3. Then $\dim(F_1(X)) \leq 2k - 2$. Furthermore,*

- if $\dim(F_1(X)) = 2k - 2$ then $X = \mathbb{P}^k$;
- if $\dim(F_1(X)) = 2k - 3$ then X is either a quadric or a 1-parameter family of \mathbb{P}^{k-1} ;
- if $\dim(F_1(X)) = 2k - 4$ then X is either a 1-parameter family of $(k - 1)$ -dimensional quadrics, a 2-parameter family of \mathbb{P}^{k-2} , or the intersection of $6 - k$ hyperplanes with the Grassmannian $\text{Gr}(1; 4) \subset \mathbb{P}^9$ in its Plücker embedding.

For lines in an arbitrary subvariety in \mathbb{P}^n one has the following fact.

Theorem 2.3. ([321, Thm. 1]; see [364] for $m = 3$) *Let X be a projective variety of dimension m in \mathbb{P}^n . Assume X is covered by lines and through a general point of X pass a finite number of lines. Then there are at most $m!$ lines passing through a general point of X .*

Notice that a line osculating to order $m + 1$ at a general point of X must be contained in X [315]. The bound $m!$ in the theorem is optimal; it is achieved for a smooth hypersurface of degree m in \mathbb{P}^{m+1} . A similar uniform bound is known for higher degree curves.

Theorem 2.4. ([247, Thm. 2]) *Let X be an irreducible projective variety of dimension m in \mathbb{P}^n , and $x \in X$ be a general point. Let $\text{Curves}_d(X, x)$ stand for the space of curves of degree d lying on X and passing through x . Then the number of components of $\text{Curves}_d(X, x)$ is bounded above by*

$$\binom{(2m+2)((2md)^m - 1)}{2m+1}^{(2m+2)(4d^2 - 4d + 2)}.$$

The exceptional locus in this theorem has codimension at least two in X [247, Thm. 3].

For the Fano schemes of h -planes in hypersurfaces we have the following facts.

Theorem 2.5. *If X is a smooth hypersurface of degree d in \mathbb{P}^n , where $n \geq 3$, then:*

- $F_h(X)$ is irreducible of the expected dimension $\delta = (h + 1)(n - h) - \binom{d+h}{h}$ provided

$$2 \binom{d+h-1}{h} \leq n - h;$$

- for $2h \geq \max\{(n - 1)/2, n + 2 - d\}$ one has [40]

$$\dim(F_h(X)) \leq \begin{cases} (m - h)(h + 1) & \text{if } n = 2m + 1 \\ (m - h - 1)(h + 1) & \text{if } n = 2m; \end{cases}$$

- if n is odd and $d > 2$ then there are at most finitely many $(n - 1)/2$ -planes contained in X . If n is even and $d > 3$ then there is at most a one-parameter family of $(n/2 - 1)$ -planes in X [40];
- assume X is general, $\binom{d+h}{h} \leq (n - h)(h + 1)$, and $d \neq 2$ or $n \geq 2h + 1$. Then there are explicit formulas for the degree of $F_h(X)$ [161, Ex. 14.7.13], [235, Thm. 1.1], [349, Thms. 3.5.18, 4.3].³

³In the case of equality $F_h(X)$ is zero-dimensional and $\deg(F_h(X))$ is the number of h -planes in X .

If X is a complete intersection of type (that is, of multidegree) $\underline{d} = (d_1, \dots, d_s)$ then the expected dimension of $F_h(X)$ is

$$(2) \quad \delta = \delta(\underline{d}, n, h) = (n - h)(h + 1) - \sum_{i=1}^s \binom{d_i + h}{h}.$$

In particular, the expected dimension of $F_1(X)$ is

$$\delta = 2(n - 1) - (d + s), \quad \text{where } d = \sum_{i=1}^s d_i.$$

Let also

$$\delta_- = \min\{\delta, n - 2h - s\}.$$

For (a) and (b) of the following theorem see [67, 161, 321, 412] and the references therein.

Theorem 2.6. *For a complete intersection $X \subset \mathbb{P}^n$ of type $\underline{d} = (d_1, \dots, d_s)$ the following holds.*

- (a) *If $\delta_- < 0$ then $F_h(X) = \emptyset$ for a general X .*
- (b) *If $\delta_- \geq 0$ then $F_h(X)$ has dimension δ , is smooth for a general X , and is irreducible if $\delta_- > 0$.*
- (c) *If $\delta_- \geq 0$ and $\delta \geq n - h - s$ then through any point $x \in X$ passes an h -plane contained in X [367].*
- (d) *For any smooth Fano complete intersection $X \subset \mathbb{P}^n$ of type $\underline{d} = (d_1, \dots, d_s)$ with $s \geq 2$ and $d = \sum_{i=1}^s d_i \leq s + 5$, $F_1(X)$ has the expected dimension $\delta = 2n - d - s - 2$ [84].*

See also [93, 266, 267, 268, 270] for the Fano schemes of toric varieties and of complete intersections in toric varieties, and [324, 325] for applications to the machine learning.

Theorem 2.7. ([24, 367]; see [349] for $s = 1$) *Let $\Sigma(\underline{d}, n)$ be the scheme which parameterizes the complete intersections of type $\underline{d} = (d_1, \dots, d_s)$ in \mathbb{P}^n , and let $\Sigma(\underline{d}, n, h)$ be the subvariety of $\Sigma(\underline{d}, n)$ of points which correspond to the complete intersections which carry h -planes. Then*

$$\dim \Sigma(\underline{d}, n) = \sum_{j=1}^s \binom{d_j + n}{d_j}.$$

If $\gamma(\underline{d}, n, h) := -\delta(\underline{d}, n, h) > 0$ then $\Sigma(\underline{d}, n, h)$ is a nonempty, irreducible and rational subvariety of codimension $\gamma(\underline{d}, n, h)$ in $\Sigma(\underline{d}, n)$. The general point of $\Sigma(\underline{d}, n, h)$ corresponds to a complete intersection which contains a unique linear subspace of dimension h and has singular locus of dimension $\max\{-1, 2h + s - n - 1\}$ along its unique h -dimensional linear subspace (in particular, it is smooth provided $n \geq 2h + s$).

To determine the degree of $\Sigma(\underline{d}, n, h)$ we propose the following receipt.

Theorem 2.8. ([114, Cor. 2.2]) *Let X be a general complete intersection of type (d_1, \dots, d_{s-1}) in \mathbb{P}^n verifying*

$$\dim(F_h(X)) > 0 \quad \text{and} \quad \dim(\Sigma(d_m, X)) > \gamma(\underline{d}, n, h) \geq 0,$$

where

$$\Sigma(d_m, X) = |\mathcal{O}_X(d_m)|.$$

Let $\Sigma(d_s, X, h)$ be the set of points in $\Sigma(d_s, X)$ which correspond to complete intersections of type $\underline{d} = (d_1, \dots, d_s)$ contained in X and containing a linear subspace of dimension h . Then the degree $\deg(\Sigma(d_m, X, h))$ equals the coefficient of the monomial $x_0^n x_1^{n-1} \dots x_h^{n-h}$ in the product of the following polynomials in x_0, \dots, x_h :

- the product $Q_{h,\underline{d}} = \prod_{i=1}^{s-1} Q_{h,d_i}$ of the polynomials

$$Q_{h,d_i} = \prod_{v_0+\dots+v_h=d_i} (v_0x_0 + \dots + v_hx_h);$$

- the homogeneous component of degree

$$\rho := \binom{d_s + h}{h} - \gamma(\underline{d}, n, h) = \dim(F_h(X))$$

of the polynomial

$$\prod_{v_0+\dots+v_h=d_m} (1 + v_0x_0 + \dots + v_hx_h);$$

- the Vandermonde polynomial $V(x_0, \dots, x_h)$.

The general point of $\Sigma(d_m, X, h)$ corresponds to a complete intersection of type $\underline{d} = (d_1, \dots, d_s)$ which contains a unique subspace of dimension h .

Notice that our assumptions hold automatically if $\gamma(\underline{d}, n, h)$ is sufficiently small, e.g., if $\gamma(\underline{d}, n, h) = 1$.

3. GEOMETRY OF THE FANO SCHEME

In this section we consider the complete intersections of type $\underline{d} = (d_1, \dots, d_s)$ in \mathbb{P}^n whose Fano schemes have positive expected dimension $\delta = \delta(\underline{d}, n, h) > 0$, see (2). We assume $d_i \geq 2$, $i = 1, \dots, s$. If also $n \geq 2h + s + 1$ then by Theorem 2.7(b), for a general complete intersection X of type \underline{d} in \mathbb{P}^n , the Fano variety $F_h(X)$ of linear subspaces of dimension h contained in X is a smooth, irreducible variety of dimension $\delta(\underline{d}, n, h)$.

Theorem 3.1. ([161, Thm. 4.3]) *In the notation and assumptions as before, the degree of the Fano scheme $F_h(X)$ under the Plücker embedding equals the coefficient of the monomial $x_0^n x_1^{n-1} \dots x_h^{n-h}$ of the product $Q_{h,\underline{d}} \cdot e^\delta \cdot V$ where*

- V stands for the Vandermonde polynomial $V(\mathbf{x}) = \prod_{0 \leq i < j \leq h} (x_i - x_j)$;
- $e(\mathbf{x}) := x_0 + \dots + x_h$ and $\delta = \delta(\underline{d}, n, h)$;
- $Q_{h,\underline{d}}$ is the product $\prod_{i=1}^s Q_{h,d_i}$ of the polynomials

$$Q_{h,d_i} = \prod_{v_0+\dots+v_h=d_i} (v_0x_0 + \dots + v_hx_h).$$

Remark 3.2. An alternative expression for $\deg(F_h(X))$ based on the Bott residue formula can be found in [114, Formula (4)], [235, Thm. 1.1], and [236, Thm. 2]; cf. also [201, Ex. 14.7.13], [348], and [349, Sect. 3.5].

There are formulas expressing certain numerical invariants of $F_h(X)$ other than the degree. If $\delta(\underline{d}, n, h) = 1$ then $F_h(X)$ is a smooth curve; its genus was computed in [236]. In the case where $F_h(X)$ is a surface, that is, $\delta(\underline{d}, n, h) = 2$, the Chern numbers of this surface and its holomorphic Euler characteristic $\chi(\mathcal{O}_{F_h(X)})$ were computed in [114]. Actually, [114] contains formulas for $c_1(F_h(X))$ and $c_2(F_h(X))$ in the general case provided $\delta(\underline{d}, n, h) > 0$. Applying these formulas to the case where the Fano scheme is a surface, one can deduce the classically known values and new ones, as in the following examples.

Examples 3.3. • In the case of the Fano surface $F = F_1(X)$ of lines on the general cubic threefold X in \mathbb{P}^4 one has [3, 335], [114, Ex. 4.1]

$$\deg(F) = 45, \quad e(F) = c_2(F) = 27, \quad c_1(F)^2 = K_F^2 = 45, \quad \text{and} \quad \chi(\mathcal{O}_F) = 6.$$

- For the Fano surface $F = F_1(X)$ of lines on a general quintic fourfold X in \mathbb{P}^5 one gets [114, Ex. 4.2]

$$\deg(F) = 6125, \quad e(F) = c_2(F) = 309375, \quad c_1^2(F) = K_F^2 = 496125, \quad \text{and} \quad \chi(\mathcal{O}_F) = 67125.$$

- For the Fano surface $F = F_1(X)$ of lines on the intersection X of two general quadrics in \mathbb{P}^5 one has [114, Ex. 4.3]

$$\deg(F) = 32, \quad e(F) = c_2(F) = 0, \quad \text{and} \quad K_F^2 = c_1(F)^2 = 0.$$

In fact, F is an abelian surface [433].

- Finally, for the Fano scheme $F = F_2(X)$ of planes on a general cubic fivefold X in \mathbb{P}^6 one gets [114, Ex. 4.4]

$$\deg(F) = 2835, \quad e(F) = c_2(F) = 1304, \quad K_F^2 = c_1(F)^2 = 25515, \quad \text{and} \quad \chi(\mathcal{O}_F) = 3213.$$

As for the Picard numbers of the Fano schemes of complete intersections, one has the following result.

Theorem 3.4. ([283, Thm. 03]; cf. also [161]) *Let X be a very general complete intersection of type $\underline{d} = (d_1, \dots, d_s)$ in \mathbb{P}^n . Assume $\delta(\underline{d}, n, h) \geq 2$. Then $\rho(F_h(X)) = 1$ except in the following cases:*

- X is a quadric in \mathbb{P}^{2h+1} , $h \geq 1$. Then $F_h(X)$ consists of two isomorphic smooth disjoint components, and the Picard number of each component is 1;
- X is a quadric in \mathbb{P}^{2h+3} , $h \geq 1$. Then $\rho(F_h(X)) = 2$;
- X is a complete intersection of two quadrics in \mathbb{P}^{2h+4} , $h \geq 1$. Then $\rho(F_h(X)) = 2h + 6$.

The assumption ‘very general’ of this theorem cannot be replaced by ‘general’; one can find corresponding examples in [283].

Next we turn to the irregular Fano schemes of general complete intersections.

Theorem 3.5. ([114, Thm. 5.1]) *Let X be a general complete intersection of type $\underline{d} = (d_1, \dots, d_s)$ in \mathbb{P}^n . Suppose that the Fano scheme $F = F_h(X)$ of h -planes in X , $h \geq 1$, is irreducible of dimension $\delta \geq 2$. Then F is irregular if and only if one of the following holds:*

- $F = F_1(X)$ is the variety of lines on a general cubic threefold X in \mathbb{P}^4 ($\dim(F) = 2$);
- $F = F_2(X)$ is the variety of planes on a general cubic fivefold X in \mathbb{P}^6 ($\dim(F) = 2$);
- $F = F_h(X)$ is the variety of h -planes on the intersection X of two general quadrics in \mathbb{P}^{2h+3} , $h \in \mathbb{N}$ ($\dim(F) = h + 1$).

Remark 3.6. 1. The Fano surface of lines $F = F_1(X)$ on a smooth cubic threefold $X \subset \mathbb{P}^4$ in (i) was studied by Fano [191] who found, in particular, that $q(F) = 5$. Using Example 3.3 we deduce $p_g(F) = 10$; cf. also [33, Thm. 4], [64, 122, 207, 335, 445, 495, 496], [433, Sect. 4.3]. There is an isomorphism $\text{Alb}(F) \simeq J(X)$ where $J(X)$ is the intermediate Jacobian, see [122] and Section 5. The latter holds as well for $F = F_2(X)$ where $X \subset \mathbb{P}^6$ is a smooth cubic fivefold as in (ii) [127], cf. Example 3.3. Thus, $q(F) > 0$ in (i) and (ii).

Consider further the Fano scheme $F = F_h(X)$ of h -planes on a smooth intersection X of two quadrics in \mathbb{P}^{2h+3} as in (iii). By a theorem of M. Reid [433, Thm. 4.8] (see also [172, Thm. 2], [499]), F is isomorphic to the Jacobian $J(C)$ of a hyperelliptic curve C of genus $g(C) = h + 1$ (of an elliptic curve if $h = 0$). Hence, one has $q(F) = \dim(F) = h + 1 > 0$ for $h \geq 0$. There is an isomorphism $F \simeq J(X)$ where $J(X)$ is the intermediate Jacobian of X [180].

2. The complete intersections in (i)-(iii) are Fano varieties. The ones in (i) are Fano threefolds of index 2 with a very ample generator of the Picard group. The Fano threefolds of index 1 with a very ample anticanonical divisor which are complete intersections are the varieties $V_{2g-2}^3 \subset \mathbb{P}^{g+1}$ of genera $g = 3, 4, 5$, that is (see [276, Ch. IV, Prop. 1.4]; cf. Section 5):

$g = 3$: the smooth quartics V_4^3 in \mathbb{P}^4 ;

$g = 4$: the smooth intersections V_6^3 of a quadric and a cubic in \mathbb{P}^5 ;

$g = 5$: the smooth intersections V_8^3 of three quadrics in \mathbb{P}^6 .

The Fano scheme of lines $F = F_1$ of a general Fano threefold V_{2g-2}^3 is a smooth curve of a positive genus $g(F) > 0$. In fact, $g(F) = 801$ for $g = 3$, $g(F) = 271$ for $g = 4$, and $g(F) = 129$ for $g = 5$ [352], [236, Ex. 1-3], [280, Thm. 4.2.7]. For $X = V_{2g-2}^3$ with $g \in \{3, 4, 5\}$, the Abel-Jacobi map $J(F) \rightarrow J(X)$ to the intermediate Jacobian is an epimorphism, see [276] and [497, Lect. 4, Sect. 1, Ex. 1 and Sect. 3].

3. Besides the Fano threefolds V_{2g-2}^3 , there are other complete intersections whose Fano scheme of lines is a curve of positive genus. This holds, for instance, for the general hypersurface of degree $2r - 4$ in \mathbb{P}^r , $r \geq 4$, and for the general complete intersections of types $\underline{d} = (n - 3, n - 2)$ and $\underline{d} = (n - 4, n - 4)$ in \mathbb{P}^n where $n \geq 5$ and $n \geq 6$, respectively [236, Ex. 1-3]. One can find in [236] a formula for the genus of F in these cases.

4. Let X be a smooth intersection of two quadrics in \mathbb{P}^{2k+2} . Then the Fano scheme $F_k(X)$ is reduced and finite of cardinality 2^{2k+2} [433, Ch. 2], whereas $F_{k-1}(X)$ is a rational Fano variety of dimension $2k$ and index 1, whose Picard number is $\rho = 2k + 4$, see [9, 91], and the references therein.

4. COUNTING CONICS IN COMPLETE INTERSECTION

A conic in \mathbb{P}^n is a curve whose Hilbert polynomial is $2t + 1$. Any conic C is contained in a unique plane and is either a smooth (reduced) plane conic, or a pair of distinct lines, or a double line (see, e.g., [165, Lem. 2.2.6]). Thus, a pair of skew lines does not fit in our terminology.

Any smooth cubic surface S in \mathbb{P}^3 contains exactly 27 pencils of conics, and any smooth conic in S belongs to a unique such pencil. Hence the variety of conics in S is reducible and consists of 27 \mathbb{P}^1 -components.

By contrast, the number of conics on a general quartic surface in \mathbb{P}^3 is finite. Furthermore, one has

Theorem 4.1. ([28]) *There exist smooth quartic surfaces in \mathbb{P}^3 which contain 432 smooth conics; 16 of these conics are mutually disjoint.*

According to [389], 16 is the maximal number of disjoint rational curves on a quartic surface. In [23] and [189] one can find constructions of two smooth quartic surfaces in \mathbb{P}^3 carrying 352 and 320 smooth conics, respectively. The maximal number of conics lying in a smooth quartic in \mathbb{P}^3 is unknown. However, given any smooth quartic surface S , a general pencil of quartic surfaces through S contains exactly 5016 surfaces with a conic, counting things with multiplicities, see Theorem 4.2. Let $\Sigma_c(d, 3)$ be the variety of those degree d surfaces in \mathbb{P}^3 which contain conics. Then a general point of $\Sigma_c(4, 3)$ corresponds to a smooth quartic surface carrying exactly two (coplanar) conics, and so, $\deg(\Sigma_c(4, 3)) = 5016/2 = 2508$.

For higher degree surfaces in \mathbb{P}^3 the following holds.

Theorem 4.2. ([348]; see also [344, Prop. 7.1])

- For $d \geq 4$, $\Sigma_c(d, 3)$ is an irreducible subvariety of codimension $2d - 7$ in $\Sigma(d, 3)$. In particular, $\Sigma_c(4, 3)$ is a hypersurface of degree 2508 in $\Sigma(4, 3)$.⁴
- For $d \geq 5$, a general point of $\Sigma_c(d, 3)$ corresponds to a surface which contains a unique (smooth) conic, and one has

$$\begin{aligned} \deg(\Sigma_c(d, 3)) = & \binom{d}{4} (d^2 - d + 8)(d^2 - d + 6)(207d^8 - 288d^7 + 498d^6 + 5068d^5 \\ & - 15693d^4 + 31732d^3 - 37332d^2 + 9280d - 47040)/967680. \end{aligned}$$

⁴Cf. [356].

For higher dimensional hypersurfaces we have the following results.

Theorem 4.3. (a) *Let X be a hypersurface of degree n in \mathbb{P}^n , $n \geq 2$. Then X is covered by a family of conics. For the general such X , the number of conics in X passing through a general point equals*

$$\frac{(2n)!}{2^{n+1}} - \frac{(n!)^2}{2}$$

[66, Prop. 3.2], see also [330, Thm. 0.1], [479] ($n = 4$), [131, Thm. 2] ($n = 5$). For $n \geq 4$ and a general X , the variety $R_2(X)$ of smooth conics in X is smooth, irreducible, of dimension $n - 2$.

- (b) *For the general hypersurface X of degree d in \mathbb{P}^n , where $n \geq 3$, $R_2(X)$ is smooth of the expected dimension $\mu(d, n) := 3n - 2d - 2$ provided $\mu(d, n) \geq 0$, and is empty otherwise. It is irreducible provided $\mu(d, n) \geq 1$ and X is not a smooth cubic surface in \mathbb{P}^3 [203, Thm. 1.1].*
- (c) *If X is general and $d \geq 3n - 1$, then X contains no reducible conic [438, Thm. 3.4].*
- (d) *Let further X be an arbitrary smooth hypersurface of degree d in \mathbb{P}^n .*
- *Assume $n \geq 6$ and $d \leq 6$ (so, X is Fano). Let $R \neq \emptyset$ be an irreducible component of $R_2(X)$ such that the plane spanned by a general conic in R is not contained in X . Then $\dim(R) = \mu(d, n) = 3n - 2d - 2$ [204].*
 - *Let X be a smooth quartic threefold in \mathbb{P}^4 ($n = d = 4$). Then $\dim R \geq \mu(4, 4) = 2$ for any irreducible component R of $R_2(X)$. Through a general point of X passes 972 conics [130, 274, 479].*

For instance, for a general sextic hypersurface X in \mathbb{P}^5 ($n = 5$, $d = 6$, and X is Calabi-Yau) the Fano scheme $F_c(X)$ of conics in X is a smooth projective curve, whose general point corresponds to a smooth conic [86, Prop. 1.3-1.4]. By (b), for a general Fano hypersurface X in \mathbb{P}^n (that is, $d \leq n$), $R_2(X)$ is smooth, irreducible, of dimension $\mu(d, n) \geq 1$ if $n \geq 4$.

For conics in Fano complete intersections we have the following facts.

Theorem 4.4. ([463, Lem. 1]) *Let $X \subset \mathbb{P}^n$ be the general complete intersection of dimension ≥ 3 . Then the Fano scheme of conics in X is a smooth projective variety.*

Theorem 4.5. ([35, 66]) *Consider a smooth complete intersection $X \subset \mathbb{P}^n$ of type $\underline{d} = (d_1, \dots, d_s)$. Assume*

$$(3) \quad 2 \sum_{i=1}^s d_i = n - s + 1.$$

Then the following holds.

- *X is a Fano variety with Picard number one of index $\iota(X) = \frac{n-s+1}{2}$. The anticanonical degree of a conic in X equals $n - s + 1$.*
- *For a general X the family of conics in X is a nonempty, smooth and irreducible component of the Chow scheme of X ⁵, of dimension $2(n - s - 1)$.*
- *Let $e(X)$ ($e_0(X)$, respectively) be the number of conics passing through a general pair of points of X (passing through a general point $x \in X$ and having a given general tangent direction at x , respectively). Then for a general X these conics are smooth, and one has*

$$e_0(X) = e(X) = \prod_{i=1}^s (d_i - 1)! d_i!$$

See [35] for formulas for the numbers of lines and conics in X meeting three general linear subspaces in \mathbb{P}^r of suitable dimensions; cf. also [335] in the case of lines, [44, Cor. 1.5], and [502].

⁵The latter holds as well if the equality in (3) is replaced by the inequality “ \leq ” [66, Cor. 2.1].

See also [398] for a study on the Kontsevich moduli spaces $\overline{\mathcal{M}}_{0,2}(X, 2)$ of conics through a pair of points in a smooth complete intersection X (see the definition below).

Set

$$\epsilon(d, n) = 2d + 2 - 3n = -\mu(d, n).$$

Consider the subvariety $\Sigma_c(d, r)$ of $\Sigma(d, r)$ whose points correspond to hypersurfaces containing plane conics.

Theorem 4.6. ([114, Thms. 6.1, 6.6]) *Assume $d \geq 2$, $n \geq 3$, and $\epsilon(d, n) \geq 0$. Then the following holds.*

- (a) $\Sigma_c(d, n)$ is irreducible of codimension $\epsilon(d, n)$ in $\Sigma(d, n)$.
- (b) If $\epsilon(d, n) > 0$ and $(d, n) \neq (4, 3)$ then the hypersurface corresponding to the general point of $\Sigma_c(d, n)$ contains a unique conic, and this conic is smooth. In the case $(d, n) = (4, 3)$ it contains exactly two distinct conics, and these conics are smooth and coplanar.
- (c) If $\epsilon(d, n) > 0$ and $(d, n) \neq (4, 3)$ then one has

$$\deg(\Sigma_c(d, n)) = -\frac{5}{32} \binom{n+1}{3} \eta(1, 1, 1),$$

where η is the homogeneous form of degree $3n-1$ in the formal power series decomposition of

$$\left(\prod_{|\mathbf{v}|=d} (1 + \langle \mathbf{v}, \mathbf{x} \rangle) \right) \cdot \left(\prod_{|\mathbf{v}|=d-2} (1 + \langle \mathbf{v}, \mathbf{x} \rangle) \right)^{-1}$$

with $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{v} = (v_1, v_2, v_3) \in (\mathbb{Z}_{\geq 0})^3$, and $|\mathbf{v}| = v_1 + v_2 + v_3$.

The latter formulas are obtained by applying Bott's residue formula; see, e.g., [68, 73, 184, 235, 359] for generalities.

Recall the definition of the Kontsevich moduli spaces of stable maps [306]. Let X be a smooth projective variety in \mathbb{P}^n . The *Kontsevich moduli space* $\overline{\mathcal{M}}_{g,r}(X, e)$ parameterizes the isomorphism classes of corteges (C, f, x_1, \dots, x_r) where

- C is a proper, connected, nodal curve of arithmetic genus g ;
- $f: C \rightarrow X$ is a morphism whose image is a curve of degree e in \mathbb{P}^n ;
- (x_1, \dots, x_r) is an ordered collection of distinct smooth points of C ;
- the cortege (C, f, x_1, \dots, x_r) admits only finitely many automorphisms.

In general, $\overline{\mathcal{M}}_{g,r}(X, e)$ is a proper Deligne-Mumford stack. The underlying variety of $\overline{\mathcal{M}}_{g,r}(X, e)$ is projective, but does not need to be smooth or irreducible. However, $\overline{\mathcal{M}}_{0,0}(X, e)$ is a compactification of the variety $R_e(X)$ of smooth rational curves in X of degree e .

Recall that a curve $C \simeq \mathbb{P}^1$ in a smooth projective variety X of dimension m is called *free* if $\mathcal{N}_{C/X} \simeq \mathcal{O}(a_1) \oplus \dots \oplus \mathcal{O}(a_{m-1})$ where $a_i \geq 0 \forall i$. Statement (a) of the next theorem follows from Theorem 4.3(b).

Theorem 4.7. (a) *Let X be a smooth hypersurface in \mathbb{P}^n of degree $d < n$.*

- *If X is general then $\overline{\mathcal{M}}_{0,0}(X, 2)$ is irreducible and of the expected dimension $\mu(d, n) = 3n - 2d - 2$ [165, 2.3.4].*
- *If X is arbitrary (but smooth) then there is a unique component of $\overline{\mathcal{M}}_{0,0}(X, 2)$ which contains a conic passing through the general point of X . Moreover, if the dimension of the variety of non-free lines on X is at most $n - 3$ then there is a unique component of $R_2(X)$ whose general point corresponds to a smooth conic through the general point of X [165, 2.3.6].*

- (b) *Let X be the Fermat hypersurface in \mathbb{P}^n of degree d . Then the moduli space $\overline{\mathcal{M}}_{0,0}(X, e)$ is irreducible of the expected dimension $e(n+1-d) + (n-4)$ in the following cases:*

- $e \geq 2$ and $ed \leq n + 1$;
- $e \geq 3$ and $ed \leq n - 1$.

It is irreducible as well if $e = 3$ and $3d \leq n + 5$ [165, 2.4.4–2.4.6, 2.4.23, 2.4.30, 2.4.36].

5. LINES AND CONICS ON FANO THREEFOLDS AND THE ABEL-JACOBI MAPPING

5.1. The Fano-Iskovskikh classification. The content of this section is partially borrowed from [280, Sect. 4.1] and [313, Sect. 2]. Let X be a smooth Fano threefold, that is, a smooth threefold with an ample anticanonical divisor $-K_X$. One attributes to X the following integers:

- the genus⁶

$$g(X) = (-K_X)^3/2 + 1 = \dim | -K_X | - 1 \geq 2;$$

- the index $\iota(X)$, that is, the maximal natural number in $\{1, \dots, 4\}$ such that $-K_X = \iota(X)H$ for an ample divisor H on X ;
- the Picard rank $\rho(X)$ such that $\text{Pic}(X) \simeq \mathbb{Z}^{\rho(X)}$;
- the degree $d(X) = H^3$;
- the Matsusaka constant $m_0 = m_0(X)$, that is, the minimal integer such that m_0H is very ample.

One has $g(X) = \iota(X)^3 d(X)/2 + 1$. Hence $d(X) = 2g(X) - 2$ if and only if $\iota(X) = 1$.

The Fano-Iskovskikh classification of the Fano threefolds with Picard rank $\rho(X) = 1$ yields

Theorem 5.1. ([276], [280, Sect. 4.1]) *Let X be a smooth Fano threefold of genus g with $\rho(X) = 1$. Then one of the following (i)–(iii) holds.*

- (i) *The anticanonical divisor is very ample, and the linear system $| -K_X |$ defines an embedding φ of X onto a projectively normal threefold $\varphi(X)$ of degree $2g - 2$ in \mathbb{P}^{g+1} with one of the following:*
 - (i3) $g = 3$ and $\varphi(X) \subset \mathbb{P}^4$ is a smooth quartic threefold;
 - (i4) $g = 4$ and $\varphi(X) \subset \mathbb{P}^5$ is complete intersection of a quadric and a cubic hypersurfaces;
 - (i5) $g = 5$ and $\varphi(X) \subset \mathbb{P}^6$ is complete intersection of three quadric hypersurfaces;
 - (ig) $g \geq 6$ and $\varphi(X) \subset \mathbb{P}^{g+1}$ is an intersection of quadric hypersurfaces.
- (ii) $g = 2$ and X is a sextic double solid, that is, $| -K_X |$ defines a double cover $X \rightarrow \mathbb{P}^3$ ramified along a smooth surface $S \subset \mathbb{P}^3$ of degree 6;
- (iii) $g = 3$ and $| -K_X |$ defines a double cover $X \rightarrow Q$ over a smooth quadric threefold $Q \subset \mathbb{P}^4$ ramified along a smooth surface $S \subset Q$ of degree 8.

The table of numerical data of the Fano threefolds with $\rho(X) = 1$ can be found in [280, Sect. 12.2]. These threefolds form 17 deformation families. According to the index, these are:

- $\iota = 1$: 10 families with genera varying from 2 to 12 excluding 11;
- $\iota = 2$: 5 families of *del Pezzo threefolds* with anticanonical degree $-K_X^3 = 8d$, $d = 1, 2, 3, 4, 5$;
- $\iota = 3$: the smooth quadric Q in \mathbb{P}^4 with anticanonical degree 54;
- $\iota = 4$: \mathbb{P}^3 with anticanonical degree 64.

The families of Fano threefolds $V_{2g-2}^3 \subset \mathbb{P}^{g+1}$ with $\rho = 1$ and $\iota = 1$ are classified according to the genus g as follows.

- $g = 2, 3, 4, 5$: the Fano threefolds listed in (i3)–(i5), (ii), and (iii);
- $g = 6$: the smooth intersections $\text{Gr}(2; 5) \cap \mathbb{P}^7 \cap Q$ of the Grassmannian $\text{Gr}(2; 5)$ with a linear subspace \mathbb{P}^7 and a quadric Q in \mathbb{P}^9 , and
- $g = 6$: the *Gushel threefolds* of genus 6, that is, the double covers of the del Pezzo threefold $Y = \text{Gr}(2; 5) \cap \mathbb{P}^6 \subset \mathbb{P}^9$ of degree 5 branched along a smooth quadric section $Q \cap Y$;

⁶The genus $g(X)$ is equal to the genus g of a general curve section of the anticanonical model of X .

- $g = 7$: the smooth linear sections $\text{OGr}_+(5; 10) \cap \mathbb{P}^8$ of a connected component $\text{OGr}_+(5; 10)$ of the orthogonal Lagrangian Grassmannian in \mathbb{P}^{15} ;
- $g = 8$: the smooth linear sections $\text{Gr}(2; 6) \cap \mathbb{P}^9$ of the Grassmannian $\text{Gr}(2; 6) \subset \mathbb{P}^{14}$;
- $g = 9$: the smooth linear sections $\text{LGr}(3; 6) \cap \mathbb{P}^{10}$ of the symplectic Lagrangian Grassmannian $\text{LGr}(3; 6) \subset \mathbb{P}^{13}$;
- $g = 10$: the smooth linear sections $\Omega^5 \cap \mathbb{P}^{11}$ of the homogeneous G_2 -fivefold $\Omega^5 \subset \mathbb{P}^{13}$ (an adjoint orbit of the group G_2);
- $g = 12$: the smooth zero loci of triplets of sections of the rank 3 vector bundle $\Lambda^2 \mathcal{E}^\vee$, where \mathcal{E} is the universal bundle over the Grassmannian $\text{Gr}(3; 7)$.

Initially, the Fano threefolds V_{2g-2}^3 with $g = 7, 9, 12$ were obtained from simpler ones via certain birational transformations (*elementary Sarkisov links*); see, e.g., [280, Prop. 3.4.1, 4.4.1, Thm. 4.3.3, 4.4.11], [477]. We use above the Mukai description [378, 379, 380, 381] of the V_{2g-2}^3 with $g = 7, \dots, 10$ as linear sections $X_{2g-2}^n \cap \mathbb{P}^{g+1}$ of certain special Grassmannians $X_{2g-2}^n = G/P$, which are flag varieties embedded in \mathbb{P}^{g+n-2} .

Notice that the family of Gushel threefolds of genus 6 (called also *special Gushel-Mukai threefolds*) is a flat specialization of the family $\text{Gr}(2; 5) \cap \mathbb{P}^7 \cap Q$ of *general Gushel-Mukai threefolds*. Thus, the Fano threefolds of genus 6 form one deformation family of Gushel-Mukai threefolds. The same holds for the Fano threefolds of genus 3; indeed, the family of smooth quartic threefolds in \mathbb{P}^4 specializes to the double covers $X \rightarrow Q$ ramified along $Q \cap Y$, where Y is a quartic in \mathbb{P}^4 .

The families of Fano threefolds with $\rho = 1$ and $\iota = 2$ are classified according to the anticanonical degree $-K_X^3 \in \{8d, d = 1, 2, 3, 4, 5\}$ as follows (Fujita [200]; see [280, Thm. 3.3.1]).

- $d = 1$: the smooth hypersurfaces of degree 6 in the weighted projective space $\mathbb{P}(3, 2, 1, 1, 1)$.
Another realization: the Veronese double cones, that is, the double covers $X \rightarrow V$, where $V \subset \mathbb{P}^6$ is the cone over the second Veronese surface in \mathbb{P}^5 , branched at the vertex v of V and along a smooth intersection of V with a cubic hypersurface which does not pass through v [274, 486], see also [214, 215, 250];
- $d = 2$: the quartic double solids, that is, the double covers $X \rightarrow \mathbb{P}^3$ branched along a smooth quartic surface $S \subset \mathbb{P}^3$;
- $d = 3$: the smooth cubic threefolds $X \subset \mathbb{P}^4$;
- $d = 4$: the smooth complete intersections of two quadrics in \mathbb{P}^5 ;
- $d = 5$: the smooth linear sections $\text{Gr}(2; 5) \cap \mathbb{P}^6$ of the Grassmannian $\text{Gr}(2; 5) \subset \mathbb{P}^9$.

5.2. Lines and conics on Fano threefolds. Let X be a Fano variety of index $\iota(X)$, and let $H = K_X/\iota(X) \in \text{Pic}(X)$. The *lines* and *conics* on X are the irreducible curves C in X satisfying $C \cdot H = 1$ and $C \cdot H = 2$, respectively. One considers the Fano schemes $F_1(X)$ of lines and $F_c(X)$ of conics in X meaning actually the unions of the components of the Hilbert schemes of X whose general points correspond to lines and conics on X , respectively.

In the case where $-K_X$ is very ample, e.g., if $\rho(X) = \iota(X) = 1$, the lines and conics on X are sent to the usual lines and conics under the anticanonical embedding $X \hookrightarrow \mathbb{P}^{2g(X)-2}$. Otherwise, consider, for instance, a *double solid*, that is, a double cover $\pi: X \rightarrow \mathbb{P}^3$ branched along a smooth surface $S \subset \mathbb{P}^3$ of degree 4 (degree 6, respectively) which does not contain any line (any conic, respectively). An irreducible curve C on X is a line (a conic, respectively) if and only if $C' = \pi(C)$ is a bitangent line of S in \mathbb{P}^3 (a conic in \mathbb{P}^3 with only even local intersection indices with S , respectively). In these cases $\pi^*(C') = C + i(C)$ has two irreducible components, where i is the involution associated to π .

The following theorem summarizes results from [92, 130, 127, 257, 276, 329, 338, 469, 477, 479].

Theorem 5.2. *Let X be a Fano threefold with $\rho(X) = 1$ of index $\iota(X) = 1$ and genus $g = g(X)$. Then the following holds.*

- Every line on X meets $l(g)$ lines counting things with multiplicities, where
 - $l(2) = 625$;
 - $l(3) = 81$ if X of genus $g = 3$ is a double cover of a quadric $Q \subset \mathbb{P}^4$;
 - $l(4) = 31$;
 - $l(5) = 17$;
 - $l(6) = 11$;
 - $l(8) = 6$ provided X of genus $g = 8$ is general [57, 352].
- If $-K_X$ is very ample and $g \geq 3$ then $F_1(X)$ is of pure dimension 1; it is smooth, reduced and irreducible for a general X . Through a point of X passes at most a finite number of lines if $g = 3$ and X is a general quartic surface in \mathbb{P}^4 , at most 6 lines if $g = 4$, and at most 4 lines if $g \geq 5$ [280, 4.2.2, 4.2.7].
- Assume $-K_X$ is very ample and $g \geq 5$. Then $F_c(X)$ is two-dimensional. Furthermore, through almost any point of X (any point if $g \geq 10$) passes at most a finite number of conics. A general conic in X meets at most a finite number of lines if $g \geq 5$; the latter is true for any conic if $g \geq 9$ [280, 4.2.5–4.2.6].

Notice that there are smooth quartic threefolds in \mathbb{P}^4 (for instance, the Fermat quartic) which contain cones over curves. However, through a general point of any smooth quartic in \mathbb{P}^4 passes exactly 972 conics [130, 274]. See also [280, 4.2.7] for the genera of the curve $F_1(X)$ for a general X as in the theorem.

The following two theorems summarize the results of [3, 72, 172, 205, 258, 260, 276, 313, 352, 419, 479].

Theorem 5.3. ([313, Thm. 1.1.1]) *Let X be a smooth Fano threefold with $\rho(X) = 1$ of index $\iota = 2$ and degree $d = d(X) \geq 3$. Then the Fano scheme of lines $F_1(X)$ is a smooth irreducible surface. In particular,*

- $d = 3$: $F_1(X)$ is a minimal surface of general type with irregularity 5, geometric genus 10, and canonical degree $K_{F_1(X)}^2 = 45$;
- $d = 4$: $F_1(X)$ is an abelian surface;
- $d = 5$: $F_1(X) \simeq \mathbb{P}^2$.

Theorem 5.4. ([313, Thm. 1.1.1]) *Let X be a smooth Fano threefold with $\rho(X) = 1$ of index 1 and genus $g = g(X) \geq 7$. Then the Fano scheme of conics $F_c(X)$ is a smooth irreducible surface. More precisely,*

- $g = 7$: $F_c(X)$ is symmetric square of a smooth curve of genus 7;
- $g = 8$: $F_c(X)$ is a minimal surface of general type with irregularity 5, geometric genus 10, and canonical degree $K_{F_c(X)}^2 = 45$;
- $g = 9$: $F_c(X)$ is a ruled surface isomorphic to the projectivization of a simple rank 2 vector bundle on a smooth curve of genus 3;
- $g = 10$: $F_c(X)$ is an abelian surface;
- $g = 12$: $F_c(X) \simeq \mathbb{P}^2$.

There exists the following duality between the Fano threefolds of indices 1 and 2 based on the Mukai construction [379, 380].

Theorem 5.5. ([314]; see also [313, Appendix B]) *For any smooth Fano threefold X with $\rho(X) = 1$, $\iota(X) = 1$, and $g(X) \in \{8, 10, 12\}$ there is a smooth Fano threefold Y with $\rho(Y) = 1$, $\iota(Y) = 2$, and $d(Y) = g(X)/2 - 1$ such that $F_1(X) \simeq F_c(Y)$ (and the derived categories of X and Y are equivalent).*

See also [476] for lines and conics on del Pezzo threefolds, [475] for polarized Fano varieties covered by linear subspaces, and [39, 157, 178, 180, 233, 342, 355, 417] for the Fano schemes of lines on some higher dimensional Fano varieties.

5.3. The Abel-Jacobi mapping. Recall [122] that the *intermediate Jacobian*

$$J(X) = H^{2,1}(X)^*/(H_3(X, \mathbb{Z}) \text{ modulo torsion})$$

of a smooth Fano threefold X is a principally polarized abelian variety. Using a fine structure of the intermediate Jacobian one can detect the non-rationality of X ; see, e.g., [38, 122, 302, 416, 513, 529], cf. also [11, 95, 414, 443]. Given a variety F we let $A(F)$ be the Albanese variety of F . For a Fano scheme $F(X)$ of a Fano threefold X the *Abel-Jacobi mapping* $A(F(X)) \rightarrow J(X)$ is defined via the *cylinder homomorphism* $H_1(F(X), \mathbb{Z}) \rightarrow H_3(X, \mathbb{Z})$, see the next section. For the Fano threefolds X one considers the Fano surface of lines $F(X) = F_1(X)$ if $\iota(X) = 2$ and of conics $F(X) = F_c(X)$ if $\iota(X) = 1$. For certain Fano threefolds X , the Abel-Jacobi mapping is known to be either an isomorphism, or an isogeny, hence $q(F) = h^{2,1}(X)$. We summarize these results in the following theorem. See Table 12.2 in [280] for the values of $h^{2,1}(X)$.

Theorem 5.6. ([280, §8.2]) *Consider a Fano threefold X with $\rho(X) = 1$ of genus $g = g(X)$, degree $d = d(X)$, and index $\iota = \iota(X) \in \{1, 2\}$.*

$\iota = 2$: *Assume $\iota = 2$, and let $F = F_1(X)$ be the Fano scheme of lines on X . Then the following holds.*

$d = 5$: $A(F)$ and $J(X)$ are both trivial [276].

$d = 2, 3, 4$: F is a smooth irreducible surface, the Abel-Jacobi mapping $A(F) \rightarrow J(X)$ is an isomorphism, and X is uniquely determined by F in the following cases:

$d = 2$: $X \rightarrow \mathbb{P}^3$ is a quartic double solid whose branching surface $S \subset \mathbb{P}^3$ has no line. One has $q(F) = 10$ [276, Ch. III, Sect. 1], [116, 121, 484, 485, 525, 526];

$d = 3$: X is a smooth cubic threefold in \mathbb{P}^4 . One has $q(F) = 5$ [122];

$d = 4$: X is a smooth complete intersection of two quadrics in \mathbb{P}^5 . One has $q(F) = 2$, $F \cong J(X)$ is an abelian surface [172, 180, 433, 499].

$d = 1$: *Let $X \rightarrow V$ be a double Veronese cone branched along a smooth surface $W \subset V$ cut out by a cubic hypersurface in \mathbb{P}^6 , and let F_0 be the Hilbert scheme of conics in V 3-tangent to W . Then F and F_0 are smooth irreducible surfaces, there is a branched double covering $\pi: F \rightarrow F_0$, the Fano scheme $\mathcal{F}(X)$ is not reduced and consists of F and the embedded ramification curve of π , and the Abel-Jacobi mapping yields an isogeny $A(F)/\pi^*A(F_0) \rightarrow J(X)$ where $\dim J(X) = h^{2,1}(X) = 21$, and $X \rightarrow V$ is uniquely determined by the pair (F, π) [486].*

$\iota = 1$: *Assume $\iota = 1$, and let $F = F_c(X)$ be the Fano scheme of conics on X . Then the following holds.*

$g = 2, 3, 4, 5$: F is a smooth irreducible surface and the Abel-Jacobi mapping $A(F) \rightarrow J(X)$ is an isomorphism in the following cases:

$g = 2$: $X \rightarrow \mathbb{P}^3$ is the general sextic double solid. One has $q(F) = 52$ [92, Thm. 3.3];

$g = 3$: X is the general quartic threefold in \mathbb{P}^4 . One has $q(F) = 30$ [130, Prop. 3.6], [329, Prop. 1], [479];

$g = 3$: $X \rightarrow Q$ is the general double cover of a smooth quadric $Q \subset \mathbb{P}^4$ branched along a smooth surface $S \subset Q$ of degree 8. One has $q(F) = 30$ [309];

$g = 4$: X is the general complete intersection of a quadric and a cubic hypersurfaces in \mathbb{P}^5 . One has $q(F) = 20$, $c_1(F)^2 = 23355$, $c_2(F) = 11961$, $p_a(F) = 2942$ [261, Cor. 18, Thm. 20], [352];

$g = 5$: X is the general complete intersection of three quadrics in \mathbb{P}^6 [33, 409, 525]. One has $q(F) = 14$, and X is uniquely determined by F [32, 149, 199, 326], [499, §5].

$g = 6$: F is irreducible for any smooth $X = \text{Gr}(2; 5) \cap \mathbb{P}^7 \cap Q \subset \mathbb{P}^9$ [156, Cor. 8.3]. For the general such X , the Abel-Jacobi mapping $A(F) \rightarrow J(X)$ is an isogeny, $q(F) = 10$, the image of F in $J(X)$ is algebraically equivalent to $2\Theta^8/8!$ where $\Theta \subset J(X)$ is a

Poincaré divisor, and X is uniquely determined by F [338, Thms. 0.15, 0.16, 0.18], [255];

- *Consider a special Gushel threefold $\pi: X \rightarrow Y$ of genus 6. Suppose X is general, that is, X is the double cover of a quintic threefold $Y = \text{Gr}(2; 5) \cap \mathbb{P}^6$ in \mathbb{P}^9 branched along a quadric section $Y \cap Q$, where $\mathbb{P}^6 \subset \mathbb{P}^9$ and Q are general. Then one has $F = F' \cup F''$ where $F' = \pi^*(F_1(Y))$ is rational and F'' is a non-normal irreducible surface with $q(F'') = 10$. For the normalization \tilde{F}'' of F'' , the Abel-Jacobi mapping $A(\tilde{F}'') \rightarrow J(X)$ is an isomorphism [256, 257].*

$g \geq 7$: F is a smooth irreducible surface with $q(F) = h^{2,1}(X)$, cf. Theorem 5.4 and [280, 12.2].

Similar results were established in [155, 157, 197, 217, 353, 420, 487, 488, 497, 509, 527] for various, possibly singular, Fano threefolds with $\rho = 1$ and for various families of curves. See also, e.g., [16, 56, 57, 127, 179, 180, 212, 289, 311, 326, 342, 343, 360, 361, 391, 409, 417, 499, 508], [280, Thm. 8.2.1] and Theorem 5.10 below for some variations and higher dimensional analogs.

5.4. The cylinder homomorphism. The results of the previous subsection are ultimately related to the studies of various cylinder homomorphisms.

Definition 5.7. Let $X \subset \mathbb{P}^n$ be a projective variety of dimension m , and let $\pi: \mathcal{C} \rightarrow S$ be a family of irreducible curves in X over an irreducible base S . Then the *cylinder homomorphism* associated with \mathcal{C} is defined as follows:

$$\Psi_{\mathcal{C}}: H_{m-2}(S, \mathbb{Z}) \rightarrow H_m(X, \mathbb{Z}), \quad \gamma \mapsto \bigcup_{s \in \gamma} C_s,$$

where γ is a topological $(m-2)$ -cycle in S and $C_s = \pi^{-1}(s)$. We let $\Psi_{\mathcal{C}, \mathbb{Q}}: H_{m-2}(S, \mathbb{Q}) \rightarrow H_m(X, \mathbb{Q})$ be the induced homomorphism.

For instance, if $S = F_1(X)$ then $\pi: \mathcal{C} \rightarrow S$ is the universal family of lines in X , and if $S = F_c(X)$ then $\pi: \mathcal{C} \rightarrow S$ is the universal family of conics in X , etc. Choosing for S the Fano scheme of lines $F_1(X)$ and letting $\Psi_1, \Psi_{1, \mathbb{Q}}$ be the associated cylinder homomorphisms, we have the following.

Theorem 5.8. ([462]) *Let X be a hypersurface in \mathbb{P}^n of degree d . Assume $n \geq 4$, $d \leq n-1$, and either X is general, or $d = 3$ and X is smooth. Then the following holds.*

Even n : Ψ_1 is an isomorphism modulo torsion;

Odd n : $\dim(\ker(\Psi_{1, \mathbb{Q}})) \leq (n-3)/4$, and Ψ_1 is surjective for $d \leq (n+5)/2$.

In particular [39, 122],

$d \leq 4$: Ψ_1 is an isomorphism modulo torsion for $n = 5$;

$d = 3$: Ψ_1 is surjective, and Ψ_1 is an isomorphism modulo torsion for even n ;

$d = 3$: Ψ_1 is an isomorphism for $n = 4, 5$.

Recall [346, Sect. 1], [463] that a smooth complete intersection $X \subset \mathbb{P}^n$ of multidegree (d_1, \dots, d_s) is a Fano variety of index ι if and only if $\iota := n+1 - \sum_{i=1}^s d_i > 0$.

Theorem 5.9. *Let $X \subset \mathbb{P}^n$ be a smooth Fano complete intersection. Then the following holds.*

- X is covered by conics [131, 132, 330];
- X is covered by lines provided $\iota(X) \geq 2$ [497, Lect. 4, Proof of Lemma 1];
- the lines in X sweep out a hypersurface provided $\iota(X) = 1$ [462].

The following theorem comprise certain results of [331, 332] for the case of Fano complete intersections of index 1.

Theorem 5.10. ([463]) *Let X be a general Fano complete intersection in \mathbb{P}^n of dimension $k \geq 3$. Then*

- both $\Psi_{1,\mathbb{Q}}$ and $\Psi_{c,\mathbb{Q}}$ are surjective;
- if $k = 2s - 1$ is odd, then the Abel-Jacobi mappings

$$J^{s-1}(F_1(X)) \rightarrow J^s(X) = J(X) \quad \text{and} \quad J^{s-1}(F_c(X)) \rightarrow J^s(X) = J(X)$$

are surjective.

See also, e.g., [57, 294, 293, 331, 332, 333, 397, 468, 513] and the literature therein for the cylinder mappings on the cycles of other intermediate dimensions.

6. COUNTING RATIONAL CURVES

6.1. Varieties of rational curves in hypersurfaces. Given a hypersurface X of degree d in \mathbb{P}^n , we let $R_e(X)$ be the space of smooth rational curves of degree e in \mathbb{P}^n lying in X . This is an open subscheme of the Hilbert scheme $\text{Hilb}^{e+1}(X)$. The number

$$(4) \quad \mu_e = \mu_e(d, n) = (n + 1 - d)e + n - 4$$

is called the *expected dimension* of $R_e(X)$. Notice that

$$\mu_2 = \mu_2(d, n) = -\epsilon(d, n) = 3n - 2d - 2.$$

Theorem 6.1. *Let X be the general hypersurface of degree $d \geq 2$ in \mathbb{P}^n , where $n \geq 3$. Then the following holds.*

- $R_e = \emptyset$ if $\mu_e < 0$ and $e \leq d + 1$ [203, Thm. 1.1].
- $R_e(X)$ is smooth of dimension μ_e if $\mu_e \geq 0$ and either $e \leq 3$, or $2e \leq d + 3$;
 - $R_e(X)$ is an integral, locally complete intersection scheme of dimension μ_e if $d \leq n - 2$ [44, 440], [203, Thm. 1.1], [225, Thm. 1.1], see also [79, 227, 472, 473];
 - $R_e(X)$ is irreducible, generically smooth and of dimension μ_e if $n \geq 4$, $d \leq n - 1$, and $e \leq d - 1$ [431, Thm. 21], [494].
- If $2d \leq n + 1$ then through any point of X passes a family of degree e rational curves of dimension $e(n + 1 - d) - 2 \geq ed$. In particular, through any point of X passes a $2(n - d)$ -dimensional family of smooth conics [225].
- Let R be a sweeping component of $R_e(X)$, that is, the corresponding rational curves sweep out an open subset of X . If $(n + 1)/2 \leq d \leq n - 3$, then R is not uniruled [42, Thm. 1.1].

Notice that the dimension of $R_e(X)$ can be strictly larger than μ_e for particular smooth hypersurfaces X . For instance [506], the family of lines on the Fermat quartic in \mathbb{P}^4 is two-dimensional, while the general quartic in \mathbb{P}^4 carries a one-parameter family of lines. See further examples in [204, Ex. 3.17–3.18]. See also [44, 141] for results on the Gromov-Witten invariants.

Concerning arbitrary smooth hypersurfaces, we have the following results.

Theorem 6.2. *Let X be a hypersurface of degree d in \mathbb{P}^n .*

- If X is smooth along C for some $C \in R_e(X)$ then $\dim_C R_e(X) \geq \mu_e = \chi(\mathcal{N}_{C/X})$ [204, Rem. 3.2], [300, II, Thm. 1.2].
- For any smooth hypersurface $X \subset \mathbb{P}^n$ of degree d where either $d = 3$ and $n \geq 5$, or $d \geq 4$ and $n \geq 2^{d-1}(5d - 4)$, the scheme $R_e(X)$ is irreducible of the expected dimension μ_e [79, 141].
- Assume X is smooth and $d + 2k - n \geq 3$. Then the quadrics of dimension k sweep out a subvariety of dimension at most $n - k - 1$ in X [40].⁷
- Assume $d = n \geq 5$. Then there is a countable set of closed, codimension two subvarieties of X such that the image of any generically finite, regular morphism from a del Pezzo surface to X is contained in one of these [47, Thm. 1.4].

⁷In particular, for $d \geq n + 1$ the conics contained in X do not cover X . The latter can be shown directly.

In [473] one can find restrictions under which the Kontsevich moduli space $\overline{\mathcal{M}}_{0,0}(X, e)$ is of general type. See also [399] for studies on the Kontsevich moduli spaces of rational curves through marked points in a Fano complete intersection variety.

Recall that a projective variety X is *rationally connected* if each pair of closed points of X is contained in a rational curve, see [151, 300]. Notice that any Fano variety X is rationally connected and covered by the rational curves of degree $\leq \dim(X) + 1$. These curves generate the Mori cone of effective 1-cycles on X [83, 304, 368], [300, Sect. IV.3, Cor. IV.1.15]. See also [162] for criteria of *simple rational connectedness*.

The following results concern rational curves and rational surfaces in smooth complete intersection Fano varieties. It is known that a general such variety of sufficiently small multidegree is unirational [11, 300, 301, 303, 401, 515].

Theorem 6.3. *Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of type $\underline{d} = (d_1, \dots, d_s)$. Then the following holds.*

- (a) *X is rationally connected if and only if $\sum_{i=1}^s d_i \leq n$, that is, X is a Fano variety [162].*
- (b) *Assume $\omega_X \simeq \mathcal{O}_X(-1)$ that is, $\sum_{i=1}^s d_i = n$. Let S be a smooth variety of dimension $2 \leq \dim(S) \leq \dim(X) - 2$ with ω_S^\vee nef. Consider a generically finite morphism $f: S \rightarrow X$. In the case $\omega_S \simeq \mathcal{O}_S$ assume further that $f(S) \subset \mathbb{P}^n$ is linearly non-degenerate. Suppose f extends to a morphism $F: \mathcal{S} \rightarrow X$, where $\mathcal{S} \rightarrow \mathcal{B}$ is a deformation family containing S as a fiber. Then the image $F(\mathcal{S})$ is contained in a subvariety of codimension at least two in X [482, Thm. 1.1].*

Notice that statement (b) generalizes Theorem 6.2(d). See also [502, 503] for count of rational and elliptic curves on rational surfaces and in projective spaces, [456, 457] for count of rational curves in Fano threefolds and Fermat hypersurfaces, and [536] for count of rational curves in del Pezzo manifolds.

6.2. Twisted cubics in complete intersections. The next results concern enumeration of twisted cubics on Fano complete intersections.

Theorem 6.4. (a) *Let $\Sigma_{tc}(d, 3) \subset \Sigma(d, 3)$ be the locus of degree d surfaces in \mathbb{P}^3 which contain twisted cubic curves. There is an explicit expression of the degree of $\Sigma_{tc}(d, 3)$ as a polynomial in d of degree 24 [344, §8].*

- (b) *Let $X \subset \mathbb{P}^n$ be a smooth complete intersection of type (d_1, \dots, d_s) where*

$$\sum_{i=1}^s (d_i - 1) = \frac{n - s}{3} + 1.$$

Then the number of twisted cubics in X passing through three general points in X equals

$$\prod_{i=1}^s ((d_i - 1)!)^2 d_i!.$$

In particular, through three general points of a smooth cubic threefold $X \subset \mathbb{P}^4$ pass exactly 24 twisted cubics in X [35].

See also [411], the survey article [410], and the references therein.

7. HYPERSURFACES WITH FEW RATIONAL CURVES

7.1. Rational curves on K3 surfaces. Recall the following theorem.

Theorem 7.1. ([370]) *Let $d > 0$ and $g \geq 0$ be integers. There is a smooth curve C of degree d and genus g lying in a smooth quartic surface X in \mathbb{P}^3 if and only if either $g = d^2/8 + 1$, or $g < d^2/8$ and $(d, g) \neq (5, 3)$.*

For surfaces in \mathbb{P}^3 containing smooth elliptic quartic curves, there are the following facts.

Theorem 7.2. *Let $\Sigma_{\text{ell},4}(d, 3)$ be the locus of surfaces of degree d in \mathbb{P}^3 containing an elliptic quartic curve. Then the following holds.*

- (a) $\Sigma_{\text{ell},4}(4, 3)$ is a hypersurface in $\Sigma(4, 3)$ of degree 38475 [146, §3.2].
- (b) There is an explicit expression of $\deg(\Sigma_{\text{ell},4}(d, 3))$ for $d \geq 5$ as a polynomial in d of degree 32 obtained via Bott's residue formula [146, §4.3].

Recall (see, e.g., [330]) that a smooth hypersurface X in \mathbb{P}^n of degree $d \geq n + 1$ cannot be covered by rational curves. This concerns, in particular, smooth quartic surfaces in \mathbb{P}^3 , and holds, more generally, for any K3 surface. Nonetheless, a projective K3 surface carries infinitely many rational curves, see (a) in the next theorem. This generalizes a previous partial result due to Bogomolov, Mumford, Mori-Mukai [372], see also [21, 59, 60, 61, 99, 230, 334, 375].

Theorem 7.3. *Let X be a projective K3 surface over an algebraically closed field. Then the following holds.*

- (a) X contains infinitely many rational curves [101, 478]; cf. also [102].
- (b) Consider the subset \mathcal{S}_g of the moduli scheme of the K3 surfaces of genus g which parameterizes the surfaces X such that the union of rational curves on X is dense in the Hausdorff topology. Then \mathcal{S}_g is of the second Baire category [103].
- (c) Any rational curve on a general K3 surface X of genus $g \geq 2$ is nodal.
- (d) Given $d \in \mathbb{N}$, for all $h \gg 0$ and for any K3 surface X of degree $2h$ over a field k of characteristic $p \neq 2, 3$ the number of rational curves in X of degree at most d does not exceed 24. This upper bound is exact for any $d \geq 3$ [366], [427, Thm. 1.1].

See also [244] for a discussion related to (b). As for the count of curves on a K3 surface in terms of the Gromov-Witten invariant, see, e.g., [36, 327, 356, 530, 531].

The generality assumption in (c) is essential; indeed, there are smooth quartic surfaces in \mathbb{P}^3 which contain the rational 3-cuspidal plane quartic [99, 100].

Notice that there are complex K3 surfaces with no algebraic curve. However, any K3 surface S carries a transcendental entire curve, that is, the image of a nonconstant holomorphic map $\mathbb{C} \rightarrow S$ [85].

7.2. Rational curves in hypersurfaces. The next theorem deals with hypersurfaces which contain few rational curves. Recall the notation $\epsilon(d, n) = 2d + 2 - 3n$.

Theorem 7.4. *Let X be a hypersurface of degree d in \mathbb{P}^n . Then the following holds.*

- (a) For $n \geq 3$ and $\epsilon(d, n) \geq 3$ any rational curve in a very general X is contained in the maximal subvariety $L(X)$ of X swept out by lines in X [438, Cor. 3.2].
- (b) In the range $n \geq 7$ and $d \in [\frac{3n+1}{2}, 2n - 3]$ the general X contains lines but no other rational curves [438, Thm. 1.3]. This is true as well for $n = 6$ and $d = 2n - 3$ [395], but fails for $n = 4$ and $\epsilon(d, n) = 0$, that is, for a general quintic threefold in \mathbb{P}^4 .
- (c) For $n = 5$ and $d \geq 2n - 3$ the general X contains just a finite number of rational curves of any given degree (that is, X does not contain any one-parameter family of rational curves) [511].
- (d) If X is very general, $n \geq 4$, and $\epsilon(d, n) \geq n - 2$ (that is, $d \geq 2n - 2$) then X contains no rational curve [511].

Notice that (d) fails for $n = 3$. Indeed, by Theorem 7.3(a), any smooth quartic surface in \mathbb{P}^3 contains a sequence of rational curves of growing degrees. Statement (c) is a strengthening of the previous results in [115, 185, 186]. The validity of an analog of (c) for $n = 4$ and $e \geq 13$ is still open; this is the famous Clemens Conjecture.

7.3. Clemens' Conjecture. This conjecture [115] suggests that a general quintic threefold in \mathbb{P}^4 contains a positive finite number of smooth rational curves of any given degree, and the scheme of such curves is reduced. The next theorem gives a brief summary of some results on Clemens' Conjecture.

Theorem 7.5. (a) *The Clemens Conjecture holds for curves of degree ≤ 12 [20, 142, 143, 286, 292].*

(b) *For any $d \geq 1$ a general quintic threefold in \mathbb{P}^4 contains a smooth rational curve C of degree d [292].*

(c) *Any smooth rational curve C in a general quintic threefold X in \mathbb{P}^4 is embedded with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. Any singular rational curve in X is a plane 6-nodal quintic [142, 143, 286, 292, 389].*

(d) *The number of smooth rational curves of degree 1, 2, 3, 4, ...10 in a general quintic threefold in \mathbb{P}^4 is, respectively,*

2875, 609250, 317206375, 242467530000, ..., 704288164978454686113488249750,

where the number 2875 of lines is due to Schubert [449]; see [410], [54, §10.6] and the references therein.

For instance [292], [467], a general hypersurface of degree $d > \frac{3}{2}n - 1$ in \mathbb{P}^n does not contain any smooth conic; however, a general quintic threefold in \mathbb{P}^4 does.

For any natural number $d \geq 1$ there is a Mirror Symmetry prediction for the number of smooth rational curves of degree d in a general quintic threefold in \mathbb{P}^4 . Actually, these virtual numbers count pseudoholomorphic curves in a general almost complex symplectic deformation of the quintic threefold via quantum cohomology, see, e.g., [54, 306, 307, 357].

The same methods work for certain smooth Calabi-Yau complete intersections (CICY, for short). Besides the quintic threefolds in \mathbb{P}^4 , there are exactly 4 types of smooth CICY threefolds of type, respectively, (3, 3) and (2, 4) in \mathbb{P}^5 , (2, 2, 3) in \mathbb{P}^6 , and (2, 2, 2, 2) in \mathbb{P}^7 . The Mirror Symmetry prediction for the number of smooth rational curves of degree $d \leq 10$ in a general CICY threefold can be found in [336]. For $d \leq 6$ this prediction gives the correct number of curves, see [237, 335]. The next theorem addresses rational and elliptic curves in general Calabi-Yau complete intersection threefolds.

Theorem 7.6. ([188, 295, 298]) *Let X be a general CICY threefold. Then for $g = 0, 1, 2, 3$ there is an integer $d_g \geq 0$, where $d_0 = 0$, such that, for any $d > d_g$, X contains an isolated smooth curve of degree d and genus g .*

In [295], some of these results are claimed to hold for any smooth CICY threefolds. However, there is a gap in the proof in [295]; see [298]. Similar facts hold for certain higher genera curves under more severe restrictions, see [298, Thm. 1.2].

The following theorem gives a short account of sporadic results for curves in hypersurfaces.

Theorem 7.7. • *A very general hypersurface of degree $d \geq 2n - 1$ in \mathbb{P}^n does not contain any smooth elliptic curve [516]; cf. [118].*

• *The degree of an elliptic curve on a very general heptic hypersurface X in \mathbb{P}^4 is a multiple of 7 [194].*

• *Let X be the general heptic hypersurface in \mathbb{P}^5 . Then X does not contain any rational curve of degree $d \in \{2, \dots, 16\}$ [144, 376, 466], any smooth elliptic curve of degree $e \leq 14$, and any smooth curve C of degree $e \leq 16$ and genus $1 \leq g \leq 3$ provided the dimension of the linear span of C is not equal to 3 [18].*

• *A general hypersurface of degree 54 in \mathbb{P}^{30} does not contain any rational quartic curve [517].*

The second statement goes in the direction of the conjecture of Griffiths and Harris [213] which says that for a very general hypersurface of degree $d \geq 6$ in \mathbb{P}^n and for any curve C in H one has $d \mid \deg(C)$. This is true for $n = 3$ due to the Noether-Lefschetz theorem; see [19] for further results.

8. COUNTING CURVES OF HIGHER GENERA AND HYPERBOLICITY

It is worthwhile to compare previous results with the following finiteness theorems related to Kobayashi hyperbolicity and the Green-Griffiths-Lang Conjecture (see below).

Theorem 8.1. (a) *Consider a projective variety $X \subset \mathbb{P}^n$. If X is Kobayashi hyperbolic then there exists $\varepsilon > 0$ such that for any curve C of geometric genus g in X one has [168]*

$$(5) \quad 2g - 2 \geq \varepsilon \deg(C).$$

Consequently, the curves of a given geometric genus in X form a bounded family.

- (b) *A general hypersurface X in \mathbb{P}^n of degree $d \geq 16(2n - 3)^5(10n - 11)$ is Kobayashi hyperbolic [52, 76, 169, 170, 362, 439, 470]. The latter holds for $n = 3$ starting with $d = 18$ [403] and for $n = 4$ starting with $d = 593$ [176].*

A weaker form of (5) called *algebraic hyperbolicity* implies the absence of rational and elliptic curves; see, e.g., [98, 105, 137, 221]. See also [39, 147] for logarithmic versions of (b). There are examples of smooth hyperbolic surfaces in \mathbb{P}^3 of any given degree $d \geq 6$ [113, 183, 458, 460, 461, 537] and of hyperbolic hypersurfaces in \mathbb{P}^n of degree $d \sim n^2/4$ [244, 459].

In the direction of algebraic hyperbolicity of general hypersurfaces, the following holds.

Theorem 8.2. ([396]) *For $n \geq 6$ and for a very general hypersurface X in \mathbb{P}^n of degree $d \geq 2n - 2$, any subvariety $Y \subset X$ is of general type.*

Cf. also [532, 533, 534].

Theorem 8.3. (a) *The number of rational and elliptic curves on a minimal smooth surface of general type with $c_1^2 > c_2$ is bounded above by an effective function of c_1 and c_2 [58, 173, 366], [322, Thm. 10.1].*

- (b) *Let X be a very general surface of degree $d \geq 5$ in \mathbb{P}^3 . Then for any curve C of geometric genus g on X one has*

$$(6) \quad 2g - 2 \geq \max \left\{ d^2 - 3d - 6, \left(d + \frac{1}{d} - 5 \right) \deg(C) \right\}.$$

In particular, for $d = 5$ one has

$$2g - 2 \geq \max \left\{ 4, \frac{1}{5} \deg(C) \right\},$$

which yields $g \geq 3$. The absolute lower bound $2g - 2 \geq d^2 - 3d - 6$ is sharp; for $d \geq 6$ it is achieved by the tritangent hyperplane sections only [105, 137, 532].

Cf. also [339, 340, 530] for (a), [138, 221] for (b).

For elliptic curves on general K3 surfaces and Fano schemes of lines on general cubic fourfolds, the following hold.

Theorem 8.4. ([387])

- (a) *Let X be a very general K3 surface with primitive curve class $\beta \in H^2(X, \mathbb{Z})$ of self-intersection $\langle \beta, \beta \rangle = 2h - 2$, $h \in \mathbb{Z}_{\geq 0}$. Then the moduli space of elliptic curves on X from the class β is a smooth curve. The number $n_{\beta, j}$ of such curves with fixed general j -invariant depends only on h . It admits an expression in terms of two particular Gromov-Witten invariants, which can be computed explicitly for any given value of h .*

- (b) *A general Fano variety of lines on a cubic fourfold in \mathbb{P}^5 contains precisely 3780 elliptic curves of minimal degree and of fixed general j -invariant.*

Notice that there are exactly 6383765416 elliptic quartics meeting 16 general lines in \mathbb{P}^3 [15]. For higher genera curves on $K3$ surfaces, we have the following theorem.

Theorem 8.5. [390, Cor. 2] *A generic projective $K3$ surface contains infinitely many g -dimensional families of irreducible immersed curves of geometric genus g , for any positive integer g .*

For the genera of curves on smooth surfaces in \mathbb{P}^3 the following is known.

Theorem 8.6. *For $d \geq 4$ let $\text{Gaps}(d)$ be the set of all the non-negative integers which cannot be realized as geometric genera of irreducible curves on a very general surface of degree d in \mathbb{P}^3 . Then $\text{Gaps}(d)$ is the union of finitely many disjoint and separated integer intervals $\text{Gaps}_j(d)$, $j = 0, 1, \dots$. One has:*

- $\text{Gaps}(5) = \{0, 1, 2\}$ [532];
- $\text{Gaps}_0(d) = \left[0, \frac{d(d-3)}{2} - 3\right]$ for all $d \geq 5$ [532];
- $\text{Gaps}_1(d) = \left[\frac{d^2-3d+4}{2}, d^2 - 2d - 9\right]$ for all $d \geq 6$ [109].

In the other direction, for the existence of subvarieties with a given geometric genus, we have the following result.

Theorem 8.7. ([97] for $s = 1$, [110, Thm. 0.1]) *Let X be a smooth irreducible projective variety of dimension $n > 1$, let L be a very ample divisor on X , and let $s \in \{1, \dots, n-1\}$. Then there is an integer $p_{X,L,s}$ (depending on X , L and s) such that for any $p \geq p_{X,L,s}$ one can find an irreducible subvariety Y of X of dimension s with at most ordinary points of multiplicity $s+1$ as singularities such that $p_g(Y) = p$. Moreover, one can choose Y to be a complete intersection $Y = D_1 \cap \dots \cap D_{n-s}$, where $D_i \in |L|$ for $i = 1, \dots, n-s-1$ are smooth and transversal and $D_{n-s} \in |mL|$ for some $m \geq 1$ is such that Y has ordinary singularities of multiplicity $s+1$.*

Corollary 8.8. (a) *For any integer $d \geq 4$, there exists an integer $c(d)$ such that, for any smooth surface S in \mathbb{P}^3 of degree d and any integer $g \geq c(d)$, S carries a reduced, irreducible nodal curve of geometric genus g , whose nodes can be prescribed generically on S [97], [110, Cor. 3.1].*

(b) *For any positive integer d and for any non-negative integer g , there is a smooth surface S in \mathbb{P}^3 of degree d and an irreducible, nodal curve C on S with geometric genus g [110, Thm. 3.3].*

Recall the famous

Green-Griffiths-Lang Conjecture. ([210, 321]) *Let X be a projective variety of general type. Then there exists a proper closed subset $Y \subset X$ which contains any subvariety $Z \subset X$ not of general type and the image of any nonconstant entire curve $\mathbb{C} \rightarrow X$.*

The conjecture is fixed in the particular case of general projective hypersurfaces.

Theorem 8.9. ([51, 52, 65, 169, 175, 470]) *For a generic projective hypersurface $X \subset \mathbb{P}^n$, $n \geq 2$, of degree $d \geq 16(n-1)^5(5n-1)$ there exists a proper subvariety $Y \subset X$ which contains the image of any nonconstant entire curve $\mathbb{C} \rightarrow X$, hence also any rational or elliptic curve on X and the images of abelian varieties.*

There are the following related conjectures.

Conjecture. (C. Ciliberto, F. Flamini, and M. Zaidenberg [111]) *There exists a strictly growing function $\varphi: \mathbb{N} \rightarrow \mathbb{N}$ such that the number of curves of geometric genus $g \leq \varphi(d)$ in any smooth surface S of degree $d \geq 5$ in \mathbb{P}^3 is finite and bounded by a function of d .*

Conjecture. (C. Voisin [512]) *Let $X \subset \mathbb{P}^n$ be a very general hypersurface of degree $d \geq n+2$. Then the degrees of rational curves in X are bounded.*

Conjecture. (P. Autissier, A. Chambert-Loir, and C. Gasbarri [14]) *Let X be a smooth projective variety of general type with canonical line bundle K_X . Then there exist real numbers A and B , and a proper Zariski closed subset $Z \subset X$ such that for any curve C of geometric genus g in X not contained in Z , one has $\deg_C(K_X) \leq A(2g - 2) + B$.*

REFERENCES

- [1] A. Albano, S. Katz. *Lines on the Fermat quintic threefold and the infinitesimal generalized Hodge conjecture.* Trans. Amer. Math. Soc. 324 (1991), 353–368
- [2] V. A. Alexeev. *Theorems about good divisors on log Fano varieties (case of index $r > n - 2$).* In: Algebraic geometry (Chicago, IL, 1989), 1–9. Lect. Notes Math. 1479, Springer, Berlin, 1991
- [3] A. B. Altman, S. L. Kleiman. *Foundations of the theory of Fano schemes.* Compositio Math. 34(1) (1977), 3–47
- [4] E. Amerik, M. Verbitsky. *Rational curves on hyperkähler manifolds.* Int. Math. Res. Not. IMRN 2015 (2015), 13009–13045
- [5] F. Anella. *Rational curves on fibered varieties.* arXiv:1809.09454 (2018), 17 pp.
- [6] F. Anella. *Rational curves on genus one fibrations.* arXiv:1809.09485 (2018), 10 pp.
- [7] C. Araujo. *Rational curves of minimal degree and characterizations of projective spaces* Math. Ann. 335 (2006), 937–951
- [8] C. Araujo, A.–M. Castravet. *Polarized minimal families of rational curves and higher Fano manifolds.* Amer. J. Math. 134 (2012), 87–107
- [9] C. Araujo, C. Casagrande. *On the Fano variety of linear spaces contained in two odd-dimensional quadrics.* Geometry and Topology 21 (2017), 3009–3045
- [10] C. Araujo, J. Kollár. *Rational curves on varieties.* In: Higher dimensional varieties and rational points (Budapest, 2001), 13–68. Bolyai Soc. Math. Stud. 12, Springer, Berlin, 2003
- [11] M. Artin, D. Mumford. *Some elementary examples of unirational varieties which are not rational.* Proc. London Math. Soc. (3) 25 (1972), 75–95
- [12] P. Aspinwall, D. Morrison. *Topological field theory and rational curves.* Comm. Math. Phys. 151 (1993), 245–262
- [13] M. Audin, J. Lafontaine, (eds). *Holomorphic Curves in Symplectic Geometry.* Birkhauser, Basel, 1994
- [14] P. Autissier, A. Chambert-Loir, and C. Gasbarri. *On the canonical degrees of curves in varieties of general type.* Geom. Funct. Anal. 22 (2012), 1051–1061
- [15] D. Avritzer, I. Vainsencher. *Compactifying the space of elliptic quartic curves.* In: Complex Algebraic Geometry 179, G. Ellingsrud, C. Peskine, G. Sacchiero, and S. A. Stroemme (Eds.), 47–58. London Math. Soc. Lect. Notes Series. Cambridge University Press, 1992
- [16] D. Avritzer, I. Vainsencher. *The Hilbert scheme component of the intersection of two quadrics.* Comm. in Algebra 27 (1999), 2995–3008
- [17] L. Badescu, M. C. Beltrametti. *Projective manifolds containing special curves.* J. Math. Soc. Japan 58 (2006), 211–230
- [18] E. Ballico. *Curves of low positive genus and low degree on a general heptic hypersurface of \mathbb{P}^5 .* Bull. Belg. Math. Soc. Simon Stevin 23 (2016), 439–464
- [19] E. Ballico, F. Catanese, and C. Ciliberto (Eds.) *Trento problems.* In: *Classification of irregular varieties. Minimal Models and Abelian Varieties*, 134–139. Proceedings of a Conference held in Trento, Italy, 17–21 December, 1990. Lect. Notes Math. 1515. Springer, 2012
- [20] E. Ballico and C. Fontanari. *Finiteness of rational curves of degree 12 on a general quintic threefold.* Pure Appl. Math. Q. 11 (2015), 537–557
- [21] A. Baragar, D. McKinnon. *$K3$ surfaces, rational curves, and rational points.* J. Number Theory 130 (2010), 1470–1479
- [22] W. Barth, A. Van de Ven. *Fano varieties of lines on hypersurfaces.* Arch. Math. (Basel) 31 (1978/79), 96–104
- [23] W. Barth, Th. Bauer. *Smooth quartic surfaces with 352 conics.* Manuscr. Math. 85 (1994), 409–417
- [24] F. Bastianelli, C. Ciliberto, F. Flamini, and P. Supino. *On complete intersections containing a linear subspace.* arXiv:1812.06682 (2018), 6 pp.
- [25] V. V. Batyrev, Y. I. Manin. *Sur le nombre des points rationnels de hauteur bornée des variétés algébriques.* Math. Ann. 286 (1990), 27–43
- [26] V. V. Batyrev, Y. Tschinkel. *Rational points on some Fano cubic bundles.* C. R. Acad. Sci. Paris Sér. I Math. 323 (1996), 41–46
- [27] V. Batyrev, D. van Straten. *Generalized hypergeometric functions and rational curves on Calabi–Yau complete intersections in toric varieties.* Comm. Math. Phys. 168 (1995), 493–533

- [28] Th. Bauer. *Quartic surfaces with 16 skew conics*. J. Reine Angew. Math. 464 (1995), 207–217.
- [29] Th. Bauer, S. Rams. *Counting lines on projective surfaces*. arXiv:1902.05133v1 (2019), 8 pp.
- [30] E. Bayer-Fluckiger, J.-P. Serre. *Lines on cubic surfaces and Witt invariants*. arXiv:1909.05312 (2019), 8 pp.
- [31] S. Basu, A. Lerario, E. Lundberg, and Ch. Peterson. *Random fields and the enumerative geometry of lines on real and complex hypersurfaces*. Math. Ann. 374 (2019), 1773–1810
- [32] A. Beauville. *Variétés de Prym et jacobiniennes intermédiaires*. Ann. Sci. É.N.S. 10 (1977), 309–391
- [33] A. Beauville. *Sous-variétés spéciales des variétés de Prym*. Comp. Math. 45 (1982), 357–383
- [34] A. Beauville. *Les singularités du diviseur thêta de la jacobienne intermédiaire de l'hypersurface cubique dans \mathbb{P}^4* . Lecture Notes Math. 947 (1982), 190–208
- [35] A. Beauville. *Quantum cohomology of complete intersections*. Mat. Fiz. Anal. Geom. 2 (1995), 384–398
- [36] A. Beauville. *Counting rational curves on K3 surfaces*. Duke Math. J. 97 (1999), 99–108
- [37] A. Beauville. *Vector bundles on the cubic threefold*. In: Proc. Conf. on Algebraic Geometry, in celebration of H. Clemens's 60th birthday (Salt-Lake City, 2000). Contemp. Mathem. 312 (2002), 71–86
- [38] A. Beauville. *The Lüroth problem*. arXiv:1507.02476 (2015), 27p.
- [39] A. Beauville, R. Donagi. *La variété des droites d'une hypersurface cubique de dimension 4*. C. R. Acad. Sci. Paris, Sér. I Math. 301 (1985), 703–706
- [40] R. Beheshti. *Linear subvarieties of hypersurfaces*. Intern. Math. Res. Notices 49 (2005), 3055–3063
- [41] R. Beheshti. *Lines on projective hypersurfaces*. J. Reine Angew. Math. 592 (2006), 1–21
- [42] R. Beheshti. *Nonuniruledness results for spaces of rational curves in hypersurfaces*. Alg. Number Theory 6 (2012), 669–687
- [43] R. Beheshti. *Hypersurfaces with too many rational curves*. Math. Ann. 360 (2014), 753–768
- [44] R. Beheshti, N. Mohan Kumar. *Spaces of rational curves on complete intersections*. Compos. Math. 149 (2013), 1041–1060
- [45] R. Beheshti, N. Mohan Kumar. *Spaces of rational curves on hypersurfaces*. J. Ramanujan Math. Soc. 28A (2013), 1–19
- [46] R. Beheshti, E. Riedl. *Linear subspaces of hypersurfaces*. arXiv:1903.02481 (2019), 17 pp.
- [47] R. Beheshti, J. Starr. *Rational surfaces in index-one Fano hypersurfaces*. J. Algebraic Geom. 17 (2008), 255–274
- [48] K. Behrend. *Gromov-Witten invariants in algebraic geometry*. Invent. Math. 127 (1997), 601–617
- [49] M. Beltrametti. *On the Chow group and the intermediate Jacobian of a conic bundle*. Ann. Math. Pura Appl. (4) 141 (1985), 331–351
- [50] O. Benoist. *Construction de courbes sur les surfaces K3*. Séminaire Bourbaki, Exposé 1081. Astérisque 367–368 (2015), 219–253
- [51] G. Berczi. *Towards the Green-Griffiths-Lang Conjecture via equivariant localization*. Proc. London Math. Soc. 118 (2019), 1057–1083
- [52] G. Berczi, F. Kirwan. *Non-reductive geometric invariant theory and hyperbolicity*. arXiv:1909.11417v2 (2020), 29 pp.
- [53] M. Bernardara. *Calabi-Yau complete intersections with infinitely many lines*. Rend. Sem. Mat. Univ. Pol. Torino 66 (2008), 1–11
- [54] J. Bertin, C. Peters. *Variations de structures de Hodge, variétés de Calabi-Yau et symétrie miroir*. In: J. Bertin, J.-P. Demailly, L. Illusie, C. Peters. *Introduction à la théorie de Hodge*, 171–255. Panoramas et synthèses 3. Société Mathématique de France, 1996
- [55] A. Bertram. *Another way to enumerate rational curves with torus actions*. Invent. Math. 142 (2000), 487–512
- [56] I. Biswas, J. Biswas, and G. V. Ravindra. *On some moduli spaces of stable vector bundles on cubic and quartic threefolds*. J. Pure Appl. Algebra 212 (2008), 2298–2306
- [57] S. Bloch, J. P. Murre. *On the Chow groups of certain types of Fano threefolds*. Compos. Math. 39 (1979), 47–105
- [58] F. A. Bogomolov. *Families of curves on a surface of general type*. Doklady Akad. Nauk SSSR (5)236 (1977), 1041–1044
- [59] F. Bogomolov, B. Hassett, and Y. Tschinkel. *Constructing rational curves on K3 surfaces*. Duke Math. J. 157 (2011), 535–550
- [60] F. Bogomolov, Y. Tschinkel. *Density of rational points on elliptic K3 surfaces*. Asian J. Math. 4 (2000), 351–368
- [61] F. Bogomolov, Y. Tschinkel. *Rational curves and points on K3 surfaces*. Amer. J. Math. 127 (2005), 825–835
- [62] S. Boissière, A. Sarti. *Counting lines on surfaces*. Ann. Sc. Norm. Sup. Pisa, Cl. Sci. 6 (2007), 39–52
- [63] S. Boissière, A. Sarti. *On the Néron-Severi group of surfaces with many lines*. Proc. Amer. Math. Soc. 136 (2008), 3861–3867
- [64] E. Bombieri, H. P. F. Swinnerton-Dyer. *On the local zeta function of a cubic threefold*. Ann. Sc. Norm. Sup. Pisa. Cl. Sci. Ser. 3, 21 (1967), 1–29

- [65] R. van Bommel, A. Javanpeykar, and L. Kamenova. *Boundedness in families with applications to arithmetic hyperbolicity*. arXiv:1907.11225 (2019), 39 pp.
- [66] L. Bonavero, A. Höring. *Counting conics in complete intersections*. Acta Math. Vietnam. 35 (2010), 23–30
- [67] C. Borcea. *Deforming varieties of k -planes of projective complete intersections*. Pacific J. Math. 143 (1990), 25–36
- [68] R. Bott. *A residue formula for holomorphic vector-fields*. J. Diff. Geom. 1 (1967), 311–330
- [69] D. Bourqui. *Asymptotic behavior of rational curves*. arXiv:1107.3824 (2011), 51 pp.
- [70] D. Bourqui. *Exemples de comptages de courbes sur les surfaces*. Math. Ann. 357 (2013), 1291–1327
- [71] D. Bourqui. *Algebraic points, non-anticanonical heights and the Severi problem on toric varieties*. Proc. Lond. Math. Soc. 113 (2016), 474–514
- [72] M. C. Brambilla, D. Faenzi. *Rank-two stable sheaves with odd determinant on Fano threefolds of genus nine*. Math. Z. 275 (2013), 185–210
- [73] M. Brion. *Equivariant cohomology and equivariant intersection theory*, arXiv:math/9802063 (2008), 49 pp.
- [74] M. Brion, S. S. Kannan. *Minimal rational curves on generalized Bott–Samelson varieties*. arXiv:1910.06197 (2019), 29 pp.
- [75] D. Brotbek. *Hyperbolicity related problems for complete intersection varieties*. Compos. Math. 150 (2014), 369–395
- [76] D. Brotbek. *On the hyperbolicity of general hypersurfaces*. Publ. Math. Inst. Hautes Études Sci. 126 (2017), 1–34
- [77] D. Brotbek, Ya Deng. *Kobayashi hyperbolicity of the complements of general hypersurfaces of high degree*. Geom. Funct. Anal. 29 (2019), 690–750
- [78] T. Browning, W. Sawin. *Free rational curves on low degree hypersurfaces and the circle method*. arXiv:1810.06882 (2018), 34 pp.
- [79] T. D. Browning, P. Vishe. *Rational curves on smooth hypersurfaces of low degree*. Algebra and Number Theory 11 (2017), 1657–1675
- [80] J. Bryan, N. C. Leung. *The Enumerative Geometry of $K3$ surfaces and Modular Forms*. J. Amer. Math. Soc. 13 (2000), 371–410
- [81] A. Cayley. *On the triple tangent planes to a surface of the third order*. Cambridge and Dublin Math. J. 4 (1849), 252–260
- [82] A. Cayley. *A memoir on cubic surfaces*. Philos. Trans. R. Soc. Lond. Ser. I (159) (1869), 231–326
- [83] F. Campana. *Connexité rationnelle des variétés de Fano*. Ann. Sci. Ecole Norm. Sup. (4) 25 (1992), 539–545
- [84] S. Canning. *On a conjecture on the variety of lines on Fano complete intersections*. arXiv:2002.04713 (2020), 6 pp.
- [85] S. Cantat. *Deux exemples concernant une conjecture de Serge Lang*. C. R. Acad. Sci. Paris Sér. I Math. 330 (2000), 581–586
- [86] Y. Cao. *Counting conics on sextic 4-folds*. arXiv:1805.04696 (2019), 10 pp.
- [87] J. Cao, A. Höring. *Rational curves on compact Kähler manifolds*. Available at: <http://math.unice.fr/hoering/articles/a27-rat-curves.pdf>
- [88] L. Caporaso, J. Harris. *Parameter spaces for curves on surfaces and enumeration of rational curves*. Compos. Math. 113 (1998), 155–208
- [89] L. Caporaso, J. Harris. *Enumerating rational curves: the rational fibration method*. Compositio Math. 113 (1998), 209–236
- [90] L. Caporaso, J. Harris. *Counting plane curves of any genus*. Invent. Math. 131 (1998), 345–392
- [91] C. Casagrande. *Rank 2 quasiparabolic vector bundles on \mathbb{P}^1 and the variety of linear subspaces contained in two odd-dimensional quadrics*. Math. Z. 280 (2015), 981–988
- [92] G. Ceresa, A. Verra. *The Abel-Jacobi isomorphism for the sextic double solid*. Pacific J. Math. 124 (1986), 85–105
- [93] M. Chan, N. Ilten. *Fano schemes of determinants and permanents*. Algebra Number Theory 9 (2015), 629–679
- [94] M. C. Chang, Z. Ran. *Divisors on some generic hypersurfaces*. J. Diff. Geom. 38 (1993), 671–678
- [95] I. Cheltsov, J. Park. *Sextic double solids*. In: Cohomological and Geometric Approaches to Rationality Problems. F. Bogomolov, Y. Tschinkel (eds.). Progr. Math. 282, 75–132. Birkhäuser Boston, Inc., Boston, MA, 2010
- [96] I. A. Cheltsov, V. V. Przhilyakovskii, and K. A. Shramov. *Hyperelliptic and trigonal Fano threefolds*. Izv. Math. 69 (2005), 365–421. Translated from: Izv. Ross. Akad. Nauk Ser. Mat. 69 (2005), 145–204
- [97] J. A. Chen. *On genera of smooth curves in higher dimensional varieties*. Proc. Amer. Math. Soc. 125 (1997), 2221–2225
- [98] X. Chen. *On algebraic hyperbolicity of log varieties*. Commun. Contemp. Math. 6 (2004), 513–559
- [99] X. Chen. *Rational Curves on $K3$ Surfaces*. J. Alg. Geom. 8 (1999), 245–278
- [100] X. Chen. *A simple proof that rational curves on $K3$ surfaces are nodal*. Math. Ann. 324 2002, 71–104

- [101] X. Chen, F. Gounelas, and C. Liedtke. *Curves on K3 surfaces*. arXiv:1907.01207 (2019), 40 pp.
- [102] X. Chen, F. Gounelas, and C. Liedtke. *Rational curves on lattice-polarized K3 surfaces*. arXiv:1907.01208 (2019), 37 pp.
- [103] X. Chen, J. D. Lewis. *Density of rational curves on K3 surfaces*. Math. Ann. 356 (2013), 331–354
- [104] Y. Cneng. *Hyperplane sections of hypersurfaces*. arXiv:2001.10983 (2020), 17 pp.
- [105] L. Chiantini, A. F. Lopez. *Focal loci of families and the genus of curves on surfaces*. Proc. Amer. Math. Soc. 127 (1999), 3451–3459
- [106] L. Chiantini, A.-F. Lopez, and Z. Ran. *Subvarieties of generic hypersurfaces in any variety*. Math. Proc. Cambr. Philos. Soc. 130 (2001), 259–268
- [107] K. Chung, J. Hong, S. H. Lee. *Geometry of moduli spaces of rational curves in linear sections of Grassmannian $\text{Gr}(2, 5)$* . J. Pure Appl. Algebra 222 (2018), 868–888.
- [108] C. Ciliberto, T. Dedieu, C. Galati, and A. L. Knutsen. *Moduli of curves on Enriques surfaces*. arXiv:1902.07142 (2020), 36 pp.
- [109] C. Ciliberto, F. Flamini, and M. Zaidenberg. *Genera of curves on a very general surface in \mathbb{P}^3* . Int. Math. Res. Not. 22 (2015), 12177–12205
- [110] C. Ciliberto, F. Flamini, and M. Zaidenberg. *Gaps for geometric genera*. Arch. Math. 106 (2016), 531–541.
- [111] C. Ciliberto, F. Flamini, and M. Zaidenberg. *A remark on the intersection of plane curves*. Differential Equations, Mathematical Physics, and Applications: Selim Grigorievich Krein Centennial. P. Kuchment, E. Semenov (eds.). Contemporary Mathematics 733, 109–128. Amer. Math. Soc., Rhode Island, 2019
- [112] C. Ciliberto, A. Lopez, and R. Miranda. *Projective degenerations of K3 surfaces, Gaussian maps, and Fano threefolds*. Invent. math. 114 (1993), 641–667
- [113] C. Ciliberto, M. Zaidenberg. *Scrolls and hyperbolicity*. Intern. J. Math. 24 (2013), 38–63
- [114] C. Ciliberto, M. Zaidenberg. *On Fano schemes of complete intersections*. arXiv:1903.11294 (2019), 32 pp.
- [115] A. Clebsch. *Zur Theorie der algebraischen Flächen*. J. Reine Angew. Math. 58 (1861), 93–108
- [116] C. H. Clemens. *Double solids*. Adv. Math. 47 (1983), 107–230
- [117] H. Clemens. *On the Surjectivity of Abel-Jacobi Mappings*. Ann. Math. 117 (1983), 71–76
- [118] H. Clemens. *Curves on generic hypersurfaces*. Ann. Sci. Ec. Norm. Sup. 19 (1986), 629–636
- [119] H. Clemens. *Curves on higher-dimensional complex projective manifolds*. In: Proc. Internat. Congr. Math. (1986: Berkeley), 634–640. Amer. Math. Soc., Providence, RI, 1987
- [120] H. Clemens. *The local geometry of the Abel-Jacobi mapping*. Proc. Symp. Pure Math. 46 (1987), 223–233
- [121] H. Clemens. *The quartic double solid revisited*. In: Proc. Sympos. in Pure Math. 53, 89–101. Amer. Math. Soc., Providence, RI, 1991
- [122] C. H. Clemens, Ph. A. Griffiths. *The Intermediate Jacobian of the Cubic Threefold*. Ann. of Math. 95 (1972), 281–356
- [123] H. Clemens, J. Kollár, S. Mori. *Higher-dimensional complex geometry*. Astérisque 166, 1988
- [124] C. H. Clemens, Z. Ran. *Twisted genus bounds for subvarieties of generic hypersurfaces*. Amer. J. Math. 126 (2004), 89–120; *Erratum*. Amer. J. Math. 127 (2005), 241–242
- [125] T. Coates, A. Givental. *Quantum Riemann-Roch, Lefschetz and Serre*. Ann. Math. 165 (2007), 15–53
- [126] A. Cohen, B. Cooperstein. *Lie incidence systems from projective varieties*. Proc. Amer. Math. Soc. 126 (1998), 2095–2102
- [127] A. Collino, *The Abel-Jacobi isomorphism for the cubic fivefold*, Pacific J. Math. **122** (1986), 43–55
- [128] A. Collino, *The fundamental group of the Fano surface I, II*. In: Algebraic threefolds (Varenna, 1981), 209–218, 219–220, Lecture Notes in Math. 947, Springer, Berlin-New York, 1982
- [129] A. Collino, *Lines on quartic threefolds*. J. London Math. Soc. (2) 19 (1979), 257–267
- [130] A. Collino, J. P. Murre, G. E. Welters. *On the family of conics lying on a quartic threefold*. Rend. Sem. Mat. Univ. Politec. Torino 38 (1980), 151–181
- [131] A. Conte, J. P. Murre. *The Hodge conjecture for fourfolds admitting a covering by rational curves*. Math. Ann. 238 (1978), 79–88
- [132] A. Conte, J. P. Murre. *The Hodge conjecture for Fano complete intersections of dimension four*. Journées de géométrie algébrique d’Angers (juillet 1979), 129–141. Sijthoff & Noordhoff, 1980
- [133] D. Coray, C. Manoil, and I. Vainsencher. *Nonminimal rational curves on K3 surfaces*. Enseign. Math. (2) 43 (1997), 299–317
- [134] J. Cordovez. *Families of lines in Fano varieties complete intersection in a Grassmannian*. Le Matematiche LVII (2002), 131–147
- [135] J. Cordovez, M. Valenzano. *On Fano scheme of k -planes in a projective complete intersection*, Università di Torino, Quaderni del Dipartimento di Matematica **21** (2005), 12 pp.
- [136] I. Coskun, J. Harris, and J. Starr. *The effective cone of the Kontsevich moduli space*. Canad. Math. Bull. 51 (2008), 519–534

- [137] I. Coskun, E. Riedl. *Algebraic hyperbolicity of the very general quintic surface in \mathbb{P}^3* . Adv. Math. 350 (2019), 1314–1323
- [138] I. Coskun, E. Riedl. *Algebraic hyperbolicity of very general surfaces*. arXiv:1912.07689 (2019), 17 pp.
- [139] I. Coskun, E. Riedl. *Normal bundles of rational curves on complete intersections*. Comm. in Contemp. Math. 21 (2019), 1850011
- [140] I. Coskun, E. Riedl. *Effective bounds on ampleness of cotangent bundles*. Bull. London Math. Soc. 00 (2020), 1–7
- [141] I. Coskun, J. Starr. *Rational curves on smooth cubic hypersurfaces*. Int. Math. Res. Not. 24 (2009), 4626–4641
- [142] E. Cotterill. *Rational curves of degree 10 on a general quintic threefold*. Comm. Algebra 33 (2005), 1833–1872
- [143] E. Cotterill. *Rational curves of degree 11 on a general quintic 3-fold*. Quart. J. Math. 63 (2012), 539–568
- [144] E. Cotterill. *Rational curves of degree 16 on a general heptic fourfold*. J. Pure Appl. Algebra 218 (2014), 121–129
- [145] D. Cox, S. Katz. *Mirror symmetry and algebraic geometry*. Math. Surveys and Monographs 68, Amer. Math. Soc., 1999
- [146] F. Cukierman, A. F. Lopez, and I. Vainsencher. *Enumeration of surfaces containing an elliptic quartic curve*. Proc. Amer. Math. Soc. 142 (2014), 3305–3313
- [147] L. Darondeau. *On the logarithmic Green-Griffiths conjecture*. Int. Math. Res. Notes 6 (2016), 1871–1923
- [148] O. Debarre. *Sur le problème de Torelli pour les variétés de Prym*. Amer. J. Math. 111 (1989), 111–134
- [149] O. Debarre. *Le théorème de Torelli pour les intersections de trois quadriques*. Invent. Math. 95 (1989), 505–528
- [150] O. Debarre. *Sur le théorème de Torelli pour les solides doubles quartiques*. Compositio Math. 73 (1990), 161–187
- [151] O. Debarre. *Higher-dimensional algebraic geometry*. Universitext. Springer-Verlag, New York, 2001
- [152] O. Debarre. *Lines on smooth hypersurfaces*. Preprint available at: https://www.math.ens.fr/~debarre/Lines_hypersurfaces.pdf (2003), 4 pp.
- [153] O. Debarre. *Rational curves on hypersurfaces*. Lecture notes for the II Latin American School of Algebraic Geometry and Applications, 1–12 June 2015. Cabo Frio, Brazil. January 4, 2016, 48 pp.
- [154] O. Debarre. *Gushel–Mukai varieties*. Lecture notes of the lecture course given at the Shanghai Center for Mathematical Sciences at Fudan University, 2019. arXiv:2001.03485 (2020), 26 pp.
- [155] O. Debarre, A. Iliev, and L. Manivel. *On nodal prime Fano threefolds of degree 10*. Sci. China Math. 54 (2011), 1591–1609
- [156] O. Debarre, A. Iliev, and L. Manivel. *On the period map for prime Fano threefolds of degree 10*. J. Algebraic Geom. 21 (2012), 21–59
- [157] O. Debarre, A. Iliev, and L. Manivel. *Special prime Fano fourfolds of degree 10 and index 2*. In: Recent Advances in Algebraic Geometry: A Volume in Honor of Rob Lazarsfeld’s 60th Birthday. C. Hacon et al. (eds.), London Math. Soc. Lect. Note Series, 123–155. Cambridge: Cambridge University Press, 2015
- [158] O. Debarre, A. Kuznetsov. *Gushel–Mukai varieties: Classification and birationalities*. Algebraic Geom. 5 (1) (2018), 15–76
- [159] O. Debarre, A. Kuznetsov. *Gushel–Mukai varieties: linear spaces and periods*. Kyoto J. Math. (2019), 57 pp. (to appear); arXiv:1605.05648 (2016), 44 pp.
- [160] O. Debarre, A. Kuznetsov. *Gushel–Mukai varieties: moduli*. arXiv:1812.09186 (2018), 51 pp.
- [161] O. Debarre, L. Manivel. *Sur la variété des espaces linéaires dans une intersection complète*. Math. Ann. 312 (1998), 549–574
- [162] A. J. de Jong, J. M. Starr. *Low degree complete intersections are rationally simply connected*. Preprint, 2006
- [163] A. Degtyarev. *Lines in supersingular quartics*. arXiv: 1604.05836 (2016), 38 pp.
- [164] A. Degtyarev, I. Itenberg, and A. S. Sertöz. *Lines on quartic surfaces*. Math. Ann. 368 (2017), 753–809
- [165] M. F. DeLand. *Geometry of rational curves on algebraic varieties*. Thesis (Ph.D.) – Columbia University. 2009. 146 pp.
- [166] M. DeLand. *Complete families of linearly non-degenerate rational curves*. Canad. Math. Bull. 54 (2011), 430–441
- [167] M. DeLand. *Relatively very free curves and rational simple connectedness*. J. Reine Angew. Math. 699 (2015), 1–33
- [168] J.-P. Demailly. *Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials*. In: Algebraic Geometry. Proceedings of the Summer Research Institute, eds. J. Kollár et al., Santa Cruz, CA, USA, July 929, 1995, 285–360. Proc. Symp. Pure Math. 62 (pt. 2), Amer. Math. Soc., Providence, RI, 1997
- [169] J.-P. Demailly. *Recent results on the Kobayashi and Green-Griffiths-Lang conjectures*. arXiv:1801.04765 (2018), 76 pp.
- [170] Ya Deng. *Effectivity in the hyperbolicity-related problems*. arXiv:1606.03831 (2016), 13 pp.

- [171] Ya Deng. *Kobayashi measure hyperbolicity for singular directed varieties of general type*. Comptes Rendus Math. 354 (2016), 920–924
- [172] U. V. Desale, S. Ramanan, *Classification of Vector Bundles of Rank 2 on Hyperelliptic Curves*. Invent. Math. 38 (1976), 161–185
- [173] M. Deschamps. *Courbes de genre géométrique borné sur une surface de type général*. Séminaire Bourbaki 20 (1977–1978), 233–247
- [174] L. E. Dickson. *The straight lines on modular cubic surfaces*. Proc. Nat. Acad. Sci. USA 1(4) (1915), 248–253
- [175] S. Diverio, J. Merker, and E. Rousseau. *Effective algebraic degeneracy*. Invent. Math. 180 (2010), 161–223
- [176] S. Diverio, S. Trapani. *A remark on the codimension of the Green-Griffiths locus of generic projective hypersurfaces of high degree*. J. Reine Angew. Math. 649 (2010), 55–61
- [177] I. V. Dolgachev. *Classical Algebraic Geometry: a modern view*. Cambridge Univ. Press, 2012
- [178] I. Dolgachev, D. Markushevich. *Lagrangian tens of planes, Enriques surfaces and holomorphic symplectic fourfolds*. arXiv:1906.01445 (2019), 71 pp.
- [179] R. Donagi. *On the geometry of Grassmannians*. Duke Math. J. 44 (1977), 795–837
- [180] R. Donagi. *Group law on the intersection of two quadrics*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 7 (1980), 217–239
- [181] R. Donagi, R. C. Smith. *The structure of the Prym map*. Acta Math. 146 (1981), 25–102
- [182] S. Druel. *Espace des modules des faisceaux semi-stables de rang 2 et de classes de Chern $c_1 = 0, c_2 = 2$ et $c_3 = 0$ sur une hypersurface lisse de \mathbb{P}^4* . Internat. Math. Res. Notices 19 (2000), 985–1004
- [183] J. Duval. *Une sextique hyperbolique dans $\mathbb{P}^3(\mathbb{C})$* . Math. Ann. 330 (2004), 473–476
- [184] D. Edidin, W. Graham. *Localization in equivariant intersection theory and the Bott residue formula*. Amer. J. Math. 120 (1998), 619–636.
- [185] L. Ein. *Subvarieties of generic complete intersections*. Invent. Math. 94 (1988), 163–169
- [186] L. Ein. *Subvarieties of generic complete intersections II*. Math. Ann. 289 (1991), 465–471
- [187] D. Eisenbud and J. Harris. *3264 and All That: A Second Course in Algebraic Geometry*. Cambridge University Press, 2016
- [188] T. Ekedahl, T. Johnsen, and D. E. Sommervoll. *Isolated rational curves on K3-fibered Calabi-Yau threefolds*. arxiv:alg-geom/9710010 (1997), 21 pp.
- [189] D. Eklund. *Curves on Heisenberg invariant quartic surfaces in projective 3-space*. European J. Math. 4 (2018), 931–952.
- [190] S. Yu. Endryushka. *The intermediate Jacobian of a three-dimensional conic bundle is a Prym variety*. Math. USSR-Sbornik 50 (1985), 269–277. Translated from: Mat. Sb. 122(164) (1983), 276–285
- [191] G. Fano. *Sul sistema ∞^2 di rette contenuto in una varietà cubica generate dello spazio a quattro dimensioni*. Atti R. Acc. Sc. Torino XXXIX (1904), 778–792
- [192] G. Fano. *Sulle varietà algebriche a tre dimensioni a curve-sezioni canoniche*. Comm. Math. Helvetici 14 (1941-1942), 23–64
- [193] G. Fano. *Sulle sezioni spaziali della varietà Grassmanniana delle rette spazio a cinque dimensioni*. Rend. R. Accad. Lincei 11 (1930), 329–356
- [194] G. Ferrarese, D. Romagnoli. *Elliptic curves on the general hypersurface of degree 7 of \mathbb{P}^4* . Indag. Math. 91 (1988), 249–252
- [195] S. Finashin, V. Kharlamov. *Abundance of real lines on real projective hypersurfaces*. Int. Math. Res. Not. (IMRN) 16 (2013), 3639–3646
- [196] S. Finashin, V. Kharlamov. *Abundance of 3-planes on real projective hypersurfaces*. Arnold Math. J. 1 (2015), 171–199
- [197] F. Flamini, E. Sernesi. *The curve of lines on a prime Fano threefold of genus 8*. Intern. J. Math 21 (2010), 1561–1584
- [198] R. Friedman, R. Smith. *The generic Torelli theorem for the Prym map*. Invent. Math. 67 (1982), 473–490
- [199] R. Friedman, R. Smith. *Degenerations of Prym varieties and intersections of three quadrics*. Invent. Math. 85 (1986), 615–635
- [200] T. Fujita. *Classification theories of polarized varieties*. Lect. Notes Series 155. Cambridge Univ. Press, 1990
- [201] W. Fulton. *Introduction to intersection theory in algebraic geometry*. Amer. Math. Soc. Providence, RI, 1980
- [202] W. Fulton, R. Pandharipande. *Notes on stable maps and quantum cohomology*. In: Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math. 62, Part 2, 45–96. Amer. Math. Soc., Providence, RI, 1997
- [203] K. Furukawa. *Rational curves on hypersurfaces*. J. Reine Angew. Math. 665 (2012), 157–188
- [204] K. Furukawa. *Dimension of the space of conics on Fano hypersurfaces*. arXiv:1702.08890 (2017), 15 pp.
- [205] M. Furushima, N. Nakayama. *The family of lines on the Fano threefold V_5* . Nagoya Math. J. 116 (1989), 111–122
- [206] G. van der Geer, A. Kouvidakis. *A note on Fano surfaces of nodal cubic threefolds*. In: Algebraic and Arithmetic Structures of Moduli Spaces (Sapporo 2007), 27–45. Advanced Studies in Pure Mathematics 58, 2010

- [207] F. Gherardelli. *Un'osservazione sulla varietà cubica di \mathbb{P}^4* . Rend. sem. mat. e fisicodi Milano 37 (1967), 157–160
- [208] L. Göttsche, V. Shende. *Refined curve counting on complex surfaces*. Geometry and Topology 18 (2014), 2245–2307
- [209] V. González-Alonso, S. Rams. *Counting Lines on Quartic Surfaces*. Taiwanese J. Math. 20 (2016), 769–785
- [210] M. Green, Ph. Griffiths. *Two applications of algebraic geometry to entire holomorphic mappings*. The Chern Symposium 1979, 41–74. Proc. Internal. Sympos. Berkeley, CA, 1979. Springer-Verlag, New York 1980
- [211] Ph. Griffiths. *Complex analysis and algebraic geometry*. Bull. Amer. Math. Soc. 1 (1979), 595–626
- [212] Ph. Griffiths, J. Harris. *Principles of algebraic geometry*. John Wiley & Sons, New York, 1978
- [213] Ph. Griffiths, J. Harris. *On the Noether–Lefschetz theorem and some remarks on codimension two cycles*. Math. Ann. 271 (1985), 31–51
- [214] M. M. Grinenko. *On a double cone over a Veronese surface*. Izv. Math. 67 (2003), 421–438. Translated from: Izv. RAN. Ser. Mat. 67 (2003), 5–22
- [215] M. M. Grinenko. *Mori Structures on a Fano Threefold of Index 2 and Degree 1*. Proc. Steklov Inst. Math. 246 (2004), 103–128
- [216] M. Gromov. *Pseudo holomorphic curves in symplectic manifolds*. Invent. Math. 82 (1985), 307–347
- [217] L. Gruson, F. Laytimi, and D. S. Nagaraj. *On prime Fano threefolds of genus 9*. Internat. J. Math. 17 (2006), 253–261
- [218] N. P. Gushel. *Fano varieties of genus 6*. Math. USSR Izv. 21 (1983), 445–459. Translated from: Izv. Akad. Nauk SSSR Ser. Mat. 46, 1159–1174
- [219] N. P. Gushel. *Fano varieties of genus 8*. Russian Math. Surv. 38 (1983), 192–193. Translated from: Usp. Mat. Nauk 38 (1983), 163–164
- [220] N. P. Gushel. *Fano 3-folds of genus 8*. St. Petersburg Math. J. 4 (1993), 115–129. Translated from: Algebra i Analiz 4 (1992), 120–134
- [221] Ch. Haase, N. Ilten. *Algebraic hyperbolicity for surfaces in toric threefolds*. arXiv:1903.0268 (2019), 30 pp.
- [222] J. Harris. *Galois groups of enumerative problems*. Duke Math. J. 46 (1979), 685–724
- [223] J. Harris, B. Mazur, and R. Pandharipande. *Hypersurfaces of low degree*. Duke Math. J. 95 (1998), 125–160
- [224] J. Harris, M. Roth, and J. Starr. *Abel–Jacobi maps associated to smooth cubic threefolds*. arXiv:math/0202080 (2002), 26p.
- [225] J. Harris, M. Roth, and J. Starr. *Rational curves on hypersurfaces of low degree*. J. Reine Angew. Math. 571 (2004), 73–106
- [226] J. Harris, M. Roth, and J. Starr. *Curves of small degree on cubic threefolds*. Rocky Mountain J. Math. 35 (2005), 761–817.
- [227] J. Harris, J. Starr. *Rational curves on hypersurfaces of low degree, II*. Compos. Math. 41 (2005), 35–92
- [228] S. Hashimoto, B. Kadets. *38406501359372282063949 & all that: Monodromy of Fano Problems*. arXiv:2002.04580 (2020), 17 pp.
- [229] S. N. Hashin. *Birational automorphisms of a double cone of Veronese of dimension 3*. Vestnik MGU. Ser. mat., mech. 1 (1983), 1984, 13–16
- [230] B. Hassett. *Rational curves on K3 surfaces*. Lectures at the 2010 Grenoble summer school. Grenoble, 2010, 25 pp. Available at: <https://www-fourier.ujf-grenoble.fr/sites/ifmaquette.ujf-grenoble.fr/files/hassett2.pdf>
- [231] B. Hassett, Yu. Tschinkel. *Rational curves on holomorphic symplectic fourfolds*. Geom. Funct. Anal. 11 (2001), 1201–1228
- [232] B. Hassett, Yu. Tschinkel, with an appendix by J.-L. Colliot-Thélène. *Rationality of complete intersections of two quadrics*. arXiv:1903.08979 (2019), 41 pp.
- [233] B. Hassett, Yu. Tschinkel. *Varieties of planes on intersections of three quadrics*. arXiv:1904.06313 (2019), 21 pp.
- [234] Dang Tuan Hiep. *Intersection Theory with Applications to the Computation of Gromov–Witten Invariants*, Thesis, Technische Universität Kaiserslautern, 2014, 125 pp.
- [235] Dang Tuan Hiep. *On the degree of the Fano schemes of linear subspaces on hypersurfaces*. Kodai Math. J. 39 (2016), 110–118
- [236] Dang Tuan Hiep. *Numerical invariants of Fano schemes of linear subspaces on complete intersections*. arXiv:1602.03659 (2017), 7 pp.
- [237] Dang Tuan Hiep. *Rational curves on Calabi–Yau threefolds: Verifying mirror symmetry predictions*. J. Symbolic Comput. 76 (2016), 65–83
- [238] M. Hochster, D. Laksov. *The linear syzygies of generic forms*. Comm. Algebra 15 (1987), 227–239
- [239] M. Homma, S. J. Kim. *A bound for the number of lines lying on a nonsingular surface in 3-space over a finite field*. arXiv:1409.6421v2 (2014), 4 pp.
- [240] A. Höring. *M-regularity of the Fano surface*. C. R. Math. Acad. Sci. Paris 344 (2007), 691–696

- [241] A. Höring. *Minimal classes on the intermediate Jacobian of a generic cubic threefold*. Commun. Contemp. Math. 12 (2010), 55–70
- [242] C. Hummel. *Gromov’s compactness theorem for pseudo-holomorphic curves*. Progress in Math. 151. Birkhäuser, 1997
- [243] D. Huybrechts. *Curves and cycles on K3 surfaces. With an appendix by C. Voisin*. Algebraic Geometry 1 (2014), 69–106
- [244] D. Huybrechts. *Lectures on K3 Surfaces*, Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 2016
- [245] Dinh Tuan Huynh. *Construction of hyperbolic hypersurfaces of low degree in $\mathbb{P}^n(\mathbb{C})$* . Internat. J. Math. 27 (2016), 1650059, 9 pp.
- [246] J. M. Hwang. *Geometry of minimal rational curves on Fano manifolds*. In: School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000). ICTP Lect. Notes 6, 335–393. Abdus Salam Int. Cent. Theoret. Phys. 2001
- [247] J.-M. Hwang. *A bound on the number of curves of a given degree through a general point of a projective variety*. Compos. Math. 141 (2005), 703–712
- [248] J. M. Hwang. *Mori geometry meets Cartan geometry: Varieties of minimal rational tangents*. In: Proc. Intern. Conf. Math. Seoul 2014, Vol. I, 369–394, Kyung Moon SA, Seoul, 2014
- [249] J. M. Hwang, S. Kebekus. *Geometry of chains of minimal rational curves*. J. Reine Angew. Math. 584 (2005), 173–194
- [250] J. M. Hwang, H. Kim. *Varieties of minimal rational tangents on Veronese double cones*. Algebraic Geometry 2 (2) (2015), 176–192
- [251] J. M. Hwang, N. Mok. *Varieties of minimal rational tangents on uniruled manifolds*. In: Several Complex Variables, M. Schneider and Y.-T. Siu (eds.), 351–389. MSRI Publications 37, Cambridge University Press, 1999
- [252] J. M. Hwang, N. Mok. *Birationality of the tangent map for minimal rational curves*. Asian J. Math. 8 (2004), 51–63
- [253] A. R. Iano-Fletcher. *Working with weighted complete intersections*. In: Explicit Birational Geometry of 3-folds, 101–174. A. Corti, M. Reid, eds. University of Warwick, 2003
- [254] A. Ikeda. *Subvarieties of generic hypersurfaces in a nonsingular projective toric variety*. Math. Z. 263 (2009), 923–937
- [255] A. Iliev. *Geometry of the Fano threefold of degree 10 of the first type*. Curves, Jacobians, and Abelian varieties (Amherst, MA, 1990). Amer. Math. Soc., Providence, RI, Contemp. Math. 136 (1992).
- [256] A. Iliev. *Lines on the Gushel threefold*. Indag. Math. 5 (1994), 307–320
- [257] A. Iliev. *The Fano surface of the Gushel threefold*. Compos. Math. 94 (1994), 81–107
- [258] A. Iliev. *The Sp_3 -Grassmannian and duality for prime Fano threefolds of genus 9*. Manuscr. Math. 112 (2003), 29–53
- [259] A. Iliev. *Pfaffian lines and vector bundles on Fano threefolds of genus 8*. J. Alg. Geom. 16 (2007), 499–530
- [260] A. Iliev, L. Manivel. *Prime Fano threefolds and integrable systems*. Math. Ann. 339 (2007), 937–955
- [261] A. Iliev, L. Manivel. *Cubic hypersurfaces and integrable systems*. Amer. J. Math. 130 (2008), 1445–1475
- [262] A. Iliev, L. Manivel. *Fano manifolds of degree ten and EPW sextics*. Ann. Sci. Éc. Norm. Supér. (4), 44 (2011), 393–426
- [263] A. Iliev, D. Markushevich. *The Abel-Jacobi map for a cubic threefold and periods of Fano threefolds of degree 14*. Documenta Math. 5 (2000), 23–47
- [264] A. Iliev, D. Markushevich. *Quartic 3-fold: Pfaffians, vector bundles, and half-canonical curves*. Michigan Math. J. 47 (2000), 385–394
- [265] A. Iliev, K. Ranestad. *Geometry of the Lagrangian Grassmannian $LG(3, 6)$ with applications to Brill–Noether loci*. Michigan Math. J. 53(2) (2005), 383–417
- [266] N. Ilten. *Fano schemes of lines on toric surfaces*. Beitr. Algebra Geom. 57 (2016), 751–763
- [267] N. Ilten, T. L. Kelly. *Fano schemes of complete intersections in toric varieties*. arXiv:1910.05593 (2019), 26 pp.
- [268] N. Ilten, H. Süß. *Fano schemes for generic sums of products of linear forms*. arXiv:1610.06770 (2019), 22 pp.
- [269] N. Ilten, M. Wrobel. *Khovanskii-finite valuations, rational curves, and torus actions*. arXiv:1807.08780 (2018), 18 pp.
- [270] N. Ilten, A. Zotine. *On Fano schemes of toric varieties*. SIAM J. Appl. Algebra Geom. 1(1) (2017), 152–174
- [271] P. Ionescu, F. Russo. *Conic-connected manifolds*. J. Reine Angew. Math. 644 (2010), 145–157
- [272] P. Ionescu, F. Russo. *Manifolds covered by lines, defective manifolds and a restricted Hartshorne Conjecture*. arXiv:0909.2763v3 (2012), 21p.
- [273] P. Ionescu, F. Russo. *Manifolds covered by lines and the Hartshorne Conjecture for quadratic manifolds*. Amer. J. Math. 135 (2013), 349–360

- [274] V. A. Iskovskikh. *Fano threefolds I, II*. Izv. Akad. Nauk SSSR Ser. Mat. 41 (1977) 516–562; 42 (1978) 506–549
- [275] V. A. Iskovskikh. *Anticanonical models of three-dimensional algebraic varieties*. J. Soviet Math. 13 (1980), 745–814. Translated from: Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat. 12, VINITI, Moscow, 1979, 59–157
- [276] V. A. Iskovskikh. *Birational automorphisms of three-dimensional algebraic varieties*. J. Soviet Math. 13 (1980), 815–868. Translated from: Itogi Nauki i Tekhniki, Sovremennye Problemy Matematiki 12 (1979), 159–236
- [277] V. A. Iskovskikh. *Algebraic threefolds (a brief survey)*. (Russian) Algebra, 46–78, Moskov. Gos. Univ., Moscow, 1982
- [278] V. A. Iskovskikh. *Double projection from a line onto Fano threefolds of the first kind*. Math. USSR-Sb. 66 (1990), 265–284
- [279] V. A. Iskovskikh, Ju. I. Manin. *Quartic threefolds and counterexamples to the Luroth problem*. Math. Sbornik 86 (1971), 140–166
- [280] V. A. Iskovskikh, Yu. G. Prokhorov. *Fano Varieties*. Algebraic Geometry V, Encyclopaedia Math. Sci. 47, Springer-Verlag, 1999
- [281] E. Izadi. *The geometric structure of A_4 , the structure of the Prym map, double solids and Γ_{00} -divisors*. J. Reine Angew. Math. 462 (1995), 93–158
- [282] A. Javanpeykar, D. Loughran. *Good reduction of Fano threefolds and sextic surfaces*. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 18 (2018), 509–535
- [283] Zhi Jiang. *A Noether-Lefschetz theorem for varieties of r -planes in complete intersections*. Nagoya Math. J. 206 (2012), 39–66
- [284] M. Jinzenji. *Coordinate change of Gauss-Manin system and generalized mirror transformation*. Int. J. Mod. Phys. A20 (2005), 2131–2156
- [285] M. Jinzenji, I. Nakamura, and Y. Suzuki. *Conics on a Generic Hypersurface*. Hokkaido University preprint series in mathematics 710. Department of Math., Hokkaido Univ., 2005. arXiv:math/0412527 (2005), 18p.
- [286] T. Johnsen, S.L. Kleiman. *Rational curves of degree at most 9 on a general quintic threefold*. Comm. Algebra 24 (1996), 2721–2753
- [287] T. Johnsen, S.L. Kleiman. *Toward Clemens’ conjecture in degrees between 10 and 24*. Serdica Math. J. 23 (1997), 131–142
- [288] T. Johnsen, A. L. Knutsen. *Rational Curves in Calabi-Yau Threefolds*. Comm. in Algebra 31 (2003), 3917–3953. Errata and Addenda: Comm. Algebra 33 (2005), 2075–2076
- [289] V. Kanev. *Intermediate Jacobians and Chow groups of threefolds with a pencil of Del Pezzo surfaces*. Annali di Matematica Pura ed Applicata (IV) 154 (1989), 13–48
- [290] M. Kapustka, K. Ranestad. *Vector bundles on Fano varieties of genus ten*. Math. Ann. 356 (2013), 439–467
- [291] J. L. Kass, K. Wickelgren. *An arithmetic count of the lines on a smooth cubic surface*. arXiv:1708.01175 (2017), 31 pp.
- [292] N. Katz. *On the finiteness of rational curves on quintic threefolds*. Compositio Math. 60 (1986), 151–162
- [293] M. Kerr, J. D. Lewis. *The Abel–Jacobi map for higher Chow groups, II*. Invent. Math. 170 (2007), 355–420
- [294] M. Kerr, J. D. Lewis, and S. Mueller-Stach. *The Abel–Jacobi map for higher Chow groups*. Compos. Math. 142 (2006), 374–396
- [295] H. P. Kley. *Rigid curves in complete intersection Calabi-Yau threefolds*. Compositio Math. 123 (2000), 185–208
- [296] B. Kim, Y. Lee, and K. Oh. *Rational curves on blowing-ups of projective spaces*. Michigan Math. J. 55 (2007), 335–345
- [297] A. L. Knutsen. *On degrees and genera of smooth curves on projective K3 surfaces*. Math. Scand. 90 (2002), 215–231
- [298] A. L. Knutsen. *On isolated smooth curves of low genera in Calabi-Yau complete intersection threefolds*. Trans. Amer. Math. Soc. 364 (2012), 5243–5264
- [299] S. Kobayashi. *Hyperbolic manifolds and holomorphic mappings: An introduction*. 2d ed., World Scientific, 2005.
- [300] J. Kollár. *Rational curves on algebraic varieties*. Ergebnisse der Mathematik und ihrer Grenzgebiete 3, vol. 32. Springer-Verlag, Berlin, 1996
- [301] J. Kollár. *Which are the simplest algebraic varieties?* Bull. Amer. Math. Soc. 38(4) (2001), 409–433
- [302] J. Kollár. *Algebraic hypersurfaces*. Bull. Amer. Math. Soc. 56 (2019), 543–568
- [303] J. Kollár, Y. Miyaoka, and Sh. Mori. *Rationally connected varieties*. J. Alg. Geom. 1 (1992), 429–448
- [304] J. Kollár, Y. Miyaoka, and Sh. Mori. *Rational connectedness and boundedness of Fano manifolds*. J. Diff. Geom. 36 (1992), 765–779
- [305] J. Kollár, Y. Miyaoka, and Sh. Mori. *Rational curves on Fano varieties*. Lect. Notes Math. 1515 (1992), 100–105

- [306] M. Kontsevich. *Enumeration of Rational Curves via Torus Actions*. The moduli space of curves, R. Dijkgraaf, C. Faber, G. van der Geer (Eds.), 335–368. Progress in Math. 129, Birkhauser, 1995
- [307] M. Kontsevich, Yu. Manin. *Gromov-Witten classes, quantum cohomology, and enumerative geometry*. Comm. Math. Phys. 164 (1994), 525–562
- [308] T. Krämer. *Cubic threefolds, Fano surfaces and the monodromy of the Gauss map*. Manuscr. Math. 149 (2016), 303–314
- [309] P. F. Kurchanov. *On conics on a double 3-dimensional quadric*. Russian Math. Surveys 41 (1986), 231–232
- [310] A. Kuznetsov. *Derived categories of Fano threefolds*. Proc. Steklov Inst. Math. 264 (2009), 110–122
- [311] A. Kuznetsov, L. Manivel, D. Markushevich. *Abel–Jacobi maps for hypersurfaces and noncommutative Calabi–Yau’s*. Comm. in Contemp. Math. 12 (2010), 373–416
- [312] A. Kuznetsov, Y. Prokhorov. *Fano threefolds of genus 12 with a \mathbb{G}_m -action*. Epijournal de Géométrie Algébrique 2:epiga:4560 (2018), 1–14
- [313] A. Kuznetsov, Y. Prokhorov, and C. Shramov. *Hilbert schemes of lines and conics and automorphism groups of Fano threefolds*. Japan. J. Math. 13 (2018), 109–185
- [314] A. Kuznetsova. *Non-rational sextic double solids*. arXiv:1901.05055v2 (2019), 22p.
- [315] J. M. Landsberg. *Is a linear space contained in a submanifold? On the number of derivatives needed to tell*. J. Reine Angew. Math. 508 (1999), 53–60
- [316] J. M. Landsberg. *Lines on projective varieties*. J. Reine Angew. Math. 562 (2003), 1–3
- [317] J. Landsberg, L. Manivel. *On the projective geometry of rational homogeneous varieties*. Comment. Math. Helv. 78 (2003), 65–100
- [318] J. M. Landsberg, C. Robles. *Fubini’s theorem in codimension two*. J. Reine Angew. Math. 631 (2009), 221–235
- [319] J. M. Landsberg, O. Tommasi. *On the Debarre–de Jong and Beheshti–Starr conjectures on hypersurfaces with too many lines*. Michigan Math. J. 59 (2010), 573–588
- [320] S. Lang. *Hyperbolic and Diophantine analysis*. Bull. Amer. Math. Soc. 14 (1986), 159–205
- [321] A. Langer. *Fano schemes of linear spaces on hypersurfaces*. Manuscr. Math. 93 (1997), 21–28
- [322] A. Langer. *Logarithmic orbifold Euler numbers of surfaces with applications*. Proc. London Math. Soc. 86 (2003), 358–396
- [323] A. Lanteri, M. Palleschi. *Projective manifolds containing many rational curves*. Indiana Univ. Math. J. 36 (1987), 857–867
- [324] P. Larsen. *Notes on Fano varieties of complete intersections*. arXiv:1211.6249 (2012), 14 pp.
- [325] P. Larsen, F. Király. *Fano schemes of generic intersections and machine learning*. Internat. J. Algebra Comput. 24(7) (2014), 923–933
- [326] Y. Laszlo. *Théorème de Torelli générique pour les intersections complètes de trois quadriques de dimension paire*. Invent. Math. 98 (1989), 247–264
- [327] J. Lee and N. C. Leung. *Counting elliptic curves in K3 surfaces*. J. Algebraic Geom. 15 (2006), 591–601
- [328] B. Lehmann, S. Tanimoto. *Geometric Manin’s conjecture and rational curves*. Compos. Math. 155 (2019), 833–862
- [329] M. Letizia. *The Abel–Jacobi mapping for the quartic threefold*. Invent. Math. 75 (1984), 477–492
- [330] J. D. Lewis. *On hypersurfaces admitting a covering by rational curves*. Comm. Algebra 13 (1985), 1917–1930
- [331] J. D. Lewis. *The cylinder correspondence for hypersurfaces of degree n in \mathbb{P}^n* . Amer. J. Math. 110 (1988), 77–114
- [332] J. D. Lewis. *The cylinder homomorphism for quintic fourfolds*. Compos. Math. 56 (2001), 315–329
- [333] J. D. Lewis. *Lectures on Hodge Theory and Algebraic Cycles*. Commun. Math. Stat. 4 (2016), 93–188
- [334] J. Li, C. Liedtke. *Rational curves on K3 surfaces*. Invent. Math. 188 (2012), 713–727
- [335] A. Libgober. *Numerical characteristics of systems of straight lines on complete intersections*. Math. Notes 13 (1973), 51–56. Translated from: Mat. Zametki 13 (1973), 87–96
- [336] A. Libgober, J Teitelbaum. *Lines on Calabi–Yau complete intersections, Mirror Symmetry, and Picard–Fuchs equations*. Internat. Math. Res. Notices 1 (1993), 29–39
- [337] D. Logachev. *Isomorphism of intermediate Jacobians of some dual sections of determinantal*. Math. USSR–Izvestiya 20 (1983), 259–265. Translated from: Izv. Akad. Nauk SSSR Ser. Mat. 46 (1982), 269–275
- [338] D. Logachev. *Fano threefolds of genus 6*. Asian J. Math. 16 (2012), 515–560
- [339] S. S.-Y. Lu, Y. Miyaoka. *Bounding curves in algebraic surfaces by genus and Chern numbers*. Math. Res. Lett. 2 (1995), 663–676
- [340] S.S.-Y. Lu, Y. Miyaoka. *Bounding codimension-one subvarieties and a general inequality between Chern numbers*. Amer. J. Math. 119 (1997), 487–502
- [341] S. Lu, De-Qi Zhang. *Positivity criteria for log canonical divisors and hyperbolicity*. J. Reine Angew. Math. 726 (2017), 173–186
- [342] E. Macrì, P. Stellari. *Fano varieties of cubic fourfolds containing a plane*. Math. Ann. 354 (2012), 1147–1176

- [343] E. Macrì, P. Stellari. *Lectures on non-commutative K3 surfaces, Bridgeland stability, and moduli spaces*. arXiv:1807.06169 (2019), 66 pp.
- [344] J. A. Maia, A. Rodrigues, F. Xavier and I. Vainsencher. *Enumeration of surfaces containing a curve of low degree*. Preprint (2011), 19 pp.
- [345] Yu. I. Manin. *Cubic Forms, Algebra, Geometry, Arithmetic*. North Holland, 1974; second edit., 1986
- [346] Yu. I. Manin. *Notes on the arithmetic of Fano threefolds*. Compositio Mathem. 85 (1993), 37–55
- [347] Yu. I. Manin. *Problems on rational points and rational curves on algebraic varieties*. In: Surveys in differential geometry II (Cambridge, MA, 1993), 214–245. International Press, Cambridge, MA, 1995
- [348] L. Manivel. *Sur les hypersurfaces contenant des espaces linéaires*. C. R. Acad. Sci. Paris Sér. I Math. 328 (1999), 307–312
- [349] L. Manivel. *Symmetric functions, Schubert polynomials and degeneracy loci*. AMS Texts and Monographs 6. Providence, RI, 2001
- [350] S. Marchesi, A. Massarenti. *Varieties connected by chains of lines*. Geom. Dedicata 168 (2014), 61–68
- [351] S. Marchesi, A. Massarenti, and S. Tafazolian. *Covered by lines and conic connected varieties*. Le Matematiche 64 (2011), 137–151
- [352] D. G. Markushevich. *Numerical invariants of families of lines on some Fano varieties*. Mat. USSR Sb. 44 (1983), 239–260. Translated from: Mat. Sb. 116 (158) (1981), 265–288
- [353] D. G. Markushevich, A. S. Tikhomirov. *The Abel–Jacobi map of a moduli component of vector bundles on the cubic threefold*. J. Algebraic Geom. 10 (2001), 37–62
- [354] D. G. Markushevich, A. S. Tikhomirov. *Parameterization of the theta divisor of the quartic double solid*. Int. Math. Res. Not. 51 (2003), 2747–2778
- [355] D. G. Markushevich, A. S. Tikhomirov. *Symplectic structure on a moduli space of sheaves on the cubic fourfold*. Izv. Math. 67 (2003), 121–144
- [356] D. Maulik, R. Pandharipande. *Gromov–Witten theory and Noether–Lefschetz theory*. arXiv:0705.1653 (2007), 48p.
- [357] D. McDuff, D. Salamon. *J-holomorphic curves and quantum cohomology*. University Lecture Series 6, Amer. Math. Soc. 1994
- [358] D. McDuff, D. Salamon. *J-holomorphic curves and symplectic topology*. Amer. Math. Soc. Colloquium Publications 52. Amer. Math. Soc., Providence, RI, 2004
- [359] A. L. Meireles Araújo, I. Vainsencher. *Equivariant intersection theory and Bott’s residue formula*. 16th School of Algebra, Part I (Portuguese) (Brasilia, 2000). Mat. Contemp. 20 (2001), 1–70
- [360] M. Mella. *Existence of good divisors on Mukai varieties*. J. Algebraic Geom. 8 (1999), 197–206
- [361] J.-Y. Mériandol. *Théorème de Torelli affine pour les intersections de deux quadriques*. Invent. Math. 80 (1985), 375–416
- [362] J. Merker, The-Anh Ta. *Degrees $d \geq (\sqrt{n} \log n)^n$ and $d \geq (n \log n)^n$ in the conjectures of Green–Griffiths and of Kobayashi*. arXiv:1901.04042, 52 pp.
- [363] P. Meurer. *The number of rational quartics on Calabi–Yau hypersurfaces in weighted projective space $\mathbb{P}(2, 1^4)$* . Math. Scand. 78 (1996), 63–83
- [364] E. Mezzetti, D. Portelli. *On threefolds covered by lines*. Abh. Math. Sem. Univ. Hamburg 70 (2000), 211–238
- [365] Y. Miyaoka. *The orbifold Miyaoka–Yau–Sakai inequality and an effective Bogomolov–McQuillan Theorem*. Publ. RIMS, Kyoto Univ. 44 (2008), 403–417
- [366] Y. Miyaoka. *Counting lines and conics on a surface*. Publ. RIMS, Kyoto Univ. 45 (2009), 919–923
- [367] C. Miyazaki. *Remarks on r -planes in complete intersections*. Tokyo J. Math. 39 (2016), 459–467
- [368] S. Mori. *Projective manifolds with ample tangent bundles*. Ann. Math. (2) 110 (1979), 593–606
- [369] S. Mori. *Lecture on extremal rays and Fano 3-folds*. Nagoya University, Fall Term 1983–84
- [370] S. Mori. *On degrees and genera of curves on smooth quartic surfaces in \mathbb{P}^3* . Nagoya Math. J. 96 (1984), 127–132
- [371] S. Mori, S. Mukai. *Classification of Fano 3-folds with $B_2 \geq 2$* . Manuscr. Math. 36 (1981), 149–162. *Erratum*. Manuscr. Math. 110 (2003), 407
- [372] S. Mori, S. Mukai. *The uniruledness of the moduli space of curves of genus 11*. In: Alg. Geometry (Tokyo/Kyoto, 1982), 334–353. Lect. Notes in Math. 1016, Springer, Berlin, 1983
- [373] S. Mori, S. Mukai. *Classification of Fano 3-folds with $B_2 \geq 2$* . I. Algebraic and Topological Theories – to the memory of Dr. Takehiko MIYATA (1985), 496–545
- [374] U. Morin. *Sull’insieme degli spazi lineari contenuti in una ipersuperficie algebrica*. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. 24 (1936), 188–190
- [375] D. R. Morrison. *The geometry of K3 surfaces*. Lectures at the Scuola Matematica Interuniversitaria, Cortona, Italy, 1988, 78 pp.
- [376] G. Mostad Hana, T. Johnsen. *Rational curves on a general heptic fourfold*. Bull. Belg. Math. Soc. Simon Stevin 16 (2009), 861–885

- [377] S. Mukai. *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface*. Invent. Math. 77 (1984), 101–116
- [378] S. Mukai. *Curves, K3 surfaces and Fano 3-folds of genus ≤ 10* . Algebraic geometry and commutative algebra I, 357–377. Kinokuniya, Tokyo, 1988
- [379] S. Mukai. *Biregular classification of Fano 3-folds and Fano manifolds of coindex 3*. Proc. Nat. Acad. Sci. 86 (1989), 3000–3002
- [380] S. Mukai. *Fano 3-folds*. In: Complex Projective Geometry, Trieste–Bergen, 1989, 255–263. London Math. Soc. Lecture Note Ser. 179, Cambridge Univ. Press, Cambridge, 1992
- [381] S. Mukai. *New development of theory of Fano 3-folds: vector bundle method and moduli problem*. Sugaku 47 (1995), 125–144
- [382] S. Mukai. *Plane quartics and Fano threefolds of genus twelve*. The Fano Conference, 563–572. Univ. Torino, Turin, 2004
- [383] S. Mukai, H. Umemura. *Minimal rational threefolds*. In: Algebraic Geometry: Proc. Japan. France Conf., Tokyo, 1982, 490–518. Lect. Notes Math. 1016. Springer, 2006
- [384] D. Mumford. *Prym varieties*. I. Contrib. Anal. New York - London (1974), 325–350
- [385] J. P. Murre. *Some results on a cubic threefold*. Lecture Notes in Math. 412, 1974
- [386] J. P. Murre. *Classification of Fano threefolds according to Fano and Iskovskih*. Algebraic Threefolds, 35–92. Lecture Notes in Math. 947, Springer-Verlag, Heidelberg, 1982
- [387] D. Nesterov, G. Oberdieck. *Elliptic curves in hyper-Kähler varieties*. arXiv:1906.05817v3 (2020), 24 pp.
- [388] P. G. J. Nijssse. *Clemens’ conjecture for octic and nonic curves*. Indag. Math. (N.S.) 6(2) (1995), 213–221
- [389] V. V. Nikulin. *On Kummer surfaces*. Math. USSR Izv. 9 (1975), 261–275
- [390] T. Nishinou. *Obstructions to deforming maps from curves to surfaces*. arXiv:1901.11239 (2019), 19 pp.
- [391] K. G. O’Grady, *The Hodge structure of the intersection of three quadrics in an odd-dimensional projective space*. Math. Ann. 273 (1986), 277–285
- [392] K. Oguiso. *Two remarks on Calabi-Yau Moishezon threefolds*. J. Reine Angew. Math. 452 (1994), 153–161
- [393] G. Occhetta, L. E. Solá Conde, and K. Watanabe. *Uniform families of minimal rational curves on Fano manifolds*. Revista Matemática Complutense 29 (2015), 423–437
- [394] Ch. Okonek, A. Teleman. *Intrinsic signs and lower bounds in real algebraic geometry*. J. Reine Angew. Math. 688 (2014), 219–241
- [395] G. Pacienza. *Rational curves on general projective hypersurfaces*. J. Alg. Geom. 12 (2003), 245–267
- [396] G. Pacienza. *Subvarieties of general type on a general projective hypersurface*. Trans. Amer. Math. Soc. 356 (2004), 2649–2661
- [397] X. Pan. *2-Cycles on higher Fano hypersurfaces*. Math. Ann. 361 (2017), 1791–1818
- [398] X. Pan. *Spaces of conics on low degree complete intersections*. Trans. Amer. Math. Soc. 370 (2018), 5381–5400
- [399] X. Pan. *Spaces of low-degree rational curves on Fano complete intersections*. Int. Math. Res. Notices, rnz026 (2019), 35 pp. doi:10.1093/imrn/rnz026
- [400] P. Pansu. *Pseudo-holomorphic curves in symplectic manifolds*. Ecole Polytechnique, Palaiseau, 1986
- [401] K. Paranjape, V. Srinivas. *Unirationality of the general complete intersection of small multidegree*. Astérisque 211 (1992), 241–248
- [402] B. Pasquier, N. Perrin. *Elliptic curves on some homogeneous spaces*. Doc. Math. 18 (2013), 679–706
- [403] M. Pauen. *Vector fields on the total space of hypersurfaces in the projective space and hyperbolicity*. Math. Ann. 340 (2008), 875–892
- [404] N. Perrin. *Courbes rationnelles sur les variétés homogènes*. Ann. Inst. Fourier (Grenoble) 52 (2002), 105–132
- [405] N. Perrin. *Rational curves on minuscule Schubert varieties*. J. Algebra 294 (2005), 431–462
- [406] E. Peyre. *Hauteurs et mesures de Tamagawa sur les variétés de Fano*. Duke Math. J. 79 (1995), 101–218
- [407] E. Peyre. *Rational points and curves on flag varieties* (joint with A. Chambert-Loir). In Mathematisches Forschungsinstitut Oberwolfach Report 12, 650–654, 2004; <https://www.mfo.de/scientific-program/publications/owr/sample-copy/report-2004.pdf/view>
- [408] E. Peyre. *Beyond heights: slopes and distribution of rational points*. arXiv:1806.11437 (2018), 53 pp.
- [409] L. Picco Botta. *On the intersection of three quadrics*. J. Reine Angew. Math. 399 (1989), 188–207
- [410] R. Piene. *On the enumeration of algebraic curves – from circles to instantons*. First European Congress of Mathematics, Vol. II (Paris, 1992), 327–353. Progr. Math. 120, Birkhäuser, Basel, 1994
- [411] R. Piene, M. Schlessinger. *On the Hilbert scheme compactification of the space of twisted cubics*. Amer. J. Math. 107 (1985), 761–774
- [412] A. Predonzan. *Intorno agli S_k giacenti sulla varietà intersezione completa di più forme*. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. 5 (1948), 238–242
- [413] Yu. G. Prokhorov. *On exotic Fano varieties*. Mosc. Univ. Bull. 45(3) (1990), 36–38. Translated from: Vestn. Mosk. Univ. Ser. I, 3 (1990), 34–37

- [414] Yu. G. Prokhorov. *Rationality constructions of some Fano fourfolds of index 2*. Moscow Univ. Math. Bull. 48 (1993), 32–35
- [415] Yu. G. Prokhorov. *The rationality problem for conic bundles*. Russian Math. Surveys 73 (2018), 375–456
- [416] Yu. G. Prokhorov. *Fano varieties of dimension three*. Lecture notes, 2017/2018, 158 pp. (Russian) <http://www.mi-ras.ru/prokhorov/teach/2017-fano/fano.pdf>
- [417] Yu. G. Prokhorov, M. Zaidenberg. *Fano-Mukai fourfolds of genus 10 as compactifications of \mathbb{C}^4* . Eur. J. Math. 4:1197 (2018), 1–67; <https://doi.org/10.1007/s40879-018-0244-y>
- [418] V. V. Przyjalkowski. *Gromov–Witten invariants of Fano threefolds of genera 6 and 8*. Sb. Math. 198 (2007), 433–446
- [419] P. J. Puts. *On some Fano-threefolds that are sections of Grassmannians*. Nederl. Akad. Wetensch. Indag. Math. 44 (1982), 77–90
- [420] D. Yu. Radionov. *Reconstruction of special double \mathbb{P}^3 from the family of lines on it*. Moscow Univ. Math. Bull. 45 (1990), 39–42. Translated from: Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1990, 69–72
- [421] D. Yu. Radionov. *On the Torelli and Schottky problems for Prym varieties*. Izv. Math. 45 (1995), 297–314. Translated from: Izv. Ross. Akad. Nauk Ser. Mat. 58 (1994), 68–84
- [422] S. Rams. *Projective surfaces with many skew lines*. Proc. Amer. Math. Soc. 133 (2004), 11–13
- [423] S. Rams, M. Schütt. *On quartics with lines of the second kind*. Adv. Geom. 14 (2014), 735–756
- [424] S. Rams, M. Schütt. *112 lines on smooth quartic surfaces (characteristic 3)*. The Quarterly J. Math. 66, (2015), 941–951
- [425] S. Rams, M. Schütt. *64 lines on smooth quartic surfaces*. Math. Ann. 362 (2015), 679–698
- [426] S. Rams, M. Schütt. *At most 64 lines on smooth quartic surfaces (characteristic 2)*. Nagoya Math. J. (to appear), 20 pp.
- [427] S. Rams, M. Schütt. *Counting lines on surfaces, especially quintics*. arXiv:1803.03548 (2018), 30 pp.
- [428] S. Rams, M. Schütt. *24 rational curves on K3 surfaces*. arXiv:1907.04182 (2019), 14 pp.
- [429] Z. Ran. *The degree of a Severi variety*. Bull. Amer. Math. Soc. 17 (1987), 125–128
- [430] Z. Ran. *Enumerative geometry of singular plane curves*. Invent. Math. 97 (1989), 447–469
- [431] Z. Ran. *Low-degree rational curves on hypersurfaces in projective spaces and their degenerations*. arXiv:1906.03747 (2019), 18 pp.
- [432] K. Ranestad, B. Sturmfels. *Twenty-seven questions about the cubic surface*. arXiv:1912.07347 (2019), 13 pp.
- [433] M. Reid. *The complete intersection of two or more quadrics*. PhD thesis, Trinity College, Cambridge, 1972.
- [434] M. Reid. *Lines on Fano 3-folds according to Shokurov*. Inst. Mittag-Leffler, 1980, report No 11, 1–41
- [435] M. Reid. *Canonical 3-folds*. In: Journées de Géométrie algébrique d’Angers, ed. A. Beauville, 273–310. Sijthoff and Noordhoff, Alphen, 1980
- [436] M. Reid. *Minimal models of canonical 3-folds*. Advanced Studies in Pure Math. 1, Algebraic Varieties and Analytic Varieties (1983), 131–180
- [437] E. Riedl, M. Woolf. *Rational curves on complete intersections in positive characteristic*. J. Algebra 494 (2018), 28–39
- [438] E. Riedl, D. Yang. *Rational curves on general type hypersurfaces*. arXiv:1605.01759 (2016), 9p. To appear in J. of Diff. Geometry.
- [439] E. Riedl, D. Yang. *Applications of a grassmannian technique in hypersurfaces*. arXiv:1806.02364 (2018), 10 pp.
- [440] E. Riedl, D. Yang. *Kontsevich spaces of rational curves on Fano hypersurfaces*. J. Reine Angew. Math. 748 (2019), 207–225
- [441] E. Rogora. *Varieties with many lines*. Manuscr. Math. 82 (1994), 207–226
- [442] L. Roth. *Algebraic varieties with canonical curve sections*. Ann. Mat. Pura Appl. 29 (1949), 91–97
- [443] L. Roth. *Algebraic threefolds with special regard to problems of rationality*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 2. Folge. Springer Verlag, 1955
- [444] X. Roulleau. *Coubes elliptiques, fibrations et automorphismes des surfaces de Fano*. C. R. Math. Acad. Sci. Paris 345 (2007), 209–212
- [445] X. Roulleau. *Elliptic curve configurations on Fano surfaces*. Manuscr. Math. 129 (2009), 381–399
- [446] G. Salmon. *A treatise on the analytic geometry of three dimensions*. Vol. II., Fifth edition, Longmans and Green, London, 1915
- [447] G. Sanna. *Rational curves and instantons on the Fano threefold Y_5* . PhD thesis, Scuola Internazionale di Studi Superiori Avanzati, 2013–2014
- [448] W. Sawin. *Freeness alone is insufficient for Manin-Peyre*. arXiv:2001.06078 (2020), 7 pp.
- [449] H. Schubert. *Kalkül der Abzählenden Geometrie*. B. G. Teubner. Leipzig, 1879
- [450] M. Schuett, T. Shioda, and R. van Luijk. *Lines on Fermat surfaces*. J. Number Theory 130 (2010), 1939–1963
- [451] F. Schur. *Über eine besondere Classe von Flächen vierter Ordnung*. Math. Ann. 20 (1882), 254–296
- [452] B. Segre. *The non-singular cubic surfaces*. The Clarendon Press, Oxford, 1942

- [453] B. Segre. *The maximum number of lines lying on a quartic surface*. Oxford Quart. J. 14 (1943), 86–96
- [454] B. Segre. *Sulle V_n contenenti più di $\infty^{n-k} S_k$* . I and II, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) 5 (1948), 193–197 and 275–280
- [455] B. Segre. *On the inflectional curve of an algebraic surface in S_4* . The Quarterly J. Math. 2:1 (1951), 216–220
- [456] M. Shen. *On the normal bundles of rational curves on Fano 3-folds*. Asian J. Math. 16 (2012), 237–270
- [457] M. Shen. *Rational curves on Fermat hypersurfaces*. Compt. Rend. Math. Acad. Sci. 350 (2012), 781–784
- [458] B. Shiffman, M. Zaidenberg. *Two classes of hyperbolic surfaces in \mathbb{P}^3* . Intern. J. Math. 11 (2000), 65–101
- [459] B. Shiffman, M. Zaidenberg. *Hyperbolic hypersurfaces in \mathbb{P}^n of Fermat–Waring type*. Proc. Amer. Math. Soc. 130 (2002), 2031–2035
- [460] B. Shiffman, M. Zaidenberg. *Constructing low degree hyperbolic surfaces in \mathbb{P}^3* . Houston J. Math. (the special issue for S.S. Chern) 28 (2002), 377–388
- [461] B. Shiffman, M. Zaidenberg. *New examples of hyperbolic surfaces in \mathbb{P}^3* . Funct. Anal. Appl. 39 (2005), 76–79
- [462] I. Shimada. *On the cylinder isomorphism associated to the family of lines on a hypersurface*. J. Fac. Sci. Univ. Tokyo Sect. IA Math. 37 (1990), 703–719
- [463] I. Shimada. *On the cylinder homomorphisms of Fano complete intersections*. J. Math. Soc. Japan 42 (1990), 719–738
- [464] I. Shimada. *On the cylinder homomorphism for a family of algebraic cycles*. Duke Math. J. 64 (1991), 201–205
- [465] I. Shimada, T. Shioda. *On a smooth quartic surface containing 56 lines which is isomorphic as a K3 surface to the Fermat quartic*. Manuscr. Math. 153 (2017), 279–297
- [466] D. Shin. *Rational curves on general hypersurfaces of degree 7 in \mathbb{P}^5* . Osaka J. Math. 44 (2007), 1–10
- [467] D. Shin. *Conics on a general hypersurface in complex projective spaces*. Bull. Korean Math. Soc. 50 (2013), 2071–2077
- [468] T. Shioda. *Algebraic cycles on hypersurfaces in \mathbb{P}^N* . Adv. Stud. Pure Math. 10, 717–732. Kinokuniya and North-Holland, 1987
- [469] V. V. Shokurov. *The existence of a straight line on Fano 3-folds*. Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), 922–964
- [470] Y.-T. Siu. *Hyperbolicity of generic high-degree hypersurfaces in complex projective space*. Invent. Math. 202 (2015), 1069–1166
- [471] F. Sottile. *Enumerative real algebraic geometry*. In: *Algorithmic and quantitative real algebraic geometry (Piscataway, NJ, 2001)*, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. 60, 139–179. Amer. Math. Soc., Providence, RI, 2003
- [472] J. Starr. *Rational curves on hypersurfaces in projective n -space*. Thesis, Harvard University, Cambridge, Massachusetts, 2000, 85 pp.
- [473] J. Starr. *The Kodaira dimension of spaces of rational curves on low degree hypersurfaces*. arXiv:math/0305432 (2003), 56 pp.
- [474] E. Strickland. *Lines in G/P* . Math. Z. 242 (2002), 227–240
- [475] T. Suzuki. *An invariant for embedded Fano manifolds covered by linear spaces*. Kyushu J. Math. 72 (2018), 223–230
- [476] H. Takagi, F. Zucconi. *Geometries of lines and conics on the quintic del Pezzo 3-fold and its application to varieties of power sums*. Michigan Math. J. 61 (2012), 19–62
- [477] K. Takeuchi. *Some birational maps of Fano 3-folds*. Contemp. Mathem. 71 (1978), 265–283
- [478] S. Tayou. *Rational curves on elliptic K3 surfaces*. arXiv:1805.07975 (2018), 11 pp.; arXiv:1805.07975v4 (2020), 9 pp.
- [479] B. R. Tennison. *On the quartic threefold*. Proc. London Math. Soc. 29 (1974), 714–734
- [480] T. Terasoma. *Varieties of lines on Fermat hypersurfaces*. Arrangements of Hyperplanes–Sapporo 2009, 459–474. Adv. Stud. Pure Math. 62, Math. Soc. Japan, Tokyo, 2012.
- [481] D. Testa. *The Severi problem for rational curves on del Pezzo surfaces*. PhD thesis, MIT 2005; arxiv.org/0609355 (2006), 99 pp.
- [482] D. Testa. *Fano varieties in index one Fano complete intersections*. Math. Z. 259 (2008), 61–64
- [483] D. Testa. *The irreducibility of the spaces of rational curves on del Pezzo surfaces*. J. Algebraic Geom. 18 (2009), 37–61
- [484] A. S. Tikhomirov. *Middle Jacobian of a double space \mathbb{P}^3 ramified at a quartic*. Izv. AN SSSR, Ser. mat. 44 (1980), 1329–1377
- [485] A. S. Tikhomirov. *The geometry of the Fano surface of the double cover of \mathbb{P}^3 branched in a quartic*. Math. USSR Izv. 16 (1981), 373–397
- [486] A. S. Tikhomirov. *The Fano surface of a Veronese double cone*. Math. USSR-Izvestiya 19 (1982), 377–443. Translated from: Izv. AN SSSR, Ser. mat. 45 (1981), 1121–1197

- [487] A. S. Tikhomirov. *Singularities of a theta divisor of a double space \mathbb{P}^3 of index 2*. Izv. AN SSSR, Ser. mat. 46 (1982), 1062–1081
- [488] A. S. Tikhomirov. *Special double space \mathbb{P}^3 of index 2*. In: Constructive algebraic geometry 200, 121–129. Yaroslavl', 1982
- [489] A. S. Tikhomirov. *The Abel-Jacobi map of sextics of genus 3 on double spaces of \mathbb{P}^3 of index two*. Soviet Math. Dokl. 33 (1986), 204–206
- [490] A. S. Tikhomirov. *New component of the moduli space $M(2; 0, 3)$ of stable vector bundles on the double space \mathbb{P}^3 of index two*. Acta Appl. Math. 75 (2003), 271–279. Translated from: Monodromy and differential equations. Moscow, 2001.
- [491] J. A. Todd. *The locus representing the lines of four-dimensional space*. Proc. London Math. Soc. 30 (1930), 513–550
- [492] S. L. Tregub. *Construction of a birational isomorphism of a cubic threefold and Fano variety of the first kind with $g = 8$, associated with a normal rational curve of degree 4*. Mosc. Univ. Math. Bull. 40 (1985), 78–80. Translated from: Vestn. Mosk. Univ. Ser. I Mat. Mekh. 40 (1985), 99–101
- [493] Y. Tschinkel. *Algebraic varieties with many rational points*. In: Arithmetic geometry, Clay Mathematics Proceedings 8, 243–334. Amer. Math. Soc., Providence, RI, 2009
- [494] D. Tseng. *A note on rational curves on general Fano hypersurfaces*. Michigan Math. J. Advance publication (2019), 20 pp.
- [495] A. N. Tyurin, *On the Fano surface of a nonsingular cubic in \mathbb{P}^4* . Math. USSR Izv. 4 (1970), 1207–1214
- [496] A. N. Tyurin. *The geometry of the Fano surface of a nonsingular cubic $F \subset \mathbb{P}^4$ and Torelli Theorems for Fano surfaces and cubics*. Math. USSR Izv. 5 (1971), 517–546
- [497] A. N. Tyurin. *Five lectures on three-dimensional varieties*. Russian Math. Surveys 27 (1972), 1–53
- [498] A. N. Tyurin. *The geometry of the Poincaré divisor of a Prym variety*. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), 1003–1043
- [499] A. N. Tyurin. *On intersection of quadrics*. Russian Math. Surveys 30 (1975), 51–105
- [500] A. N. Tyurin. *Middle Jacobian of threefolds*. in: Contemporary problems of mathematics. Itogi nauki i tehniki, VINITI, Moscow, 12 (1979), 5–57
- [501] I. Vainsencher, F. Xavier. *A compactification of the space of twisted cubics*. Math. Scand. 91 (2002), 221–243
- [502] R. Vakil. *Counting curves on rational surfaces*. Manuscr. Math. 102 (2000), 53–84
- [503] R. Vakil. *The enumerative geometry of rational and elliptic curves in projective space*. J. Reine Angew. Math. 529 (2000), 101–153
- [504] B. L. van der Waerden. *Zur algebraischen Geometrie. II, die geraden Linien auf den Hyperflächen des \mathbb{P}^n* . Math. Ann. 108 (1933), 253–259
- [505] D. C. Veniani. *The maximum number of lines lying on a K3 quartic surface*. Math. Z. 285 (2017), 1141–1166
- [506] A. Verra. *Problemi di razionalità e unirazionalità: da Ugo Morin ai giorni nostri*. Rend. Istit. Mat. Univ. Trieste XL (2008), 165–184
- [507] I. Vincent. *Lines on intersections of three quadrics*. PhD thesis, warwick.ac.uk/lib-publications, 2016, 85 pp. <http://wrap.warwick.ac.uk/90063>
- [508] C. Voisin. *Théorème de Torelli pour les cubiques de \mathbb{P}^5* . Invent. Math. 86 (1986), 577–601
- [509] C. Voisin. *Sur la jacobienne intermédiaire du double solide d'indice deux*. Duke Math. J. 57 (1988), 629–646
- [510] C. Voisin. *Relations dans l'anneau de Chow de la surface des coniques des variétés de Fano et applications*. Amer. J. Math. 112 (1990), 877–898
- [511] C. Voisin. *On a conjecture of Clemens on rational curves on hypersurfaces*. J. Diff. Geom. 44 (1996), 200–213. *A correction*. ibid., 49 (1998), 601–611
- [512] C. Voisin. *On some problems of Kobayashi and Lang; algebraic approaches*. Current Developments in Mathematics 2003, 53–125
- [513] C. Voisin. *Abel–Jacobi map, integral Hodge classes and decomposition of the diagonal*. J. Algebraic Geom. 1056 (3011) (2012), 1–34
- [514] C. Voisin. *Hodge theory and complex algebraic geometry*. I, II. Cambridge Studies in Advanced Mathematics 76, 77. Cambridge University Press, Cambridge, 2007
- [515] A. Waldron. *Fano varieties of low-degree smooth hypersurfaces and unirationality*. Master thesis, Harvard University, 2008
- [516] B. Wang. *Genus of curves in generic hypersurfaces*. arXiv:1110.0185v3 (2011), 14p.
- [517] B. Wang. *First order deformations of pairs and non-existence of rational curves*. Rocky Mountain J. Math. 46 (2016), 663–678
- [518] L.-C. Wang. *A remark on divisors of Calabi-Yau hypersurfaces*. Asian J. Math. 4 (2000), 369–372
- [519] L.-C. Wang. *Divisors of generic hypersurfaces of general type*. Taiwanese J. Math. 6 (2002), 507–513
- [520] X. Wang. *Maximal linear spaces contained in the base loci of pencils of quadrics*. Algebraic Geometry 5 (3) (2018), 359–397

- [521] Y. Wang. *Rational curves on hypersurfaces*. Michigan Math. J. 66 (2017), 625–635
- [522] K. Watanabe. *Classification of polarized manifolds admitting homogeneous varieties as ample divisors*. Math. Ann. 342 (2008), 557–563
- [523] K. Watanabe. *Actions of linear algebraic groups of exceptional type on projective varieties*. Pacific J. Math. 239 (2009), 391–395
- [524] K. Watanabe. *Lengths of chains of minimal rational curves on Fano manifolds*. J. Algebra 325 (2011), 163–176
- [525] G. E. Welters. *Abel-Jacobi isogenies for certain types of Fano threefolds* (thesis). Mathematical Centre Tracts 141, CWI, Amsterdam, 1981, 139p.
- [526] G. E. Welters. *The Fano surface of lines on a double \mathbb{P}^3 with 4th order discriminant locus*. I. Univ. Utrecht. Preprint No 123, 60 p.
- [527] G. E. Welters. *Divisor varieties, Prym varieties and a conjecture of Tyurin*. Univ. Utrecht. Preprint No 139, 57 p.
- [528] G. E. Welters. *Recovering the curve data from a general Prym variety*. Amer. J. Math. 109 (1987), 165–182
- [529] P. M. H. Wilson. *Towards birational classification of algebraic varieties*. Bull. London Math. Soc. 19 (1987), 1–48
- [530] B. Wu. *The number of rational curves on K3 surfaces*. Asian J. Math. 11 (2007), 635–650
- [531] S.-T. Yau, E. Zaslow. *BPS States, String Duality, and Nodal Curves on K3*. Nuclear Physics B 471(3) (1996), 503–512
- [532] G. Xu. *Subvarieties of general hypersurfaces in projective space*. J. Differential Geom. 39 (1994), 139–172
- [533] G. Xu. *Divisors on hypersurfaces*. Math. Z. 219 (1995), 581–589
- [534] G. Xu. *Divisors on generic complete intersections in projective space*. Trans. Amer. Math. Soc. 348 (1996), 2725–2736
- [535] Z. Xu. *A remark on the Abel-Jacobi morphism for the cubic threefold*. C. R. Acad. Sci. Paris, Ser. I 351 (2013), 63–67
- [536] A. Zahariuc. *Rational curves on Del Pezzo manifolds*. Advances in Geometry 18 (2018), 451–465
- [537] M. Zaidenberg. *Hyperbolicity of general deformations*. Funct. Anal. Appl. 43 (2009), 113–118

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DEGLI STUDI DI ROMA “TOR VERGATA”, VIA DELLA RICERCA SCIENTIFICA, 00133 ROMA, ITALY

E-mail address: `cilibert@mat.uniroma2.it`

UNIV. GRENOBLE ALPES, CNRS, INSTITUT FOURIER, F-38000 GRENOBLE, FRANCE

E-mail address: `Mikhail.Zaidenberg@univ-grenoble-alpes.fr`