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The Hamilton-Jacobi Equation : an intuitive approach.

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The Hamilton-Jacobi equation (HJE) is one of the most elegant approach to Lagrangian systems such as geometrical optics and classical mechanics, establishing the duality between trajectories and waves and paving the way naturally for the quantum mechanics. Usually, this formalism is taught at the end of a course on analytical mechanics through its technical aspects and its relation to canonical transformations. I propose that the teaching of this subject be centered on this duality along the lines proposed here, and the canonical transformations be taught only after some familiarity with the HJE has been gained by the students.

I. INTRODUCTION.

There are three different formalization of classical mechanics : the Lagrangian, the Hamiltonian and the Hamilton-Jacobi formalism. Usually, textbooks on mechanics (see for example [1–5]) begin with the Lagrangian formalism and the variational principle, where students discover the beauty of post-Newtonian mechanics. Historically, this formalism was developed in analogy with optics and the principle of Fermat[1]. Then, after a Legendre transform, the Hamiltonian approach is introduced where students discover the beauty of the phase space and the geometry herein. The mathematics behind these two methods is fairly standard and more or less easily digested by students. Finally, students come to the Hamilton-Jacobi equation (HJE). The HJE is usually introduced after a heavy passage through canonical transformations to uncover a first-order non-linear partial differential equation that does not seem any more useful to students at first glance than the former approaches.

The aim of this short note is to make an intuitive approach to the HJE by reversing how it is generally taught. The beauty of the HJ approach is to uncover the duality between trajectories and wavefronts. This duality was known in optics[6] where light could be either investigated by rays and geometric optics (Fermat's principle) or by wavefront (Huygens principle)[7], much before interference and the electromagnetic nature of light was discovered. Hamilton showed that this duality can be extended to any system described by a Lagrangian formalism, including and foremost, mechanics. I believe that this duality and its various extensions, specifically to quantum mechanics, are what should be taught first and foremost to students, studied in depth. Only when the students are familiarized with these concepts, one should introduce the canonical transformations and the technical aspects that make this approach, in the words of Arnold[8], “[...] the most powerful method known for the exact integration [of Hamilton equations]”. At the undergraduate level, specifically to physics students, these technical aspects seem less relevant : Arnold[8] quotes Felix Klein, who had great respect for the work of Hamilton[9], about HJ method “*that does not bring*

anything to the engineer and very little to the physicist”. Indeed, many examples of HJE treated in the above mentioned textbooks of analytical mechanics can be as easily treated by the Lagrangian and Hamiltonian approach.

II. GEOMETRICAL OPTICS.

Eighteen century physics saw a raging debate between the particle theory and wave theory of light[10]. In the first description, light is made of particles whose trajectories can be followed and are called the “ray paths”. In the second description, light is made of *waves*, and the “wave front” can be followed exactly as we follow waves on the surface of a liquid or sounds. This second approach was developed first by Huygens around 1680 AD[7]. In the limit of geometrical optics, when the wave length can be considered small, these two approaches are equivalent : knowing the wave fronts, one can deduce the ray paths and vice versa. We will detail this derivation below, but let us first define more precisely what a wave front is in optics.

Consider light emitted from a point \mathbf{r}_0 at time t_0 . The boundary \mathcal{C}_{t,t_0} of the domain that the light has covered at time t is called the “*wave front*” (figure 1) at time t . If the propagation medium is homogeneous, the wave front is a sphere given by the equation

$$\|\mathbf{r} - \mathbf{r}_0\| = \frac{c}{n}(t - t_0)$$

where c is the speed of light and n the index of the propagating medium. We can rewrite this equation as

$$S(\mathbf{r}, t) = -(c/n)t_0$$

Where the function $S(\mathbf{r}, t) = \|\mathbf{r} - \mathbf{r}_0\| - (c/n)t$. The relation $S(\mathbf{r}, t) = -(c/n)t_0$ defines the collection of points that the light (emitted at \mathbf{r}_0, t_0) has reached at time t .

We don't need to suppose that the light is emitted by a single point, we can as well describe the wave front of the light emitted by a line or a surface (or any at most $n - 1$ dimensional object). In fact, Huygens discovered that the wave front at time t can be described by the light emitted by the wave front at time $t - t_\alpha$. This is

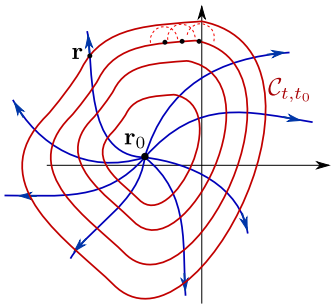


Figure 1. Wave fronts \mathcal{C} (in red) of light emitted at point \mathbf{r}_0 at time t_0 . Blue lines are the rays path. The Huygens principle states that the wave front at time t can be seen as the wave front of light emitted at time $t - t_\alpha$ by the wave front at this time (red dashed lines).

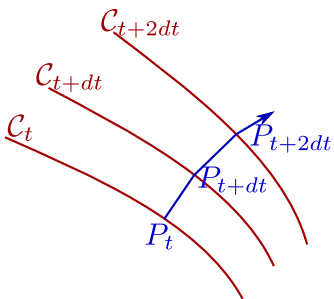


Figure 2. In geometrical optics in isotropic media, trajectories P_t of the light rays and wave fronts are orthogonal. Therefore, trajectories can be recovered from the wave front: from the point P_t on the wave front \mathcal{C}_t , draw the orthogonal to the wave front and recover the point P_{t+dt} at which it intercepts the wave front \mathcal{C}_{t+dt} . Proceeds by recurrence.

called the Huygens principle. Finally, note that if $r_0 \gg r$, $\|\mathbf{r} - \mathbf{r}_0\| \approx r_0 - (\mathbf{r}_0/r_0) \cdot \mathbf{r}$ and we can approximate the spherical wave by a plane one of the form $S(\mathbf{r}, t) = \mathbf{u} \cdot \mathbf{r} - (c/n)t$ where $\mathbf{u} = -(\mathbf{r}_0/r_0)$ is the direction of the plane wave propagation.

If the medium is not homogeneous ($n = n(\mathbf{r})$), the wave fronts are not spherical any more. The principle of Fermat states that the path taken by a ray to go from a point A to a point B is the one that minimizes the traveling time :

$$T = \frac{1}{c} \int_A^B n ds$$

where ds is the element of arc length along a path. In order to compute a wave front now, one has to compute the ray paths and collect points along the path that have been reached at a given time t . If the medium is isotropic (*i.e.* not like a crystal with particular directions of propagation), it can be shown that ray paths and wave fronts are orthogonal (see below). In this case, deducing the wave fronts from the ray paths is simple. On the other hand, *if* we knew the wave fronts, we could compute the ray paths (figure 2). Paths and wave fronts are dual ob-

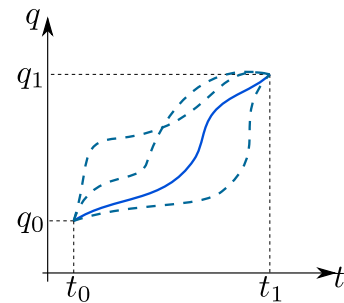


Figure 3. The trajectory chosen by an object (solid line) optimizes the action compared to all other possible trajectories (dashed lines) (1)

jects linked together through an orthogonality.

Even if the medium is *not* isotropic, we can still compute the wave front from the rays, and vice versa. All we need is a relation between the tangent to the ray path (let's call it $\dot{\mathbf{q}}$) at a point and the normal to the wave front (call it \mathbf{p}) at the same point. We will come to this subject in more general detail in the next sections.

III. BASIC NOTIONS OF ANALYTICAL MECHANICS.

Very soon after the publication of *Principia* by Newton (1684), Bernoulli challenged (1696) the scientific community to find the fastest path that, under gravity, brings a mass from point A to point B . The analogy with optics and the Fermat's principle was not lost on the mathematicians who responded to the challenge[11]. This analogy was then fully developed in subsequent years [12] and took its definitive form under the name of Euler-Lagrange equation.

The foundation of analytical mechanics is based on a variational principles: Given a Lagrangian $\mathcal{L}(\dot{q}, q, t)$, an object (be it a particle or a ray of light) *chooses* the trajectory $q(t)$ that makes the *action*

$$S = \int_{t_0, q_0}^{t_1, q_1} \mathcal{L}(\dot{q}, q, t) dt \quad (1)$$

stationary (figure 3). The action depends on the *end points* (t_0, q_0) and (t_1, q_1) and the trajectory $q(t)$ must obey the Euler-Lagrange equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0 \quad (2)$$

For a classical particle, the Lagrangian is the difference between the kinetic and the potential energy $\mathcal{L} = T - V$, while for geometrical optics, the Lagrangian is the traveling time.

We can reformulate equation (2) by making a Legendre transform. Defining the momentum

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}} \quad (3)$$

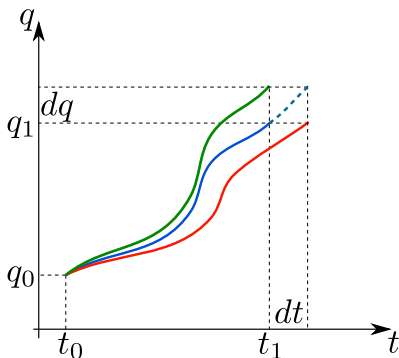


Figure 4. Varying the end points of a movement.

expressing \dot{q} as a function of p and defining $H(p, q, t) = p\dot{q} - \mathcal{L}$, we obtain the Hamilton equations

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} ; \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q} \quad (4)$$

which allows us to move to the phase space and have a more geometrical view of the trajectories. One consequence of the above equation is the variation of H as a function of time along a trajectory:

$$dH = \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial t} dt = \frac{\partial H}{\partial t} dt \quad (5)$$

Therefore, if the Hamiltonian does not depend explicitly on time, the Hamiltonian is conserved along a trajectory: $H = E$.

In the above two formulation of analytical mechanics, the action $S()$ itself plays little explicit role; what is important is the differential equations (2) or (4) whose solution determines the trajectory. However, Let us have a closer look at the action itself. By action S here we mean the integral expression (1) when the particle moves along the *optimal* path. Even though the absolute value of S can be hard to compute analytically, we can compute its *variation* if we vary the end points (figure 4). We will keep here the initial point fixed and vary the final end point either by dt or dq .

We begin by keeping the final time fixed at t_1 but move the final position by dq (figure 4). The trajectory $q(t)$ will vary by $\delta q(t)$ where $\delta q(t_0) = 0$ and $\delta q(t_1) = dq$. The variation in S is

$$\delta S = \int_{t_0}^{t_1} \left\{ \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta \dot{q} + \frac{\partial \mathcal{L}}{\partial q} \delta q \right\} dt \quad (6)$$

However, the trajectories obey the Euler-Lagrange equation (2) and we must have

$$\frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

On the other hand, $\delta \dot{q} = d(\delta q)/dt$. Using these relations, we can rewrite equation (6) as

$$\delta S = \int_{t_0}^{t_1} \left\{ \frac{\partial \mathcal{L}}{\partial \dot{q}} \frac{d(\delta q)}{dt} + \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \delta q \right\} dt$$

$$\begin{aligned} &= \int_{t_0}^{t_1} \frac{d}{dt} \left\{ \frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q \right\} dt \\ &= \left[\frac{\partial \mathcal{L}}{\partial \dot{q}} \delta q \right]_{t_0}^{t_1} = \frac{\partial \mathcal{L}}{\partial \dot{q}} \Big|_{t_1} dq \end{aligned}$$

As we have kept the final time fixed, $\delta S = (\partial S / \partial q) dq$ and therefore

$$\frac{\partial S}{\partial q} = \frac{\partial \mathcal{L}}{\partial \dot{q}} \Big|_{t_1} = p(t_1) \quad (7)$$

If we vary the end point q_1 , the relative variation in S is the *momentum* p at the end point.

To compute the variation of S as a function of the end point's time, consider letting the original trajectory to continue along its optimal path. Then $dS = \mathcal{L} dt$. On the other hand

$$dS = \mathcal{L} dt = \frac{\partial S}{\partial q} dq + \frac{\partial S}{\partial t} dt$$

Using our previous result (7), we have

$$\mathcal{L} dt = p dq + \frac{\partial S}{\partial t} dt = \left(p \dot{q} + \frac{\partial S}{\partial t} \right) dt$$

and therefore

$$\frac{\partial S}{\partial t} = \mathcal{L} - p \dot{q} = -H \quad (8)$$

Relation (7,8) are very general results of variational calculus with varying end points and are not restricted to mechanics. The contact angle of a liquid droplet on a solid surface is obtained for example by these computations. Note also that even though we derived these equations in one dimension of space, they are trivially generalized to any dimension.

IV. GENERAL WAVE FRONTS.

In geometrical optics, we had used the traveling time to define the wave front. But the traveling time is just one example of action and variational principles. In analogy with optics, let us define the function $S_{\mathbf{q}_0, t_0}(\mathbf{q}, t)$ as the action of a particle that arrives at (\mathbf{q}, t) after leaving (\mathbf{q}_0, t_0) , following its optimal path. By this function, we can associate to each point (\mathbf{q}, t) a value in space-time. Then, $S(\mathbf{q}, t) = C$ defines an $n-1$ dimensional surface \mathcal{C}_t , *i.e.* the collection of points \mathbf{q} that have the same value C of action at time t . Figure 1 that illustrated wave front in optics illustrates similarly the general wavefronts of action.

Consider for example a classical free particle, whose trajectories are straight lines with constant speed $v = \|\mathbf{q} - \mathbf{q}_0\| / (t - t_0)$. The action is therefore

$$S(\mathbf{q}, t) = \frac{m}{2} v^2 (t - t_0) = \frac{m}{2} \|\mathbf{q} - \mathbf{q}_0\|^2 / (t - t_0)$$

and the curves C_t are spheres of radius proportional to $\sqrt{2(t-t_0)}/m$. If the initial point is far away from the region of interest ($|t| \ll |t_0|$, $q \ll q_0$), we can develop the above expression and write it, to the first order in q, t :

$$\begin{aligned} S(\mathbf{q}, t) &\approx \frac{m}{2t_0} (q_0^2 - 2\mathbf{q}_0 \cdot \mathbf{q}) (-1 - t/t_0) \\ &= S_0 + \mathbf{p} \cdot \mathbf{q} - Et \end{aligned} \quad (9)$$

where we have defined the constants $\mathbf{p} = m\mathbf{q}_0/t_0$ and $E = (1/2)mq_0^2/t_0^2$. In this case, the action is a plane wave.

We have defined the wave front as the collection of points \mathbf{q} at time t for which $S(\mathbf{q}, t) = \text{const}$. To compute the wavefronts however, we have relied on the knowledge of trajectories. To go further, we need to derive an independent equation from which $S()$ can be computed directly, without any a priori knowledge of trajectories. For this purpose, we just have to recall from the last section (7,8) that we can compute the *variation* of S as a function of the variation of its end points:

$$\frac{\partial S}{\partial \mathbf{q}} = \mathbf{p} ; \quad \frac{\partial S}{\partial t} = -H \quad (10)$$

where $\partial S/\partial \mathbf{q} = (\partial_{q_1} S, \partial_{q_2} S, \dots)$. Note that this a generalization of the free particle case where (according to 9), $dS = \mathbf{p}d\mathbf{q} - E dt$. Now, we know that $H = H(\mathbf{q}, \mathbf{p}, t)$, therefore combining the above two expressions, we have

$$\frac{\partial S}{\partial t} + H\left(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}, t\right) = 0 \quad (11)$$

which is a *first order* PDE and called the Hamilton-Jacobi equation (HJE). If we can solve this equation and find the wave fronts, then we can deduce the trajectories from the wavefronts. The procedure is similar to what we did in geometrical optics : At each time t , we know the wave front S , and therefore, we can compute the momentum at points \mathbf{q} : $\mathbf{p}(\mathbf{q}) = \partial S/\partial \mathbf{q}$ (figure 5). This vector is related to the tangent to a trajectory $\dot{\mathbf{q}}$ through the relation

$$\mathbf{p} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}$$

By resolving the above relation, we can compute $\dot{\mathbf{q}}$ at each point of space at each time :

$$\dot{\mathbf{q}} = f(\mathbf{q}, t) \quad (12)$$

If we knew the wave fronts, the second order differential equations of Euler-Lagrange (equation 2) are transformed into ordinary first order differential equations (12) as above. For the simplest mechanical systems with one particle and a potential $V(\mathbf{q}, t)$, \mathbf{p} and $\dot{\mathbf{q}}$ are co-linear and the construction is really similar to optics.

We can further simplify the HJE (eq. 11) if the function H does not contain t explicitly. In this case, we can *separate* the function S into

$$S(\mathbf{q}, t) = W(\mathbf{q}) - Et \quad (13)$$

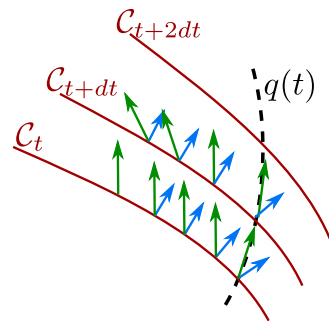


Figure 5. From known wave fronts C_t (in red) to trajectories : at each point, the normal to the wave front $\mathbf{p} = \partial S/\partial \mathbf{q}$ (in blue) can be computed ; knowing \mathbf{p} , we can compute the tangent to the trajectory $\dot{\mathbf{q}}$ (in green) and find a trajectory following a given line of tangents. The procedure is trivially generalized to higher dimensional space where \mathbf{q} collects the coordinates of many particles.

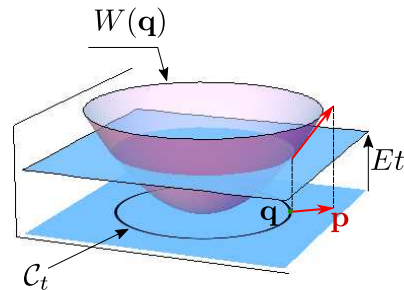


Figure 6. An illustration of the wave front in a two dimensional space where the function $W(\mathbf{q})$ is represented as a surface in three dimension. The wave front C_S is the contour plot of the function $W(\mathbf{q})$. At any given point \mathbf{q} , the momentum is given by $\mathbf{p} = \nabla W$

where the function $W()$ (often called Hamilton principal function) obeys the relation

$$H\left(\mathbf{q}, \frac{\partial W}{\partial \mathbf{q}}\right) = E$$

Once $W()$ is solved for, we can find the wave fronts by slicing the function $W()$ at different “heights” : at a given time t , we collect all points \mathbf{q} such that $W(\mathbf{q}) = Et + \text{const}$. into the wave front C_t (figure 6).

V. EXAMPLES.

A. One particle.

Consider one classical free particle with the Lagrangian $\mathcal{L} = (1/2)m\dot{\mathbf{q}}^2$, $\mathbf{p} = m\dot{\mathbf{q}}$ and $H = \mathbf{p}^2/2m$ where we use the square of a vector as a shorthand: $\mathbf{u}^2 = \mathbf{u} \cdot \mathbf{u}$. Therefore, the HJE is simply

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \left(\frac{\partial S}{\partial \mathbf{q}} \right)^2 = 0 \quad (14)$$

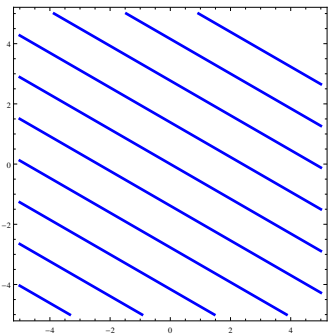


Figure 7. Contour plot of $W(q_1, q_2)$ of free particle in 2 dimensions (relation 16) for $u_1 = \cos \theta = 1/2$.

It is straightforward to check that the spherical wave $S = m(\mathbf{q} - \mathbf{q}_0)^2/2(t - t_0)$ is a solution of the above equation, where \mathbf{q}_0 and t_0 are some constants. We can also look for a *separable* solution of the form $S = W - Et$, in which case

$$\frac{1}{2m} \left(\frac{\partial W}{\partial \mathbf{q}} \right)^2 = E$$

To solve this PDE, we can search for further separability in the form of

$$W(\mathbf{q}) = \sqrt{2m} \sum_i w_i(q_i) \quad (15)$$

and solve the equations $dw_i/dq_i = \sqrt{e_i}$ where e_i are integration constants. The solution of these equations are $w_i(q_i) = \sqrt{e_i}q_i + C_i$ with the constraints $\sum_i e_i = E$ and C_i another set of integration constants. The complete solution is then a plane wave with (figure 7)

$$W(\mathbf{q}) = \sqrt{2mE} \sum_i u_i q_i + C_i' \quad (16)$$

where $u_i = \sqrt{e_i/E}$ are the integration constants. We collect the constants u_i into a constant vector \mathbf{v} such that $v_i = vu_i$, $E = (1/2)mv^2$ and write (figure 7)

$$W(\mathbf{q}) = m\mathbf{v} \cdot \mathbf{q} + C$$

Now that we know the wave front, if we wish so, we can deduce the trajectories : the moment is given by $\mathbf{p} = \partial W/\partial \mathbf{q} = m\mathbf{v}$. From the Lagrangian, we know that $\dot{\mathbf{q}} = \mathbf{p}/m$, and therefore $\dot{\mathbf{q}} = \mathbf{v}$ and $\mathbf{q} = \mathbf{v}t + \mathbf{q}_0$ where \mathbf{q}_0 is another integration constant.

For a classical free particle, the HJE is obviously an overkill. The purpose of this example is to illustrate how the solution of the HJE with the integration constant \mathbf{v} leads to the trajectories. It is straightforward to check that the spherical wave solution leads to the same result for trajectories.

Adding a potential $V(\mathbf{q})$ to the problem give rise to the HJE

$$\frac{\partial S}{\partial t} + \frac{1}{2m} \left(\frac{\partial S}{\partial \mathbf{q}} \right)^2 = -V(\mathbf{q}) \quad (17)$$

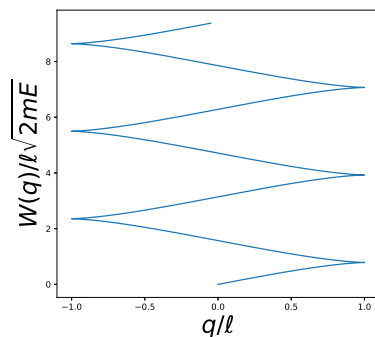


Figure 8. The Hamilton principal function $W(q)$ for the uni-dimensional harmonic oscillator.

There exist a systematic method to search for the solution of this equation, called *canonical transformations* (see for example [4, section 10.4]). If however, the potential is itself separable $V(\mathbf{q}) = \sum_i V_i(q_i)$, we can look for a separable solution of the HJE as before. As an illustration, consider the simple one dimensional harmonic oscillator with $V(q) = (1/2)kq^2$. Extension to higher dimensional case is trivial but harder to present graphically. Setting $S = W - Et$ we have

$$\frac{dW}{dq} = \sqrt{2mE} \sqrt{1 - \frac{x^2}{\ell^2}}$$

where $\ell^2 = 2E/k$. Setting $q = \ell \sin \theta$ transforms the above equation into

$$\frac{dW}{d\theta} = \ell \sqrt{2mE} \cos^2 \theta$$

that integrates directly

$$W(\theta) = \frac{1}{2} \ell \sqrt{2mE} (\theta + \frac{1}{2} \sin 2\theta) + C$$

Figure 8 displays a plot of $W(q)$ as a function of q . It can be observed that the function $W()$ is multivalued and at its ‘‘turning points’’, $p = \partial W/\partial q = 0$, a fact that is common to all bounded mechanical systems.

B. Relativistic particle.

We distinguish here explicitly between time and space coordinate for more clarity at the expense of elegance. Consider a free relativistic particle whose action is given by its Minkowski arc length

$$S = -m \int_A^B ds$$

where (in natural units $c = 1$) $\mathcal{L}dt = -m ds = -m \sqrt{dt^2 - d\mathbf{x}d\mathbf{x}} = -m \sqrt{1 - \dot{\mathbf{x}}^2} dt$. We have

$$\mathbf{p} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}} = \frac{m\dot{\mathbf{x}}}{\sqrt{1 - \dot{\mathbf{x}}^2}}$$

and therefore

$$H = \mathbf{p}\dot{\mathbf{x}} - \mathcal{L} = \sqrt{m^2 + \mathbf{p}^2}$$

The HJ equation is therefore

$$\left(\frac{\partial S}{\partial t}\right) = -\sqrt{m^2 + \left(\frac{\partial S}{\partial \mathbf{x}}\right)^2}$$

or

$$\left(\frac{\partial S}{\partial t}\right)^2 - \left(\frac{\partial S}{\partial \mathbf{x}}\right)^2 = m^2 \quad (18)$$

Note that the parabolic PDE of a classical dynamics becomes a wave equation when we consider the relativistic dynamics. This is exactly how the Schrodinger equation transforms into the Klein-Gordon one, *i.e.* the relativistic wave equation for spineless particles. This can be extended to the case of a particle with in an electromagnetic field by considering

$$\mathcal{L}dt = -m ds - q ds \cdot \mathbf{A}$$

where the four vector $\mathbf{A} = (-\phi, \vec{A})$, ϕ is the electromagnetic potential and \vec{A} the (three) vector potential.

C. Geometrical optics.

Consider light propagating in an isotropic medium. The action is the total traveling time

$$S = \int_A^B n ds$$

where $n(\mathbf{q})$ is the index of the medium at position \mathbf{q} , ds is the arc length along a trajectory and we have set the speed of light in vacuum $c = 1$. This is called the principle of Fermat. For simplicity, we will consider a two-dimensional medium where x is used as the integration variable and $ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + y'^2} dx$; the Lagrangian is

$$\mathcal{L} = n(x, y) \sqrt{1 + y'^2}$$

and by definition,

$$p = \frac{\partial \mathcal{L}}{\partial y'} = n \frac{y'}{\sqrt{1 + y'^2}} \quad (19)$$

if we set θ as the angle between the tangent to the trajectory and the x axis, the above relation is simply $p = n \sin \theta$, which is the conserved quantity if $n = n(x)$ (Snell's law). Solving relation 19 in y' , we have $y' = p/\sqrt{n^2 - p^2}$ and therefore the Hamiltonian is

$$H = py' - \mathcal{L} = -\sqrt{n^2 - p^2}$$

The HJE is then

$$\left(\frac{\partial S}{\partial x}\right) - \sqrt{n^2 - \left(\frac{\partial S}{\partial y}\right)^2} = 0$$

or in other words,

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 = n^2 \quad (20)$$

The above expression, called the eikonal equation, is the fundamental equation of geometrical optics. In the Hamilton-Jacobi approach, its resemblance to relativistic particle is obvious. We will see below that the eikonal equation can be obtained through approximation of the wave equation.

VI. WAVES AND PARTICLES.

For about 50 years after its introduction, the Hamilton-Jacobi equation was considered a beautiful but useless tool. With the advent of quantum mechanics, Schrodinger realized that this equation is the natural road to formulating a "wave" equation for particles. The approach was as follow : geometrical optic is an approximation of the Maxwell equations that neglects interference effect. We know the Maxwell equation and the approximation procedure to get to geometrical optics. Schrodinger realized that classical mechanics can be such an approximation of a more complicated theory and reverse engineered the geometrical optics approximation to get to his famous equation in 1926. The detail of this procedure and its connection to Hamilton-Jacobi equation is beautifully written by Massoliver and Ros[13] and we don't develop it here. However, it is very simple to show that classical mechanics is an approximation of the quantum mechanics.

Consider the Schrodinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi$$

using a standard change of function

$$\psi = e^{iS/\hbar} \quad (21)$$

the Schrodinger equation transforms into

$$-\frac{\partial S}{\partial t} = -\frac{i\hbar}{2m} \frac{\partial^2 S}{\partial x^2} + \frac{1}{2m} \left(\frac{\partial S}{\partial x}\right)^2 + V(x) \quad (22)$$

we see that the above equation, when we neglect the term in \hbar , reduces exactly to the classical HJE (17): the classical mechanics is indeed the limit of quantum mechanics when $\hbar \rightarrow 0$.

The transformation (21), called the ansatz of Sommerfield and Runge[14], was nothing unusual at the time of Schrodinger and is used to recover the geometrical optics

from the wave equation (see[15] for a review). Consider the equation of an electromagnetic wave propagating through space, where the index of refraction is not supposed to be constant :

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \nabla^2 \psi \quad (23)$$

where ψ is any component of the electromagnetic tensor or the vector potential and $v = c/n$ where c is the speed of light and n the index of the medium. We look for a solution of the form

$$\psi(t) = A(\mathbf{r}) \exp(ik_0(\phi(\mathbf{r}) - ct)) \quad (24)$$

in analogy with plane waves when $n = \text{const.}$ $k_0 = 2\pi/\lambda_0$ is the wave number and λ_0 is the wave length in vacuum ; A (the amplitude) and ϕ (the phase) are real functions. Note that the total phase

$$\Phi(\mathbf{r}) = \phi(\mathbf{r}) - ct$$

has the same structure as the function S in relation (13) and $\phi()$ plays the same role as the function $W()$.

Plugging expression (24) into (23), separating the real and the imaginary part, we have:

$$\nabla^2 A - Ak_0^2 (\nabla\phi)^2 = -k_0^2 n^2 A \quad (25)$$

$$2(\nabla\phi)(\nabla A) + A\nabla^2\phi = 0 \quad (26)$$

The geometrical optics is obtained from the wave equation by letting $\lambda_0 \rightarrow 0$, *i.e.* when we assume that the

scale of variation in the index is large compared to the wave length, or equivalently, when $|\nabla^2 A/A| \ll k_0^2$. Neglecting the $\nabla^2 A$ term in relation (25), we obtain an equation for the phase ϕ alone:

$$(\nabla\phi)^2 = n^2 \quad (27)$$

which is the eikonal equation we had already obtained from the principle of Fermat (eq. 20).

VII. CONCLUSION.

The Hamilton-Jacobi equation is one of the most elegant and beautiful approach to mechanics with far reaching consequences in many adjacent fields such as quantum mechanics and probability theory. Unfortunately, its beauty is lost to many students learning the basics of analytical mechanics. An informal and statistically non-significant inquiry of practicing physicists suggests that even among scientists, Hamilton-Jacobi brings up mostly (if any) memories of arcane transformations with no observable use.

The materials developed in this short article, which does not contain the usual mathematical complexity found in most textbooks, can be covered in one or two lectures and I hope help students to get a basic understanding of the Hamilton-Jacobi approach to variational systems.

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