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# MEAN NUMBER AND CORRELATION FUNCTION OF CRITICAL POINTS OF ISOTROPIC GAUSSIAN FIELDS AND SOME RESULTS ON GOE RANDOM MATRICES

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Let  $\mathcal{X} = \{X(t) : t \in \mathbb{R}^N\}$  be an isotropic Gaussian random field with real values. In a first part we study the mean number of critical points of  $\mathcal{X}$  with index  $k$  using random matrices tools. We obtain an exact expression for the probability density of the  $k$ th eigenvalue of a  $N$ -GOE matrix. We deduce some exact expressions for the mean number of critical points with a given index. In a second part we study attraction or repulsion between these critical points. A measure is the correlation function. We prove attraction between critical points when  $N > 2$ , neutrality for  $N = 2$  and repulsion for  $N = 1$ . The attraction between critical points that occurs when the dimension is greater than two is due to critical points with adjacent indexes. A strong repulsion between maxima and minima is proved. The correlation function between maxima (or minima) depends on the dimension of the ambient space.

**1. Introduction.** Critical points of random fields play an important role in small or large dimension. In large dimension data they appear in the study of algorithms of maximisation of the likelihood [13, 29]. In smaller dimension they play a role in many applications from various areas: detection of peaks in a random field [16, 32, 33, 34], optimisation of a response modelled by a random field [19], modelling of random sea waves [24, 7, 28]. Critical points and their indexes are an important element of the geometry of random fields. They appear in the computation of the Euler characteristic [2, 18]. They are a subject of study on their own as in [27, 3, 11, 17].

Let  $\mathcal{X} = \{X(t) : t \in \mathbb{R}^N\}$  be an isotropic Gaussian random field with real values. If we look at the repartition of the critical points of  $\mathcal{X}$  as a function of their index in dimension two, considerations of symmetry and of Euler characteristic (see, for example, [1], lemma 6.1.1) imply that if  $\mathcal{N}^c(S)$ ,  $\mathcal{N}_0^c(S)$ ,  $\mathcal{N}_2^c(S)$ ,  $\mathcal{N}_1^c(S)$  are respectively the number of critical, minimum, maximum and saddle points on a given set  $S$

$$(1) \quad \mathbb{E}(\mathcal{N}_0^c(S)) = \mathbb{E}(\mathcal{N}_2^c(S)) = \frac{1}{2} \mathbb{E}(\mathcal{N}_1^c(S)) = \frac{1}{4} \mathbb{E}(\mathcal{N}^c(S)).$$

In higher dimensions simulations suggest that such a simple result does not hold true. The purpose of Section 3 is to compute these expectations using random matrices tools. With this objective in mind, we obtain an exact expression for the probability density of the  $k$ th ordered eigenvalue of a  $N$ -GOE matrix (see Theorem 3.5 and (22)–(28)). We deduce exact expressions for the mean number of critical points with a given index (Propositions 3.2, 3.7 and 3.8). In particular, for  $N = 3$ , if we denote by  $\mathcal{N}_0^c(S)$ ,  $\mathcal{N}_1^c(S)$ ,  $\mathcal{N}_2^c(S)$  and  $\mathcal{N}_3^c(S)$  respectively the number of minimum, the number of critical points with index 1 and 2 (see Section 3, equation (9) for the definition) and the number of maximum on a given set  $S$ , we

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obtain (see Proposition 3.7)

$$\begin{aligned} \mathbb{E}(\mathcal{N}_0^c(S)) &= \mathbb{E}(\mathcal{N}_3^c(S)) = \frac{29 - 6\sqrt{6}}{116} \mathbb{E}(\mathcal{N}^c(S)), \\ \mathbb{E}(\mathcal{N}_1^c(S)) &= \mathbb{E}(\mathcal{N}_2^c(S)) = \frac{29 + 6\sqrt{6}}{116} \mathbb{E}(\mathcal{N}^c(S)). \end{aligned}$$

Proposition 3.8 gives these expectations for  $N = 4$ .

On the other hand Section 4 studies how the critical points are spread in the space. In a pioneering work, Belyaev, Cammarota and Wigman [11] study the attraction or repulsion (see Section 4.2 for a precise definition) of the point process of critical points of a particular random wave model in dimension 2. In the case of random processes ( $N = 1$ ), it was generally admitted that repulsion between crossings or critical points occurs. In fact this result has never been written explicitly so it is the object of Section 4.4 with Proposition 4.4. With respect to this behaviour the result of [11] is unexpected since no repulsion occurs between critical points. The object of Section 4.5 is to explore the phenomenon of attraction or repulsion for a large class of random fields, in all dimensions and for each type of indexes. A precise definition of attraction and repulsion is given in Section 4.2. Theorem 4.6 proves attraction between critical points when  $N > 2$ , neutrality for  $N = 2$  and repulsion for  $N = 1$ . Theorem 4.9 shows that the attraction between critical points that occurs when the dimension is greater than 2 is due to the attraction between critical points with adjacent indexes. In Theorem 4.11 we prove a strong repulsion, growing with  $N$ , between maxima and minima. Finally Theorem 4.12 gives an upper bound to the correlation function between maxima (or equivalently minima) proving repulsion for  $N < 5$ .

In Appendix A we prove the validity of all the Kac-Rice formulas used in the paper. Appendix B is devoted to the proofs of Lemmas 2.1, 4.1 and 4.2.

## 2. Notation, main assumptions and background.

### 2.1. Notation.

- $\varphi(\cdot)$  and  $\Phi(\cdot)$  are respectively the probability density and the cumulative distribution function of a standard Gaussian variable,  $\bar{\Phi}(\cdot) := 1 - \Phi(\cdot)$ .
- If  $X$  and  $Y$  are random vectors,  $\text{Var}(X)$  denotes the variance-covariance matrix of  $X$  and

$$\text{Cov}(X, Y) := \mathbb{E} \left( (X - \mathbb{E}(X))(Y - \mathbb{E}(Y))^{\top} \right).$$

- For  $X$  and  $Y$  two random vectors  $p_X(x)$  is the probability density of  $X$  at  $x$  and  $p_X(x/Y = y)$  the probability density of  $X$  conditionally to  $Y = y$  when these densities exist.
- For a random field  $\mathcal{X} = \{X(t) : t \in \mathbb{R}^N\}$ , we denote by  $\nabla X(t)$  the gradient of  $X(t)$  and by  $\nabla^2 X(t)$  its Hessian, when they exist.
- $i(M)$  is the index of the matrix  $M$ : the number of its negative eigenvalues.
- $z^+$  and  $z^-$  denote respectively  $\sup(0, z)$  and  $-\inf(0, z)$ .
- $X_{i_1 i_2 \dots i_n}(t)$  denotes  $\frac{\partial^n X(t)}{\partial t_{i_1} \partial t_{i_2} \dots \partial t_{i_n}}$ .

- $Id_n$  is the identity matrix of size  $n$ ;  $J_{n,p}$  the  $(n \times p)$ -matrix with all elements equal to 1.
- $|S|$  is the Lebesgue measure of the Borel set  $S$ .
- For a random field  $\mathcal{X} = \{X(t) : t \in \mathbb{R}^N\}$  the number of critical points is precisely defined for every Borel set  $S$  as

$$\mathcal{N}^c(S) := \#\{t \in S : \nabla X(t) = 0\}.$$

2.2. *Main assumptions.* In the sequel we will use the following assumptions (Ak), with  $k = 2, 3$  or 4.

ASSUMPTION (Ak). •  $\mathcal{X} = \{X(t) : t \in \mathbb{R}^N\}$  is a stationary and isotropic Gaussian field with real values. We assume, without loss of generality, that it is centred with variance 1. We set

$$\mathbb{E}(X(s)X(t)) = c(s, t) = \mathbf{r}(\|s - t\|^2).$$

- $\mathbf{r}$  is of class  $\mathcal{C}^k$ . This is equivalent to the existence of a finite  $2k$ th spectral moment  $\lambda_{2k}$  and it implies that  $\mathcal{X}$  is  $k$  times differentiable in quadratic mean. Note that  $\forall \ell = 1, \dots, N$  and  $n = 1, \dots, k$

$$(2) \quad \begin{aligned} 1 &= \text{Var}(X(t)) = \mathbf{r}(0), \\ \lambda_{2n} &:= \text{Var}\left(\frac{\partial^n X(t)}{\partial t_\ell^n}\right) = (-1)^n \frac{(2n)!}{n!} \mathbf{r}^{(n)}(0). \end{aligned}$$

- To avoid the trivial case of a constant random field, we assume that  $\lambda_2 = -2\mathbf{r}'(0) > 0$ .
- When  $N > 1$  we know from Lemma 2.1 below that  $\lambda_{2n}\lambda_{2n-4} > \lambda_{2n-2}^2$  for  $n = 2, \dots, k$ . When  $N = 1$ , we have to assume these relations to avoid the particular case of the sine-cosine process.
- When  $k = 2$  we have to assume, in addition for the validity of the Kac-Rice formula, that  $\mathcal{X}$  is  $C^2$  which is slightly stronger than the finiteness of  $\lambda_4$ . When  $k \geq 3$  the fact that  $\mathcal{X}$  is  $C^2$  is a consequence of the finiteness of  $\lambda_6$ .

Note that, of course, (A4) is stronger than (A3) which is itself stronger than (A2).

2.3. *Background.* We give two lemmas that will be needed in the next sections. The first one gives universal properties for  $\mathcal{X}$ . The second one is more technical.

LEMMA 2.1. *Let  $\mathcal{X}$  be a stationary and isotropic Gaussian field. When  $\lambda_{2k}$  is finite, the spectral moments satisfy for  $n = 2, \dots, k$*

$$(3) \quad \lambda_{2n}\lambda_{2n-4} \geq \frac{(2n-1)(2n-4+N)}{(2n-3)(2n-2+N)} \lambda_{2n-2}^2.$$

Moreover when  $i_1 + j_1, \dots, i_N + j_N$  are all even and when all the quantities hereunder are well defined

$$(4) \quad \begin{aligned} \mathbb{E}\left(\frac{\partial^{|i|} X(t)}{\partial t_1^{i_1}, \dots, \partial t_N^{i_N}} \frac{\partial^{|j|} X(t)}{\partial t_1^{j_1}, \dots, \partial t_N^{j_N}}\right) &= \frac{\partial^{|i|+|j|} c(s, t)}{\partial s_1^{i_1}, \dots, \partial s_N^{i_N} \partial t_1^{j_1}, \dots, \partial t_N^{j_N}} \Big|_{s=t} \\ &= (-1)^{|j|} \mathbf{r}^{(|\beta|)}(0) \prod_{\ell=1}^N \frac{(2\beta_\ell)!}{\beta_\ell!} = (-1)^{|\beta|+|j|} \lambda_{2|\beta|} \frac{|\beta|!}{(2|\beta|)!} \prod_{\ell=1}^N \frac{(2\beta_\ell)!}{\beta_\ell!}, \end{aligned}$$

where  $|\beta| = \frac{|i| + |j|}{2}$  and  $\beta_\ell = \frac{i_\ell + j_\ell}{2}$  for  $\ell = 1, \dots, N$ . In the other cases the covariances vanish.

REMARK 2.2. Note that the coefficients  $K(n, N) := \frac{(2n-1)(2n-4+N)}{(2n-3)(2n-2+N)}$  are greater than 1 when  $N > 1$  and take the value 1 when  $N = 1$ .

The proof of Lemma 2.1 is given in Appendix B.

In the sequel we need a precise description of the distribution of  $X(t), \nabla X(t), \nabla^2 X(t)$  and, for technical reasons in the last section, we need to add  $X_{1ii}, X_{11\ell}, X_{1111}$  for  $1 \leq i \leq N$  and  $2 \leq \ell \leq N$ . To get independence we have to partition this vector as follows. Set

$$\begin{aligned}\zeta_1 &:= (X_2(t), \dots, X_{\ell-1}(t), X_{\ell+1}(t), X_N(t))^\top, \\ \zeta_2 &:= (X_{ij}(t), i, j = 1, \dots, N \text{ and } i \neq j)^\top, \\ \zeta_3 &:= (X(t), X_{1111}(t), X_{11}(t), X_{22}(t), X_{33}(t), \dots, X_{NN}(t))^\top, \\ \zeta_4 &:= (X_1(t), X_{111}(t), X_{122}(t), X_{133}(t), \dots, X_{1NN}(t))^\top, \\ \zeta_5 &:= (X_\ell(t), X_{11\ell}(t))^\top.\end{aligned}$$

LEMMA 2.3. Under Assumption (A4) the random vectors  $\zeta_1, \zeta_2, \zeta_3, \zeta_4$  and  $\zeta_5$  defined above are Gaussian, independent, centred with variance matrices respectively given by

$$\text{Var}(\zeta_1) = \lambda_2 \text{Id}_{N-2}, \quad \text{Var}(\zeta_2) = \frac{\lambda_4}{3} \text{Id}_{N(N-1)/2}, \quad \text{Var}(\zeta_3) = M_{(N+2)},$$

$$\text{Var}(\zeta_4) = \tilde{M}_{(N+1)}, \quad \text{Var}(\zeta_5) = \begin{pmatrix} \lambda_2 & -\lambda_4/3 \\ -\lambda_4/3 & \lambda_6/5 \end{pmatrix},$$

where

$$M_{(N+2)} = \begin{pmatrix} 1 & \lambda_4 & -\lambda_2 & -\lambda_2 \dots -\lambda_2 \\ \lambda_4 & \lambda_8 & -\lambda_6 & -\lambda_6/5 \dots -\lambda_6/5 \\ -\lambda_2 & -\lambda_6 & \lambda_4 & \lambda_4/3 \dots \lambda_4/3 \\ -\lambda_2 & -\lambda_6/5 & \lambda_4/3 & \\ \vdots & \vdots & \vdots & \left( \frac{2\lambda_4}{3} \text{Id}_{N-1} + \frac{\lambda_4}{3} J_{N-1} \right) \\ -\lambda_2 & -\lambda_6/5 & \lambda_4/3 & \end{pmatrix}$$

and

$$\tilde{M}_{(N+1)} = \begin{pmatrix} \lambda_2 & -\lambda_4 & -\lambda_4/3 \dots -\lambda_4/3 \\ -\lambda_4 & \lambda_6 & \lambda_6/5 \dots \lambda_6/5 \\ -\lambda_4/3 & \lambda_6/5 & \\ \vdots & \vdots & \left( \frac{2\lambda_6}{15} \text{Id}_{N-1} + \frac{\lambda_6}{15} J_{N-1} \right) \\ -\lambda_4/3 & \lambda_6/5 & \end{pmatrix}.$$

Moreover we have

$$(5) \quad \det(\text{Var}(X_{11}(t), X_{22}(t), \dots, X_{NN}(t))) = (N+2)2^{N-1} \left( \frac{\lambda_4}{3} \right)^N,$$

$$(6) \quad \det(\text{Var}(X_1(t), X_{122}(t), \dots, X_{1NN}(t))) = \left( \frac{2\lambda_6}{15} \right)^{N-2} \frac{3(N+1)\lambda_2\lambda_6 - 5(N-1)\lambda_4^2}{45}.$$

PROOF OF LEMMA 2.3. The joint distribution of  $\zeta_1, \zeta_2, \zeta_3, \zeta_4$  is a direct consequence of (4). Now let us prove (5) and (6). We have

$$\text{Var}(X_{11}(t), X_{22}(t), \dots, X_{NN}(t)) = \frac{2\lambda_4}{3} Id_N + \frac{\lambda_4}{3} J_{N,N}.$$

For  $x, y \in \mathbb{R}$  it is well known that

$$(7) \quad \det(xId_N + yJ_{N,N}) = x^{N-1}(x + Ny).$$

So we obtain (5). We have

$$\det(\text{Var}(X_1(t), X_{122}(t), \dots, X_{1NN}(t))) = \det \begin{bmatrix} \lambda_2 & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix},$$

with  $\tilde{A}_{12} = [-\lambda_4/3 \dots -\lambda_4/3]$ ,  $\tilde{A}_{21} = \tilde{A}_{12}^\top$  and  $\tilde{A}_{22} = \frac{2\lambda_6}{15} Id_{N-1} + \frac{\lambda_6}{15} J_{N-1, N-1}$ .

Then using the fact that for a partitioned matrix  $\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}$  we have (see [30] p.46)

$$(8) \quad \det \tilde{A} = \det \tilde{A}_{11} \times \det(\tilde{A}_{22} - \tilde{A}_{21} \tilde{A}_{11}^{-1} \tilde{A}_{12}),$$

we obtain

$$\begin{aligned} \det(\text{Var}(X_1(t), X_{122}(t), \dots, X_{1NN}(t))) \\ = \lambda_2 \det \left( \frac{2\lambda_6}{15} Id_{N-1} + J_{N-1, N-1} \left( \frac{3\lambda_2\lambda_6 - 5\lambda_4^2}{45\lambda_2} \right) \right). \end{aligned}$$

Finally using (7) we obtain (6).  $\square$

**3. Mean number of critical points with a given index.** In this section  $\mathcal{X}$  is a random field satisfying Assumption (A2). Then it is proved in Appendix A that the sample paths are almost surely Morse and this implies that the number of critical points with index  $k$  in a Borel set  $S$ ,  $\mathcal{N}_k^c(S)$ , is well defined. More precisely

$$(9) \quad \mathcal{N}_k^c(S) := \#\{t \in S : \nabla X(t) = 0, i(\nabla^2 X(t)) = k\}.$$

We define also the number of critical points of index  $k$  above the level  $u$  by

$$(10) \quad \mathcal{N}_k^c(u, S) := \#\{t \in S : \nabla X(t) = 0, i(\nabla^2 X(t)) = k, X(t) > u\}.$$

The aim of this section is to derive explicit and exact expressions for the expectation of (9) and (10).

3.1. *The general case.* By Kac-Rice formulas (68) and (69) and Lemma 2.3 we get

$$(11) \quad \mathbb{E}(\mathcal{N}_k^c(S)) = \frac{|S|}{(2\pi\lambda_2)^{N/2}} \mathbb{E}(|\det(\nabla^2 X(t))| \mathbb{1}_{i(\nabla^2 X(t))=k}).$$

$$(12) \quad \begin{aligned} \mathbb{E}(\mathcal{N}_k^c(u, S)) &= \frac{|S|}{\lambda_2^{N/2} (2\pi)^{(N+1)/2}} \int_u^{+\infty} \exp\left(-\frac{x^2}{2}\right) \\ &\quad \times \mathbb{E}(|\det(\nabla^2 X(t))| \mathbb{1}_{i(\nabla^2 X(t))=k} / X(t) = x) dx. \end{aligned}$$

Our main tool will be the use of random matrices theory and more precisely the GOE distribution. We recall that  $G_n$  is said to have the GOE (Gaussian Orthogonal Ensemble) distribution if it is symmetric and all its elements are independent centred Gaussian variables satisfying  $\mathbb{E}(G_{ii}^2) = 1$  and  $\mathbb{E}(G_{ij}^2) = \frac{1}{2}$ . The relation between GOE matrices and the study of critical points of stationary isotropic Gaussian fields is due to the following lemma initially due to [8] and that can be derived from Lemma 2.3.

LEMMA 3.1. *Let  $G_N$  be a size  $N$  GOE matrix and  $\Lambda$  a  $\mathcal{N}(0, 1/2)$  random variable independent of  $G_N$ . Then  $\nabla^2 X(t)$  is equal in distribution to*

$$\sqrt{\frac{2\lambda_4}{3}} \left( G_N - \Lambda Id_N \right),$$

and under the assumption that  $\lambda_4 \geq 3\lambda_2^2$ ,  $\nabla^2 X(t)$  conditionally to  $X(t) = x$ , is distributed as

$$\sqrt{\frac{2\lambda_4}{3}} \left( G_N - \tilde{\Lambda} Id_N \right),$$

where  $\tilde{\Lambda}$  is a  $\mathcal{N} \left( \lambda_2 x \sqrt{\frac{3}{2\lambda_4}}, \frac{\lambda_4 - 3\lambda_2^2}{2\lambda_4} \right)$  random variable independent of  $G_N$ .

We recall that the joint density  $f_N$  of the eigenvalues  $(\mu_1, \dots, \mu_N)$  of a  $N$ -GOE matrix (see Theorem 3.3.1 of [25]) is given by:

$$(13) \quad f_N(\mu_1, \dots, \mu_N) = k_N \exp \left( -\frac{\sum_{i=1}^N \mu_i^2}{2} \right) \prod_{1 \leq i < k \leq N} |\mu_k - \mu_i|,$$

where:

$$(14) \quad k_N := (2\pi)^{-N/2} (\Gamma(3/2))^N \left( \prod_{i=1}^N \Gamma(1 + i/2) \right)^{-1}.$$

Using Lemma 3.1, (11), (12) and (13) we get the following proposition.

PROPOSITION 3.2. *Let  $L_p$  be the  $p$ th ordered eigenvalue of a  $(N + 1)$ -GOE matrix ( $L_1 < L_2 < \dots < L_{N+1}$ ). For  $\mathcal{X}$  and  $S$  as above, under Assumption (A2)*

$$(15) \quad \mathbb{E}(\mathcal{N}_k^c(S)) = \frac{|S|}{\pi^{(N+1)/2}} \left( \frac{\lambda_4}{3\lambda_2} \right)^{N/2} \frac{k_N}{k_{N+1}} \frac{1}{N+1} \mathbb{E} \left( \exp \left( -\frac{L_{k+1}^2}{2} \right) \right).$$

When  $\lambda_4 - 3\lambda_2^2 > 0$ ,

$$(16) \quad \mathbb{E}(\mathcal{N}_k^c(u, S)) = \frac{|S|}{\pi^{(N+1)/2}} \left( \frac{\lambda_4}{3\lambda_2} \right)^{N/2} \frac{k_N}{k_{N+1}} \frac{1}{N+1} \times \mathbb{E} \left\{ \exp \left( -\frac{L_{k+1}^2}{2} \right) \bar{\Phi} \left[ \sqrt{\frac{\lambda_4}{\lambda_4 - 3\lambda_2^2}} \left( u - L_{k+1} \frac{\sqrt{6}\lambda_2}{\sqrt{\lambda_4}} \right) \right] \right\},$$

and when  $\lambda_4 - 3\lambda_2^2 = 0$

$$(17) \quad \mathbb{E}(\mathcal{N}_k^c(u, S)) = \frac{|S|}{\pi^{(N+1)/2}} \frac{k_N}{k_{N+1}} \frac{\lambda_2^{N/2}}{N+1} \mathbb{E} \left( \exp \left( -\frac{L_{k+1}^2}{2} \right) \mathbb{1}_{L_{k+1} > u/\sqrt{2}} \right).$$

REMARK 3.3. Such kind of result was first obtained by [4] for the  $p$ -spin spherical spin glass model defined on the Euclidean sphere of radius  $\sqrt{N}$  of  $\mathbb{R}^N$ . This result can also be found in [17] (Proposition 3.9) written in a slightly different way. In this paper we go further: in Theorem 3.5 we obtain an exact expression for the probability density of the  $k$ th ordered eigenvalue of a  $N$ -GOE matrix, denoted by  $q_N^k(l)$  (see (20) and (22)–(28)). We can deduce exact expressions for (15), (16) and (17) as in Propositions 3.7, and 3.8.

REMARK 3.4. As remarked in [10] the condition  $\lambda_4 \geq 3\lambda_2^2$  is met if the covariance function  $\mathbf{r}$  is a ‘‘Schoenberg covariance’’: it is a valid covariance function in every dimension. Note that more general cases have been studied by [17].

PROOF OF PROPOSITION 3.2. Using (11), Lemma 3.1 and (13)

$$\begin{aligned} & \mathbb{E} \left( |\det \nabla^2 X(t)| \mathbb{1}_{i(\nabla^2 X(t))=k} \right) \\ &= \left( \frac{2\lambda_4}{3} \right)^{N/2} k_N N! \int_{\mu_1 < \mu_2 < \dots < \mu_k < \lambda < \mu_{k+1} < \dots < \mu_N} \prod_{1 \leq i < k \leq N} |\mu_k - \mu_i| \prod_{1 \leq i \leq N} |\lambda - \mu_i| \\ & \quad \exp \left( -\frac{\sum_{i=1}^N \mu_i^2}{2} \right) \pi^{-1/2} \exp(-\lambda^2/2) \exp(-\lambda^2/2) d\mu_1 \dots d\mu_N d\lambda \\ &= \left( \frac{2\lambda_4}{3} \right)^{N/2} \pi^{-1/2} \frac{k_N}{k_{N+1}} \frac{N!}{(N+1)!} \mathbb{E} \left( \exp \left( -\frac{L_{k+1}^2}{2} \right) \right), \end{aligned}$$

which proves (15). Using Lemma 3.1, by (12) and (13) we can write

$$\begin{aligned} \mathbb{E}(\mathcal{N}_k^c(u, S)) &= \frac{|S|}{\lambda_2^{N/2} (2\pi)^{(N+1)/2}} \int_u^{+\infty} \exp \left( -\frac{x^2}{2} \right) \left( \frac{2\lambda_4}{3} \right)^{N/2} k_N N! \\ & \quad \times \int_{\mu_1 < \mu_2 < \dots < \mu_k < y < \mu_{k+1} < \dots < \mu_N} \prod_{1 \leq i < k \leq N} |\mu_k - \mu_i| \prod_{1 \leq i \leq N} |y - \mu_i| \exp \left( -\frac{\sum_{i=1}^N \mu_i^2}{2} \right) \\ & \quad \times \exp(-y^2/2) \exp(y^2/2) p_Y(y) d\mu_1 \dots d\mu_N dy dx. \end{aligned}$$

Integrating first with respect to  $x$  we obtain (16). In the same way, when  $\lambda_4 - 3\lambda_2^2 = 0$ ,  $\nabla^2 X(t)/X(t) = x$  is distributed as  $\sqrt{2\lambda_2^2} G_N - \lambda_2 x I d_N$  and following the same lines as previously we obtain (17).  $\square$

3.2. *Exact expressions.* Before giving exact expressions for  $\mathbb{E}(\mathcal{N}_k^c(S))$  and  $\mathbb{E}(\mathcal{N}_k^c(u, S))$  we obtain, in the following paragraph 3.2.1, some results concerning the probability density of the  $k$ th ordered eigenvalue of a  $N$ -GOE matrix. These results will be used in paragraph 3.2.2, with Propositions 3.2, to derive exact expressions for  $\mathbb{E}(\mathcal{N}_k^c(S))$  and  $\mathbb{E}(\mathcal{N}_k^c(u, S))$ .

3.2.1. *Probability density of the  $k$ th ordered eigenvalue of a  $N$ -GOE matrix.* We denote by  $L_1 \leq L_2 \leq \dots \leq L_N$  the ordered eigenvalues of a  $N$ -GOE matrix. We denote by  $q_N^k(\ell)$  the probability density of  $L_k$ . In this section we obtain an expression for  $q_N^k(\ell)$ . We give explicit expressions of  $q_N^k(\ell)$  for  $N = 2, 3, 4, 5$  and  $k = 1, \dots, N$ . In particular we obtain that  $q_3^2(\ell)$  is the density of a  $\mathcal{N}(0, 1/2)$  random variable.

Let  $\mathcal{F}_p(n)$  be the set of the parts of  $\{1, 2, \dots, n\}$  of size  $p$ . Let  $I \in \mathcal{F}_p(n)$ , let  $\ell \in \mathbb{R}$ , we define  $\mathcal{D}_i^I(\ell) = (-\infty, \ell)$  when  $i \in I$  and  $\mathcal{D}_i^I(\ell) = (\ell, +\infty)$  when  $i \notin I$ . We denote by  $\mathbf{s}(\cdot)$  the sign function. We define the matrix  $\mathcal{A}^\alpha(I, \ell)$ , for  $\alpha = 1$  or  $2$ , as follows. When  $n$  is even,  $\mathcal{A}^\alpha(I, \ell)$  is the  $n \times n$  skew matrix whose elements are,  $\forall i, j = 1 \dots, n$ :

$$(18) \quad a_{i,j}^\alpha(I, \ell) = \int_{\mathcal{D}_i^I(\ell)} dx \int_{\mathcal{D}_j^I(\ell)} dy \quad \mathbf{s}(y-x) x^{i-1} y^{j-1} (x-\ell)^\alpha (y-\ell)^\alpha \exp \left( -\frac{x^2 + y^2}{2} \right).$$



When  $n$  is odd,  $\mathcal{A}^\alpha(I, \ell)$  is the  $(n+1) \times (n+1)$  skew matrix whose elements are defined by (18)  $\forall i, j = 1, \dots, n$  and we add the extra terms, for  $i = 1, \dots, n$ :

$$(19) \quad a_{i, n+1}^\alpha(I, \ell) = -a_{n+1, i}^\alpha(I, \ell) = \int_{\mathcal{D}_i^\alpha(\ell)} x^{i-1} (x - \ell)^\alpha \exp\left(-\frac{x^2}{2}\right) dx \text{ and } a_{n+1, n+1}^I(\ell) = 0.$$

We are now able to state the following theorem that gives an exact expression for  $q_N^k(\ell)$ , the probability density of  $L_k$ , the  $k$ th ordered eigenvalue of a  $N$ -GOE matrix.

**THEOREM 3.5.** *Using notations above, for  $\ell \in \mathbb{R}$  and for  $k = 1, \dots, N$  we have*

$$(20) \quad q_N^k(\ell) = k_N N! (-1)^{k-1} \exp\left(-\frac{\ell^2}{2}\right) \sum_{I \in \mathcal{F}_{k-1}(N-1)} \text{Pf}(\mathcal{A}^1(I, \ell))$$

where  $k_N$  is given by (14) and  $\text{Pf}(\mathcal{A}^1(I, \ell))$  is the Pfaffian of the skew matrix  $\mathcal{A}^1(I, \ell)$  defined by (18) and (19) with  $n = N - 1$  and  $\alpha = 1$ .

**REMARK 3.6.** The probability density of the  $k$ th ordered eigenvalue of a  $N$ -GOE matrix was only known for  $k = N$ , the largest eigenvalue (see [20]). Theorem 3.5 gives an expression for all  $k = 1, \dots, N$ .

**PROOF OF THEOREM 3.5.** Let  $G_{N-1}$  be a  $(N-1)$ -GOE matrix with eigenvalues denoted by  $\mu_1, \dots, \mu_{N-1}$ . For  $\ell \in \mathbb{R}$ , we set for  $k = 2, \dots, N-1$ ,  $\mathcal{O}_k(\ell) = \{(\mu_i, i = 1, \dots, N-1) \in \mathbb{R}^{N-1} : \mu_1 \leq \mu_2 \leq \dots \leq \mu_{k-1} \leq \ell \leq \mu_k \leq \dots \leq \mu_{N-1}\}$ , with trivial adaptation when  $k = 1$  and  $N$ .  $L_1 < L_2 < \dots < L_N$  denote the ordered eigenvalues of a  $N$ -GOE matrix. Using (13), the probability density of  $L_k$  is given by:

$$\begin{aligned} q_N^k(\ell) &= N! \int_{\mathcal{O}_k(\ell)} f_N(\mu_1, \dots, \mu_{N-1}, \ell) d\mu_1 \dots d\mu_{N-1} \\ &= \frac{k_N N!}{k_{N-1}} \int_{\mathcal{O}_k(\ell)} (-1)^{k-1} \det(G_{N-1} - \ell Id_{N-1}) f_{N-1}(\mu_1, \dots, \mu_{N-1}) \\ &\quad \times \exp\left(-\frac{\ell^2}{2}\right) d\mu_1 \dots d\mu_{N-1}. \end{aligned}$$

Thus  $q_N^k(\ell) = \frac{k_N}{k_{N-1}} N (-1)^{k-1} \exp\left(-\frac{\ell^2}{2}\right) \gamma_{N-1,1}^{k-1}(\ell)$  where

$$\begin{aligned} \gamma_{N-1,1}^{k-1}(\ell) &= k_{N-1} (N-1)! \\ &\quad \times \int_{\mathcal{O}_k(\ell)} \prod_{i=1}^{N-1} (\mu_i - \ell) \prod_{1 \leq i < j \leq N-1} (\mu_j - \mu_i) \exp\left(-\frac{\sum_{i=1}^{N-1} \mu_i^2}{2}\right) d\mu_1 \dots d\mu_{N-1}. \end{aligned}$$

We set  $h_m(\mu_j, \ell) = \mu_j^{m-1} (\mu_j - \ell) \exp\left(-\mu_j^2/2\right)$ . We denote by  $\mathbf{H}(\boldsymbol{\mu}, \ell)$  the matrix  $\{h_m(\mu_j, \ell), m, j = 1, \dots, N-1\}$ . Then calculating  $\det(\mathbf{H}(\boldsymbol{\mu}, \ell))$  using a Vandermonde determinant we obtain:

$$\gamma_{N-1,1}^{k-1}(\ell) = k_{N-1} (N-1)! \int_{\mathcal{O}_k(\ell)} \det(\mathbf{H}(\boldsymbol{\mu}, \ell)) d\mu_1 \dots d\mu_{N-1}.$$

We denote by  $\mathbf{I}(\boldsymbol{\mu}, \ell, k)$  the  $(N-1) \times (N-1)$  diagonal matrix defined by

$$\mathbf{I}(\boldsymbol{\mu}, \ell, k) = \text{diag}(\mathbb{1}_{\mu_1 \leq \ell}, \dots, \mathbb{1}_{\mu_{k-1} \leq \ell}, \mathbb{1}_{\mu_k \geq \ell}, \dots, \mathbb{1}_{\mu_{N-1} \leq \ell})$$

with trivial adaptations when  $k = 1$  or  $k = N$ . We set

$$\mathbf{H}^*(\boldsymbol{\mu}, \ell, k) := \mathbf{H}(\boldsymbol{\mu}, \ell) \mathbf{I}(\boldsymbol{\mu}, \ell, k).$$

Then

$$\gamma_{N-1,1}^{k-1}(\ell) = k_{N-1}(N-1)! \int_{\mu_1 \leq \dots \leq \mu_{N-1}} \det(\mathbf{H}^*(\boldsymbol{\mu}, \ell, k)) d\mu_1 \cdots \mu_{N-1}.$$

Now we need to introduce some more notation. Let  $I$  and  $J \in \mathcal{F}_p(n)$  and  $M$  a  $n \times n$  matrix, we set  $\Delta_{I,J}(M)$  for  $\det(M_{I,J})$ . We use the convention  $\Delta_{\emptyset,\emptyset}(M) = 1$ .  $\bar{K} = \{1, \dots, n\} \setminus K$  for any subset  $K$ . Let  $I \in \mathcal{F}_p(n)$ ,  $\sigma_I$  is the permutation of  $(1, 2, \dots, n)$  such that  $\sigma_I|_{(1,2,\dots,p)}$  (resp.  $\sigma_I|_{(p+1,\dots,n)}$ ) is an increasing one-to-one mapping from  $(1, 2, \dots, p)$  on  $I$  (resp. from  $(p+1, \dots, n)$  on  $\bar{I}$ ). Finally we denote by  $\epsilon(\sigma)$  the signature of the permutation  $\sigma$  of  $(1, 2, \dots, n)$ .

For a  $(n+p) \times (n+p)$  matrix  $D$  we have, for any  $J \in \mathcal{F}_p(n+p)$  fixed,

$$(21) \quad \det D = \sum_{I \in \mathcal{F}_p(n+p)} \epsilon(\sigma_I) \epsilon(\sigma_J) \Delta_{I,J}(D) \Delta_{\bar{I},\bar{J}}(D).$$

We apply it for  $p = k-1$ ,  $n = N-k$  and  $J = \{1, \dots, k-1\}$ . We get  $\epsilon(\sigma_J) = 1$  and

$$\begin{aligned} \gamma_{N-1,1}^{k-1}(\ell) &= k_{N-1}(N-1)! \sum_{I \in \mathcal{F}_{k-1}(N-1)} \epsilon(\sigma_I) \\ &\quad \times \int_{\mu_1 \leq \dots \leq \mu_{N-1}} \Delta_{I,J}(\mathbf{H}^*(\boldsymbol{\mu}, \ell, k)) \Delta_{\bar{I},\bar{J}}(\mathbf{H}^*(\boldsymbol{\mu}, \ell, k)) d\mu_1 \cdots \mu_{N-1}. \end{aligned}$$

We denote by  $\mathbf{H}^I(\boldsymbol{\mu}, \ell)$  the matrix  $\{h_m(\mu_j, \ell) \mathbb{1}_{\mu_j \in \mathcal{D}_m^I(\ell)}, m, j = 1, \dots, N-1\}$ . We have

$$\Delta_{I,J}(\mathbf{H}^*(\boldsymbol{\mu}, \ell)) = \Delta_{I,J}(\mathbf{H}^I(\boldsymbol{\mu}, \ell)) \text{ and } \Delta_{\bar{I},\bar{J}}(\mathbf{H}^*(\boldsymbol{\mu}, \ell)) = \Delta_{\bar{I},\bar{J}}(\mathbf{H}^I(\boldsymbol{\mu}, \ell)).$$

To check this, note that, for example, for  $j \leq k-1$ , the indicator function appearing in the entry  $(m, j)$  of  $\mathbf{H}^*(\boldsymbol{\mu}, \ell, k)$  is  $\mathbb{1}_{\mu_j \leq \ell}$ . For every  $I$  and for  $m \in I$ , this quantity is also equal to  $\mathbb{1}_{\mu_j \in \mathcal{D}_m^I(\ell)}$ .

Then using (21) with  $p = k-1$ ,  $n = N-k$  and  $J = \{1, \dots, k-1\}$ , we have  $\Delta_{\bar{I},\bar{J}}(\mathbf{H}^I(\boldsymbol{\mu}, \ell)) = 0$  when  $\bar{I} \neq I$  and

$$\det(\mathbf{H}^I(\boldsymbol{\mu}, \ell)) \mathbb{1}_{\mu_1 \leq \dots \leq \mu_{N-1}} = \epsilon(\sigma_I) \Delta_{I,J}(\mathbf{H}^I(\boldsymbol{\mu}, \ell)) \Delta_{\bar{I},\bar{J}}(\mathbf{H}^I(\boldsymbol{\mu}, \ell)) \mathbb{1}_{\mu_1 \leq \dots \leq \mu_{N-1}}.$$

We conclude that

$$\gamma_{N-1,1}^{k-1}(\ell) = k_{N-1}(N-1)! \sum_{I \in \mathcal{F}_{k-1}(N-1)} \int_{\mu_1 \leq \dots \leq \mu_{N-1}} \det(\mathbf{H}^I(\boldsymbol{\mu}, \ell)) d\mu_1 \cdots \mu_{N-1}.$$

Since  $\text{Pf}(\mathcal{A}^1(I, \ell)) = \int_{\mu_1 \leq \dots \leq \mu_{N-1}} \det(\mathbf{H}^I(\boldsymbol{\mu}, \ell)) d\mu_1 \cdots \mu_{N-1}$  (see [15], [26] or equation (13) in [20]) we obtain (20). This concludes the proof.  $\square$

The major drawback of the result above is its complicated form. However, for small values of  $N$ , we are able to get an explicit expression for  $q_N^k(\ell)$  and consequently (using (15), (16) and (17)) for  $E(\mathcal{N}_k^c(S))$  and  $E(\mathcal{N}_k^c(u, S))$ . We give some examples below and we derive

Propositions 3.7 and 3.8 which are new results.

**Examples:** After tedious calculations we obtain

1. For  $N = 2$ :  $q_2^1(l) = q_2^2(-l)$  and

$$(22) \quad q_2^2(l) = \frac{\exp\left(-\frac{l^2}{2}\right)}{2\sqrt{\pi}} \left[ \exp\left(-\frac{l^2}{2}\right) + \sqrt{2\pi}l\Phi(l) \right].$$

2. For  $N = 3$ :  $q_3^1(l) = q_3^3(-l)$ ,  $q_3^2(l) = \frac{\exp(-l^2)}{\sqrt{\pi}}$  and

$$(23) \quad q_3^3(l) = \frac{\exp\left(-\frac{l^2}{2}\right)}{\pi\sqrt{2}} \left[ \sqrt{\pi}(2l^2 - 1)\Phi(l\sqrt{2}) + \sqrt{2\pi} \exp\left(-\frac{l^2}{2}\right) \Phi(l) + l \exp(-l^2) \right].$$

3. For  $N = 4$ :  $q_4^1(l) = q_4^4(-l)$ ,  $q_4^2(l) = q_4^3(-l)$  and

$$(24) \quad q_4^3(l) = \frac{\exp\left(-\frac{l^2}{2}\right)}{2\pi} \left[ \frac{3l}{2} \exp\left(-\frac{3l^2}{2}\right) + \sqrt{2\pi} \left(1 - \frac{l^2}{2}\right) \bar{\Phi}(l) \exp(-l^2) \right. \\ \left. - \frac{\pi(2l^3 - 3l)}{\sqrt{2}} \Phi(l\sqrt{2}) \bar{\Phi}(l) + \frac{3\sqrt{\pi}(1 + 2l^2)}{2} \Phi(l\sqrt{2}) \exp\left(-\frac{l^2}{2}\right) \right].$$

$$(25) \quad q_4^4(l) = \frac{\exp\left(-\frac{l^2}{2}\right)}{2\pi} \left[ \frac{3l}{2} \exp\left(-\frac{3l^2}{2}\right) - \sqrt{2\pi} \left(1 - \frac{l^2}{2}\right) \Phi(l) \exp(-l^2) \right. \\ \left. + \frac{\pi(2l^3 - 3l)}{\sqrt{2}} \Phi(l\sqrt{2}) \Phi(l) + \frac{3\sqrt{\pi}(1 + 2l^2)}{2} \Phi(l\sqrt{2}) \exp\left(-\frac{l^2}{2}\right) \right].$$

4. For  $N = 5$ :  $q_5^1(l) = q_5^5(-l)$ ,  $q_5^2(l) = q_5^4(-l)$  and

$$(26) \quad q_5^3(l) = q_5^4(l) - 2q_5^5(l) + \frac{\sqrt{2} \exp\left(-\frac{l^2}{2}\right)}{3\pi^{3/2}} \left[ \pi(4l^4 - 12l^2 + 3) \Phi(l\sqrt{2}) \right. \\ \left. + \sqrt{\pi} \left(3l^3 - \frac{13l}{2}\right) \exp(-l^2) + \sqrt{2\pi} \left(l^4 + 3l^2 + \frac{3}{4}\right) \exp\left(-\frac{l^2}{2}\right) \Phi(l) \right].$$

$$(27) \quad q_5^4(l) = \frac{\sqrt{2} \exp\left(-\frac{l^2}{2}\right)}{3\pi^{3/2}} \left[ \sqrt{2\pi} \exp\left(-\frac{3l^2}{2}\right) \left(\frac{l^3}{2} + \frac{5l}{4}\right) \right. \\ \left. + \sqrt{2\pi} \Phi(l\sqrt{2}) \exp\left(-\frac{l^2}{2}\right) \left(l^4 + 3l^2 + \frac{3}{4}\right) \right].$$

$$(28) \quad q_5^5(l) = \frac{\sqrt{2} \exp\left(-\frac{l^2}{2}\right)}{3\pi^{3/2}} \left[ \left(2l^4 - 6l^2 + \frac{3}{2}\right) \pi \Phi^2(l\sqrt{2}) \right. \\ \left. + \left(l^4 + 3l^2 + \frac{3}{4}\right) \sqrt{2\pi} \Phi(l\sqrt{2}) \Phi(l) \exp\left(-\frac{l^2}{2}\right) + \sqrt{2\pi} \left(\frac{l^3}{2} + \frac{5l}{4}\right) \Phi(l) \exp\left(-\frac{3l^2}{2}\right) \right].$$

$$+ \left( 3l^3 - \frac{13l}{2} \right) \sqrt{\pi} \Phi \left( l\sqrt{2} \right) \exp(-l^2) + (l^2 - 2) \exp(-2l^2) \Big].$$

These probability densities are plotted in Figure 1. They all seem very close to a Gaussian density. But only one,  $q_3^2(l)$ , is exactly Gaussian.

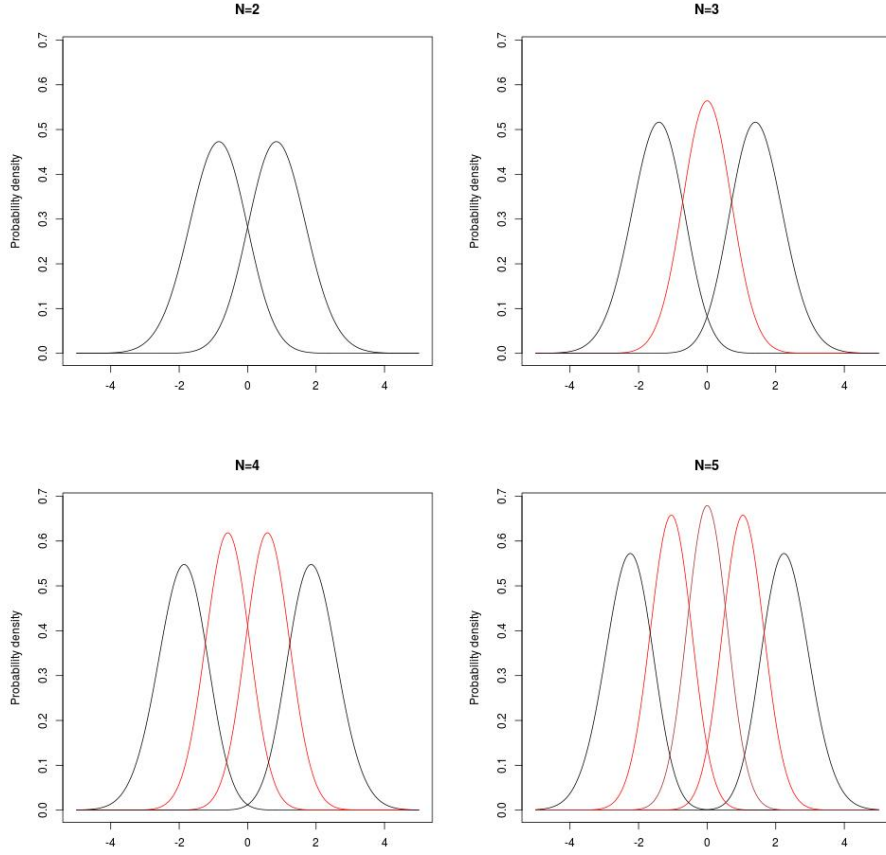


FIG 1. Probability densities  $q_N^k(l)$ ,  $k = 1, \dots, N$ , of the ordered eigenvalues of a  $N$  GOE matrix for  $N = 2, 3, 4, 5$ .

3.2.2. *Mean number of critical points.* For  $N = 2$ , using (15) and (23), after some calculations, we retrieve (1) with

$$E(\mathcal{N}^c(S)) = \frac{2|S|}{\sqrt{3}\pi} \left( \frac{\lambda_4}{3\lambda_2} \right).$$

In Proposition 3.7 we give the exact expression of  $E(\mathcal{N}_k^c(S))$  when  $N = 3$  for  $k = 0, 1, 2, 3$ . This proposition is a new result.

PROPOSITION 3.7. *Under the conditions of Proposition 3.2 when  $N = 3$*

$$E(\mathcal{N}_0^c(S)) = E(\mathcal{N}_3^c(S)) = \frac{|S|}{\pi^2\sqrt{2}} \left( \frac{\lambda_4}{3\lambda_2} \right)^{3/2} \left( \frac{29 - 6\sqrt{6}}{12\sqrt{6}} \right)$$

$$\mathbb{E}(\mathcal{N}_1^c(S)) = \mathbb{E}(\mathcal{N}_2^c(S)) = \frac{|S|}{\pi^2 \sqrt{2}} \left( \frac{\lambda_4}{3\lambda_2} \right)^{3/2} \left( \frac{29 + 6\sqrt{6}}{12\sqrt{6}} \right).$$

Consequently:

$$\frac{\mathbb{E}(\mathcal{N}_0^c(S))}{\mathbb{E}(\mathcal{N}^c(S))} = \frac{\mathbb{E}(\mathcal{N}_3^c(S))}{\mathbb{E}(\mathcal{N}^c(S))} = \frac{29 - 6\sqrt{6}}{116} \simeq 0.1233$$

$$\frac{\mathbb{E}(\mathcal{N}_1^c(S))}{\mathbb{E}(\mathcal{N}^c(S))} = \frac{\mathbb{E}(\mathcal{N}_2^c(S))}{\mathbb{E}(\mathcal{N}^c(S))} = \frac{29 + 6\sqrt{6}}{116} \simeq 0.3767.$$

In the same way for  $N = 4$ , we obtain the expressions given in Proposition 3.8.

**PROPOSITION 3.8.** *Set  $\mathcal{I} := \mathbb{E}(\Phi(Y)\Phi(\sqrt{2}Y))$ ,  $Y$  being a Gaussian centred variable with variance  $1/3$ . Under the conditions of Proposition 3.2 when  $N = 4$*

$$\mathbb{E}(\mathcal{N}_1^c(S)) = \mathbb{E}(\mathcal{N}_3^c(S)) = \frac{|S|}{\pi^2} \left( \frac{\lambda_4}{3\lambda_2} \right)^2 \left( \frac{25}{24\sqrt{3}} \right),$$

$$\mathbb{E}(\mathcal{N}_0^c(S)) = \mathbb{E}(\mathcal{N}_4^c(S)) = \frac{|S|}{\pi^2} \left( \frac{\lambda_4}{3\lambda_2} \right)^2 \left( \frac{\mathcal{I} \times 100\pi - 57}{48\sqrt{3}\pi} \right),$$

$$\mathbb{E}(\mathcal{N}_2^c(S)) = \frac{|S|}{\pi^2} \left( \frac{\lambda_4}{3\lambda_2} \right)^2 \left( \frac{50\pi(1 - 2\mathcal{I}) + 57}{24\sqrt{3}\pi} \right).$$

Consequently,

$$\frac{\mathbb{E}(\mathcal{N}_1^c(S))}{\mathbb{E}(\mathcal{N}^c(S))} = \frac{\mathbb{E}(\mathcal{N}_3^c(S))}{\mathbb{E}(\mathcal{N}^c(S))} = \frac{1}{4}, \quad \frac{\mathbb{E}(\mathcal{N}_0^c(S))}{\mathbb{E}(\mathcal{N}^c(S))} = \frac{\mathbb{E}(\mathcal{N}_4^c(S))}{\mathbb{E}(\mathcal{N}^c(S))} = \frac{\mathcal{I} \times 100\pi - 57}{200\pi} \simeq 0.060,$$

$$\frac{\mathbb{E}(\mathcal{N}_2^c(S))}{\mathbb{E}(\mathcal{N}^c(S))} = \frac{50\pi(1 - 2\mathcal{I}) + 57}{100\pi} \simeq 0.380.$$

Note that, in the same way, it is possible to obtain exact expressions for the mean number of critical points above a level  $u$  for  $N = 2, 3$  or  $4$  using (23), (24), (25) and (26), (27), (28) with Proposition 3.2.

**4. Correlation function between critical points.** In this section  $\mathcal{X}$  is a random field satisfying Assumption (A2).

4.1. *Correlation function, two points function.* The correlation function of a point process  $\mathcal{P}$  is defined by [14]

$$A(s, t) := \lim_{\epsilon \rightarrow 0} \frac{1}{V^2(\epsilon)} \mathbb{E}(\mathcal{P}(B(s, \epsilon))\mathcal{P}(B(t, \epsilon))) \text{ when it exists,}$$

where  $\mathcal{P}(B(s, \epsilon))$  is the number of points in the ball with center  $s$  and radius  $\epsilon$  and  $V(\epsilon)$  its volume of this ball. When the considered process is stationary and isotropic, this function depends only on the norm  $\rho := \|s - t\|$  and by a small abuse of notation we will denote it by  $A(\rho)$ .

Suppose now that the point process  $\mathcal{P}$  is the process of critical points of  $\mathcal{X}$ . Under our conditions, the Kac-Rice formula of order two in sufficiently small sets is valid (see Appendix A, Proposition A.3, equation (70)). In particular if  $S$  and  $T$  are disjoint, sufficiently close and

sufficiently small (as  $B(s, \epsilon)$  and  $B(t, \epsilon)$  for  $s \neq t$  and  $\|s - t\|$  and  $\epsilon$  sufficiently small), the Kac-Rice formula of order two (70) yields

$$(29) \quad \begin{aligned} & \mathbb{E}[\mathcal{N}^c(S)\mathcal{N}^c(T)] \\ &= \int_{S \times T} \mathbb{E}(|\det(\nabla^2 X(s)) \det(\nabla^2 X(t))| / \nabla X(s) = \nabla X(t) = 0) p_{\nabla X(s), \nabla X(t)}(0, 0) ds dt, \end{aligned}$$

proving that for  $\rho$  sufficiently small,  $A(\rho)$  is well defined and given by

$$(30) \quad A(\rho) = \mathbb{E}(|\det(\nabla^2 X(0)) \det(\nabla^2 X(t))| / \nabla X(0) = \nabla X(t) = 0) p_{\nabla X(0), \nabla X(t)}(0, 0)$$

with, for example,  $t = \rho e_1$  and  $e_1 := (1, 0, \dots, 0)$ .

Some papers, as [11] consider the behaviour as  $\rho \rightarrow 0$  of

$$T(\rho) := \mathbb{E}(\mathcal{N}^c(B_\rho)(\mathcal{N}^c(B_\rho) - 1)), \quad B_\rho \text{ is any ball with radius } \rho.$$

It is elementary to see that if  $A(\rho) \simeq C\rho^d$  then  $T(\rho) \simeq CV_N^2 \rho^{d+2N}$ , where  $\simeq$  means, as in the rest of the paper, equivalence as  $\rho \rightarrow 0$  and  $V_N$  is the volume of the unit ball.

#### 4.2. Attraction, neutrality, repulsion.

- The reference of neutral point process is the Poisson process for which the correlation function  $A(\rho)$  is constant and  $T(\rho)$  behaves as  $\rho^{2N}$ .
- The repulsion is defined by the fact that the correlation function  $A(\rho)$  tends to zero as  $\rho \rightarrow 0$ . Equivalently  $T(\rho) = o(\rho^{2N})$ . Note that determinantal processes [23] are a way of constructing repulsive point processes.
- The attraction is just the contrary: as  $\rho \rightarrow 0$ ,  $A(\rho) \rightarrow +\infty$ ,  $\frac{T(\rho)}{\rho^{2N}} \rightarrow +\infty$ .

As already mentioned in the preceding section, under our assumptions the sample paths are a.s. Morse, and the index of each critical point can a.s. be defined. We can generalise the definition of the correlation function by

$$(31) \quad A^{i_1, i_2}(s, t) := \lim_{\epsilon \rightarrow 0} \frac{1}{V^2(\epsilon)} \mathbb{E}(\mathcal{N}_{i_1}^c(B(s, \epsilon))\mathcal{N}_{i_2}^c(B(t, \epsilon))), \text{ when it exists.}$$

The Kac-Rice formula (71) yields that for  $\rho$  sufficiently small this function is well defined and

$$(32) \quad \begin{aligned} & A^{i_1, i_2}(\rho) = p_{\nabla X(0), \nabla X(t)}(0, 0) \\ & \times \mathbb{E}(|\det(\nabla^2 X(0)) \mathbb{1}_{i(\nabla^2 X(0))=i_1} \det(\nabla^2 X(t)) \mathbb{1}_{i(\nabla^2 X(t))=i_2}| / \nabla X(0) = \nabla X(t) = 0), \end{aligned}$$

with again  $t = \rho e_1$ . In the same way, we can consider attraction, neutrality or repulsion between critical points with indexes  $i_1$  and  $i_2$ .

Before giving our main results concerning correlation functions between critical points, we give in the following paragraph some results concerning the conditional distribution of  $\nabla^2 X(0), \nabla^2 X(t)$  given  $\nabla X(0) = \nabla X(t) = 0$ .

4.3. *Conditional distribution of  $\nabla^2 X(0), \nabla^2 X(t)$ .* In this section, for short,  $\mathbf{r}(\rho^2)$ ,  $\mathbf{r}'(\rho^2)$ ,  $\mathbf{r}''(\rho^2)$ ,  $\mathbf{r}'''(\rho^2)$  and  $\mathbf{r}^{(4)}(\rho^2)$  are denoted by  $\mathbf{r}$ ,  $\mathbf{r}'$ ,  $\mathbf{r}''$ ,  $\mathbf{r}'''$  and  $\mathbf{r}^{(4)}$ . We recall that  $t = \rho e_1$ .

LEMMA 4.1. *Let  $\xi(0)$  and  $\xi(t)$  be a representation of the distribution of  $\nabla^2 X(0)$  and  $\nabla^2 X(t)$  given  $\nabla X(0) = \nabla X(t) = 0$ . We note  $\xi_d(t)$  the vector  $(\xi_{11}(t), \dots, \xi_{NN}(t))$  and  $\xi_u(t)$  the vector  $(\xi_{12}(t), \xi_{13}(t), \dots, \xi_{(N-1)N}(t))$ . The joint distribution of  $(\xi_d(0), \xi_u(0), \xi_d(t), \xi_u(t))$  is centred Gaussian with variance-covariance matrix:*

$$\begin{bmatrix} \Gamma_1 & 0 & \Gamma_3 & 0 \\ 0 & \Gamma_2 & 0 & \Gamma_4 \\ \Gamma_3 & 0 & \Gamma_1 & 0 \\ 0 & \Gamma_4 & 0 & \Gamma_2 \end{bmatrix}.$$

$\Gamma_1$  is the  $N \times N$  matrix:

$$\Gamma_1 = \begin{bmatrix} 12\mathbf{r}''(0) & 4\mathbf{r}''(0) & \cdots & 4\mathbf{r}''(0) \\ 4\mathbf{r}''(0) & \ddots & 4\mathbf{r}''(0) & \vdots \\ \vdots & 4\mathbf{r}''(0) & \ddots & 4\mathbf{r}''(0) \\ 4\mathbf{r}''(0) & \cdots & 4\mathbf{r}''(0) & 12\mathbf{r}''(0) \end{bmatrix} + \frac{\rho^2 \mathbf{r}'(0)}{2[\mathbf{r}'(0)^2 - (\mathbf{r}' + 2\mathbf{r}''\rho^2)^2]} \times M,$$

with

$$M = \begin{bmatrix} (12\mathbf{r}'' + 8\rho^2 \mathbf{r}''')^2 & 4\mathbf{r}''(12\mathbf{r}'' + 8\rho^2 \mathbf{r}''') & \cdots & 4\mathbf{r}''(12\mathbf{r}'' + 8\rho^2 \mathbf{r}''') \\ 4\mathbf{r}''(12\mathbf{r}'' + 8\rho^2 \mathbf{r}''') & 16\mathbf{r}''^2 & \cdots & 16\mathbf{r}''^2 \\ \vdots & \vdots & \ddots & \vdots \\ 4\mathbf{r}''(12\mathbf{r}'' + 8\rho^2 \mathbf{r}''') & 16\mathbf{r}''^2 & \cdots & 16\mathbf{r}''^2 \end{bmatrix}.$$

$\Gamma_2$  is the  $\frac{N(N-1)}{2} \times \frac{N(N-1)}{2}$  diagonal matrix:

$$\Gamma_2 = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \text{ with } D_1 = \left( 4\mathbf{r}''(0) + \frac{8\rho^2(\mathbf{r}'')^2 \mathbf{r}'(0)}{\mathbf{r}'(0)^2 - (\mathbf{r}')^2} \right) Id_{N-1} \text{ and } D_2 = (4\mathbf{r}''(0)) Id_{\frac{(N-1)(N-2)}{2}}.$$

We set  $a := 4\mathbf{r}'' + 8\rho^2 \mathbf{r}'''$  and  $d := 12\mathbf{r}'' + 48\rho^2 \mathbf{r}''' + 16\rho^4 \mathbf{r}^{(4)}$ .  $\Gamma_3$  is the  $N \times N$  matrix:

$$\Gamma_3 = \begin{bmatrix} d & a & \cdots & \cdots & a \\ a & 12\mathbf{r}'' & 4\mathbf{r}'' & \cdots & 4\mathbf{r}'' \\ \vdots & 4\mathbf{r}'' & \ddots & 4\mathbf{r}'' & \vdots \\ \vdots & \vdots & 4\mathbf{r}'' & \ddots & 4\mathbf{r}'' \\ a & 4\mathbf{r}'' & \cdots & 4\mathbf{r}'' & 12\mathbf{r}'' \end{bmatrix} + \frac{\rho^2(\mathbf{r}' + 2\mathbf{r}''\rho^2)}{2[\mathbf{r}'(0)^2 - (\mathbf{r}' + 2\mathbf{r}''\rho^2)^2]} \times M.$$

$\Gamma_4$  is the  $\frac{N(N-1)}{2} \times \frac{N(N-1)}{2}$  diagonal matrix:

$$\Gamma_4 = \begin{bmatrix} \tilde{D}_1 & 0 \\ 0 & \tilde{D}_2 \end{bmatrix} \text{ with } \tilde{D}_1 = \left( a + \frac{8\rho^2(\mathbf{r}'')^2 \mathbf{r}'}{\mathbf{r}'(0)^2 - \mathbf{r}'^2} \right) Id_{N-1} \text{ and } \tilde{D}_2 = 4\mathbf{r}'' Id_{\frac{(N-1)(N-2)}{2}}.$$

The proof of Lemma 4.1 is given in Appendix B.

In the following lemma, we give the equivalent of the variance-covariance matrix of  $\nabla^2 X(0), \nabla^2 X(t)$  given  $\nabla X(0) = \nabla X(t) = 0$  as  $\rho \rightarrow 0$ .

LEMMA 4.2. *Using the same notation as in Lemma 4.1, for  $j \neq k$  and  $j, k \neq 1$ , as  $\rho \rightarrow 0$*

$$(33) \quad \det(\text{Var}(\nabla X(0), \nabla X(t))) \simeq \rho^{2N} \frac{\lambda_2^N \lambda_4^N}{3^{N-1}},$$

$$(34) \quad \text{Var}(\xi_{11}(t)) \simeq \frac{\rho^2}{4} \frac{(\lambda_2 \lambda_6 - \lambda_4^2)}{\lambda_2}, \quad \text{Var}(\xi_{1j}(t)) \simeq \frac{\rho^2}{4} \frac{(9\lambda_2 \lambda_6 - 5\lambda_4^2)}{45\lambda_2},$$

$$(35) \quad \text{Var}(\xi_{jj}(t)) \simeq \frac{8\lambda_4}{9}, \quad \text{Var}(\xi_{jk}(t)) \simeq \frac{\lambda_4}{3},$$

$$(36) \quad \text{Cov}(\xi_{11}(t), \xi_{jj}(t)) = \rho^2 \frac{11\lambda_2 \lambda_6 - 15\lambda_4^2}{180\lambda_2} + o(\rho^2), \quad \text{Cov}(\xi_{jj}(t), \xi_{kk}(t)) \simeq \frac{2\lambda_4}{9}.$$

(37) *All the other covariances  $\text{Cov}(\xi_{il}(t), \xi_{mn}(t))$  are zero,  $\forall i, l, m, n \in \{1, \dots, N\}$ . We have of course the same relations for  $\xi(0)$ .*

Moreover we have  $\forall j, k \in \{2, \dots, N\}$  and  $\forall i \in \{1, \dots, N\}$ ; as  $\rho \rightarrow 0$ ,

$$(38) \quad \text{Cov}(\xi_{jk}(0), \xi_{jk}(t)) \simeq \text{Var}(\xi_{jk}(t)), \quad \text{Cov}(\xi_{1i}(0), \xi_{1i}(t)) \simeq -\text{Var}(\xi_{1i}(t)),$$

$$(39) \quad \text{Cov}(\xi_{11}(0), \xi_{jj}(t)) = \rho^2 \frac{15\lambda_4^2 - 7\lambda_2 \lambda_6}{180\lambda_2} + o(\rho^2), \quad \text{Cov}(\xi_{jj}(0), \xi_{kk}(t)) \simeq \frac{2\lambda_4}{9} \text{ for } j \neq k,$$

(40) *All the other covariances  $\text{Cov}(\xi_{il}(0), \xi_{mn}(t))$  are zero,  $\forall i, l, m, n \in \{1, \dots, N\}$ .*

Finally we also have

$$(41) \quad \det(\text{Var}(\xi_{11}(t), \xi_{11}(0))) \simeq \frac{\rho^6}{144} \frac{(\lambda_4 \lambda_8 - \lambda_6^2)(\lambda_2 \lambda_6 - \lambda_4^2)}{\lambda_2 \lambda_4}.$$

The proof of Lemma 4.2 is given in Appendix B. It is based on some lengthy computations but uses a trick that simplify drastically computations. We present it here in the example of the proof of (33).

TRICK 4.3. *Because a determinant is invariant by adding to some row (resp. column) a linear combination of other rows (resp. columns) as  $\rho \rightarrow 0$ ,*

$$\begin{aligned} \det(\text{Var}(\nabla X(0), \nabla X(t))) &= \det(\text{Var}(X'_1(0), \dots, X'_N(0), X'_1(t), \dots, X'_N(t))) \\ &= \det(\text{Var}(X'_1(0), \dots, X'_N(0), X'_1(t) - X'_1(0), \dots, X'_N(t) - X'_N(0))) \\ &\simeq \rho^{2N} \det(\text{Var}(X'_1(0), \dots, X'_N(0), X''_{11}(0), \dots, X''_{1N}(0))). \end{aligned}$$

Using Lemma 2.3 we obtain (33).

4.4. *Correlation function between critical points for random processes.* The result below appears in a hidden way in Proposition 4.5 or Section 5.2 of [9] or in [5]. It is stated in term of crossings of a level and can be applied directly to critical points that are simply crossings of the level zero of the derivative.

PROPOSITION 4.4. *Let  $\mathcal{X} = \{X(t) : t \in \mathbb{R}\}$  be a stationary and isotropic Gaussian process with real values satisfying Assumption (A3). Let  $A(\rho)$  be the correlation function between critical points of  $\mathcal{X}$  defined in (30). Then, as  $\rho \rightarrow 0$*

$$(42) \quad A(\rho) \simeq \frac{(\lambda_2 \lambda_6 - \lambda_4^2)}{8\pi \sqrt{\lambda_4 \lambda_2^3}} \rho.$$

Under Assumption (A4), let  $A^{1,1}(\rho)$  (resp.  $A^{0,0}(\rho)$ ) be the correlation function between maxima (resp. minima) defined in (32). Then as  $\rho \rightarrow 0$

$$(43) \quad A^{1,1}(\rho) = A^{0,0}(\rho) \simeq \frac{(\lambda_4 \lambda_8 - \lambda_6^2)^{3/2}}{1296\pi^2 (\lambda_4^2) (\lambda_2 \lambda_6 - \lambda_4^2)^{1/2}} \rho^4.$$



Note that all the coefficients above are positive. The interpretation of the proposition is that we have always repulsion between critical points and a very strong repulsion between maxima or between minima. As we will see, the surprising result is that does not remain true in higher dimension.

Before beginning the proof we need to give the following lemma .

LEMMA 4.5. *Let  $X, Y$  be two jointly Gaussian variables with common variance  $\sigma^2$  and correlation  $c$ . Let  $r$  a positive real. Then as  $c \rightarrow -1$*

$$\mathbb{E}((X+Y^+)^r) \simeq K_r \sigma^{-2(1+r)} (\det(\text{Var}(X, Y)))^{(2r+1)/2},$$

where

$$K_r = \frac{1}{2\pi} \int_0^{+\infty} \int_0^{+\infty} x^r y^r \exp\left(-\frac{(x+y)^2}{2}\right) dx dy < +\infty, \quad K_1 = \frac{1}{6\pi}.$$

Moreover as  $c \rightarrow -1$

$$\mathbb{E}(X+Y^-) \simeq \frac{\sigma^2}{2}.$$

PROOF OF LEMMA 4.5. We set  $\Sigma := \text{Var}(X, Y)$ . Then

$$\begin{aligned} \mathbb{E}((X+Y^+)^r) &= \int_0^{+\infty} \int_0^{+\infty} x^r y^r \frac{1}{2\pi\sqrt{\det \Sigma}} \exp\left(-\frac{\sigma^2(x^2 - 2cxy + y^2)}{2\det \Sigma}\right) dx dy \\ &\simeq \frac{1}{2\pi} (\det \Sigma)^{(2r+1)/2} \sigma^{-2(1+r)} \int_0^{+\infty} \int_0^{+\infty} x^r y^r \exp\left(-\frac{(x+y)^2}{2}\right) dx dy, \end{aligned}$$

where we have made the change of variables  $x = x' \sqrt{\frac{\det \Sigma}{\sigma^2}}$ ,  $y = y' \sqrt{\frac{\det \Sigma}{\sigma^2}}$  and the passage to the limit is justified because the integrand is a monotone function of  $c$ . It is easy to check the convergence of the integral.

$$\mathbb{E}(X+Y^-) = - \int_0^{+\infty} \int_{-\infty}^0 \frac{xy}{2\pi\sqrt{\det \Sigma}} \exp\left(-\frac{\sigma^2(x^2 - 2cxy + y^2)}{2\det \Sigma}\right) dy dx.$$

Integrating first with respect to  $y$  we obtain

$$\mathbb{E}(X+Y^-) = \frac{(\det \Sigma)^{3/2}}{2\pi\sigma^4} - c \int_0^{+\infty} \frac{x^2}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \Phi\left(-cx\sqrt{\frac{\sigma^2}{\det \Sigma}}\right) dx.$$

As  $c \rightarrow -1$ ,  $\frac{\det \Sigma}{\sigma^2} = \sigma^2(1 - c^2) \rightarrow 0$ , therefore

$$\int_0^{+\infty} \frac{x^2}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right) \Phi\left(-cx\sqrt{\frac{\sigma^2}{\det \Sigma}}\right) dx \rightarrow \frac{\sigma^2}{2}.$$

Moreover  $\frac{(\det \Sigma)^{3/2}}{\sigma^6} \rightarrow 0$ . That concludes the proof.  $\square$

PROOF OF PROPOSITION 4.4. The first relation (42) is a particular case of Theorem 4.6 below. Let us prove the second relation (43). By (32)

$$A^{0,0}(\rho) = p_{X'(\rho), X'(0)}(0, 0) \mathbb{E}(X''(0)^+ X''(\rho)^+ / X'(0) = X'(\rho) = 0).$$

By (33) we have

$$p_{X'(\rho), X'(0)}(0, 0) \simeq \rho^{-1} \left( 2\pi \sqrt{\lambda_4 \lambda_2} \right)^{-1}.$$

Define

$$\begin{aligned} w &:= \text{Cov} \left( X''(0), X''(\rho) / X'(0) = X'(\rho) = 0 \right), \\ v &:= \text{Var} \left( X''(0) / X'(0) = X'(\rho) = 0 \right) = \text{Var} \left( X''(\rho) / X'(0) = X'(\rho) = 0 \right), \\ D &:= \det \left( \text{Var} \left( X''(0), X''(\rho) / X'(0) = X'(\rho) = 0 \right) \right) = v^2 - w^2. \end{aligned}$$

By (38), (34) and (41) we have  $w \simeq -v$ ,

$$v \simeq \rho^2 \frac{\lambda_2 \lambda_6 - \lambda_4^2}{4\lambda_2} \text{ and } D \simeq \rho^6 \frac{(\lambda_4 \lambda_8 - \lambda_6^2)(\lambda_2 \lambda_6 - \lambda_4^2)}{144\lambda_2 \lambda_4}.$$

Now using Lemma 4.5 we get

$$\mathbb{E} \left( X''(0)^+ X''(\rho)^+ / X'(0) = X'(\rho) = 0 \right) \simeq \frac{D^{3/2}}{6\pi v^2}.$$

That concludes the proof.  $\square$

#### 4.5. Correlation function between critical points for random fields.

4.5.1. *All the critical points.* Consider  $\mathcal{X} = \{X(t) : t \in \mathbb{R}^N\}$  a stationary and isotropic Gaussian field with real values. Let us consider two points  $s, t \in \mathbb{R}^N$ . Theorem 4.6 below, gives the asymptotic expression (as  $\rho \rightarrow 0$ ) of the correlation function between critical points (30) of any isotropic Gaussian field. It generalizes the result of [11] and of [12] to general fields in any dimension.

**THEOREM 4.6.** *Let  $\mathcal{X} = \{X(t) : t \in \mathbb{R}^N\}$  be a stationary and isotropic Gaussian field with real values satisfying Assumption (A3). Let  $A(\rho)$  be the correlation function between critical points (30). Then as  $\rho \rightarrow 0$ ,*

$$(44) \quad A(\rho) \simeq \rho^{2-N} \frac{\gamma_{N-1}}{2^3 3^{(N-1)/2} \pi^N} \left( \sqrt{\frac{\lambda_4}{\lambda_2}} \right)^N \frac{(\lambda_2 \lambda_6 - \lambda_4^2)}{\lambda_2 \lambda_4},$$

where  $\gamma_{N-1}$  is defined by

$$\gamma_{N-1} := \mathbb{E} \left( \det^2 (G_{N-1} - \Lambda \text{Id}_{N-1}) \right),$$

with  $G_{N-1}$  a  $(N-1)$ -GOE matrix and  $\Lambda$  an independent Gaussian random variable with variance  $1/3$ .

**REMARK 4.7.** We set  $\gamma_{N-1,2}(x) := \mathbb{E} \left( \det^2 (G_{N-1} - x \text{Id}_{N-1}) \right)$ . Note that formulas 2.2.16 and 2.2.17 in [26] give explicit expressions for the calculation of  $\gamma_{N-1,2}(x)$  for  $x \in \mathbb{R}$ .

**REMARK 4.8.** • Of course when  $N = 1$ , we retrieve (42).

- In the particular case  $N = 2$ , our result agrees with the result of [12], Theorem 1.2.
- Theorem 4.6 means that, for  $N = 2$ , there is a neutrality between critical points and for  $N > 2$  there is even attraction! This is quite different from the case  $N = 1$ . The next theorem will give an interpretation of this phenomenon.

- A first important consequence is finiteness of the second moment of the number of critical points. Indeed if  $S$  is a compact set of  $\mathbb{R}^N$  we can write the Kac-Rice formulas of order 1 and 2, (11), (29), (32). If  $\mathcal{N}^c(S)$  is the number of critical points that belong to  $S$  then

$$\mathbb{E}(\mathcal{N}^c(S)) = |S| \mathbb{E}(|\det \nabla^2 X(t)|) p_{\nabla X(t)}(0)$$

and

$$(45) \quad \mathbb{E}((\mathcal{N}^c(S))(\mathcal{N}^c(S) - 1)) = \int_{S^2} A(\|s - t\|) ds dt.$$

Since  $\lambda_4$  is assumed to be finite, the expectation is always finite. As for the second factorial moment, its finiteness is in general not systematic. In our case (44) implies the convergence of the integral in (45) on the diagonal implying in turn the finiteness of the second moment. Nevertheless our conditions are less general than [22] or [6].

**PROOF OF THEOREM 4.6.** For the simplicity of the exposition we admit in a first step all passages to the limit. They will be justified at the end of the proof. By (30)

$$(46) \quad A(\rho) = \mathbb{E}(|\det(\xi(0)) \det(\xi(t))|) p_{\nabla X(0), \nabla X(t)}(0, 0),$$

with  $t = \rho e_1$ . We recall that  $\xi(0)$  and  $\xi(t)$  are a representation of the conditional distribution of the Hessian at 0 and  $t$ . Because of (33),

$$(47) \quad p_{\nabla X(0), \nabla X(t)}(0, 0) \simeq \rho^{-N} 3^{(N-1)/2} \lambda_2^{-N/2} \lambda_4^{-N/2} (2\pi)^{-N}.$$

It remains to study the expectation in (46). We denote by  $\xi_{-1}(t)$  the  $(N-1) \times (N-1)$  matrix corresponding to the Hessian matrix  $\xi(t)$  without its first row and column.

Let us develop  $\det(\xi(t))$  with respect to the first row, for example. The first term is  $\xi_{11}(t) \det(\xi_{-1}(t))$ .

Consider another term and develop it now with respect to the first column. Each of the  $N-1$  terms that appear are products that include a factor  $\xi_{1j} \xi_{j'1}$  with  $j, j' \neq 1$  so because of (34) they are  $O_p(\rho^2)$ : divided by  $\rho^2$  they are bounded in probability. As a consequence we have proved that

$$\det(\xi(t)) = \xi_{11}(t) \det(\xi_{-1}(t)) + O_p(\rho^2).$$

By (35) and (34) we have

$$(48) \quad \xi_{-1}(t) \simeq \sqrt{\frac{2\lambda_4}{3}} (G_{N-1} - \Lambda Id_{N-1}),$$

where  $\Lambda$  is a  $\mathcal{N}(0, 1/3)$  random variable,  $G_{N-1}$  is a size  $(N-1)$  GOE matrix defined previously,  $\Lambda$  and  $G_{N-1}$  are independent. So

$$(49) \quad \det(\xi_{-1}(t)) \simeq \left(\frac{2\lambda_4}{3}\right)^{(N-1)/2} \det(G_{N-1} - \Lambda Id_{N-1}).$$

The order of magnitude of the first term in the development of the determinant is then  $\rho$ . In conclusion we have proved that

$$(50) \quad \det(\xi(t)) = O_p(\rho) = \xi_{11}(t) \det(\xi_{-1}(t)) + O_p(\rho^2)$$

$$\det(\xi(0)) \det(\xi(t)) = O_p(\rho^2) = \xi_{11}(0) \xi_{11}(t) \det(\xi_{-1}(0)) \det(\xi_{-1}(t)) + O_p(\rho^3).$$

By (38), (36) and (37):

$$\text{Corr}(\xi_{11}(0), \xi_{11}(t)) \rightarrow -1, \quad \text{Corr}(\xi_{jk}(t), \xi_{jk}(0)) \rightarrow 1, \quad \forall j, k \in 2, \dots, N,$$

$$\text{Cov}(\xi_{11}(t), \xi_{jj}(t)) = O(\rho^2), \quad \text{Cov}(\xi_{11}(t), \xi_{jk}(t)) = 0 \quad \forall j \neq k \in 1, \dots, N.$$

Then

$$(51) \quad \det(\xi(0)) \det(\xi(t)) \simeq -\xi_{11}(t)^2 \det^2(\xi_{-1}(t))$$

$$E(|\det(\xi(0)) \det(\xi(t))|) \simeq \text{Var}(\xi_{11}(t)) E(\det^2(\xi_{-1}(t))),$$

Finally equations (34) and (49) with (47) and (46) give (44).

Justification of the passages to the limit: Since we are in the Gaussian space generated by the random field  $\mathcal{X}$  all the variables considered above are jointly Gaussian. So their absolute values are bounded by the maximum in absolute value of a bounded random field. General results about the maximum of random fields, for example the Borell-Sudakov-Tsirelson theorem ([9] Section 2.4) implies that the maximum of its absolute value has moments of every order, giving all the dominations needed.  $\square$

#### 4.5.2. Correlation function between critical points with adjacent indexes.

**THEOREM 4.9.** *Let  $\mathcal{X} = \{X(t) : t \in \mathbb{R}^N\}$  be a stationary and isotropic Gaussian field with real values satisfying Assumption (A3). Let  $A^{k,k+1}(\rho)$  be the correlation function (32) between critical points with index  $k$  and critical points with index  $k+1$ ,  $k = 0, \dots, N-1$ . Then as  $\rho \rightarrow 0$ ,*

$$(52) \quad A^{k,k+1}(\rho) \simeq \rho^{2-N} \frac{\gamma_{N-1}^k}{2^4 3^{(N-1)/2} \pi^N} \left( \sqrt{\frac{\lambda_4}{\lambda_2}} \right)^N \frac{(\lambda_2 \lambda_6 - \lambda_4^2)}{\lambda_2 \lambda_4},$$

where  $\gamma_{N-1}^k$  is defined by

$$(53) \quad \gamma_{N-1}^k := E(\det^2(G_{N-1} - \Lambda Id_{N-1}) \mathbb{1}_{\{i(G_{N-1} - \Lambda Id_{N-1})=k\}}),$$

with  $G_{N-1}$  a  $(N-1)$ -GOE matrix and  $\Lambda$  an independent centred Gaussian random variable with variance  $1/3$ .

This theorem gives an interpretation to Theorem 4.6: the attraction for  $N \geq 3$  is in fact due to attraction between critical points with adjacent indexes.

Again, in the particular case of the random plane wave ( $N = 2$ ), our result agrees with the result of [11], formula (9).

We set  $\gamma_{N-1,2}^k(x) := E(\det^2(G_{N-1} - x Id_{N-1}) \mathbb{1}_{\{i(G_{N-1} - x Id_{N-1})=k\}})$  for  $x \in \mathbb{R}$ . Before proving Theorem 4.9 we give in the following lemma an exact expression for the calculation of  $\gamma_{N-1,2}^k(x)$ .

**LEMMA 4.10.** *Using notation above and notation introduced in paragraph 3.2.1 we have*

$$\gamma_{N-1,2}^k(x) = k_{N-1}(N-1)! \sum_{I \in \mathcal{F}_k(N-1)} \text{Pf}(\mathcal{A}^2(I, \ell)),$$

where  $k_{N-1}$  is given by (14) and  $\text{Pf}(\mathcal{A}^2(I, \ell))$  is the Pfaffian of the skew matrix  $\mathcal{A}^2(I, \ell)$  defined by (18) and (19) with  $n = N-1$  and  $\alpha = 2$ .

PROOF OF LEMMA 4.10. The proof follows the same lines as in Theorem 3.5 with

$$\begin{aligned} \gamma_{N-1,2}^k(\ell) &= k_{N-1}(N-1)! \int_{\mathcal{O}_{k+1}(\ell)} \prod_{i=1}^{N-1} (\mu_i - \ell)^2 \\ &\quad \prod_{1 \leq i < j \leq N-1} (\mu_j - \mu_i) \exp\left(-\frac{\sum_{i=1}^{N-1} \mu_i^2}{2}\right) d\mu_1 \dots d\mu_{N-1}. \end{aligned}$$

Then we set  $\mathbf{H}(\boldsymbol{\mu}, \ell)$  the matrix  $\{h_m(\mu_j, \ell), m, j = 1, \dots, N-1\}$  with  $h_m(\mu_j, \ell) = \mu_j^{m-1}(\mu_j - \ell)^2 \exp(-\mu_j^2/2)$ .  $\square$

PROOF OF THEOREM 4.9. By (32) we have

$$\begin{aligned} A^{k,k+1}(\rho) &= p_{\nabla X(0), \nabla X(t)}(0, 0) \mathbb{E}(|\det \xi(0)| \mathbb{1}_{\{i(\xi(0))=k\}} |\det \xi(t)| \mathbb{1}_{\{i(\xi(t))=k+1\}}) \\ &= -p_{\nabla X(0), \nabla X(t)}(0, 0) \mathbb{E}(\det(\xi(0)) \mathbb{1}_{\{i(\xi(0))=k\}} \det \xi(t) \mathbb{1}_{\{i(\xi(t))=k+1\}}). \end{aligned}$$

We can modify the computation of the determinant of  $\xi(0)$  in (50) to compute in place  $\det(\xi(0) - \lambda Id_N)$ :

$$(54) \quad \det(\xi(0) - \lambda Id_N) = (\xi_{11}(0) - \lambda) \det(\xi_{-1}(0) - \lambda Id_{N-1}) + O_p(\rho^2).$$

Note that as  $\rho \rightarrow 0$ ,  $\xi(0)$  converges in  $L^2$  (or a.s. if we use a Skorohod imbedding argument) to

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & \xi_{-1}(0) & & \\ 0 & & & \end{pmatrix}.$$

Its eigenvectors converge to those of this last matrix. These eigenvectors are associated to different eigenvalues so we can define properly the eigenvalue  $\mu_1$  that tends to zero. Because of (54) with  $\lambda = \mu_1$ :

$$(\xi_{11}(0) - \mu_1) \det(\xi_{-1}(0) - \mu_1 Id_{N-1}) = O_p(\rho^2),$$

implying in turn that

$$\mu_1 = \xi_{11}(0) + O_p(\rho^2).$$

On the other hand, the others eigenvalues, say  $\mu_2, \dots, \mu_N$ , converge in distribution to that given by the right-hand side of (48). This implies in turn that

$$\mathbb{1}_{\{i(\xi(0))=k\}} \simeq \mathbb{1}_{\{\xi_{11}(0) < 0; i(\xi_{-1}(0))=k-1\}} + \mathbb{1}_{\{\xi_{11}(0) > 0; i(\xi_{-1}(0))=k\}}.$$

As a consequence, when computing  $\mathbb{E}(\det(\xi(0)) \mathbb{1}_{\{i(\xi(0))=k\}} \det \xi(t) \mathbb{1}_{\{i(\xi(t))=k+1\}})$ , we have four cases to consider depending on the signs of  $\xi_{11}(0), \xi_{11}(t)$ . First we consider the case  $\xi_{11}(0) > 0, \xi_{11}(t) < 0$ . We have the equivalent of (51)

$$\begin{aligned} &\det(\xi(0)) \det(\xi(t)) \mathbb{1}_{\{\xi_{11}(0) > 0, \xi_{11}(t) < 0\}} \mathbb{1}_{\{i(\xi_{-1}(t))=i(\xi_{-1}(0))=k\}} \\ &= -\xi_{11}(t)^2 \mathbb{1}_{\{\xi_{11}(t) < 0\}} \det^2(\xi_{-1}(t)) \mathbb{1}_{\{i(\xi_{-1}(t))=k\}} + O_p(\rho^3), \end{aligned}$$

giving

$$\begin{aligned} & \mathbb{E} \left( |\det(\xi(0)) \det(\xi(t))| \mathbb{1}_{\{\xi_{11}(0) > 0, \xi_{11}(t) < 0\}} \mathbb{1}_{\{i(\xi_{-1}(t)) = i(\xi_{-1}(0)) = k\}} \right) \\ & = 1/2 \operatorname{Var}(\xi_{11}(t)) \mathbb{E} \left( \det^2(\xi_{-1}(t)) \mathbb{1}_{\{i(\xi_{-1}(t)) = k\}} \right) + O(\rho^3). \end{aligned}$$

This gives (52) as soon as we have checked that the three other cases give smaller contributions which is direct.  $\square$

4.5.3. *Correlation function between critical points with extreme indexes.* First, we give a bound for the correlation function between maxima and minima (Theorem 4.11) then a bound for the correlation function between maxima (or between minima) (Theorem 4.12).

**THEOREM 4.11.** *Let  $\mathcal{X} = \{X(t) : t \in \mathbb{R}^N, N > 1\}$  be a stationary and isotropic Gaussian field with real values satisfying Assumption (A3). Let  $A^{0,N}(\rho)$  be the correlation function (32) between minima and maxima. Then for all  $\epsilon > 0$  there exists a positive constant  $K(\epsilon) > 0$  such that as  $\rho \rightarrow 0$ ,*

$$(55) \quad A^{0,N}(\rho) \leq K(\epsilon) \times \rho^{2N-1-\epsilon}.$$

Upper bound (55) proves strong repulsion between maxima and minima.

**THEOREM 4.12.** *Let  $\mathcal{X} = \{X(t) : t \in \mathbb{R}^N, N > 1\}$  be a stationary and isotropic Gaussian field with real values satisfying Assumption (A4). Let  $A^{N,N}(\rho)$  (respectively  $A^{0,0}(\rho)$ ) be the correlation function (32) between maxima (respectively minima). Then  $\forall \epsilon > 0$  there exists a constant  $\tilde{K}(\epsilon) > 0$  such that as  $\rho \rightarrow 0$ ,*

$$(56) \quad A^{0,0}(\rho) = A^{N,N}(\rho) \leq \tilde{K}(\epsilon) \times \rho^{5-N-\epsilon}.$$

Upper bound (56) proves repulsion between maxima (or minima) for  $N < 5$ .

In the particular case  $N = 1$ , note that, strictly speaking, Proposition 4.4 gives no equivalent of Theorem 4.11. In fact, since a maxima is followed by a minima and reciprocally it is easy to deduce from (42) that we have (55) with  $\epsilon = 0$ . On the other hand (43) implies that the result of Theorem 4.12 holds true with  $\epsilon = 0$ .

As this work was achieved we have discovered the work [12] which is similar to the result of this section but limited to the dimension  $N = 2$ . In this last particular case Theorem 4.6 is the analogue of Theorem 1.2 of [12]; Theorem 4.9 has no equivalent in [12]; Theorems 4.11 and 4.12 are similar but a little weaker than Theorems 1.3 and 1.4 of [12] that are obtained by a difficult diagonalization of the Hessian which is absolutely impossible in the case  $N > 2$ .

**PROOF OF THEOREM 4.11.** We use Hadamard's inequality. If  $M$  is a positive semidefinite matrix of size  $N$

$$\det(M) \leq M_{11} \times \cdots \times M_{NN}.$$

As a consequence for any symmetric matrix  $M'$

$$(57) \quad \det(M') \mathbb{1}_{\{i(M')=0\}} \leq (M'_{11})^+ \times \cdots \times (M'_{NN})^+.$$

By Kac-Rice formula (32)

$$A^{0,N}(\rho) = p_{\nabla X(0), \nabla X(t)}(0, 0) \mathbb{E} \left( |\det \xi(0)| \mathbb{1}_{\{i(\nabla^2 \xi(0))=0\}} |\det \xi(t)| \mathbb{1}_{\{i(\nabla^2 \xi(t))=N\}} \right).$$

We have

$$A^{0,N}(\rho) \leq (-1)^N p_{\nabla X(0), \nabla X(t)}(0, 0) \mathbb{E} (\xi_{11}^+(0) \xi_{22}^+(0) \dots \xi_{NN}^+(0) \xi_{11}^-(t) \xi_{22}^-(t) \dots \xi_{NN}^-(t)).$$

By Cauchy-Schwarz inequality and symmetry of the role of 0 and  $t$  we obtain

$$A^{0,N}(\rho) \leq (-1)^{N-1} p_{\nabla X(0), \nabla X(t)}(0, 0) \mathbb{E} (\xi_{11}^+(0)^2 \xi_{22}^+(0) \dots \xi_{NN}^+(0) \xi_{22}^-(t) \dots \xi_{NN}^-(t)).$$

By Hölder's inequality for every  $p > 1$

$$A^{0,N}(\rho) \leq p_{\nabla X(0), \nabla X(t)}(0, 0) \times I_\rho \times \left( \mathbb{E} \left( \xi_{11}^+(0)^{2p/(p-1)} \right) \right)^{(p-1)/p}.$$

where  $I_\rho := \left( \mathbb{E} (|\xi_{22}^+(0) \dots \xi_{NN}^+(0) \xi_{22}^-(t) \dots \xi_{NN}^-(t)|^p) \right)^{1/p}$ . Since  $\xi_{11}(0)$  is centred Gaussian with variance  $\rho^2 \frac{\lambda_2 \lambda_6 - \lambda_4^2}{4\lambda_2}$ , there exists a constant  $K(p) > 0$  depending on  $p$  such that

$$\left( \mathbb{E} \left( \xi_{11}^+(0)^{2p/(p-1)} \right) \right)^{(p-1)/p} \leq \rho^2 K(p).$$

By (47) we have  $p_{\nabla X(0), \nabla X(t)}(0, 0) = O(\rho^{-N})$ . Now let us prove that

$$(58) \quad I_\rho = O \left( \rho^{(N-1)(2+1/p)} \right).$$

We set  $\tilde{\xi} := (\xi_{22}(0), \dots, \xi_{NN}(0), \xi_{22}(t), \dots, \xi_{NN}(t))$ ,  $\tilde{\xi}_i$  the  $i$ th coordinate of  $\tilde{\xi}$  for  $i = 1, \dots, 2N-2$ ,  $\xi_{-i}$  (resp.  $\xi_{-(i,j)}$ ) the vector  $\xi$  without its  $i$ th (resp.  $i$ th and  $j$ th) coordinate(s). We set  $\Sigma_\rho := \text{Var}(\tilde{\xi})$ . We define  $\mathcal{S} = \{x \in \mathbb{R}^{2N-2} : x_1 > 0, \dots, x_{N-1} > 0, x_N < 0, \dots, x_{2N-2} < 0\}$ . Then

$$(59) \quad I_\rho = \left( \int_{\mathcal{S}} \frac{|\tilde{\xi}_1 \dots \tilde{\xi}_{2N-2}|^p}{(2\pi)^{(2N-2)/2} \sqrt{\det \Sigma_\rho}} \exp \left( -\frac{1}{2} \tilde{\xi} \Sigma_\rho^{-1} \tilde{\xi}^\top \right) d\tilde{\xi} \right)^{1/p}.$$

• Equivalent of  $\det(\Sigma_\rho)$  as  $\rho \rightarrow 0$

For short we set  $X_i := X_i(0)$  and  $X_{ij} := X_{ij}(0)$ .

$$\det \Sigma_\rho = \frac{\det \text{Var}(X_{22}, \dots, X_{NN}, X_{22}(t), \dots, X_{NN}(t), X_1, \dots, X_N, X_1(t), \dots, X_N(t))}{\det \text{Var}(X_1, \dots, X_N, X_1(t), \dots, X_N(t))}.$$

Using Trick 4.3, we obtain as  $\rho \rightarrow 0$

$$\det \Sigma_\rho \simeq \rho^{2(N-1)} \frac{\det \text{Var}(X_{22}, \dots, X_{NN}, X_{122}, \dots, X_{1NN}, X_1, \dots, X_N, X_{11}, \dots, X_{1N})}{\det \text{Var}(X_1, \dots, X_N, X_{11}, \dots, X_{1N})}.$$

Using Lemma 2.3, (5) and (6) we get, as  $\rho \rightarrow 0$

$$\begin{aligned} \det \Sigma_\rho &\simeq \rho^{2(N-1)} \frac{\det \text{Var}(X_{11}, \dots, X_{NN}) \det \text{Var}(X_1, X_{122}, \dots, X_{1NN})}{\text{Var}(X_1) \text{Var}(X_{11})} \\ &\simeq \rho^{2(N-1)} (N+2) \left( \frac{2\lambda_4}{3} \right)^{N-1} \left( \frac{2\lambda_6}{15} \right)^{N-2} \left( \frac{3(N+1)\lambda_2\lambda_6 - 5(N-1)\lambda_4^2}{135\lambda_2} \right). \end{aligned}$$

So

$$\rho^{2(1-N)} \det \Sigma_\rho \simeq g := (N+2) \left( \frac{2\lambda_4}{3} \right)^{N-1} \left( \frac{2\lambda_6}{15} \right)^{N-2} \left( \frac{3(N+1)\lambda_2\lambda_6 - 5(N-1)\lambda_4^2}{135\lambda_2} \right).$$

By (3) we have  $\lambda_2\lambda_6 \geq \frac{5(2+N)}{3(4+N)}\lambda_4^2$ . Since  $\frac{5(N-1)}{3(N+1)} < \frac{5(2+N)}{3(4+N)}$  we deduce that  $g > 0$  under our hypotheses.

- Equivalent of  $\Sigma_\rho^{-1}$  as  $\rho \rightarrow 0$ .

Because of the exchangeability of the coordinates  $2, \dots, N$

$$\Sigma_\rho^{-1} := \left( \begin{array}{c|c} \alpha_\rho Id_{N-1} + \beta_\rho J_{N-1, N-1} & \gamma_\rho Id_{N-1} + \delta_\rho J_{N-1, N-1} \\ \hline \gamma_\rho Id_{N-1} + \delta_\rho J_{N-1, N-1} & \alpha_\rho Id_{N-1} + \beta_\rho J_{N-1, N-1} \end{array} \right).$$

Using the classical expression of conditional variance and covariance

$$\begin{aligned} (\Sigma_\rho^{-1})_{ii} &= \frac{1}{\text{Var}(\tilde{\xi}_i/\tilde{\xi}_{-i})} = \frac{\det \text{Var}(\tilde{\xi}_{-i})}{\det \text{Var}(\tilde{\xi})} \\ (\Sigma_\rho^{-1})_{ij} &= \frac{-\text{Cov}(\tilde{\xi}_i, \tilde{\xi}_j/\tilde{\xi}_{-(i,j)})}{\det \text{Var}(\tilde{\xi}_i, \tilde{\xi}_j/\tilde{\xi}_{-(i,j)})} = \frac{\text{Var}(\tilde{\xi}_i/\tilde{\xi}_{-(i,j)}) + \text{Var}(\tilde{\xi}_j/\tilde{\xi}_{-(i,j)}) - \text{Var}(\tilde{\xi}_i + \tilde{\xi}_j/\tilde{\xi}_{-(i,j)})}{2 \det \text{Var}(\tilde{\xi}_i, \tilde{\xi}_j/\tilde{\xi}_{-(i,j)})} \\ &= \frac{\det \text{Var}(\tilde{\xi}_{-j}) + \det \text{Var}(\tilde{\xi}_{-i}) - \det \text{Var}(\tilde{\xi}_i + \tilde{\xi}_j, \tilde{\xi}_{-(i,j)})}{2 \det \text{Var}(\tilde{\xi})}, \end{aligned}$$

and using the same techniques as those previously used for the calculation of the equivalent of  $\det(\Sigma_\rho)$  and for the proof of Lemma 4.2, we obtain the equivalents of  $(\Sigma_\rho^{-1})_{11}$ ,  $(\Sigma_\rho^{-1})_{12}$ ,  $(\Sigma_\rho^{-1})_{1N}$  and  $(\Sigma_\rho^{-1})_{1(N+1)}$ .

We get  $\rho^2\alpha_\rho \simeq \alpha := \frac{15}{2\lambda_6}$ ,  $\rho^2\beta_\rho \simeq \beta := \frac{15}{2\lambda_6} \frac{5\lambda_4^2 - 3\lambda_2\lambda_6}{3(N+1)\lambda_2\lambda_6 - 5(N-1)\lambda_4^2}$ ,  $\rho^2\gamma_\rho \simeq -\alpha$  and  $\rho^2\delta_\rho \simeq -\beta$ . So  $\rho^2\Sigma_\rho^{-1}$  converges to

$$\tilde{\Sigma} := \left( \begin{array}{c|c} \alpha Id_{N-1} + \beta J_{N-1, N-1} & -\alpha Id_{N-1} - \beta J_{N-1, N-1} \\ \hline -\alpha Id_{N-1} - \beta J_{N-1, N-1} & \alpha Id_{N-1} + \beta J_{N-1, N-1} \end{array} \right)$$

where the matrix  $\alpha Id_{N-1} + \beta J_{N-1, N-1}$  is positive definite under our conditions. Indeed by (7)

$$\det(\alpha Id_n + \beta J_{n,n}) = \alpha^{n-1}(\alpha + n\beta),$$

and we have  $\alpha > 0$  and

$$\alpha + n\beta = \frac{15}{2\lambda_6} \frac{3(N+1-n)\lambda_2\lambda_6 - 5(N-1-n)\lambda_4^2}{3(N+1)\lambda_2\lambda_6 - 5(N-1)\lambda_4^2} > 0 \forall n = 1, \dots, N-1$$

under our conditions.

- Formal finite limit

We make the change of variables  $\tilde{\xi} = \rho z$  in (59) to obtain

$$(60) \quad I_\rho = \rho^{(2N-2)+(N-1)/p} \times \left( \int_{\mathcal{S}} \frac{|z_1 \dots z_{2N-2}|^p}{(2\pi)^{(2N-2)/2} \sqrt{\rho^{2(1-N)} \det \Sigma_\rho}} \exp\left(-\frac{1}{2} z (\rho^2 \Sigma_\rho^{-1}) z^\top\right) dz \right)^{1/p}.$$



We set  $\bar{z}_1 := (z_1, \dots, z_{N-1})$ ,  $\bar{z}_2 := (z_N, \dots, z_{2N-2})$  and

$$f_\rho(z) := \frac{|z_1 \dots z_{2N-2}|^p}{(2\pi)^{(2N-2)/2} \sqrt{\rho^{2(1-N)} \det \Sigma_\rho}} \exp\left(-\frac{1}{2} z (\rho^2 \Sigma_\rho^{-1}) z^\top\right).$$

As  $\rho \rightarrow 0$ ,  $f_\rho(z)$  converges to  $\frac{|z_1 \dots z_{2N-2}|^p}{(2\pi)^{(2N-2)/2} \sqrt{g}} \exp\left(-\frac{1}{2} z \tilde{\Sigma} z^\top\right)$  equals to

$$\frac{|z_1 \dots z_{2N-2}|^p}{(2\pi)^{(2N-2)/2} \sqrt{g}} \exp\left(-\frac{1}{2} (\bar{z}_1 - \bar{z}_2) (\alpha Id_{N-1} + \beta J_{N-1, N-1}) (\bar{z}_1 - \bar{z}_2)^\top\right)$$

that is integrable on  $\mathcal{S}$  since  $g > 0$  and  $\alpha Id_{N-1} + \beta J_{N-1, N-1}$  is positive definite.

• Domination

Let us choose  $g_0$ ,  $\alpha_0$  and  $\beta_0$  such that  $0 < g_0 < g$ ,  $0 < \alpha_0 < \alpha$  and  $\beta_0 < \beta$  such that  $0 < \alpha_0 + n\beta_0 \forall n = 1, \dots, N-1$  which is always possible. Then for  $\rho$  sufficiently small (61)

$$f_\rho(z) \leq \frac{|z_1 \dots z_{2N-2}|^p}{(2\pi)^{(2N-2)/2} \sqrt{g_0}} \exp\left(-\frac{1}{2} (\bar{z}_1 - \bar{z}_2) (\alpha_0 Id_{N-1} + \beta_0 J_{N-1, N-1}) (\bar{z}_1 - \bar{z}_2)^\top\right).$$

Since  $\alpha_0 > 0$  and  $\alpha_0 + n\beta_0 > 0 \forall n = 1, \dots, N-1$ ,  $\alpha_0 Id_{N-1} + \beta_0 J_{N-1, N-1}$  is positive definite and the right hand side of (61) is integrable on  $\mathcal{S}$ . Finally we conclude the proof taking  $p = 1 - \epsilon$ .  $\square$

PROOF OF THEOREM 4.12. We follow the same lines as in the proof of Theorem 4.11. By Kac-Rice formula (32):

$$A^{0,0}(\rho) = p_{\nabla X(0), \nabla X(t)}(0, 0) \mathbb{E}(|\det \xi(0)| \mathbb{1}_{\{i(\nabla^2 \xi(0))=0\}} |\det \xi(t)| \mathbb{1}_{\{i(\nabla^2 \xi(t))=0\}}).$$

Using (57), Cauchy-Schwarz inequality and symmetry of the role of 0 and  $t$  we get

$$A^{0,0}(\rho) \leq p_{\nabla X(0), \nabla X(t)}(0, 0) \mathbb{E}(\xi_{11}^+(0) \xi_{11}^+(t) \xi_{22}^+(0)^2 \dots \xi_{NN}^+(0)^2).$$

By (47) we have  $p_{\nabla X(0), \nabla X(t)}(0, 0) = O(\rho^{-N})$ . Now let us prove that

$$(62) \quad \check{I}_\rho := \mathbb{E}(\xi_{11}^+(0) \xi_{11}^+(t) \xi_{22}^+(0)^2 \dots \xi_{NN}^+(0)^2) = O(\rho^{4+1/p}).$$

By Hölder's inequality for every  $p > 1$

$$(63) \quad \check{I}_\rho \leq \left(\mathbb{E}(\xi_{11}^+(0)^p \xi_{11}^+(t)^p)\right)^{1/p} \left(\mathbb{E}(\xi_{22}^+(0)^{2p/(p-1)} \dots \xi_{NN}^+(0)^{2p/(p-1)})\right)^{(p-1)/p}.$$

Since  $\text{Cov}(\xi_{11}(0), \xi_{11}(t)) \simeq -\text{Var}(\xi_{11}(t))$  (see Lemma 4.2), by Lemma 4.5

$$(64) \quad \mathbb{E}(\xi_{11}^+(0)^p \xi_{11}^+(t)^p) \simeq (\text{const}) \frac{1}{(\text{Var}(\xi_{11}(t)))^{1+p}} (\det(\text{Var}(\xi_{11}(0), \xi_{11}(t))))^{p+1/2}.$$

By Lemma 4.2 we have

$$\begin{aligned} \text{Var}(\xi_{11}(t)) &\simeq \frac{\rho^2 (\lambda_2 \lambda_6 - \lambda_4^2)}{4 \lambda_2}, \\ \det(\text{Var}(\xi_{11}(0), \xi_{11}(t))) &\simeq \frac{\rho^6 (\lambda_2 \lambda_6 - \lambda_4^2) (\lambda_4 \lambda_8 - \lambda_6^2)}{144 \lambda_2 \lambda_4}, \end{aligned}$$

and by (64) we deduce

$$(65) \quad \left( \mathbb{E} \left( \xi_{11}^+(0)^p \xi_{11}^+(t)^p \right) \right)^{1/p} \simeq (\text{const}) \rho^{4+1/p}.$$

We now consider the term  $\left( \mathbb{E} \left( \xi_{22}^+(0)^{2p/(p-1)} \dots \xi_{NN}^+(0)^{2p/(p-1)} \right) \right)^{(p-1)/p}$  in (63).

We set  $\check{\xi} := (\xi_{22}(0), \xi_{33}(0), \dots, \xi_{NN}(0))$ .  $\check{\xi}$  is a Gaussian vector centred with variance-covariance matrix

$$\text{Var}(\check{\xi}) = (a_\rho I_{N-1} + b_\rho J_{N-1, N-1})$$

explicitly given in Lemma 4.1. According to Lemma 4.2, as  $\rho \rightarrow 0$ ,

$$a_\rho \rightarrow \frac{2\lambda_4}{3} \text{ and } b_\rho \rightarrow \frac{2\lambda_4}{9}.$$

Using Lebesgue's dominated convergence theorem we can deduce that there exists a constant  $C(p) > 0$  such that, as  $\rho \rightarrow 0$

$$(66) \quad \mathbb{E} \left( \xi_{22}^+(0)^{2p/(p-1)} \dots \xi_{NN}^+(0)^{2p/(p-1)} \right) \simeq C(p).$$

(66) and (65) give (62). Finally we conclude the proof taking  $p = 1 - \epsilon$ .  $\square$

#### APPENDIX A: VALIDITY OF KAC-RICE FORMULAS FOR STATIONARY GAUSSIAN PROCESSES AND FIELDS

Let  $\mathcal{X} = \{X(t) : t \in U \subset \mathbb{R}^N\}$  be a zero-mean, stationary Gaussian random field defined on an open set  $U \subset \mathbb{R}^N$ . We follow chapter 6 of [9]. So we need  $\nabla X(\cdot)$  to be  $C^1$ . Recall that  $\mathcal{N}^c(S)$ ,  $\mathcal{N}_k^c(S)$  and  $\mathcal{N}_k^c(u, S)$  are respectively the number of critical points, the number of critical points with index  $k$  and the number of critical points with index  $k$  above the level  $u$  of the random field  $X(\cdot)$  in a Borel set  $S \subset U$ .

**PROPOSITION A.1.** *Let  $\mathcal{X} = \{X(t) : t \in U \subset \mathbb{R}^N\}$  be a zero-mean, stationary Gaussian random field defined on an open set  $U \subset \mathbb{R}^N$ . Suppose that  $\mathcal{X}$  is  $C^2$  and that the distribution of  $\nabla^2 X(t)$  does not degenerate, then the Kac-Rice formula of order one is always true in the sense that for every Borel set  $S \subset U$*

$$(67) \quad \mathbb{E}(\mathcal{N}^c(S)) = |S| p_{\nabla X(t)}(0) \mathbb{E}(|\det(\nabla^2 X(t))|)$$

$$(68) \quad \mathbb{E}(\mathcal{N}_k^c(S)) = |S| p_{\nabla X(t)}(0) \mathbb{E}(|\det(\nabla^2 X(t))| \mathbb{1}_{i(\nabla^2 X(t))=k})$$

$$(69) \quad \mathbb{E}(\mathcal{N}_k^c(u, S)) = |S| p_{\nabla X(t)}(0) \times \int_u^{+\infty} p_{X(t)}(x) \mathbb{E}(|\det(\nabla^2 X(t))| \mathbb{1}_{i(\nabla^2 X(t))=k} / X(t) = x).$$

*In addition the sample paths are a.s. Morse functions.*

Note that Lemma 2.3 implies that the conditions of Proposition A.1 are met for a random field satisfying (A2).

**REMARK A.2.** In the particular case  $N = 1$ , the Kac-Rice formulas (67), (68) and (69) are always true (both sides are equal, finite or infinite) when  $\mathcal{X}$  is a zero-mean stationary Gaussian process with  $C^1$  covariance function. This can be proved as in [21]. These conditions are weaker than those in Proposition A.1.

PROOF OF PROPOSITION A.1. We first consider the mean number of critical points. We need to apply Theorem 6.2 of [9]. The conditions are clearly verified in our case except for the condition (iv) :

$$P\{\exists t \in S : \nabla X(t) = 0, \det(\nabla^2 X(t)) = 0\} = 0.$$

This last condition is equivalent to the fact that the sample paths are a.s. Morse functions. In the case of stationary random fields the simplest is to use Proposition 6.5 of [9] with condition b). Note that because of stationarity, by an extension argument, we can get rid of the compactness condition. Since  $\nabla X(t)$  and  $\nabla^2 X(t)$  are independent, this condition is equivalent to

$$P\{|\det(\nabla^2 X(t))| < \delta\} \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

which is equivalent to

$$P\{|\det(\nabla^2 X(t))| = 0\} = 0.$$

This is implied by assuming the non degeneracy of  $\nabla^2 X(t)$ .

We consider now the case of Kac-Rice formula for the mean number of critical points with a given index and above the level  $u$ . We have to apply Theorem 6.4 of [9]. More precisely we have to write their formula (6.6) with

$$g(t, Y^t) = g(\nabla^2 X(t), X(t)),$$

here the process  $Y^t$  is simply the Gaussian vector  $\nabla^2 X(t), X(t)$  which is Gaussian as required in Theorem 6.4 of [9]. On the other hand,  $g$  is the indicator of the index multiplied by the indicator function that  $X(t) > u$ . It is not continuous and we must apply an approximation with a continuous function, followed by a monotone convergence argument. Then, since we have independence between  $X(t)$  and  $\nabla X(t)$  and between  $\nabla X(t)$  and  $\nabla^2 X(t)$  (see Lemma 2.3), we obtain (68) and (69).  $\square$

For simplification we limit our second proposition to the isotropic case

PROPOSITION A.3. *Under Assumption (A2), the Kac-Rice formula of order 2 is valid for a sufficiently small set  $S$  in the sense that if  $S_1$  and  $S_2$  are sufficiently small and sufficiently close to each other,*

$$(70) \quad E(\mathcal{N}^c(S_1)\mathcal{N}^c(S_2)) = \int_{S_1 \times S_2} dt_1 dt_2 p_{\nabla X(t_1)\nabla X(t_2)}(0, 0) \\ E(|\det(\nabla^2 X(t_1)) \det(\nabla^2 X(t_2))| / \nabla X(t_1) = \nabla X(t_2) = 0)$$

$$(71) \quad E(\mathcal{N}_k^c(S_1)\mathcal{N}_{k'}^c(S_2)) = \int_{S_1 \times S_2} dt_1 dt_2 p_{\nabla X(t_1)\nabla X(t_2)}(0, 0) \\ E(|\det(\nabla^2 X(t_1)) \mathbb{1}_{i(\nabla^2 X(t_1))=k} \det(\nabla^2 X(t_2)) \mathbb{1}_{i(\nabla^2 X(t_2))=k'}| / \nabla X(t_1) = \nabla X(t_2) = 0)$$

PROOF. We need to apply Theorem 6.3 from [9]. With respect to Proposition A.1, we need in addition the distribution of  $\nabla X(0), \nabla X(\rho e_1)$  to be non degenerated in  $\mathbb{R}^{2N}$  (Condition (iii') of [9]). From section B.2.1.1 we have

$$\text{Var}(\nabla X(0)) = \text{Var}(\nabla X(\rho e_1)) = \lambda_2 I d_N$$

$$E(\nabla X(0)\nabla X(\rho e_1)^T) = \text{diag}\left(-2\mathbf{r}'(\rho^2) - 4\rho^2 \mathbf{r}''(\rho^2), -2\mathbf{r}'(\rho^2), \dots, -2\mathbf{r}'(\rho^2)\right),$$

which implies that the  $N$  vectors  $(X_i(0), X_i(\rho e_1))$ ,  $i = 1, \dots, N$  are independent. It remains to check that the correlation between  $X_i(0)$  and  $X_i(\rho e_1)$  cannot be 1. This is done by the Taylor expansions

$$\begin{aligned} -2\mathbf{r}'(\rho^2) &= -2\mathbf{r}'(0) - 2\rho^2\mathbf{r}''(0) + o(\rho^2) \\ -2\mathbf{r}'(\rho^2) - 4\rho^2\mathbf{r}''(\rho^2) &= -2\mathbf{r}'(0) - 6\rho^2\mathbf{r}''(0) + o(\rho^2). \end{aligned}$$

Since by hypothesis  $0 < \mathbf{r}''(0) = \lambda_4/12 < +\infty$  the result follows.

We consider now the case of Kac-Rice formula for the mean number of critical points with a given index. We proceed using Theorem 6.4 of [9] as in the preceding case.  $\square$

## APPENDIX B: PROOFS OF LEMMAS

**B.1. Proof of Lemma 2.1.** We start by proving (3). The spectral moment of order  $2p$  has the following expressions

$$(72) \quad \lambda_{2p} = \int_{\mathbb{R}^N} \langle e_1, \lambda \rangle^{2p} dF(\lambda) = \int_0^{+\infty} x^{2p} dG(x) \int_{S^{N-1}} (\langle e_1, u \rangle)^{2p} d\sigma(u),$$

where  $F$  is the spectral measure;  $\sigma$  is the uniform probability on the sphere  $S^{N-1}$  and  $G$  is a measure deduced from  $F$  by change in polar coordinates. Using the fact that a standard normal variable is the independent product of a uniform variable on the sphere by a  $\chi(N)$  distribution and the fact that those two distributions have well known moments, we obtain

$$\mu_{2p} := \int_{S^{N-1}} (\langle e_1, u \rangle)^{2p} d\sigma(u) = \frac{(2p-1)!!\Gamma(N/2)}{2^p\Gamma(N/2+p)} = \frac{\Gamma(N/2)\Gamma(1/2+p)}{\sqrt{\pi}\Gamma(N/2+p)}.$$

We have

$$\mu_{2n}\mu_{2n-4} = \frac{(2n-4+N)(2n-1)}{(2n-2+N)(2n-3)}\mu_{2n-2}^2.$$

Using Cauchy-Schwarz inequality we conclude that

$$\begin{aligned} \lambda_{2n}\lambda_{2n-4} &= \mu_{2n}\mu_{2n-4} \int_0^{+\infty} x^{2n} dG(x) \int_0^{+\infty} y^{2n-4} dG(y) \\ &= K(n, N)\mu_{2n-2}^2 \int_0^{+\infty} x^{2n} dG(x) \int_0^{+\infty} y^{2n-4} dG(y) \\ &\geq K(n, N)\mu_{2n-2}^2 \int_0^{+\infty} x^{2n-2} dG(x) \int_0^{+\infty} y^{2n-2} dG(y) = K(n, N)\lambda_{2n-2}^2 \end{aligned}$$

giving (3). Now let us prove (4). By (72), see also [31] and [35],  $\mathbf{r}(\|s-t\|^2)$  can be written as

$$(73) \quad \begin{aligned} \mathbf{r}(\|s-t\|^2) &= \int_0^{+\infty} dG(x) \int_{S^{N-1}} \cos(\langle \|s-t\|e_1, xu \rangle) d\sigma(u) \\ &= \int_0^{+\infty} \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k} (\|s-t\|^2)^k}{2^{2k+(N-2)/2} k! \Gamma(k+1+(N-2)/2)} dG(x). \end{aligned}$$

Thus setting  $|\beta| = \frac{|i|+|j|}{2}$  and  $\beta_k = \frac{i_k+j_k}{2}$  for  $k = 1, \dots, N$ , we have

$$\left. \frac{\partial^{|i|+|j|} \mathbf{r}(\|s-t\|^2)}{\partial s_1^{i_1}, \dots, \partial s_N^{i_N} \partial t_1^{j_1}, \dots, \partial t_N^{j_N}} \right|_{s=t} = \int_0^{+\infty} \frac{(-1)^{|\beta|} x^{2|\beta|}}{2^{2|\beta|+(N-2)/2} |\beta|! \Gamma(|\beta|+1+(N-2)/2)}$$

$$\times \frac{\partial^{|i|+|j|} (\|s-t\|^2)^{|\beta|}}{\partial s_1^{i_1}, \dots, \partial s_N^{i_N} \partial t_1^{j_1}, \dots, \partial t_N^{j_N}} \Big|_{s=t} dG(x).$$

By the multinomial formula

$$(74) \quad (\|s-t\|^2)^{|\beta|} = \sum_{k_1+\dots+k_N=|\beta|} \frac{|\beta|!}{k_1! \times \dots \times k_N!} ((s_1-t_1)^2)^{k_1} \times \dots \times ((s_N-t_N)^2)^{k_N}.$$

Thus

$$\frac{\partial^{|i|+|j|} (\|s-t\|^2)^{|\beta|}}{\partial s_1^{i_1}, \dots, \partial s_N^{i_N} \partial t_1^{j_1}, \dots, \partial t_N^{j_N}} \Big|_{s=t} = (-1)^{|j|} |\beta|! \prod_{k=1}^N \frac{(2\beta_k)!}{\beta_k!},$$

since the only term in the sum in the right side of (74) whose derivative is not null is the one for which  $k_1 = \frac{i_1+j_1}{2}, \dots, k_N = \frac{i_N+j_N}{2}$ . We obtain

$$\begin{aligned} \frac{\partial^{|i|+|j|} \mathbf{r}(\|s-t\|^2)}{\partial s_1^{i_1}, \dots, \partial s_N^{i_N} \partial t_1^{j_1}, \dots, \partial t_N^{j_N}} \Big|_{s=t} &= \frac{(-1)^{|\beta|} (-1)^{|j|} |\beta|!}{2^{2|\beta|+(N-2)/2} |\beta|! \Gamma(|\beta|+1+(N-2)/2)} \\ &\quad \times \prod_{k=1}^N \frac{(2\beta_k)!}{\beta_k!} \int_0^{+\infty} x^{2|\beta|} dG(x). \end{aligned}$$

Furthermore, using (73), we have

$$\mathbf{r}^{(|\beta|)}(0) = \frac{(-1)^{|\beta|} |\beta|!}{2^{2|\beta|+(N-2)/2} |\beta|! \Gamma(|\beta|+1+(N-2)/2)} \int_0^{+\infty} x^{2|\beta|} dG(x).$$

That gives the first equality in (4). The second equality is obtained using (2).

**B.2. Proof of Lemma 4.1.** We give the steps for the computation of the conditional variance of  $\nabla^2 X(0)$ ,  $\nabla^2 X(t)$  given  $\nabla X(0) = \nabla X(t) = 0$  with  $t = \rho e_1$ . Some tedious but easy calculations are not detailed.  $\mathbf{r}(\rho^2)$ ,  $\mathbf{r}'(\rho^2)$ ,  $\mathbf{r}''(\rho^2)$ ,  $\mathbf{r}'''(\rho^2)$  and  $\mathbf{r}^{(4)}(\rho^2)$  are denoted by  $\mathbf{r}$ ,  $\mathbf{r}'$ ,  $\mathbf{r}''$ ,  $\mathbf{r}'''$  and  $\mathbf{r}^{(4)}$  for short. The results below extend Lemma 2.1 to two times.

### B.2.1. Unconditional distribution.

#### B.2.1.1. Gradient.

$$\text{Var}(\nabla X(t)) = \text{Var}(\nabla X(0)) = -2\mathbf{r}'(0)Id_N,$$

$$\text{Cov}(X_1(0), X_1(t)) = -2\mathbf{r}' - 4\rho^2\mathbf{r}'' \text{ and } \text{Cov}(X_i(0), X_i(t)) = -2\mathbf{r}' \text{ for } i \neq 1.$$

Any other covariance is zero.

**B.2.1.2. Hessian.** We define  $X_d''(t)$  as  $(X_{11}(t), \dots, X_{NN}(t))$  and  $X_u''(t)$  as  $\{X_{ij}(t), 1 \leq i < j \leq N\}$ . These two vectors are independent and

$$\text{Var}(X_d''(0)) = 4\mathbf{r}''(0)(2Id_N + J_{N,N}),$$

$$\text{Cov}(X_d''(0), X_d''(t)) = \begin{pmatrix} 12\mathbf{r}'' + 48\rho^2\mathbf{r}''' + 16\rho^4\mathbf{r}^{(4)} & (4\mathbf{r}'' + 8\rho^2\mathbf{r}''')J_{1,(N-1)} \\ (4\mathbf{r}'' + 8\rho^2\mathbf{r}''')J_{(N-1),1} & 4\mathbf{r}''(2Id_{N-1} + J_{(N-1),(N-1)}) \end{pmatrix},$$

$$\text{Var}(X_u''(0)) = 4\mathbf{r}''(0)Id_{N(N-1)/2},$$

$$\text{Cov}(X_u''(0), X_u''(t)) = \text{diag}((4\mathbf{r}'' + 8\rho^2\mathbf{r}''')Id_{N-1}, 4\mathbf{r}''Id_{(N-1)(N-2)/2}).$$

*B.2.1.3. Relation between gradient and Hessian.*

$$\begin{aligned}
& \mathbb{E}(X_{11}(0)/X_1(0), X_1(t)) \\
&= \frac{1}{4\mathbf{r}'(0)^2 - (2\mathbf{r}' + 4\rho^2\mathbf{r}'')^2} (0, 12\rho\mathbf{r}'' + 8\rho^3\mathbf{r}''') \begin{pmatrix} -2\mathbf{r}'(0) & 2\mathbf{r}' + 4\rho^2\mathbf{r}'' \\ 2\mathbf{r}' + 4\rho^2\mathbf{r}'' & -2\mathbf{r}'(0) \end{pmatrix} \begin{pmatrix} X_1(0) \\ X_1(t) \end{pmatrix} \\
&= (12\rho\mathbf{r}'' + 8\rho^3\mathbf{r}''') \frac{(2\mathbf{r}' + 4\rho^2\mathbf{r}'')X_1(0) - 2\mathbf{r}'(0)X_1(t)}{4\mathbf{r}'(0)^2 - (2\mathbf{r}' + 4\rho^2\mathbf{r}'')^2} \\
&=: (12\rho\mathbf{r}'' + 8\rho^3\mathbf{r}''')K_1(t).
\end{aligned}$$

For  $i \neq 1$

$$\begin{aligned}
\mathbb{E}(X_{1i}(0)/X_i(0), X_i(t)) &= \frac{1}{4\mathbf{r}'(0)^2 - 4(\mathbf{r}')^2} (0, 4\rho\mathbf{r}'') \begin{pmatrix} -2\mathbf{r}'(0) & 2\mathbf{r}' \\ 2\mathbf{r}' & -2\mathbf{r}'(0) \end{pmatrix} \begin{pmatrix} X_i(0) \\ X_i(t) \end{pmatrix} \\
&= 4\rho\mathbf{r}'' \frac{2\mathbf{r}'X_i(0) - 2\mathbf{r}'(0)X_i(t)}{4\mathbf{r}'(0)^2 - 4(\mathbf{r}')^2} =: 4\rho\mathbf{r}''K_i(t).
\end{aligned}$$

$$\mathbb{E}(X_{ii}(0)/X_1(0), X_1(t)) = 4\rho\mathbf{r}''K_1(t).$$

We have equivalent formulas, reversing time, for example:

$$\begin{aligned}
\mathbb{E}(X_{11}(t)/X_1(t), X_1(0)) &= (12\rho\mathbf{r}'' + 8\rho^3\mathbf{r}''') \frac{-(2\mathbf{r}' + 4\rho^2\mathbf{r}'')X_1(t) + 2\mathbf{r}'(0)X_1(0)}{4(\mathbf{r}'(0))^2 - (2\mathbf{r}' + 4\rho^2\mathbf{r}'')^2} \\
&=: (12\rho\mathbf{r}'' + 8\rho^3\mathbf{r}''')\bar{K}_1(t).
\end{aligned}$$

Any other case corresponds to independence between gradient and Hessian.

*B.2.2. Conditional distribution.* Since the conditional variance-covariance matrix is equal to the unconditional variance-covariance matrix diminished by the variance of the conditional expectation, we compute this last term only.

Let us consider for example the two terms  $X_{11}(0)$  and  $X_{11}(t)$  given  $\nabla X(0) = \nabla X(t) = 0$ . The  $2 \times 2$  matrix to subtract is  $(12\rho\mathbf{r}'' + 8\rho^3\mathbf{r}''')^2 \text{Var}(K_1(t), \bar{K}_1(t))$  with

$$\text{Var}(K_1(t), \bar{K}_1(t)) = \frac{1}{4(\mathbf{r}'(0))^2 - (2\mathbf{r}' + 4\rho^2\mathbf{r}'')^2} \begin{pmatrix} -2\mathbf{r}'(0) & -(2\mathbf{r}' + 4\rho^2\mathbf{r}'') \\ -(2\mathbf{r}' + 4\rho^2\mathbf{r}'') & -2\mathbf{r}'(0) \end{pmatrix}.$$

In the same way for  $i \neq 1$ :

$$\text{Var}(K_i(t); \bar{K}_i(t)) = \frac{1}{4\mathbf{r}'(0)^2 - 4(\mathbf{r}')^2} \begin{pmatrix} -2\mathbf{r}'(0) & -2\mathbf{r}' \\ -2\mathbf{r}' & -2\mathbf{r}'(0) \end{pmatrix}.$$

Giving the extra term to subtract to get Lemma 4.1.

**B.3. Proof of Lemma 4.2.** First note that (37) and (40) are deduced from Lemma 4.1.

*B.3.1. Proof of (34), (35) and (41).* We now consider the case of  $\xi_{11}(t)$ . Using Trick 4.3 we have

$$\begin{aligned}
\text{Var}(\xi_{11}(t)) &= \frac{\det(\text{Var}(X_1(0), \dots, X_N(0), X_{11}(0), X_1(t), \dots, X_N(t)))}{\det(\text{Var}(\nabla X(0), \nabla X(t)))} \\
&\simeq \det(\text{Var}(X_1(0), \dots, X_N(0), X_{11}(0), X_1(t) - X_1(0) - \rho X_{11}(0), \\
&\quad X_2(t) - X_2(0), \dots, X_N(t) - X_N(0))) / \det(\text{Var}(\nabla X(0), \nabla X(t))) \\
&\simeq \frac{\rho^2 3^{N-1}}{4\lambda_2^N \lambda_4^N} \det(\text{Var}(X_1(0), \dots, X_N(0), X_{11}(0), \dots, X_{1N}(0), X_{111}(0))).
\end{aligned}$$

By Lemma 2.3 we obtain  $\text{Var}(\xi_{11}(t)) \simeq \frac{\rho^2 (\lambda_2 \lambda_6 - \lambda_4^2)}{4 \lambda_2}$ . The other variances in (34) and (35) and the determinant (41) are obtained in the same way.

B.3.2. *Proof of (36), (38) and (39).*

- We consider  $\text{Cov}(\xi_{jj}(t), \xi_{kk}(t))$  for  $j \neq k$  and  $j, k \neq 1$ . Since

$$\text{Cov}(\xi_{jj}(t), \xi_{kk}(t)) = \frac{1}{2} (\text{Var}(\xi_{jj}(t) + \xi_{kk}(t)) - \text{Var}(\xi_{jj}(t)) - \text{Var}(\xi_{kk}(t)))$$

and since, by (34),  $\text{Var}(\xi_{jj}(t)) = \text{Var}(\xi_{kk}(t)) = \frac{8\lambda_4}{9}$ , we just need to compute

$$\text{Var}(\xi_{jj}(t) + \xi_{kk}(t)) = \frac{\det(\text{Var}(X_{jj}(t) + X_{kk}(t), X_1(0), \dots, X_N(0), X_1(t), \dots, X_N(t)))}{\det(\text{Var}(\nabla X(0), \nabla X(t)))}.$$

Using Trick 4.3 we obtain

$$\begin{aligned} & \text{Var}(\xi_{jj}(t) + \xi_{kk}(t)) \\ & \simeq \frac{\det(\text{Var}(X_{jj}(0) + X_{kk}(0), X_1(0), \dots, X_N(0), X_{11}(0), X_{12}(0), \dots, X_{1N}(0)))}{\det(\text{Var}(X_1(0), \dots, X_N(0), X_{11}(0), X_{12}(0), \dots, X_{1N}(0)))}. \end{aligned}$$

By Lemma 2.3 we have

$$\begin{aligned} & \det(\text{Var}(X_{jj}(0) + X_{kk}(0), X_1(0), \dots, X_N(0), X_{11}(0), X_{12}(0), \dots, X_{1N}(0))) \\ & = \left(\frac{\lambda_4}{3}\right)^{N-1} \lambda_2^N \det(\text{Var}(X_{11}(0), X_{jj}(0) + X_{kk}(0))), \end{aligned}$$

$$\det(\text{Var}(X_{11}(0), X_{jj}(0) + X_{kk}(0))) = \frac{20\lambda_4^2}{9}$$

and

$$\det(\text{Var}(X_1(0), \dots, X_N(0), X_{11}(0), X_{12}(0), \dots, X_{1N}(0))) = \frac{\lambda_2^N \lambda_4^N}{3^{N-1}}.$$

Finally we obtain  $\text{Var}(\xi_{jj}(t) + \xi_{kk}(t)) \simeq \frac{20\lambda_4}{9}$ , implying  $\text{Cov}(\xi_{jj}(t), \xi_{kk}(t)) \simeq \frac{2\lambda_4}{9}$ . In the same way we obtain  $\text{Cov}(\xi_{jj}(0), \xi_{kk}(t)) \simeq \frac{2\lambda_4}{9}$ ,  $\text{Cov}(\xi_{jk}(0), \xi_{jk}(t)) \simeq \text{Var}(\xi_{jk}(t))$  for  $j \neq k$  and  $\text{Cov}(\xi_{1i}(0), \xi_{1i}(t)) \simeq -\text{Var}(\xi_{1i}(t)) \forall i \in \{1, \dots, N\}$ .

- Now let us prove that  $\text{Cov}(\xi_{11}(t), \xi_{jj}(t)) = \rho^2 \frac{11\lambda_2 \lambda_6 - 15\lambda_4^2}{180\lambda_2} + o(\rho^2)$  for  $j \neq 1$ . We have, (see Lemma 2.3 and Sections B.2.1.2 and B.2.1.3)

$$\text{Cov}(X_{jj}(0), X_i(0)) = 0, \forall i, j \in \{1, \dots, N\}$$

$$\text{Cov}(X_{jj}(0), X_i(t)) = -\text{Cov}(X_{jj}(t), X_i(0)) = 4\rho \mathbf{r}''(\rho^2) \delta_{1i} \text{ for } j \neq i,$$

$$\text{Cov}(X_{ii}(0), X_i(t)) = -\text{Cov}(X_{ii}(t), X_i(0)) = (12\rho \mathbf{r}''(\rho^2) + 8\rho^3 \mathbf{r}'''(\rho^2)) \delta_{1i} \text{ and}$$

$$\text{Cov}(X_{11}(0), X_{jj}(t)) = 4\mathbf{r}''(\rho^2) + 8\rho^2 \mathbf{r}'''(\rho^2).$$

With Section B.2.1.1 we deduce

$$\begin{aligned} \text{Var}(\xi_{jj}(0)) &= \frac{\det(\text{Var}(X_1(0), \dots, X_N(0), X_1(t), \dots, X_N(t), X_{jj}(0)))}{\det(\text{Var}(\nabla X(0), \nabla X(t)))} \\ &= \frac{\det(\text{Var}(\nabla_{-(1)} X(0), \nabla_{-(1)} X(t))) \times \det(\text{Var}(X_1(0), X_1(t), X_{jj}(0)))}{\det(\text{Var}(\nabla X(0), \nabla X(t)))}, \end{aligned}$$

where  $\nabla_{-(1)}X$  denotes the gradient without its first coordinate. Similarly

$$\text{Var}(\xi_{11}(0)) = \frac{\det(\text{Var}(\nabla_{-(1)}X(0), \nabla_{-(1)}X(t))) \times \det(\text{Var}(X_1(0), X_1(t), X_{11}(0)))}{\det(\text{Var}(\nabla X(0), \nabla X(t)))}$$

and

$$\begin{aligned} \text{Var}(\xi_{11}(0) + \xi_{jj}(0)) &= \\ \frac{\det(\text{Var}(\nabla_{-(1)}X(0), \nabla_{-(1)}X(t))) \times \det(\text{Var}(X_1(0), X_1(t), X_{11}(0) + X_{jj}(0)))}{\det(\text{Var}(\nabla X(0), \nabla X(t)))}. \end{aligned}$$

Using Lemma 2.3 we can verify that

$$\begin{aligned} \det(\text{Var}(X_1(0), X_1(t), X_{11}(0) + X_{jj}(0))) &= \frac{8}{3}\lambda_4\lambda_2^2 - \lambda_2(\alpha_1 + \alpha_2)^2 - \frac{8}{3}\lambda_4\beta_1^2 \\ \det(\text{Var}(X_1(0), X_1(t), X_{11}(0))) &= \lambda_4\lambda_2^2 - \lambda_2\alpha_1^2 - \lambda_4\beta_1^2 \\ \det(\text{Var}(X_1(0), X_1(t), X_{jj}(0))) &= \lambda_4\lambda_2^2 - \lambda_2\alpha_2^2 - \lambda_4\beta_1^2, \end{aligned}$$

where  $\beta_1 = -2\mathbf{r}'(\rho^2) - 4\rho^2\mathbf{r}''(\rho^2)$ ,  $\alpha_1 = 12\rho\mathbf{r}''(\rho^2) + 8\rho^3\mathbf{r}'''(\rho^2)$  and  $\alpha_2 = 4\rho\mathbf{r}''(\rho^2)$ . We deduce

$$\text{Cov}(\xi_{11}(0), \xi_{jj}(0)) = \frac{\det(\text{Var}(\nabla_{-(1)}X(0), \nabla_{-(1)}X(t))) \times \left( \frac{1}{3}\lambda_4\lambda_2^2 - \lambda_2\alpha_1\alpha_2 - \frac{1}{3}\lambda_4\beta_1^2 \right)}{\det(\text{Var}(\nabla X(0), \nabla X(t)))}.$$

We can check that

$$\frac{1}{3}\lambda_4\lambda_2^2 - \lambda_2\alpha_1\alpha_2 - \frac{1}{3}\lambda_4\beta_1^2 \simeq \rho^4 \left( \frac{11}{180}\lambda_2\lambda_4\lambda_6 - \frac{\lambda_4^3}{12} \right).$$

We deduce, using Trick 4.3 and Lemma 2.3, that

$$\text{Cov}(\xi_{11}(0), \xi_{jj}(0)) = \rho^2 \frac{11\lambda_2\lambda_6 - 15\lambda_4^2}{180\lambda_2} + o(\rho^2).$$

In the same way we prove  $\text{Cov}(\xi_{11}(0), \xi_{jj}(t)) = \rho^2 \frac{15\lambda_4^2 - 7\lambda_2\lambda_6}{180\lambda_2} + o(\rho^2)$ .

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