



**HAL**  
open science

# Mean number and correlation function of critical points of isotropic Gaussian fields

Jean-Marc Azaïs, Céline Delmas

► **To cite this version:**

Jean-Marc Azaïs, Céline Delmas. Mean number and correlation function of critical points of isotropic Gaussian fields. 2019. hal-02317445v1

**HAL Id: hal-02317445**

**<https://hal.science/hal-02317445v1>**

Preprint submitted on 5 Nov 2019 (v1), last revised 26 Nov 2020 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Mean number and correlation function of critical points of isotropic Gaussian fields

Jean-Marc Azaïs<sup>1</sup> and Céline Delmas<sup>2</sup>

<sup>1</sup>*Institut de Mathématiques de Toulouse  
Université Paul Sabatier, 118 route de Narbonne, 31062 Toulouse, France  
e-mail: [jean-marc.azais@math.univ-toulouse.fr](mailto:jean-marc.azais@math.univ-toulouse.fr)*

<sup>2</sup>*MIAT, UR 875, Université de Toulouse, INRA, 31320 Castanet-Tolosan, France  
e-mail: [celine.delmas@inra.fr](mailto:celine.delmas@inra.fr)*

**Abstract:** Let  $\mathcal{X} = \{X(t) : t \in \mathbb{R}^N\}$  be an isotropic Gaussian random field with real values. In a first part we study the mean number of critical points of  $\mathcal{X}$  with index  $k$  using random matrices tools. We obtain an exact expression for the probability density of the eigenvalue of rank  $k$  of a  $N$ -GOE matrix. We deduce some exact expressions for the mean number of critical points with a given index. In a second part we study attraction or repulsion between these critical points. A measure is the correlation function. We prove attraction between critical points when  $N > 2$ , neutrality for  $N = 2$  and repulsion for  $N = 1$ . The attraction between critical points that occurs when the dimension is greater than two is due to critical points with adjacent indexes. A strong repulsion between maxima and minima is observed. The correlation function between maxima (or minima) depends on the dimension of the ambient space.

**Keywords and phrases:** Critical points, Isotropic Gaussian fields, GOE matrices, Kac-Rice formula, Point processes, Random matrices.

## 1. Introduction

Critical points of random field play an important role in small or large dimension. In large dimension data they appear in the study of algorithms of maximisation of the likelihood [12, 28]. In smaller dimension they play a role in many applications from various areas: detection of peaks in a random field [16, 30, 31, 32], optimisation of a response modelled by a random field [19], modelling of random sea waves [23, 6, 27]. Critical points and their indexes are an important element of the geometry of random fields. They appear in the computation of the Euler characteristic [2, 18]. They are a subject of study on their own as in [26, 3, 10, 17].

Let  $\mathcal{X} = \{X(t) : t \in \mathbb{R}^N\}$  be an isotropic Gaussian random field with real values. If we look at the repartition of the critical points of  $\mathcal{X}$  as a function of their index in dimension two, considerations of symmetry and of Euler characteristic imply that if  $\mathcal{N}^c$ ,  $\mathcal{N}_{min}^c$ ,  $\mathcal{N}_{max}^c$ ,  $\mathcal{N}_{saddle}^c$  are respectively the number of critical, minimum, maximum and saddle points on a given set:

$$\mathbb{E}(\mathcal{N}_{min}^c) = \mathbb{E}(\mathcal{N}_{max}^c) = \frac{1}{2} \mathbb{E}(\mathcal{N}_{saddle}^c) = \frac{1}{4} \mathbb{E}(\mathcal{N}^c).$$

In higher dimensions simulations suggest that such a simple result does not hold true (see Figure 1). The purpose of Section 3 is to compute these expectations using random matrices tools. With this objective in mind, we obtain an exact expression for the probability density of the eigenvalue of rank  $k$  of a  $N$ -GOE matrix (see Theorem 3.5 and (18)–(24)). We deduce exact expressions for the mean number of critical points with a given index (Propositions 3.3–3.7). In particular, if we note  $\mathcal{N}_1^c$  and  $\mathcal{N}_2^c$  the number of critical points respectively with index 1 and 2 (see Section 2.3 for the definition), we obtain, for  $N = 3$

(see Proposition 3.6):

$$\begin{aligned} \mathbb{E}(\mathcal{N}_{min}^c) = \mathbb{E}(\mathcal{N}_{max}^c) &= \frac{29 - 6\sqrt{6}}{116} \mathbb{E}(\mathcal{N}^c) \\ \mathbb{E}(\mathcal{N}_1^c) = \mathbb{E}(\mathcal{N}_2^c) &= \frac{29 + 6\sqrt{6}}{116} \mathbb{E}(\mathcal{N}^c). \end{aligned}$$

Proposition 3.7 gives these expectations for  $N = 4$ .

On the other hand we can consider how the critical points are spread in the space. In a pioneering work, Belyaev, Cammarota and Wigman [10] study the attraction or repulsion (see Section 2 for a precise definition) of the point process of critical points of a particular random wave model in dimension 2. In the case of random processes ( $N = 1$ ), it was generally admitted that repulsion between crossings or critical points occurs. In fact this result has never been written explicitly so it is the object of Section 4 with Propositions 4.1 and 4.3. With respect to this behaviour the result of [10] is unexpected since no repulsion occurs between critical points. The object of Section 5 is to explore the phenomenon of attraction or repulsion for a large class of random fields, in all dimensions and for each type of indexes. Our hypotheses as well as a precise definition of attraction and repulsion are given in Section 2. Theorem 5.2 proves attraction between critical points when  $N > 2$ , neutrality for  $N = 2$  and repulsion for  $N = 1$ . Theorem 5.3 shows that the attraction between critical points that occurs when the dimension is greater than 2 is due to attraction between critical points with adjacent indexes. In Theorem 5.4 we prove a strong repulsion, growing with  $N$ , between maxima and minima. Finally Theorem 5.5 gives an upper bound to the correlation function between maxima (or equivalently minima) suggesting neutrality for  $N = 5$ , repulsion for  $N < 5$  and attraction for  $N > 5$ . All these results generalise the results of [10].

In Appendix A we present some results on GOE matrices. Appendix B is devoted to the calculation of the conditional joint distribution of the Hessian matrices of  $X(t)$  and  $X(0)$  given the nullity of the gradients of  $X(t)$  and  $X(0)$  and to the proof of Lemma 5.1 giving some of its asymptotic equivalents when  $t \rightarrow 0$ .

## 2. The point process of critical points

### 2.1. Assumptions (Ak) , $k = 2, 3, 4$ and basic notations

In all the sequel  $\mathcal{X} = \{X(t) : t \in \mathbb{R}^N\}$  is stationary and isotropic Gaussian field with real values. We assume, without loss of generality, that it is centred with variance 1. We set

$$\mathbb{E}(X(s)X(t)) = r(\|s - t\|^2).$$

We assume that  $r$  is of class  $\mathcal{C}^k$ . This is equivalent to the existence of a finite  $2k$ th spectral moment  $\lambda_{2k}$ . It implies that  $\mathcal{X}$  is  $k$  times differentiable in quadratic mean.

We denote by  $\nabla X(t) = (X'_1(t), \dots, X'_N(t))$  the gradient of  $X(t)$ , and by  $\nabla^2 X(t) = (X''_{ij}(t), i \leq j = 1, \dots, N)$  its Hessian. We have for every  $i = 1, \dots, N$

$$\begin{aligned} 1 = \text{Var}(X(t)) &= r(0) & \lambda_2 &:= \text{Var}(X'_i(t)) = -2r'(0) \\ \lambda_4 &:= \text{Var}(X''_{ii}(t)) = 12r''(0) & \lambda_6 &:= \text{Var}(X'''_{iii}(t)) = -120r'''(0) \text{ if } k = 3, 4 \\ \lambda_8 &:= \text{Var}(X^{(4)}_{iii}(t)) = 1680r^{(4)}(0), \text{ if } k = 4. \end{aligned}$$

that are all finite because of our hypotheses.  $X^{(4)}_{iii}(0)$  is, for example, the fourth derivative in the  $i$ th direction.

We assume that the joint distribution of  $X(0), X_1'(0), \dots, X_1^{(k)}(0)$  is non-degenerated where  $X_1^{(k)}(0)$  denotes here the  $k$ th derivative of  $X$  in the first direction. Note that the choice of the first direction, that is arbitrary, plays no role. A direct algebra shows that this non-degeneracy condition is equivalent to

$$\begin{aligned}\lambda_2 &> 0, & \lambda_4 &> \lambda_2^2, \\ \lambda_2\lambda_6 &> \lambda_4^2, & \text{if } k &> 2 \\ \lambda_4\lambda_8 &> \lambda_6^2, & \text{if } k &= 4.\end{aligned}$$

Finally  $p_{\nabla X(t), \nabla X(s)}(x, y)$  denotes the density probability of  $(\nabla X(t), \nabla X(s))$  at  $(x, y)$  when it exists;  $\varphi(\cdot)$  and  $\Phi(\cdot)$  denotes respectively the probability density and the cumulative distribution function of a standard Gaussian variable and  $\bar{\Phi}(\cdot) = 1 - \Phi(\cdot)$ . Let  $S$  be a compact subset of  $\mathbb{R}^N$ , we denote by  $|S|$  the Lebesgue measure of  $S$ . If  $X$  is a random vector,  $\text{Var}(V)$  will denote the variance-covariance matrix. We keep the notation  $\text{Cov}$  for the matrix

$$\text{Cov}(X, Y) := \mathbb{E} \left( (X - \mathbb{E}(X))(Y - \mathbb{E}(Y))^{\top} \right).$$

## 2.2. Correlation function, two points function

The correlation function of a point process  $\mathcal{P}$  is defined by [13]

$$A(s, t) := \lim_{\epsilon \rightarrow 0} \frac{1}{V^2(\epsilon)} \mathbb{E}(\mathcal{P}(B(s, \epsilon))\mathcal{P}(B(t, \epsilon))),$$

where  $B(s, \epsilon)$  is the ball with center  $s$  and radius  $\epsilon$  and  $V(\epsilon)$  its volume. Since the process is stationary and isotropic, this function depends only on the norm  $\rho := \|s - t\|$  and by a small abuse of notation we will denote it by  $A(\rho)$ .

Suppose now that the point process  $\mathcal{P}$  is the process of critical points of  $\mathcal{X}$  and let us define for  $S \in \mathbb{R}^N$  compact

$$\mathcal{N}^c(S) = \#\{t \in S : \nabla X(t) = 0\}.$$

Under our conditions, the derivatives of  $\mathcal{X}$  satisfies the Kac-Rice formula of order two. In particular if  $S$  and  $T$  are disjoint (as  $B(s, \epsilon)$  and  $B(t, \epsilon)$  for  $s \neq t$  and  $\epsilon$  sufficiently small), the Kac-Rice formula of order two yields

$$\mathbb{E}[\mathcal{N}^c(S)\mathcal{N}^c(T)] = \int_{S \times T} \mathbb{E}(|\det(\nabla^2 X(s)) \det(\nabla^2 X(t))| / |\nabla X(s) = \nabla X(t) = 0|) p_{\nabla X(s), \nabla X(t)}(0, 0) ds dt; \quad (1)$$

proving that

$$A(\rho) = \mathbb{E}(|\det(\nabla^2 X(0)) \det(\nabla^2 X(t))| / |\nabla X(0) = \nabla X(t) = 0|) p_{\nabla X(0), \nabla X(t)}(0, 0), \quad (2)$$

with, for example,  $t = \rho e_1$  with  $e_1 := (1, 0, \dots, 0)$ .

Some papers, as [10] consider the behaviour as  $\rho \rightarrow 0$  of

$$T(\rho) := \mathbb{E}(\mathcal{N}^c(B_\rho)(\mathcal{N}^c(B_\rho) - 1)), \quad B_\rho \text{ is any ball with radius } \rho.$$

It is elementary to see that if  $A(\rho) \simeq C\rho^d$  then  $T(\rho) \simeq CV_N^2 \rho^{d+2N}$ , where  $\simeq$  means, as in the rest of the paper, equivalence as  $\rho \rightarrow 0$  and  $V_N$  is the volume of the unit ball.

### 2.3. Attraction, neutrality, repulsion

- The reference of neutral point process is the Poisson process for which the correlation function  $A(\rho)$  is constant and  $T(\rho)$  behaves as  $\rho^{2N}$ .
- The repulsion is defined by the fact that the correlation function  $A(\rho)$  tends to zero as  $\rho \rightarrow 0$ . Equivalently  $T(\rho) = o(\rho^{2N})$ . Determinantal processes [22] is a way of constructing repulsive point processes.
- The attraction is just the contrary: as  $\rho \rightarrow 0$ ,  $A(\rho) \rightarrow +\infty$ ,  $\frac{T(\rho)}{\rho^{2N}} \rightarrow +\infty$ .

In many situation, assuming that the function is Morse, i.e. there is no critical point with a degenerated Hessian, one considers the index  $i$  of the critical point defined as the number of negative eigenvalues of the Hessian. We define

$$\mathcal{N}_k^c(S) := \#\{t \in S : \nabla X(t) = 0; i(\nabla^2 X(t)) = k\}. \quad (3)$$

All the quantities above have direct generalization when, instead of considering all the critical points, one considers the relation between the number of critical points with indexes  $i_1$  and  $i_2$ . As in (1) and (2) the Kac-Rice formula of order two yields

$$A^{i_1, i_2}(\rho) = \mathbb{E}(|\det(\nabla^2 X(0))\mathbb{1}_{i(\nabla^2 X(0))=i_1} \det(\nabla^2 X(t))\mathbb{1}_{i(\nabla^2 X(t))=i_2}| / \nabla X(0) = \nabla X(t) = 0) p_{\nabla X(0), \nabla X(t)}(0, 0), \quad (4)$$

with again  $t = \rho e_1$ . In the same way, we can consider attraction neutrality or repulsion between critical points with indexes  $i_1$  and  $i_2$ .

## 3. Mean number of critical points with a given index

### 3.1. The particular case of random processes ( $N=1$ )

Suppose that the random process  $\mathcal{X}$  satisfies Assumption (A2), and let  $\mathcal{M}_1(S)$  and  $\mathcal{M}_2(S)$  be respectively the total number of maxima and minima of  $\mathcal{X}$  on  $S$  then by direct application of the Kac-Rice formula :

$$\mathbb{E}(\mathcal{M}_1(S)) = \mathbb{E}(\mathcal{M}_2(S)) = \frac{|S|}{2\pi} \sqrt{\frac{\lambda_4}{\lambda_2}}.$$

More interesting is the case when we consider the numbers  $\mathcal{M}_1(u, S)$  (respectively  $\mathcal{M}_2(u, S)$ ) of maxima (respectively minima) of  $\mathcal{X}$  above a certain level  $u$  on  $S$  a compact subset of  $\mathbb{R}$ . We can write the Kac-Rice formula of order one:

$$\mathbb{E}(\mathcal{M}_1(u, S)) = \int_S \int_{-\infty}^0 \int_u^{+\infty} |x''| p_{X''(t), X'(t), X(t)}(x'', 0, x) dx dx'' dt,$$

where  $p_{X''(t), X'(t), X(t)}(x'', 0, x)$  denotes the density probability of  $(X''(t), X'(t), X(t))$  at  $(x'', 0, x)$ . We deduce immediately the following proposition.

**Proposition 3.1.** *With notations above, under Assumption (A2):*

$$\begin{aligned} \mathbb{E}(\mathcal{M}_1(u, S)) &= \frac{|S|}{2\pi} \sqrt{\frac{\lambda_4}{\lambda_2}} \bar{\Phi} \left( u \sqrt{\frac{\lambda_4}{\lambda_4 - \lambda_2^2}} \right) + \frac{|S| \sqrt{\lambda_2}}{\sqrt{2\pi}} \varphi(u) \Phi \left( \frac{u \lambda_2}{\sqrt{\lambda_4 - \lambda_2^2}} \right), \\ \mathbb{E}(\mathcal{M}_2(u, S)) &= \frac{|S|}{2\pi} \sqrt{\frac{\lambda_4}{\lambda_2}} \bar{\Phi} \left( u \sqrt{\frac{\lambda_4}{\lambda_4 - \lambda_2^2}} \right) - \frac{|S| \sqrt{\lambda_2}}{\sqrt{2\pi}} \varphi(u) \bar{\Phi} \left( \frac{u \lambda_2}{\sqrt{\lambda_4 - \lambda_2^2}} \right). \end{aligned}$$

Let us set  $\alpha := \lambda_4/\lambda_2^2$ . Under Assumption (A2),  $\alpha > 1$  and

$$\frac{\mathbb{E}(\mathcal{M}_1(u, S))}{\mathbb{E}(\mathcal{M}_1(u, S) + \mathcal{M}_2(u, S))} = \frac{\sqrt{\alpha}\bar{\Phi}\left(u\sqrt{\frac{\alpha}{\alpha-1}}\right) + \sqrt{2\pi}\varphi(u)\Phi\left(\frac{u}{\sqrt{\alpha-1}}\right)}{2\left(\sqrt{\alpha}\bar{\Phi}\left(u\sqrt{\frac{\alpha}{\alpha-1}}\right) + \sqrt{2\pi}\varphi(u)\Phi\left(\frac{u}{\sqrt{\alpha-1}}\right)\right) - \sqrt{2\pi}\varphi(u)},$$

depends only on  $\alpha$ .

### 3.2. A first class of expressions

#### 3.2.1. Mean number of critical points

$\mathcal{N}^c(S)$  and  $\mathcal{N}_k^c(S)$  are respectively the number of critical points and the number of critical points with index  $k$  that belong to  $S$ . To compute their mean number we use the Kac-Rice formula of order one.

$$\mathbb{E}(\mathcal{N}^c(S)) = \int_S \mathbb{E}(|\det(\nabla^2 X(t))|/\nabla X(t) = 0) p_{\nabla X(t)}(0) dt = \frac{|S|}{(2\pi\lambda_2)^{N/2}} \mathbb{E}(|\det(\nabla^2 X(t))|), \quad (5)$$

$$\mathbb{E}(\mathcal{N}_k^c(S)) = \frac{|S|}{(2\pi\lambda_2)^{N/2}} \mathbb{E}(|\det(\nabla^2 X(t))| \mathbf{1}_{i(\nabla^2 X(t))=k}). \quad (6)$$

Let  $G_n$  be a size  $n$  GOE matrix. We recall that  $G_n$  is said to have the GOE (Gaussian Orthogonal Ensemble) distribution if it is symmetric and all its elements are independent centred Gaussian variables satisfying  $\mathbb{E}(G_{ii}^2) = 1$  and  $\mathbb{E}(G_{ij}^2) = \frac{1}{2}$ .

The relation between GOE matrices and the study of critical points of stationary isotropic Gaussian fields is due to the following lemma derived from [21] and Lemma 8.5 of [7].

**Lemma 3.2.**  $\nabla^2 X(t)$  is equal in distribution to

$$\sqrt{\frac{2\lambda_4}{3}} \left( G_N - \Lambda Id_N \right),$$

where  $G_N$  is a size  $N$  GOE matrix and  $\Lambda$  follows a  $N(0, 1/2)$  independent of  $G_N$ .

*Proof.* The proof is a direct consequence of the following identity that can be found, for example, in Lemma 8.5 of [7]:

$$\mathbb{E}(X''_{ik}(t)X''_{i'k'}(t)) = \frac{\lambda_4}{3} [\delta_{ii'}\delta_{kk'} + \delta_{i'k}\delta_{ik'} + \delta_{ik}\delta_{i'k'}], \quad (7)$$

where  $\delta$  is the Kronecker symbol. □

We recall that the joint density  $f_N$  of the eigenvalues  $(\mu_1, \dots, \mu_N)$  of a  $N$ -GOE matrix (see Theorem 3.3.1 of [24]) is given by:

$$f_N(\mu_1, \dots, \mu_N) = k_N \exp\left(-\frac{\sum_{i=1}^N \mu_i^2}{2}\right) \prod_{1 \leq i < k \leq N} |\mu_k - \mu_i|, \quad (8)$$

where:

$$k_N := (2\pi)^{-N/2} (\Gamma(3/2))^N \left( \prod_{i=1}^N \Gamma(1 + i/2) \right)^{-1}. \quad (9)$$

Using Lemma 3.2, (5), (6) and (8) we get directly the following proposition.

**Proposition 3.3.** *With notations above, under Assumption (A2):*

$$\mathbb{E}(\mathcal{N}_k^c(S)) = \frac{|S|}{\pi^{(N+1)/2}} \left( \frac{\lambda_4}{3\lambda_2} \right)^{N/2} \frac{k_N}{k_{N+1}} \frac{1}{N+1} \mathbb{E} \left( \exp \left( -\frac{L_{k+1}^2}{2} \right) \right), \quad (10)$$

$$\mathbb{E}(\mathcal{N}^c(S)) = \frac{|S|}{\pi^{(N+1)/2}} \left( \frac{\lambda_4}{3\lambda_2} \right)^{N/2} \frac{k_N}{k_{N+1}} \mathbb{E} \left( \exp \left( -\frac{L^2}{2} \right) \right), \quad (11)$$

where  $L$  is an eigenvalue “at random” and  $L_p$  is the eigenvalue of rank  $p$  of a  $(N+1)$ -GOE matrix.

*Proof of Proposition 3.3.* We give the proof of the first equality only. Using Lemma 3.2 and (8)

$$\begin{aligned} & \mathbb{E}(|\det \nabla^2 X(t)| \mathbf{1}_{i(\nabla^2 X(t))=k}) \\ &= \left( \frac{2\lambda_4}{3} \right)^{N/2} k_N N! \int_{\mu_1 < \mu_2 < \dots < \mu_k < \lambda < \mu_{k+1} < \dots < \mu_N} \prod_{1 \leq i < k \leq N} |\mu_k - \mu_i| \prod_{1 \leq i \leq N} |\lambda - \mu_i| \\ & \quad \exp \left( -\frac{\sum_{i=1}^N \mu_i^2}{2} \right) \pi^{-1/2} \exp(-\lambda^2/2) \exp(-\lambda^2/2) d\mu_1 \dots d\mu_N d\lambda \\ &= \left( \frac{2\lambda_4}{3} \right)^{N/2} \pi^{-1/2} \frac{k_N}{k_{N+1}} \frac{N!}{(N+1)!} \mathbb{E} \left( \exp \left( -\frac{L_{k+1}^2}{2} \right) \right). \end{aligned}$$

The rest is plain. □

**Remarks:**

- The expected number of critical points depends on the spectral moments of the random field whereas the ratio  $\frac{\mathbb{E}(\mathcal{N}_k^c(S))}{\mathbb{E}(\mathcal{N}^c(S))}$  is universal.
- This result can be found in [17] (Proposition 3.9). In this paper we go further: In Theorem 3.5 we obtain an exact expression for the probability density of the eigenvalue of rank  $k$  of a  $N$ -GOE matrix, denoted  $q_N^k(l)$  (see (17)–(24)). We can deduce exact expressions for (10) and (11) as in Propositions 3.6 and 3.7.
- The density probability of  $L$ , denoted  $\frac{R_1(l)}{N+1}$ , can be found in [24] chapter 7 formulas 7.2.30 and 7.2.32.
- For  $N = 2$  it is known, using symmetry and mean Euler characteristic considerations, that

$$\mathbb{E}(\mathcal{N}_0^c(S)) = \mathbb{E}(\mathcal{N}_2^c(S)) = \frac{1}{2} \mathbb{E}(\mathcal{N}_1^c(S)) = \frac{1}{4} \mathbb{E}(\mathcal{N}^c(S)).$$

In that particular case, the ratio  $\frac{\mathbb{E}(\mathcal{N}_k^c(S))}{\mathbb{E}(\mathcal{N}^c(S))}$  is equal to the binomial coefficient  $\frac{N!}{2^N k!(N-k)!}$ . In higher dimension, by simulations, we can affirm that this equality is no longer true as shown in Figure 1 for  $N = 10$ .

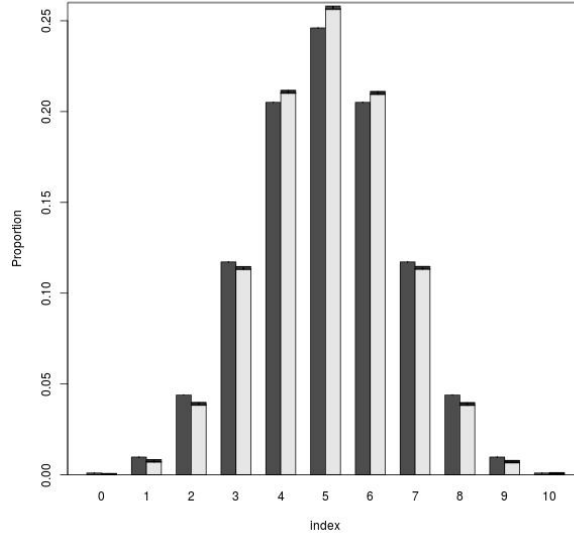


FIGURE 1.  $N = 10$ . In white : Estimation of  $E(\mathcal{N}_k^c(S)) / E(\mathcal{N}^c(S))$  as a function of the index  $k$  (mean over 1000 samples; each sample is the mean of 10000 simulations). Confidence intervals in black at the top. In black : Binomial coefficients  $\frac{N!}{2^N k!(N-k)!}$ .

### 3.2.2. Mean number of critical points above a level

We consider the mean number of critical points above a level  $u$ . We define

$$\begin{aligned} \mathcal{N}^c(u, S) &= \#\{t \in S : \nabla X(t) = 0, X(t) > u\} \\ \mathcal{N}_k^c(u, S) &= \#\{t \in S : \nabla X(t) = 0, i(\nabla^2 X(t)) = k, X(t) > u\}. \end{aligned}$$

The Kac-Rice formula implies that

$$E(\mathcal{N}_k^c(u, S)) = \frac{|S|}{\lambda_2^{N/2} (2\pi)^{(N+1)/2}} \int_u^{+\infty} \exp\left(-\frac{x^2}{2}\right) E(|\det(\nabla^2 X(t))| \mathbb{1}_{i(\nabla^2 X(t))=k} / X(t) = x) dx.$$

To simplify the presentation we limit our attention to the case  $\lambda_4 \geq 3\lambda_2$ . As remarked in [9] this condition is met if the covariance function  $r$  is a ‘‘Shoenberg covariance’’: it is a valid covariance function in every dimension. Note that more general cases have been studied by [17]. Following [21] and Lemma 8.5 of [7], by regression formulas we have that under the condition  $X(t) = x$ ,  $\nabla^2 X(t)$  can be written in distribution as the independent sum:

$$\nabla^2 X(t) = \left[ \mathcal{N}\left(-\lambda_2 x, \left(\frac{\lambda_4}{3} - \lambda_2^2\right)\right) \right] Id_N + \sqrt{\frac{2\lambda_4}{3}} G_N,$$

where  $G_N$  is a  $N$ -GOE matrix. Then using (8), by direct calculations we get:



**Proposition 3.4.** *With notations above, under Assumption (A2), when  $\lambda_4 - 3\lambda_2^2 > 0$ ,*

$$\begin{aligned} & \mathbb{E}(\mathcal{N}_k^c(u, S)) \\ &= \frac{|S|}{\pi^{(N+1)/2}} \left( \frac{\lambda_4}{3\lambda_2} \right)^{N/2} \frac{k_N}{k_{N+1}} \frac{1}{N+1} \mathbb{E} \left\{ \exp \left( -\frac{L_{k+1}^2}{2} \right) \bar{\Phi} \left[ \sqrt{\frac{\lambda_4}{\lambda_4 - 3\lambda_2^2}} \left( u - L_{k+1} \frac{\sqrt{6}\lambda_2}{\sqrt{\lambda_4}} \right) \right] \right\}. \end{aligned} \quad (12)$$

When  $\lambda_4 - 3\lambda_2^2 = 0$

$$\mathbb{E}(\mathcal{N}_k^c(u, S)) = \frac{|S|}{\pi^{(N+1)/2}} \frac{k_N}{k_{N+1}} \frac{\lambda_2^{N/2}}{N+1} \mathbb{E} \left( \exp \left( -\frac{L_{k+1}^2}{2} \right) \mathbb{1}_{L_{k+1} > u/\sqrt{2}} \right), \quad (13)$$

where  $L_p$  is the eigenvalue of rank  $p$  of a  $(N+1)$ -GOE matrix.

### 3.3. Explicit expressions

#### 3.3.1. Probability density of the eigenvalue of rank $k$ of a $N$ -GOE matrix

The aim of this section is to give an exact expression for the probability density of the eigenvalue of rank  $k$  of a  $N$ -GOE matrix and to deduce exact expressions for  $\mathbb{E}(\mathcal{N}_k^c(S))$  and  $\mathbb{E}(\mathcal{N}_k^c(u, S))$ .

For simplicity it is conducted using the simplest family of polynomials:  $x^i$ ,  $i = 0, 1, 2, \dots$  in equations (15), (16) and (17). In fact it is valid using any family  $P_i(x)$  of monic polynomials:  $P_i$  has degree  $i$  and its highest coefficient is one.

We must introduce some further notations.

We recall that the Pfaffian of an  $n \times n$  skew matrix  $A$  ( $a_{ij} = -a_{ji}$ ,  $a_{ii} = 0$ ), for  $n$  even  $n = 2m$ , is defined by:

$$\text{Pf}(A) = \frac{1}{2^m m!} \sum_{j_1=1}^n \cdots \sum_{j_n=1}^n \sigma(j_1, \dots, j_n) a_{j_1 j_2} a_{j_3 j_4} \cdots a_{j_{2m-1} j_{2m}} \quad (14)$$

with  $\sigma(j_1, \dots, j_n)$  the signature of the permutation  $j_1, \dots, j_n$  with  $\sigma(j_1, \dots, j_n) = 0$  if two of the  $j$ 's are equal.

Let  $J = \{m_1 < \dots < m_n\}$  a set of integers and  $K$  a subset of  $\{1, \dots, n\}$ . The  $n \times n$  ( $n$  even) or  $(n+1) \times (n+1)$  ( $n$  odd) skew matrix  $B_J^K(l)$  that depends on the real  $l$  is defined by its entries  $b_{ij}$ .

When  $n$  is even

$$b_{ij} = \int_{\mathcal{D}_i^K(l)} dx \int_{\mathcal{D}_j^K(l)} dy \mathbf{s}(y-x) x^{m_i-1} y^{m_j-1} \exp \left( -\frac{x^2 + y^2}{2} \right), \quad (i, j = 1, \dots, n), \quad (15)$$

where  $\mathbf{s}(\cdot)$  is the sign function,  $\mathcal{D}_i^K(l) = (-\infty, l)$  when  $i \in K$ ,  $\mathcal{D}_i^K(l) = (l, +\infty)$  when  $i \notin K$ .

When  $n$  is odd, we use also (15) and we add the extra terms

$$b_{i, n+1}(l) = -b_{n+1, i}(l) = \int_{\mathcal{D}_i^K(l)} x^{m_i-1} e^{-\frac{x^2}{2}} dx, \quad (i = 1, \dots, n); \quad b_{n+1, n+1}(l) = 0. \quad (16)$$

We are now able to state the following theorem that gives an exact expression for  $q_N^k(l)$ : the probability density of the eigenvalue of rank  $k$  of a  $N$ -GOE matrix.

**Theorem 3.5.** Let  $q_N^k(l)$  be the probability density of the eigenvalue of rank  $k$  of a  $N$ -GOE matrix. Using notations above, for  $l \in \mathbb{R}$  we have:

$$q_N^k(l) = k_N N! \exp\left(-\frac{l^2}{2}\right) \sum_{i=1}^N (-1)^{k+i} l^{i-1} \sum_{K \in \mathcal{F}_{k-1}(N-1)} \text{Pf}(B_{\bar{I}}^K(l)) \quad (17)$$

where  $\bar{I} = \{1, \dots, N\} \setminus \{i\}$ ,  $k_N$  is given in (9),  $\mathcal{F}_{k-1}(N-1)$  is the set of parts of size  $k-1$  of  $\{1, \dots, N-1\}$  and  $B_{\bar{I}}^K(l)$  is the matrix  $B_J^K(l)$  defined in (15) and (16) with  $n = N-1$  and  $J = \bar{I}$ .

*Proof of Theorem 3.5.* Let  $G_N$  be a  $N$ -GOE matrix with eigenvalues denoted  $\mu_1, \dots, \mu_N$ . It is easy to check that the probability density of the eigenvalue of rank  $k+1$  of a  $(N+1)$  GOE matrix is given by:

$$\begin{aligned} q_{N+1}^{k+1}(l) &= \frac{k_{N+1}}{k_N} (N+1) \int_{\mathbb{R}^n} (-1)^k \det(G_N - lId_N) f_N(\mu_1, \dots, \mu_N) \exp\left(-\frac{l^2}{2}\right) \mathbb{1}_{\{i(\nabla^2(G_N - lId_N))=k\}} d\mu_1 \dots d\mu_N \\ &= \frac{k_{N+1}}{k_N} (N+1) (-1)^k \gamma_{N,1}^k(l) \exp\left(-\frac{l^2}{2}\right), \end{aligned}$$

where  $f_N$  is given in (8) and  $\gamma_{N,1}^k(l) := \mathbb{E}(\det(G_N - lId_N) \mathbb{1}_{\{i(\nabla^2(G_N - lId_N))=k\}})$ . We prove in Appendix A, Lemma A.2, equation (56), that:

$$\gamma_{N,1}^k(l) = k_N N! \sum_{i=1}^{N+1} (-1)^{i+1} l^{i-1} \times \sum_{K \in \mathcal{F}_k(N)} \text{Pf}(B_{\bar{I}}^K(l)),$$

where  $\bar{I} = \{1, \dots, N+1\} \setminus \{i\}$ ;  $B_{\bar{I}}^K(l)$  is the matrix  $B_J^K(l)$  defined in (15) and (16) with  $n = N$  and  $J = \bar{I}$ . That concludes the proof.  $\square$

The major drawback of the result above is its complicated form. However, for small values of  $N$ , we manage to get an explicit expression for  $q_N^k(l)$  and consequently (using (10), (12) and (13)) for  $E(\mathcal{N}_k^c(S))$  and  $E(\mathcal{N}_k^c(u, S))$ . We give some examples below and we derive Propositions 3.6 and 3.7 which are new results.

**Examples:** We denote  $\mu_{(k)}$ ,  $k = 1, \dots, N$ , the ordered eigenvalues of a  $N$ -GOE matrix. After tedious calculations we obtain:

1. For  $N = 2$ :  $q_2^1(l) = q_2^2(-l)$  and

$$q_2^2(l) = \frac{\exp\left(-\frac{l^2}{2}\right)}{2\sqrt{\pi}} \left[ \exp\left(-\frac{l^2}{2}\right) + \sqrt{2\pi} l \Phi(l) \right]. \quad (18)$$

$$\mathbb{E}(\mu_{(2)}) = -\mathbb{E}(\mu_{(1)}) = \frac{\sqrt{\pi}}{2} \text{ and } \text{Var}(\mu_{(2)}) = \text{Var}(\mu_{(1)}) = \frac{3}{2} - \frac{\pi}{4}.$$

2. For  $N = 3$ :  $q_3^1(l) = q_3^3(-l)$ ,  $q_3^2(l) = \frac{\exp(-l^2)}{\sqrt{\pi}}$  and

$$q_3^3(l) = \frac{\exp\left(-\frac{l^2}{2}\right)}{\pi\sqrt{2}} \left[ \sqrt{\pi}(2l^2 - 1)\Phi(l\sqrt{2}) + \sqrt{2\pi} \exp\left(-\frac{l^2}{2}\right) \Phi(l) + l \exp(-l^2) \right]. \quad (19)$$

$$\mathbb{E}(\mu_{(3)}) = -\mathbb{E}(\mu_{(1)}) = \frac{9}{2\sqrt{3\pi}} \text{ and } \text{Var}(\mu_{(3)}) = \text{Var}(\mu_{(1)}) = \frac{33\pi - 81}{12\pi}.$$

3. For  $N = 4$ :  $q_4^1(l) = q_4^4(-l)$ ,  $q_4^2(l) = q_4^3(-l)$  and

$$q_4^3(l) = \frac{\exp\left(-\frac{l^2}{2}\right)}{2\pi} \left[ \frac{3l}{2} \exp\left(-\frac{3l^2}{2}\right) + \sqrt{2\pi} \left(1 - \frac{l^2}{2}\right) \bar{\Phi}(l) \exp(-l^2) - \frac{\pi(2l^3 - 3l)}{\sqrt{2}} \Phi(l\sqrt{2}) \bar{\Phi}(l) + \frac{3\sqrt{\pi}(1 + 2l^2)}{2} \Phi(l\sqrt{2}) \exp\left(-\frac{l^2}{2}\right) \right]. \quad (20)$$

$$q_4^4(l) = \frac{\exp\left(-\frac{l^2}{2}\right)}{2\pi} \left[ \frac{3l}{2} \exp\left(-\frac{3l^2}{2}\right) - \sqrt{2\pi} \left(1 - \frac{l^2}{2}\right) \Phi(l) \exp(-l^2) + \frac{\pi(2l^3 - 3l)}{\sqrt{2}} \Phi(l\sqrt{2}) \Phi(l) + \frac{3\sqrt{\pi}(1 + 2l^2)}{2} \Phi(l\sqrt{2}) \exp\left(-\frac{l^2}{2}\right) \right]. \quad (21)$$

4. For  $N = 5$ :  $q_5^1(l) = q_5^5(-l)$ ,  $q_5^2(l) = q_5^4(-l)$  and

$$q_5^3(l) = q_5^4(l) - 2q_5^5(l) + \frac{\sqrt{2} \exp\left(-\frac{l^2}{2}\right)}{3\pi^{3/2}} \left[ \pi(4l^4 - 12l^2 + 3) \Phi(l\sqrt{2}) + \sqrt{\pi} \left(3l^3 - \frac{13l}{2}\right) \exp(-l^2) + \sqrt{2\pi} \left(l^4 + 3l^2 + \frac{3}{4}\right) \exp\left(-\frac{l^2}{2}\right) \Phi(l) \right]. \quad (22)$$

$$q_5^4(l) = \frac{\sqrt{2} \exp\left(-\frac{l^2}{2}\right)}{3\pi^{3/2}} \left[ \sqrt{2\pi} \exp\left(-\frac{3l^2}{2}\right) \left(\frac{l^3}{2} + \frac{5l}{4}\right) + \sqrt{2\pi} \Phi(l\sqrt{2}) \exp\left(-\frac{l^2}{2}\right) \left(l^4 + 3l^2 + \frac{3}{4}\right) \right]. \quad (23)$$

$$q_5^5(l) = \frac{\sqrt{2} \exp\left(-\frac{l^2}{2}\right)}{3\pi^{3/2}} \left[ \left(2l^4 - 6l^2 + \frac{3}{2}\right) \pi \Phi^2(l\sqrt{2}) + \left(l^4 + 3l^2 + \frac{3}{4}\right) \sqrt{2\pi} \Phi(l\sqrt{2}) \Phi(l) \exp\left(-\frac{l^2}{2}\right) + \sqrt{2\pi} \left(\frac{l^3}{2} + \frac{5l}{4}\right) \Phi(l) \exp\left(-\frac{3l^2}{2}\right) + \left(3l^3 - \frac{13l}{2}\right) \sqrt{\pi} \Phi(l\sqrt{2}) \exp(-l^2) + (l^2 - 2) \exp(-2l^2) \right]. \quad (24)$$

These probability densities are plotted in Figure 2. They all seem very close to a Gaussian density. But only one,  $q_5^2(l)$ , is exactly Gaussian.

### 3.3.2. Mean number of critical points

For  $N = 2$ , using (19) and (10), after some calculations, we check that the mean number of maxima is given by

$$\mathbb{E}(\mathcal{N}_2^c(S)) = \mathbb{E}(\mathcal{N}_0^c(S)) = \frac{1}{4} \mathbb{E}(\mathcal{N}^c(S)) = \frac{1}{2} \mathbb{E}(\mathcal{N}_1^c(S)) = \frac{|S|}{2\sqrt{3}\pi} \left( \frac{\lambda_4}{3\lambda_2} \right),$$

which coincides with Proposition 1.1 of [10].

In Proposition 3.6 we give the exact expression of  $E(\mathcal{N}_k^c(S))$  when  $N = 3$  for  $k = 0, 1, 2, 3$ . This proposition is a new result.

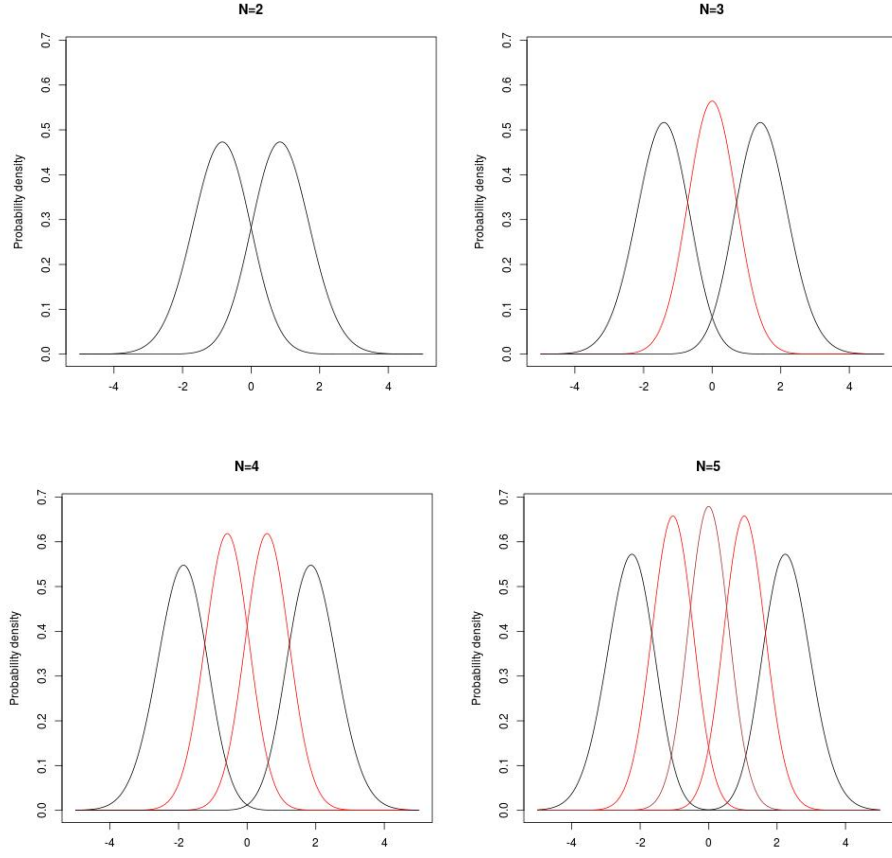


FIGURE 2. Probability densities  $q_N^k(l)$ ,  $k = 1, \dots, N$ , of the ordered eigenvalues of a  $N$  GOE matrix for  $N = 2, 3, 4, 5$ .

**Proposition 3.6.** *Under Assumption (A2) when  $N = 3$ :*

$$\begin{aligned} \mathbb{E}(\mathcal{N}_0^c(S)) &= \mathbb{E}(\mathcal{N}_3^c(S)) = \frac{|S|}{\pi^2 \sqrt{2}} \left( \frac{\lambda_4}{3\lambda_2} \right)^{3/2} \left( \frac{29 - 6\sqrt{6}}{12\sqrt{6}} \right) \\ \mathbb{E}(\mathcal{N}_1^c(S)) &= \mathbb{E}(\mathcal{N}_2^c(S)) = \frac{|S|}{\pi^2 \sqrt{2}} \left( \frac{\lambda_4}{3\lambda_2} \right)^{3/2} \left( \frac{29 + 6\sqrt{6}}{12\sqrt{6}} \right). \end{aligned}$$

Consequently:

$$\frac{\mathbb{E}(\mathcal{N}_0^c(S))}{\mathbb{E}(\mathcal{N}^c(S))} = \frac{\mathbb{E}(\mathcal{N}_3^c(S))}{\mathbb{E}(\mathcal{N}^c(S))} = \frac{29 - 6\sqrt{6}}{116} \simeq 0.1233 \quad \text{and} \quad \frac{\mathbb{E}(\mathcal{N}_1^c(S))}{\mathbb{E}(\mathcal{N}^c(S))} = \frac{\mathbb{E}(\mathcal{N}_2^c(S))}{\mathbb{E}(\mathcal{N}^c(S))} = \frac{29 + 6\sqrt{6}}{116} \simeq 0.3767.$$

In the same way for  $N = 4$ , we obtain the expressions given in Proposition 3.7.

**Proposition 3.7.** *We denote  $\mathcal{I} := \mathbb{E}(\Phi(X)\Phi(\sqrt{2}X))$  where  $X$  is a Gaussian centred variable with*

variance  $1/3$ . Under Assumption (A1) when  $N = 4$ :

$$\begin{aligned} \mathbb{E}(\mathcal{N}_1^c(S)) &= \mathbb{E}(\mathcal{N}_3^c(S)) = \frac{|S|}{\pi^2} \left( \frac{\lambda_4}{3\lambda_2} \right)^2 \left( \frac{25}{24\sqrt{3}} \right), \quad \mathbb{E}(\mathcal{N}_2^c(S)) = \frac{|S|}{\pi^2} \left( \frac{\lambda_4}{3\lambda_2} \right)^2 \left( \frac{50\pi(1-2\mathcal{I}) + 57}{24\sqrt{3}\pi} \right) \\ \mathbb{E}(\mathcal{N}_0^c(S)) &= \mathbb{E}(\mathcal{N}_4^c(S)) = \frac{|S|}{\pi^2} \left( \frac{\lambda_4}{3\lambda_2} \right)^2 \left( \frac{\mathcal{I} \times 100\pi - 57}{48\sqrt{3}\pi} \right). \end{aligned}$$

Consequently:

$$\begin{aligned} \frac{\mathbb{E}(\mathcal{N}_1^c(S))}{\mathbb{E}(\mathcal{N}^c(S))} &= \frac{\mathbb{E}(\mathcal{N}_3^c(S))}{\mathbb{E}(\mathcal{N}^c(S))} = \frac{1}{4}, \quad \frac{\mathbb{E}(\mathcal{N}_0^c(S))}{\mathbb{E}(\mathcal{N}^c(S))} = \frac{\mathbb{E}(\mathcal{N}_4^c(S))}{\mathbb{E}(\mathcal{N}^c(S))} = \frac{\mathcal{I} \times 100\pi - 57}{200\pi} \simeq 0.060, \\ \frac{\mathbb{E}(\mathcal{N}_2^c(S))}{\mathbb{E}(\mathcal{N}^c(S))} &= \frac{50\pi(1-2\mathcal{I}) + 57}{100\pi} \simeq 0.380. \end{aligned}$$

#### 4. Repulsion for random processes ( $\mathbf{N}=1$ )

In this section we start by considering crossings of a level and then interpret critical points as crossings of zero of the derivative.

##### 4.1. Crossings

The process  $\mathcal{Y} = \{Y(t) : t \in \mathbb{R}\}$  that we consider in this subsection will be, in a second step, the derivative of  $\mathcal{X}$ . For this reason we can, in this subsection only, weaken our assumption by assuming only that  $\lambda_6$  is finite. In addition we cannot assume that the variance  $\lambda_0$  is equal to 1. We define respectively the number of crossings  $\mathcal{N}_0$  and the number of up-crossings  $\mathcal{U}_0$  of  $\mathcal{Y}$  on  $[0, T]$ :

$$\begin{aligned} \mathcal{N}_0 &= \#\{t \in [0, T] : Y(t) = 0\} \\ \mathcal{U}_0 &= \#\{t \in [0, T] : Y(t) = 0; Y'(t) > 0\}. \end{aligned}$$

We note  $A(\rho)$  the correlation function of crossings and  $A^u(\rho)$  the correlation function of up-crossings (or down-crossings equivalently). By the Kac-Rice formula we have:

$$\begin{aligned} A(\rho) &= \mathbb{E}(|Y'(0)Y'(\rho)|/Y(0) = Y(\rho) = 0) p_{Y(0), Y(\rho)}(0, 0), \\ A^u(\rho) &= \mathbb{E}(Y'(0)^+ Y'(\rho)^+ / Y(0) = Y(\rho) = 0) p_{Y(0), Y(\rho)}(0, 0). \end{aligned}$$

The result below appears in a hidden way in Proposition 4.5 or Section 5.2 of [7] or in [4].

**Proposition 4.1.** *Under the hypotheses above and as  $\rho \rightarrow 0$*

$$\begin{aligned} A(\rho) &\simeq \frac{(\lambda_4 \lambda_0 - \lambda_2^2)}{8\pi \sqrt{\lambda_2 \lambda_0^3}} \rho, \\ A^u(\rho) &\simeq \frac{(\lambda_6 \lambda_2 - \lambda_4^2)^{3/2}}{1296\pi^2 (\lambda_2^2) (\lambda_4 \lambda_0 - \lambda_2^2)^{1/2}} \rho^4. \end{aligned}$$

*Proof of Proposition 4.1.* We give only a sketch of the proof since most of the ingredients can be found in the cited literature and since a proof will be given for higher dimension in Theorem 5.2 .

Define

$$\begin{aligned} v &:= \text{Var}(Y'(0)/Y(0) = Y(\rho) = 0) = \text{Var}(Y'(\rho)/Y(0) = Y(\rho) = 0), \\ w &:= \text{Cov}(Y'(0), Y'(\rho)/Y(0) = Y(\rho) = 0), \\ D &:= v^2 - w^2. \end{aligned}$$

We have for example from the expression of conditional density

$$\text{Var}(Y'(0)/Y(0) = Y(\rho) = 0) = \frac{\det(\text{Var}(Y(0), Y(\rho), Y'(0)))}{\det(\text{Var}(Y(0), Y(\rho)))}.$$

Recall that  $\text{Var}$  denotes the variance-covariance matrix. And we use the following method (see [11]): because a determinant is invariant by adding to some row (resp. column) a linear combination of other rows (resp. columns) as  $\rho \rightarrow 0$ ,

$$\begin{aligned} \det(\text{Var}(Y(0), Y(\rho), Y'(0))) &= \det(\text{Var}(Y(0), Y'(0), Y(\rho) - Y(0) - \rho Y'(0))) \\ &\simeq \frac{\rho^4}{4} \det(\text{Var}(Y(0), Y'(0), Y''(0))). \end{aligned}$$

In the same fashion

$$\det(\text{Var}(Y(0), Y(\rho))) \simeq \rho^2 \det(\text{Var}(Y(0), Y'(0)))$$

yielding

$$v \simeq \rho^2 \frac{1}{4} \frac{\det(\text{Var}(Y(0), Y'(0), Y''(0)))}{\det(\text{Var}(Y(0), Y'(0)))} \simeq \rho^2 \frac{\lambda_0 \lambda_4 - \lambda_2^2}{4\lambda_0}.$$

By similar calculations

$$D \simeq \rho^6 \frac{1}{144} \frac{\det \text{Var}(Y(0), Y'(0), Y''(0), Y'''(0))}{\det \text{Var}(Y(0), Y'(0))} \simeq \rho^6 \frac{(\lambda_6 \lambda_2 - \lambda_4^2)(\lambda_4 \lambda_0 - \lambda_2^2)}{144 \lambda_0 \lambda_2},$$

$$p_{Y(\rho), Y(0)}(0, 0) \simeq \rho^{-1} \left( 2\pi \sqrt{\det[\text{Var}(Y(\rho), Y'(0))]} \right)^{-1} \simeq \rho^{-1} \left( 2\pi \sqrt{\lambda_2 \lambda_0} \right)^{-1}.$$

By direct calculations we have  $w \simeq -v$ . Using lemma 4.2 hereafter

$$\mathbb{E}(Y'(0)^+ Y'(\rho)^+ / Y(0) = Y(\rho) = 0) \simeq \frac{D^{3/2}}{6\pi v^2}$$

and

$$\begin{aligned} \mathbb{E}(|Y'(0)Y'(\rho)| / Y(0) = Y(\rho) = 0) &= \\ 2 \int_0^{+\infty} \int_0^{+\infty} y'_1 y'_2 p(y'_1, y'_2) dy'_1 dy'_2 &- 2 \int_{-\infty}^0 \int_0^{+\infty} y'_1 y'_2 p(y'_1, y'_2) dy'_1 dy'_2 \simeq v, \end{aligned}$$

where  $p$  is the density of the centred Gaussian vector with variance  $\begin{pmatrix} v & w \\ w & v \end{pmatrix}$ . This achieves the proof.  $\square$

**Lemma 4.2.** *Let  $X, Y$  be two jointly Gaussian variables with common variance  $\sigma^2$  and correlation  $c$ . Let  $r$  a positive real. Then as  $c \rightarrow -1$*

$$\mathbb{E}((X^+ Y^+)^r) \simeq K_r \sigma^{2r} (1 - c^2)^{(2r+1)/2} = K_r \sigma^{-2(1+r)} (\det(\text{Var}(X, Y)))^{(2r+1)/2},$$

where

$$K_r = \frac{1}{2\pi} \int_0^{+\infty} \int_0^{+\infty} x^r y^r \exp\left(-\frac{(x+y)^2}{2}\right) dx dy < +\infty, \quad K_1 = 1/6\pi.$$

*Proof of Lemma 4.2.* By homogeneity we can assume that  $\sigma^2 = 1$ . Then

$$\begin{aligned} E((X^+Y^+)^r) &= \int_0^{+\infty} \int_0^{+\infty} x^r y^r \frac{1}{2\pi\sqrt{1-c^2}} \exp\left(-\frac{x^2 - 2cxy + y^2}{2(1-c^2)}\right) dx dy \\ &\simeq \frac{1}{2\pi} (1-c^2)^{(2r+1)/2} \int_0^{+\infty} \int_0^{+\infty} x^r y^r \exp\left(-\frac{(x+y)^2}{2}\right) dx dy, \end{aligned}$$

where we have made the change of variables  $x = x'(1-c^2)^{1/2}$ ,  $y = y'(1-c^2)^{1/2}$  and the passage to the limit is justified because the integrand is a monotone function of  $c$ . It is easy to check the convergence of the integral.  $\square$

#### 4.2. Critical points

By applying Proposition 4.1 to the derivatives of  $\mathcal{X}$  we get directly the following proposition

**Proposition 4.3.** *Let  $A^c(\rho)$  be the correlation function between critical points and  $A^e(\rho)$  the correlation function between maxima (or minima equivalently). Under Assumption (A4) and as  $\rho \rightarrow 0$*

$$A^c(\rho) \simeq \frac{(\lambda_2\lambda_6 - \lambda_4^2)}{8\pi\sqrt{\lambda_4\lambda_2^3}} \rho, \quad (25)$$

$$A^e(\rho) \simeq \frac{(\lambda_8\lambda_4 - \lambda_6^2)^{3/2}}{1296\pi^2(\lambda_4^2)(\lambda_6\lambda_2 - \lambda_4^2)^{1/2}} \rho^4. \quad (26)$$

The interpretation of the proposition is that we have always repulsion between critical points and a very strong repulsion between extrema. As we will see, the surprising result is that does not remain true in higher dimension.

### 5. Correlation function between critical points for random fields

#### 5.1. All the critical points

Consider  $\mathcal{X} = \{X(t) : t \in \mathbb{R}^N\}$  a stationary and isotropic Gaussian field with real values as defined in Section 2. Let us consider two points  $s, t \in \mathbb{R}^N$ . Because of stationarity and isotropy we can limit our attention to  $s = 0, t = \rho e_1$ , where  $e_1$  is the first vector of the basis. Let  $\xi(0), \xi(t)$  a representation of the distribution of  $\nabla^2 X(0), \nabla^2 X(t)$  conditional to  $\nabla X(t) = \nabla X(s) = 0$ . Before giving our first main result we present in Lemma 5.1 the result that we need on the distribution of  $(\xi_{ij}(t), \xi_{kl}(s), i, j, k, l = 1 \cdots, N)$ . The proof of this lemma is detailed in the appendix.

**Lemma 5.1.** *Under Assumption (A3), for  $j \neq k$  and  $j, k \neq 1$ , as  $\rho = \|t\| \rightarrow 0$ :*

$$\det(\text{Var}(\nabla X(0), \nabla X(t))) \simeq \rho^{2N} \frac{\lambda_2^N \lambda_4^N}{3^{N-1}}, \quad (27)$$

$$\text{Var}(\xi_{11}(t)) \simeq \frac{\rho^2}{4} \frac{(\lambda_6\lambda_2 - \lambda_4^2)}{\lambda_2}, \quad \text{Var}(\xi_{1j}(t)) \simeq \frac{\rho^2}{4} \frac{(9\lambda_6\lambda_2 - 5\lambda_4^2)}{45\lambda_2}, \quad \text{Var}(\xi_{jj}(t)) \simeq \frac{8\lambda_4}{9}, \quad \text{Var}(\xi_{jk}(t)) \simeq \frac{\lambda_4}{3}, \quad (28)$$

$$\text{Cov}(\xi_{11}(t), \xi_{jj}(t)) \simeq \rho^2 \frac{11\lambda_2\lambda_6 - 15\lambda_4^2}{180\lambda_2}, \quad \text{Cov}(\xi_{jj}(t), \xi_{kk}(t)) \simeq \frac{2\lambda_4}{9}, \quad (29)$$

$$\text{All the other covariances } \text{Cov}(\xi_{il}(t), \xi_{mn}(t)) \text{ are zero, } \forall i, l, m, n \in \{1, \dots, N\}. \quad (30)$$

We have of course the same relations for  $\xi(0)$ .

Moreover we have  $\forall j, k \in \{2, \dots, N\}$  and  $\forall i \in \{1, \dots, N\}$ ; as  $\rho \rightarrow 0$ ,

$$\text{Cov}(\xi_{jk}(0), \xi_{jk}(t)) \simeq \text{Var}(\xi_{jk}(t)), \quad \text{Cov}(\xi_{1i}(0), \xi_{1i}(t)) \simeq -\text{Var}(\xi_{1i}(t)), \quad (31)$$

$$\text{Cov}(\xi_{11}(0), \xi_{jj}(t)) \simeq \rho^2 \frac{15\lambda_4^2 - \lambda_2\lambda_6}{180\lambda_2}, \quad \text{Cov}(\xi_{jj}(0), \xi_{kk}(t)) \simeq \frac{2\lambda_4}{9} \text{ for } j \neq k, \quad (32)$$

$$\text{All the other covariances } \text{Cov}(\xi_{il}(0), \xi_{mn}(t)) \text{ are zero, } \forall i, l, m, n \in \{1, \dots, N\} \quad (33)$$

Our first main result, Theorem 5.2 below, gives the asymptotic expression (as  $\rho \rightarrow 0$ ) of the correlation function between critical points of any isotropic Gaussian field. It generalizes the result of [10] to general fields and any dimension.

**Theorem 5.2.** *Let  $A^c(\rho)$  be the correlation function between critical points. Under Assumption (A3) and as  $\rho \rightarrow 0$ ,*

$$A^c(\rho) \simeq \rho^{2-N} \frac{\gamma_{N-1}}{2^3 3^{(N-1)/2} \pi^N} \left( \sqrt{\frac{\lambda_4}{\lambda_2}} \right)^N \frac{(\lambda_2\lambda_6 - \lambda_4^2)}{\lambda_2\lambda_4}, \quad (34)$$

where  $\gamma_{N-1}$  is defined by

$$\gamma_{N-1} := \mathbb{E}(\det^2(G_{N-1} - \Lambda Id_{N-1})) = \mathbb{E}(\gamma_{N-1,2}(\Lambda)),$$

with  $G_{N-1}$  a  $(N-1)$ -GOE matrix,  $\Lambda$  an independent Gaussian random variable with variance  $1/3$  and  $\gamma_{N-1,2}$  is defined in (54). Expressions for  $\gamma_{N-1,2}$  are given in Appendix A, Lemma A.1.

**Remarks:**

- Of course when  $N = 1$ , we retrieve (25).
- In the particular case of the random plane wave ( $N = 2$ ), our result coincides with the result of [10], formula (4).
- Theorem 5.2 means that, for  $N = 2$ , there is a neutrality between critical points and for  $N > 2$  there is even attraction! This is quite different from the case  $N = 1$ . The next theorem will give an interpretation of this phenomenon.
- A first important consequence is finiteness of the second moment of the number of critical points. Indeed if  $S$  is a compact set of  $\mathbb{R}^N$  we can write the Kac-Rice formulas of order 1 and 2, (5), (1), (2). If  $\mathcal{N}^c(S)$  is the number of critical points that belong to  $S$  then

$$\mathbb{E}(\mathcal{N}^c(S)) = |S| \mathbb{E}(|\det \nabla^2 X(t)|) p_{\nabla X(t)}(0)$$

and

$$\mathbb{E}((\mathcal{N}^c(S))(\mathcal{N}^c(S) - 1)) = \int_{S^2} A^c(|s - t|) ds dt. \quad (35)$$



Since  $\lambda_4$  is assumed to be finite, the expectation is always finite. As for the second factorial moment, its finiteness is in general not systematic. In our case (34) implies the convergence of the integral in (35) on the diagonal implying in turn the finiteness of the second moment. Nevertheless our conditions are less general than [20] or [5].

*Proof of Theorem 5.2.* For the simplicity of the exposition we admit in a first step all passage-to-the-limits. They will be justified at the end of the proof.

By (2)

$$A^c(\rho) = \mathbb{E}(|\det(\xi(0)) \det(\xi(t))|) p_{\nabla X(0), \nabla X(t)}(0, 0), \quad (36)$$

with  $t = \rho e_1$ . We recall that  $\xi(0)$  and  $\xi(t)$  are a representation of the conditional distribution of the Hessian at 0 and  $t$ . Because of (27),

$$p_{\nabla X(0), \nabla X(t)}(0, 0) \simeq \rho^{-N} 3^{(N-1)/2} \lambda_2^{-N/2} \lambda_4^{-N/2} (2\pi)^{-N}. \quad (37)$$

It remains to study the expectation in (36). We denote by  $\xi_{-1}(t)$  the  $(N-1) \times (N-1)$  matrix corresponding to the Hessian matrix  $\xi(t)$  without its first row and column.

Let us develop  $\det(\xi(t))$  with respect to the first row, for example. The first term is  $\xi_{11}(t) \det(\xi_{-1}(t))$ . Consider another term and develop it now with respect to the first column. Each of the  $N-1$  terms that appear are products that include a factor  $\xi_{1j} \xi_{j'1}$  with  $j, j' \neq 1$  so because of (28) they are  $O_p(\rho^2)$ . As a consequence we have proved that

$$\det(\xi(t)) = \xi_{11}(t) \det(\xi_{-1}(t)) + O_p(\rho^2).$$

By (29) we have:

$$\xi_{-1}(t) \simeq \sqrt{\frac{2\lambda_4}{3}} (G_{N-1} + \Lambda Id_{N-1}), \quad (38)$$

where  $\Lambda$  is a  $\mathcal{N}(0, 1/3)$  random variable and  $G_{N-1}$  is a size  $(N-1)$  GOE matrix defined previously. So:

$$\det(\xi_{-1}(t)) \simeq \left(\frac{2\lambda_4}{3}\right)^{(N-1)/2} \det(G_{N-1} - \Lambda Id_{N-1}). \quad (39)$$

The order of magnitude of the first term in the development of the determinant is then  $\rho$ .

In conclusion we have proved that

$$\begin{aligned} \det(\xi(t)) &= O_p(\rho) = \xi_{11}(t) \det(\xi_{-1}(t)) + O_p(\rho^2) \\ \det(\xi(0)) \det(\xi(t)) &= O_p(\rho^2) = \xi_{11}(0) \xi_{11}(t) \det(\xi_{-1}(0)) \det(\xi_{-1}(t)) + O_p(\rho^3). \end{aligned} \quad (40)$$

By (31) and (32)

$$\begin{aligned} \text{Corr}(\xi_{11}(0), \xi_{11}(t)) &\rightarrow -1, \quad \text{Corr}(\xi_{jk}(t), \xi_{jk}(0)) \rightarrow 1, \quad \forall j, k \in 2, \dots, N, \\ \text{Cov}(\xi_{11}(t), \xi_{jj}(t)) &= O(\rho^2), \quad \text{Cov}(\xi_{11}(t), \xi_{jk}(t)) = 0 \quad \forall j \neq k \in 1, \dots, N. \end{aligned}$$

Then:

$$\begin{aligned} \det(\xi(0)) \det(\xi(t)) &\simeq -\xi_{11}(t)^2 \det^2(\xi_{-1}(t)) \\ \mathbb{E}(|\det(\xi(0)) \det(\xi(t))|) &\simeq \text{Var}(\xi_{11}(t)) \mathbb{E}(\det^2(\xi_{-1}(t))), \end{aligned} \quad (41)$$

where the second equality uses (29) and (30). Eventually equations (28), (39) and (54) give (34).

Passage-to-the-limit: Since we are in the Gaussian space generated by the random field  $\mathcal{X}$  all the variables considered above are jointly Gaussian. So their absolute values are bounded by the maximum in absolute value of a complicated bounded random field. General results about the maximum of random fields, for example the Borell-Sudakov-Tsirelson theorem ([7] Section 2.4) implies that the maximum of its absolute value has moments of every order, giving all the dominations needed.  $\square$

### 5.2. Correlation function between critical points with a given index

**Theorem 5.3.** *Let  $A^{k,k+1}(\rho)$  be the correlation function between critical points with index  $k$  and critical points with index  $k+1$ ,  $k=0, \dots, N-1$ . Under Assumption (A3), as  $\rho \rightarrow 0$ ,*

$$A^{k,k+1}(\rho) \simeq \rho^{2-N} \frac{\gamma_{N-1}^k}{2^4 3^{(N-1)/2} \pi^N} \left( \sqrt{\frac{\lambda_4}{\lambda_2}} \right)^N \frac{(\lambda_2 \lambda_6 - \lambda_4^2)}{\lambda_2 \lambda_4}, \quad (42)$$

where  $\gamma_{N-1}^k$  is defined by

$$\gamma_{N-1}^k := \mathbb{E} \left( \det^2 (G_{N-1} - \Lambda \text{Id}_{N-1}) \mathbb{1}_{\{i(G_{N-1} - \Lambda \text{Id}_{N-1})=k\}} \right) = \mathbb{E} \left( \gamma_{N-1,2}^k(\Lambda) \right),$$

with  $G_{N-1}$  a  $(N-1)$ -GOE matrix,  $\Lambda$  an independent Gaussian random variable with variance  $1/3$  and  $\gamma_{N-1,2}^k$  is defined in (55). Expressions for  $\gamma_{N-1,2}^k$  are given in Appendix A, Lemma A.2, equation (57).

This theorem gives an interpretation to Theorem 5.2: the attraction for  $N \geq 3$  is in fact due to attraction between critical points with adjacent indexes.

Again, in the particular case of the random plane wave ( $N=2$ ), our result coincides with the result of [10], formula (9).

*Proof of Theorem 5.3.* We can adapt the application of the Kac-Rice formula that leads to (2) to get that

$$\begin{aligned} A^{k,k+1}(\rho) &= p_{\nabla X(0), \nabla X(t)}(0,0) \mathbb{E} \left( |\det \xi(0)| \mathbb{1}_{\{i(\xi(0))=k\}} |\det \xi(t)| \mathbb{1}_{\{i(\xi(t))=k+1\}} \right) \\ &= -p_{\nabla X(0), \nabla X(t)}(0,0) \mathbb{E} \left( \det(\xi(0)) \mathbb{1}_{\{i(\xi(0))=k\}} \det \xi(t) \mathbb{1}_{\{i(\xi(t))=k+1\}} \right). \end{aligned}$$

We can modify the computation of the determinant of  $\xi(0)$  in (40) to compute in place  $\det(\xi(0) - \lambda \text{Id}_N)$ :

$$\det(\xi(0) - \lambda \text{Id}_N) = (\xi_{11}(0) - \lambda) \det(\xi_{-1}(0) - \lambda \text{Id}_{N-1}) + o_p(\rho^2). \quad (43)$$

Note that as  $\rho \rightarrow 0$ ,  $\xi(0)$  converges in  $L^2$  (or a.s. if we use a Skorohod imbedding argument) to

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \xi_{-1}(0) & \\ 0 & & & \end{pmatrix}.$$

Its eigenvectors converge to those of this last matrix. These eigenvectors are associated to different eigenvalues so we can define properly the eigenvalue  $\mu_1$  that tends to zero. Because of (43) with  $\lambda = \mu_1$ :

$$(\xi_{11}(0) - \mu_1) \det(\xi_{-1}(0) - \lambda \text{Id}_{N-1}) = o_p(\rho^2),$$

implying in turn that

$$\mu_1 = \xi_{11}(0) + o_p(\rho^2).$$

On the other hand, the others eigenvalues, say  $\mu_2, \dots, \mu_N$ , converge in distribution to that given by the right-hand side of (38). This implies in turn that

$$\mathbb{1}_{\{i(\xi(0))=k\}} \simeq \mathbb{1}_{\{\xi_{11}(0) < 0; i(\xi_{-1}(0))=k-1\}} + \mathbb{1}_{\{\xi_{11}(0) > 0; i(\xi_{-1}(0))=k\}}.$$

As a consequence, when computing  $E(\det(\xi(0)) \mathbb{1}_{\{i(\xi(0))=k\}} \det(\xi(t)) | \mathbb{1}_{\{i(\xi(t))=k+1\}})$ , we have four cases to consider depending on the signs of  $\xi_{11}(0), \xi_{11}(t)$ . First we consider the case  $\xi_{11}(0) > 0, \xi_{11}(t) < 0$ . We have the equivalent of (41)

$$\det(\xi(0)) \det(\xi(t)) \mathbb{1}_{\{\xi_{11}(0) > 0, \xi_{11}(t) < 0\}} = -\xi_{11}(t)^2 \mathbb{1}_{\{\xi_{11}(t) < 0\}} \det^2(\xi_{-1}(t)) + o_p(\rho^3),$$

giving

$$E(|\det(\xi(t)) \det(\xi(s))| \mathbb{1}_{\{\xi_{11}(0) > 0, \xi_{11}(t) < 0\}}) = 1/2 \text{Var}(\xi_{11}(t)) E(\det^2(\xi_{-1}(t))) + o(\rho^3).$$

This gives (42) as soon as we have checked that the three other cases give smaller contributions which is direct.  $\square$

We give now some bound for the case of extremes indexes: maxima and minima. We believe that, in fact, this bound is sharp but we were unable to find a simple proof of that.

- Assumption (B1): The distribution of  $(X_1(0), X_{111}(0), X_{221}(0), X_{331}(0), \dots, X_{NN1}(0))$  is non degenerated. This condition is equivalent to (see (50)):

$$\mathcal{B}_1 := 3(N+4)\lambda_2\lambda_6 - 5(N+2)\lambda_4^2 > 0.$$

- Assumption (B2): The distribution of  $(X_{1111}(0), X_{11}(0), X_{22}(0), \dots, X_{NN}(0))$  is non degenerated. This condition is equivalent to (see (53)):

$$\mathcal{B}_2 := 25(N+2)\lambda_4\lambda_8 - 3(9N+16)\lambda_6^2 > 0.$$

**Theorem 5.4.** *Let  $A^{0,N}(\rho)$  be the correlation function between minima and maxima. Under Assumptions (A4) and (B1), as  $\rho \rightarrow 0$ ,*

$$A^{0,N}(\rho) \leq K_1(N) \times \rho^{2N-1}, \quad (44)$$

where  $K_1(N)$  is a constant depending on  $N$ .

**Theorem 5.5.** *Let  $A^{N,N}(\rho)$  (or  $A^{0,0}(\rho)$ ) be the correlation function between maxima (or minima). Under Assumption (A4) and (B2), as  $\rho \rightarrow 0$ ,*

$$A^{0,0}(\rho) = A^{N,N}(\rho) \leq K_2(N) \times \rho^{5-N}, \quad (45)$$

where  $K_2(N)$  is a constant depending on  $N$ .

Note that, for  $N = 1$ , (44) and (45) retrieve the behavior in  $\rho$  and  $\rho^4$  obtained in (25) and (26).

In the particular case of the random plane wave ( $N = 2$ ), the bound (44) is weaker than [10], formula (8), while (45) is sharper than [10], formula (5).

Before giving the proofs of Theorems 5.4 and 5.5 we give the following lemma which is obtained by derivation as in Lemma 3.2

**Lemma 5.6.** *Under Assumption (A4)*

$$\begin{aligned} & \text{Var}(X_2, \dots, X_N, X_{12}, X_{13}, \dots, X_{1N}, X_{22}, X_{33}, \dots, X_{NN}, X_{11}, X_{1111}, X_1, X_{122}, X_{133}, \dots, X_{1NN}, X_{111}) \\ &= \text{diag} \left( \lambda_2 Id_{N-1}, \frac{\lambda_4}{3} Id_{N-1}, M_{(N+1)}, \tilde{M}_{(N+1)} \right), \end{aligned}$$

where

$$M_{(N+1)} = \text{Var}(X_{22}, X_{33}, \dots, X_{NN}, X_{11}, X_{1111}) = \begin{pmatrix} & & & & -\lambda_6/5 \\ & \left(\frac{2\lambda_4}{3} Id_N + \frac{\lambda_4}{3} J_{N,N}\right) & & & \vdots \\ & & & & -\lambda_6/5 \\ -\lambda_6/5 & \dots & -\lambda_6/5 & -\lambda_6 & \\ & & & & \lambda_8 \end{pmatrix}$$

and

$$\tilde{M}_{(N+1)} = \text{Var}(X_1, X_{122}, X_{133}, \dots, X_{1NN}, X_{111}) = \begin{pmatrix} \lambda_2 & -\lambda_4/3 & \dots & -\lambda_4/3 & -\lambda_4 \\ -\lambda_4/3 & & & & \lambda_6/5 \\ \vdots & \left(\frac{2\lambda_6}{15} Id_N + \frac{\lambda_6}{15} J_{N,N}\right) & & & \vdots \\ -\lambda_4/3 & & & & \lambda_6/5 \\ -\lambda_4 & \lambda_6/5 & \dots & \lambda_6/5 & \lambda_6 \end{pmatrix}.$$

*Proof.* The result is obtained by derivation as for establishing (7).  $\square$

*Proof of Theorem 5.4.* We use the classical result of linear algebra. If  $M$  is a positive semidefinite matrix of size  $N$ :

$$\det(M) \leq M_{11} \times \dots \times M_{NN}.$$

As a consequence for any symmetric matrix  $M'$ :

$$\det(M') \mathbf{1}_{\{i(M')=0\}} \leq (M_{11})^+ \times \dots \times (M_{NN})^+. \quad (46)$$

By Kac-Rice formula (4):

$$A^{0,N}(\rho) = p_{\nabla X(0), \nabla X(t)}(0, 0) \mathbb{E} \left( |\det \xi(0)| \mathbf{1}_{\{i(\nabla^2 \xi(0))=0\}} |\det \xi(t)| \mathbf{1}_{\{i(\nabla^2 \xi(t))=N\}} \right).$$

We have:

$$A^{0,N}(\rho) \leq (-1)^N p_{\nabla X(0), \nabla X(t)}(0, 0) \mathbb{E} \left( \xi_{11}^+(0) \xi_{22}^+(0) \dots \xi_{NN}^+(0) \xi_{11}^-(t) \xi_{22}^-(t) \dots \xi_{NN}^-(t) \right).$$

By Cauchy-Schwarz inequality and symmetry of the role of 0 and  $t$  we obtain:

$$A^{0,N}(\rho) \leq (-1)^{N-1} p_{\nabla X(0), \nabla X(t)}(0, 0) \mathbb{E} \left( \xi_{11}^+(0)^2 \xi_{22}^+(0) \dots \xi_{NN}^+(0) \xi_{22}^-(t) \dots \xi_{NN}^-(t) \right).$$

By (37) we have  $p_{\nabla X(0), \nabla X(t)}(0, 0) = O(\rho^{-N})$ . Let us now prove that

$$I_\rho := \mathbb{E} \left( \xi_{11}^+(0)^2 \xi_{22}^+(0) \dots \xi_{NN}^+(0) \xi_{22}^-(t) \dots \xi_{NN}^-(t) \right) = O(\rho^{3N-1}). \quad (47)$$

We set  $\tilde{\xi} := (\xi_{11}(0), \xi_{22}(0), \xi_{22}(t), \xi_{33}(0), \xi_{33}(t), \dots, \xi_{NN}(0), \xi_{NN}(t))$ ,  $\tilde{\xi}_i$  the  $i$ th coordinate of  $\tilde{\xi}$  for  $i = 1, \dots, 2N-1$ ,  $\tilde{\xi}_{-i}$  (resp.  $\tilde{\xi}_{-(i,j)}$ ) the vector  $\tilde{\xi}$  without its  $i$ th (resp.  $i$ th and  $j$ th) coordinate(s). We set  $\Sigma_\rho := \text{Var}(\tilde{\xi})$ .

• Equivalent of  $\det(\Sigma_\rho)$  as  $\rho \rightarrow 0$

$$\begin{aligned} \det \Sigma_\rho &= \det \text{Var}(\xi_{11}(0), \xi_{22}(0), \xi_{22}(t), \xi_{33}(0), \xi_{33}(t), \dots, \xi_{NN}(0), \xi_{NN}(t)) \\ &= \frac{\det \text{Var}(X_{11}(0), X_{22}(0), X_{22}(t), \dots, X_{NN}(0), X_{NN}(t), X_1(0), \dots, X_N(0), X_1(t), \dots, X_N(t))}{\det \text{Var}(X_1(0), \dots, X_N(0), X_1(t), \dots, X_N(t))}. \end{aligned}$$

For short we set  $X_i := X_i(0)$  and  $X_{ij} := X_{ij}(0)$ . By analogous techniques as those used in the proof of Lemma 5.1 we obtain:

$$\begin{aligned} \det \Sigma_\rho &\simeq \frac{\rho^{2N}}{4} \frac{\det \text{Var}(X_1, \dots, X_N, X_{11}, X_{22}, \dots, X_{NN}, X_{21}, \dots, X_{N1}, X_{111}, X_{221}, \dots, X_{NN1})}{\det \text{Var}(X_1, \dots, X_N, X_{11}, \dots, X_{N1})} \\ &= \frac{\rho^{2N}}{4} \frac{\det \text{Var}(X_2, \dots, X_N, X_{21}, \dots, X_{N1}, X_{11}, \dots, X_{NN}) \times \det \text{Var}(X_{221}, \dots, X_{NN1}, X_1, X_{111})}{\det \text{Var}(X_1, \dots, X_N, X_{11}, \dots, X_{N1})}. \end{aligned}$$

By Lemma 5.6 we have:

$$\begin{aligned} \det \text{Var}(X_1, \dots, X_N, X_{11}, \dots, X_{N1}) &= \lambda_2^N \lambda_4 \left( \frac{\lambda_4}{3} \right)^{N-1} \\ \det \text{Var}(X_2, \dots, X_N, X_{21}, \dots, X_{N1}, X_{11}, \dots, X_{NN}) &= (\lambda_2)^{N-1} \left( \frac{\lambda_4}{3} \right)^{N-1} \times \det \text{Var}(X_{11}, \dots, X_{NN}). \end{aligned}$$

Let  $J_N$  be the  $(N \times N)$ -matrix with all elements equal to 1. By Lemma 5.6 we have:

$$\text{Var}(X_{11}, X_{22}, \dots, X_{NN}) = \frac{2\lambda_4}{3} Id_N + \frac{\lambda_4}{3} J_N.$$

For  $x, y \in \mathbb{R}$  it is well known that:

$$\det(xId_N + yJ_N) = x^{N-1}(x + Ny). \quad (48)$$

So we have:

$$\det \text{Var}(X_{11}, X_{22}, \dots, X_{NN}) = \left( \frac{2\lambda_4}{3} \right)^{N-1} \left( \frac{2\lambda_4}{3} + N \frac{\lambda_4}{3} \right).$$

By Lemma 5.6 we have:

$$\det \text{Var}(X_1, X_{111}, X_{221}, \dots, X_{NN1}) = \det \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}$$

$$\text{with } \tilde{A}_{11} = \begin{bmatrix} \lambda_2 & -\lambda_4 \\ -\lambda_4 & \lambda_6 \end{bmatrix}, \tilde{A}_{12} = \begin{bmatrix} -\lambda_4/3 & \cdots & -\lambda_4/3 \\ \lambda_6/5 & \cdots & \lambda_6/5 \end{bmatrix}, \tilde{A}_{21} = \tilde{A}_{12}^\top \text{ and } \tilde{A}_{22} = \frac{2\lambda_6}{15} Id_{N-1} + \frac{\lambda_6}{15} J_{N-1}.$$

Then using the fact that for a partitioned matrix  $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$  we have (see [29] p.46):

$$\det A = \det A_{11} \times \det(A_{22} - A_{21}A_{11}^{-1}A_{12}) \quad (49)$$

we obtain

$$\det \text{Var}(X_1, X_{111}, X_{221}, \dots, X_{NN1}) = (\lambda_2 \lambda_6 - \lambda_4^2) \times \det \left( \frac{2\lambda_6}{15} Id_{N-1} + J_{N-1} \left( \frac{6\lambda_2 \lambda_6^2 - 10\lambda_6 \lambda_4^2}{225(\lambda_2 \lambda_6 - \lambda_4^2)} \right) \right)$$

and finally using (48)

$$\det \text{Var}(X_1, X_{111}, X_{221}, \dots, X_{NN1}) = \left( \frac{2\lambda_6}{15} \right)^{N-2} \frac{6(N+4)\lambda_2 \lambda_6^2 - 10(N+2)\lambda_6 \lambda_4^2}{225}. \quad (50)$$

Therefore:

$$\det \Sigma_\rho \simeq \rho^{2N} \frac{(N+2)2^{2N-4}}{9^N 5^N} \frac{\lambda_6^{N-1} \lambda_4^{N-1}}{\lambda_2} (3(N+4)\lambda_2 \lambda_6 - 5(N+2)\lambda_4^2) =: \rho^{2N} g.$$

Note that, under Assumption (B1),  $g > 0$ .

- Equivalent of  $\Sigma_\rho^{-1}$  as  $\rho \rightarrow 0$

Using the classical expression of conditional variance and covariance

$$\begin{aligned} (\Sigma_\rho^{-1})_{ii} &= \frac{1}{\text{Var}(\tilde{\xi}_i/\tilde{\xi}_{-i})} = \frac{\det \text{Var}(\tilde{\xi}_{-i})}{\det \text{Var}(\tilde{\xi})} \\ (\Sigma_\rho^{-1})_{ij} &= \frac{-\text{Cov}(\tilde{\xi}_i, \tilde{\xi}_j/\tilde{\xi}_{-(i,j)})}{\det \text{Var}(\tilde{\xi}_i, \tilde{\xi}_j/\tilde{\xi}_{-(i,j)})} = \frac{\text{Var}(\tilde{\xi}_i/\tilde{\xi}_{-(i,j)}) + \text{Var}(\tilde{\xi}_j/\tilde{\xi}_{-(i,j)}) - \text{Var}(\tilde{\xi}_i + \tilde{\xi}_j/\tilde{\xi}_{-(i,j)})}{2 \det \text{Var}(\tilde{\xi}_i, \tilde{\xi}_j/\tilde{\xi}_{-(i,j)})} \\ &= \frac{\det \text{Var}(\tilde{\xi}_{-j}) + \det \text{Var}(\tilde{\xi}_{-i}) - \det \text{Var}(\tilde{\xi}_i + \tilde{\xi}_j, \tilde{\xi}_{-(i,j)})}{2 \det \text{Var}(\tilde{\xi})}. \end{aligned}$$

We set  $\tilde{X} := (X_{11}(0), X_{22}(0), X_{22}(t), X_{33}(0), X_{33}(t), \dots, X_{NN}(0), X_{NN}(t))$ . We obtain:

$$\begin{aligned} (\Sigma_\rho^{-1})_{ii} &= \frac{\det \text{Var}(\tilde{X}_{-i}/\nabla X(0), \nabla X(t))}{\det \text{Var}(\tilde{X}/\nabla X(0), \nabla X(t))} \\ &= \frac{\det \text{Var}(\tilde{X}_{-i}, \nabla X(0), \nabla X(t))}{\det \text{Var}(X, \nabla X(0), \nabla X(t))}. \end{aligned}$$

Since  $\tilde{X}$  is independent of  $X_2(0), \dots, X_N(0), X_2(t), \dots, X_N(t)$  (see Section B.1.3) we get:

$$(\Sigma_\rho^{-1})_{ii} = \frac{\det \text{Var}(\tilde{X}_{-i}, X_1(0), X_1(t))}{\det \text{Var}(\tilde{X}, X_1(0), X_1(t))}$$

and:

$$(\Sigma_\rho^{-1})_{ij} = \frac{\det \text{Var}(\tilde{X}_{-j}, X_1(0), X_1(t)) + \det \text{Var}(\tilde{X}_{-i}, X_1(0), X_1(t)) - \det \text{Var}(\tilde{X}_i + \tilde{X}_j, \tilde{X}_{-(i,j)}, X_1(0), X_1(t))}{2 \det \text{Var}(\tilde{X}, X_1(0), X_1(t))}.$$

By analogous techniques as those used in the proof of Lemma 5.1 we obtain for  $j = 2, \dots, N$ ; as  $\rho \rightarrow 0$ :

$$\begin{aligned} (\Sigma_\rho^{-1})_{11} &\simeq \frac{1}{\rho^2} \frac{4 \det \text{Var}(X_1, X_{221}, \dots, X_{NN1})}{\det \text{Var}(X_1, X_{111}, X_{221}, \dots, X_{NN1})} =: \frac{1}{\rho^2} a \\ (\Sigma_\rho^{-1})_{jj} &\simeq \frac{1}{\rho^2} \frac{\det \text{Var}(X_1, X_{331}, \dots, X_{NN1}, X_{111})}{\det \text{Var}(X_1, X_{111}, X_{221}, \dots, X_{NN1})} =: \frac{1}{\rho^2} b. \end{aligned}$$

Note that  $a$  and  $b$  are positive because of the non-degeneracy hypothesis (B1). For the covariances:

$$\begin{aligned} (\Sigma_\rho^{-1})_{23} &\simeq -\frac{1}{\rho^2} b \\ (\Sigma_\rho^{-1})_{12} &\simeq \frac{2}{\rho^2} \left( \frac{a+b}{4} - \frac{\det \text{Var}(X_1, X_{331}, \dots, X_{NN1}, X_{221} + X_{111}/2)}{\det \text{Var}(X_1, X_{221}, \dots, X_{NN1}, X_{111})} \right) =: \frac{1}{\rho^2} d \\ (\Sigma_\rho^{-1})_{13} &\simeq \frac{2}{\rho^2} \left( \frac{a+b}{4} - \frac{\det \text{Var}(X_1, X_{331}, \dots, X_{NN1}, X_{221} - X_{111}/2)}{\det \text{Var}(X_1, X_{221}, \dots, X_{NN1}, X_{111})} \right) =: \frac{1}{\rho^2} c \\ (\Sigma_\rho^{-1})_{24} &\simeq \frac{1}{\rho^2} \left( b - \frac{1}{2} \frac{\det \text{Var}(X_1, X_{441}, \dots, X_{NN1}, X_{111}, X_{221} + X_{331})}{\det \text{Var}(X_1, X_{221}, \dots, X_{NN1}, X_{111})} \right) =: \frac{1}{\rho^2} e \\ (\Sigma_\rho^{-1})_{25} &\simeq \frac{1}{\rho^2} \left( b - \frac{1}{2} \frac{\det \text{Var}(X_1, X_{441}, \dots, X_{NN1}, X_{111}, X_{221} - X_{331})}{\det \text{Var}(X_1, X_{221}, \dots, X_{NN1}, X_{111})} \right) =: \frac{1}{\rho^2} f \end{aligned}$$

Moreover it is easy to verify that for  $i = 1, \dots, N - 1$ :

$$(\Sigma_\rho^{-1})_{2i,2i+1} = (\Sigma_\rho^{-1})_{23},$$

for  $i = 2, \dots, N$ :

$$(\Sigma_\rho^{-1})_{1,2i-2} = (\Sigma_\rho^{-1})_{12}, (\Sigma_\rho^{-1})_{1,2i-1} = (\Sigma_\rho^{-1})_{13},$$

for  $i = 1, \dots, N - 2$ :

$$(\Sigma_\rho^{-1})_{2i,2i+2} = (\Sigma_\rho^{-1})_{24}, (\Sigma_\rho^{-1})_{2i,2i+3} = (\Sigma_\rho^{-1})_{25}, (\Sigma_\rho^{-1})_{2i+1,2i+2} = (\Sigma_\rho^{-1})_{2i,2i+3}, (\Sigma_\rho^{-1})_{2i+1,2i+3} = (\Sigma_\rho^{-1})_{2i,2i+2}.$$

Now, we compute  $a, b, c, d, e, f$ . For  $a$ , we have:

$$\det \text{Var}(X_1, X_{221}, \dots, X_{NN1}) = \det \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^\top & A_{22} \end{bmatrix}$$

with  $A_{11} = \lambda_2$ ,  $A_{12} = [ -\lambda_4/3 \quad \dots \quad -\lambda_4/3 ]$  and  $A_{22} = \frac{2\lambda_6}{15} Id_N + \frac{\lambda_6}{15} J_N$ . Using (48) and (49) we obtain:

$$\det \text{Var}(X_1, X_{221}, \dots, X_{NN1}) = \left( \frac{2\lambda_6}{15} \right)^{N-2} \frac{3(N+1)\lambda_2\lambda_6 - 5(N-1)\lambda_4^2}{45}.$$

Using (50) we deduce:

$$a = \frac{10(3(N+1)\lambda_2\lambda_6 - 5(N-1)\lambda_4^2)}{\lambda_6(3(N+4)\lambda_2\lambda_6 - 5(N+2)\lambda_4^2)}.$$

Following the same lines we obtain:

$$b = \frac{15(3(N+3)\lambda_2\lambda_6 - 5(N+1)\lambda_4^2)}{2\lambda_6(3(N+4)\lambda_2\lambda_6 - 5(N+2)\lambda_4^2)},$$

and:

$$c = -d = -2e = 2f = \frac{15(3\lambda_2\lambda_6 - 5\lambda_4^2)}{\lambda_6(3(N+4)\lambda_2\lambda_6 - 5(N+2)\lambda_4^2)}.$$

Because of our hypotheses  $a, b > 0$ , the sign of  $c$  is not determined. Therefore:

$$\Sigma_\rho^{-1} \simeq \frac{1}{\rho^2} \tilde{\Sigma}^{-1}$$

where:

$$\tilde{\Sigma}^{-1} = \begin{bmatrix} a & -c & c & -c & c & \dots & \dots & -c & c \\ -c & b & -b & -c/2 & c/2 & \dots & \dots & -c/2 & c/2 \\ c & -b & b & c/2 & -c/2 & \dots & \dots & c/2 & -c/2 \\ -c & -c/2 & c/2 & b & -b & & & -c/2 & c/2 \\ c & c/2 & -c/2 & -b & b & & & c/2 & -c/2 \\ & & & & & \ddots & & & \\ -c & -c/2 & c/2 & \dots & \dots & \dots & \dots & b & -b \\ c & c/2 & -c/2 & \dots & \dots & \dots & \dots & -b & b \end{bmatrix}$$

• Formal passage to the limit

We denote by  $p_{\tilde{\xi}}(z)$  the probability density of  $\tilde{\xi}$  at  $z$ . We have

$$p_{\tilde{\xi}}(z) = \frac{1}{(\sqrt{2\pi})^{2N-1} \sqrt{\det \Sigma_\rho}} \exp \left( -\frac{1}{2} z \Sigma_\rho^{-1} z^\top \right)$$

and by equation (47), for  $k = 1, \dots, N - 1$ :

$$I_\rho = \int_{z_1 > 0, z_{2k} > 0, z_{2k+1} < 0} z_1^2 z_2 |z_3| \dots |z_{2N-1}| p_\xi(z) dz.$$

We make the change of variables  $z = \rho \tilde{z}$  to get that  $I_\rho = \rho^{3N-1} \tilde{I}_\rho$  with:

$$\tilde{I}_\rho = \int_{\tilde{z}_1 > 0, \tilde{z}_{2k} > 0, \tilde{z}_{2k+1} < 0} \frac{\tilde{z}_1^2 \tilde{z}_2 \dots \tilde{z}_{2N-1}}{(\sqrt{2\pi})^{2N-1} \sqrt{\left(\frac{\det(\Sigma_\rho)}{\rho^{2N}}\right)}} \exp\left(-\frac{1}{2} \tilde{z} (\rho^2 \Sigma_\rho^{-1}) \tilde{z}^\top\right) d\tilde{z}.$$

$\rho^2 \Sigma_\rho^{-1}$  converges to  $\tilde{\Sigma}^{-1}$  and  $\left(\frac{\det(\Sigma_\rho)}{\rho^{2N}}\right)$  converges to  $g$ , positive, under our hypotheses. So the formal limit is

$$J := (\text{const}) g^{-1/2} \int_{\tilde{z}_1 > 0, \tilde{z}_{2k} > 0, \tilde{z}_{2k+1} < 0} \tilde{z}_1^2 \tilde{z}_2 \dots \tilde{z}_{2N-1} \exp\left(-\frac{1}{2} \tilde{z} \tilde{\Sigma}^{-1} \tilde{z}^\top\right) d\tilde{z}.$$

Performing the change of variable  $t_j = \tilde{z}_{2j} - \tilde{z}_{2j+1}$  and remarking that

$$\int_{z > 0, z' < 0, z - z' = t} z |z'| f(z - z') dz dz' = \int_0^{+\infty} \frac{t^3}{6} f(t) dt,$$

we get

$$J \leq (\text{const}) \int_{z_1 > 0, t_1 > 0, \dots, t_{N-1} > 0} z_1^2 t_1^3 t_2^3 \dots t_{N-1}^3 \exp\left(-\frac{1}{2} z^* \Sigma^* z^{*\top}\right) dz_1 dt_1 \dots dt_{N-1}$$

where  $z^* := (z_1, t_1, \dots, t_{N-1})$  and  $\Sigma^* := \begin{pmatrix} a & -c \mathbf{1}_{N-1}^\top \\ -c \mathbf{1}_{N-1} & M \end{pmatrix}$  with  $M = (b + c/2) Id_{N-1} - (c/2) J_{N-1}$ .

Note that the eigenvalues of  $M$  are

- $(b + c/2)$  which is positive because when developing its expression, its numerator and denominator are proportional to  $\mathcal{B}_1 > 0$ , see Assumption (B1).
- $(b + c/2) - (N - 1)c/2$  which is positive because its numerator is equal to  $225/2(\lambda_2 \lambda_6 - \lambda_4^2) > 0$ ,

proving that  $M$  is definite positive.

It is easy to prove that its Shur's complement  $(a - c^2 \mathbf{1}_{N-1}^\top M^{-1} \mathbf{1}_{N-1})$  is equal to  $\frac{4\lambda_2}{\lambda_2 \lambda_6 - \lambda_4^2} > 0$ , so  $\Sigma^*$  is definite positive. Finally

$$\begin{aligned} \int_{z_1 > 0, t_1 > 0, \dots, t_{N-1} > 0} z_1^2 t_1^3 t_2^3 \dots t_{N-1}^3 \exp\left(-\frac{1}{2} z^* \Sigma^* z^{*\top}\right) dz_1 dt_1 \dots dt_{N-1} \\ \leq \int_{\mathbb{R}^N} z_1^2 |t_1|^3 |t_2|^3 \dots |t_{N-1}|^3 \exp\left(-\frac{1}{2} z^* \Sigma^* z^{*\top}\right) dz_1 dt_1 \dots dt_{N-1}. \end{aligned}$$

This last expression is finite because it is, up to some multiplicative constant, a moment of a Gaussian variable.

### • Domination

Performing the same change of variables but now with the exact expression, we get

$$\tilde{I}_\rho = (\text{const}) \int_{z_1 > 0, t_1 > 0, \dots, t_{N-1} > 0} z_1^2 t_1^3 t_2^3 \dots t_{N-1}^3 \exp\left(-\frac{1}{2} z^* \Sigma_\rho^* z^{*\top}\right) dz_1 dt_1 \dots dt_{N-1}, \quad (51)$$



with  $\Sigma_\rho^* \rightarrow \Sigma^*$ .

As  $\Sigma^*$  is definite positive, for  $\rho$  small enough, the smallest eigenvalue of  $\Sigma_\rho^*$  is bounded-below by a positive quantity, say  $Q$ . Then a domination of the integrand in (51) is given by

$$z_1^2 t_1^3 t_2^3 \dots t_{N-1}^3 \exp -\frac{Q \|z^*\|^2}{2},$$

giving the result. □

*Proof of Theorem 5.5.* We follow the same lines as in the proof of Theorem 5.4. By Kac-Rice formula (4):

$$A^{0,0}(\rho) = p_{\nabla X(0), \nabla X(t)}(0, 0) \mathbb{E} \left( |\det \xi(0)| \mathbb{1}_{\{\langle \nabla^2 \xi(0) \rangle = 0\}} |\det \xi(t)| \mathbb{1}_{\{\langle \nabla^2 \xi(t) \rangle = 0\}} \right).$$

Using (46), Cauchy-Schwarz inequality and symmetry of the role of 0 and  $t$  we get:

$$A^{0,0}(\rho) \leq p_{\nabla X(0), \nabla X(t)}(0, 0) \mathbb{E} \left( \xi_{11}^+(0) \xi_{11}^+(t) \xi_{22}^+(0)^2 \dots \xi_{NN}^+(0)^2 \right).$$

By (37) we have  $p_{\nabla X(0), \nabla X(t)}(0, 0) = O(\rho^{-N})$ . Now let us prove that

$$\check{I}_\rho := \mathbb{E} \left( \xi_{11}^+(0) \xi_{11}^+(t) \xi_{22}^+(0)^2 \dots \xi_{NN}^+(0)^2 \right) = O(\rho^5). \quad (52)$$

We set  $\check{\xi} := (\xi_{11}(0), \xi_{11}(t), \xi_{22}(0), \xi_{33}(0), \dots, \xi_{NN}(0))$ ,  $\check{\xi}_i$  the  $i$ th coordinate of  $\check{\xi}$  for  $i = 1, \dots, N+1$ . We set  $\check{\Sigma}_\rho := \text{Var}(\check{\xi})$ .

• Equivalent of  $\det(\check{\Sigma}_\rho)$  as  $\rho \rightarrow 0$

By the same arguments as those used in the proof of Theorem 5.4 for the calculation of the equivalent of  $\det(\Sigma_\rho)$  (see (48) and (49)) we get:

$$\begin{aligned} \det(\check{\Sigma}_\rho) &\simeq \frac{\rho^6}{144} \frac{\det \text{Var}(X_1, X_{111}) \det \text{Var}(X_{11}, X_{22}, \dots, X_{NN}, X_{1111})}{\det \text{Var}(X_1, X_{11})} \\ \det(\check{\Sigma}_\rho) &\simeq \rho^6 \frac{(\lambda_2 \lambda_6 - \lambda_4^2)}{\lambda_2 \lambda_4} \left( \frac{2\lambda_4}{3} \right)^{N-1} \left( \frac{25(N+2)\lambda_4 \lambda_8 - 3(9N+16)\lambda_6^2}{10800} \right) := \rho^6 \check{g}. \end{aligned} \quad (53)$$

Note that  $\check{g} > 0$  under Assumption (B2).

• Equivalent of  $\check{\Sigma}_\rho^{-1}$  as  $\rho \rightarrow 0$

We set  $\check{\Sigma}_\rho = \begin{pmatrix} \check{A}_\rho & \check{B}_\rho \\ \check{B}_\rho^\top & \check{C}_\rho \end{pmatrix}$  and  $\check{\Sigma}_\rho^{-1} = \begin{pmatrix} A_\rho & B_\rho \\ B_\rho^\top & C_\rho \end{pmatrix}$  where  $\check{A}_\rho$  and  $A_\rho$  are  $2 \times 2$ ,  $\check{B}_\rho$  and  $B_\rho$  are  $2 \times (N-1)$ ,  $\check{C}_\rho$  and  $C_\rho$  are  $(N-1) \times (N-1)$ . We have:

$$\check{\Sigma}_\rho^{-1} = \begin{bmatrix} (\check{A}_\rho - \check{B}_\rho \check{C}_\rho^{-1} \check{B}_\rho^\top)^{-1} & -(\check{A}_\rho - \check{B}_\rho \check{C}_\rho^{-1} \check{B}_\rho^\top)^{-1} \check{B}_\rho \check{C}_\rho^{-1} \\ -\check{C}_\rho^{-1} \check{B}_\rho^\top (\check{A}_\rho - \check{B}_\rho \check{C}_\rho^{-1} \check{B}_\rho^\top)^{-1} & (\check{C}_\rho - \check{B}_\rho^\top \check{A}_\rho^{-1} \check{B}_\rho)^{-1} \end{bmatrix}.$$

All the elements of  $\check{A}_\rho$ ,  $\check{B}_\rho$  and  $\check{C}_\rho$  are given by Lemma B.1. By direct calculations we obtain:

$$\check{\Sigma}_\rho^{-1} \simeq \begin{pmatrix} A/\rho^4 & B/\rho^2 \\ B^\top/\rho^2 & C \end{pmatrix},$$

where  $A$  is the  $2 \times 2$  matrix with all elements equal to  $\check{a}$ ,  $B$  is the  $2 \times (N-1)$  matrix with all elements equal to  $\check{d}$ ,  $C$  is the  $(N-1) \times (N-1)$  matrix  $C = (\check{b} - \check{c})Id_{N-1} + \check{c}J_{N-1}$  with:

$$\begin{aligned}\check{a} &:= \frac{900\lambda_4(N+2)}{25(N+2)\lambda_4\lambda_8 - 3(9N+16)\lambda_6^2} \\ \check{b} &:= \frac{3}{2\lambda_4} \left( \frac{25(N+1)\lambda_4\lambda_8 - 3(9N+7)\lambda_6^2}{25(N+2)\lambda_4\lambda_8 - 3(9N+16)\lambda_6^2} \right) \\ \check{c} &:= \frac{-3}{2\lambda_4} \left( \frac{25\lambda_4\lambda_8 - 27\lambda_6^2}{25(N+2)\lambda_4\lambda_8 - 3(9N+16)\lambda_6^2} \right) \\ \check{d} &:= \frac{-90\lambda_6}{25(N+2)\lambda_4\lambda_8 - 3(9N+16)\lambda_6^2}.\end{aligned}$$

- For the passage to the limit we proceed as in the previous theorem:

We perform the change of variable  $\check{\xi}_1 = \rho^2 z_1$ ,  $\check{\xi}_2 = \rho^2 z_2$ ,  $\check{\xi}_k = z_k$  for  $k = 3, \dots, N+1$ , to get that:

$$\check{I}_\rho = \rho^5(\text{const}) \int_{z_k > 0, k \geq 1} z_1 z_2 z_3^2 \dots z_{N+1}^2 \exp\left(-\frac{1}{2} z^T \check{\Sigma}^{-1} z\right) dz_1 \dots dz_{N+1}$$

where

$$\check{\Sigma}^{-1} := \begin{pmatrix} A & B \\ B^\top & C \end{pmatrix}.$$

Performing the change of variables  $t = z_1 + z_2$  we get:

$$\check{I}_\rho = \rho^5(\text{const}) \int_{t > 0, z_3 > 0, \dots, z_{N+1} > 0} t^3 z_3^2 \dots z_{N+1}^2 \exp\left(-\frac{1}{2} \check{z}^* \check{\Sigma}^* \check{z}^{*\top}\right) dt dz_3 \dots dz_{N+1}$$

where  $\check{z}^* := (t, z_3, \dots, z_{N+1})$  and  $\check{\Sigma}^* := \begin{pmatrix} \check{a} & \check{d}\mathbf{1}_N^\top \\ \check{d}\mathbf{1}_N & C \end{pmatrix}$ .

First we have to check that the matrix  $C$  is definite positive. This is equivalent to

- $\check{b} - \check{c} > 0$
- $\check{b} - \check{c} + (N-1)\check{c} > 0$ ,

which are both a consequence of the positivity of  $\mathcal{B}_2$ , see Assumption (B2).

Second it is easy to check that the Shur's complement  $\left(\check{a} - \check{d}^2 \mathbf{1}_{N-1}^\top C^{-1} \mathbf{1}_{N-1}\right)$  is positive. We deduce that  $\check{\Sigma}^*$  is definite positive. The proof ends as in the previous theorem.  $\square$

## Appendix A: Some results on random matrices

For  $l \in \mathbb{R}$ ,  $p = 1, 2$  let us define

$$\gamma_{n,p}(l) := \mathbb{E}(\det^p(G_n - lId_n)) \quad (54)$$

and:

$$\gamma_{n,p}^k(l) := \mathbb{E}(\det^p(G_n - lId_n) \mathbf{1}_{\{i(\nabla^2(G_n - lId_n))=k\}}). \quad (55)$$

Expressions for the calculation of (54) are given by [25] and [24]. In this section, we recall their results and generalize them for the calculation of (55).

Let  $H_n(x)$  be the  $n$ th "physician" Hermite polynomial:  $H_n(x) := e^{x^2} (-d/dx)^n e^{-x^2}$ . Define  $R_n$  the polynomial of degree  $n$  such that:

$$R_{2n}(x) = \frac{H_{2n}(x)}{2^{2n}}$$

$$R_{2n+1}(x) = \frac{xH_{2n}(x) - H'_{2n}(x)}{2^{2n}}.$$

Define also

$$r_n := 2^{1-2n} (2n)! \sqrt{\pi}$$

$$g_m := \int R_m(x) \exp\left(-\frac{x^2}{2}\right) dx.$$

Now we state the following lemma due to [25], formulas 2.2.16 and 2.2.17:

**Lemma A.1.** *Using the notations above, for  $l \in \mathbb{R}$ :*

1. *For  $p = 1$  and  $n = 2L$ :*

$$\gamma_{2L,1}(l) = k_{2L}(2L)! \times r_0 \times \dots \times r_{L-1} \times \frac{H_{2L}(l)}{2^{2L}}.$$

*Recall that  $k_{2L}$  is given by (9).*

*For  $p = 1$  and  $n = 2L + 1$ ,*

$$\gamma_{2L+1,1}(l) = -k_{2L+1}(2L+1)! \times \sum_{j=0}^L (g_{2j} R_{2j+1}(l) - g_{2j+1} R_{2j}(l)) \times \prod_{i \in \{0, \dots, L\} \setminus j} r_i.$$

2. *For  $p = 2$  and  $n = 2L$ :*

$$\gamma_{2L,2}(l) = k_{2L}(2L)! \times \sum_{j=0}^L \left( \frac{H_{2j}(l)^2 + H'_{2j}(l)^2 - H_{2j}(l)H''_{2j}(l)}{2^{4j}} \right) \times \prod_{i \in \{0, \dots, L\} \setminus j} r_i.$$

*and for  $p = 2$  and  $n = 2L + 1$ :*

$$\gamma_{2L+1,2}(l) = k_{2L+1}(2L+1)! \times \sum_{j=0}^L \left( \prod_{i \in \{0, \dots, L\} \setminus j} r_i \right) \det \begin{bmatrix} g_{2j} & R_{2j}(l) & R'_{2j}(l) \\ g_{2j+1} & R_{2j+1}(l) & R'_{2j+1}(l) \\ g_{2L+2} & R_{2L+2}(l) & R'_{2L+2}(l) \end{bmatrix}.$$

Now we generalize Lemma A.1 to obtain an expression for  $\gamma_{n,p}^k(l)$  for  $p = 1$  and  $p = 2$ . For simplicity we give these expressions, (56) and (57), using the simplest family of polynomials  $P_i(x) = x^i$ ,  $i = 0, 1, 2, \dots$ . Note that equation (58) gives the general expression for  $\gamma_{n,p}^k(l)$  for  $p = 1$  and  $p = 2$ , valid for any family of monic polynomials  $P_i(x)$ . We recall that a monic polynomial of degree  $i$  is such that the coefficient of the highest power  $i$  is equal to one.

**Lemma A.2.** *For  $l \in \mathbb{R}$  we have:*

$$\gamma_{n,1}^k(l) = k_n n! \sum_{i=1}^{n+1} (-1)^{i+1} l^{i-1} \times \sum_{K \in \mathcal{F}_k(n)} Pf(B_F^K(l)) \quad (56)$$

where  $\bar{I} = \{1, \dots, n+1\} \setminus \{i\}$ ;  $B_{\bar{I}}^K(l)$  is the matrix  $B_J^K(l)$  defined in (15) and (16) with  $J = \bar{I}$ ;  $\text{Pf}(B_{\bar{I}}^K(l))$  is the Pfaffian of the matrix  $B_{\bar{I}}^K(l)$  defined in (14);  $\mathcal{F}_k(n)$  is the set of the parts of  $\{1, 2, \dots, n\}$  of size  $k$  and  $k_n$  is given in (9). Moreover we have:

$$\gamma_{n,2}^k(l) = k_n n! \sum_{I \in \mathcal{F}_2(n+2)} \epsilon(\sigma_I) (i_2 - i_1) l^{i_1+i_2-3} \sum_{K \in \mathcal{F}_k(n)} \text{Pf}(B_{\bar{I}}^K(l)) \quad (57)$$

where  $\bar{I} = \{1, \dots, n+2\} \setminus I$ ,  $\sigma_I$  is the permutation of  $(1, 2, \dots, n+2)$  such that  $\sigma_I|_{(1,2)}$  (resp.  $\sigma_I|_{(3, \dots, n+2)}$ ) is an increasing one-to-one mapping from  $(1, 2)$  on  $I$  (resp. from  $(3, \dots, n+2)$  on  $\bar{I}$ ),  $\epsilon(\sigma_I)$  is the signature of the permutation  $\sigma_I$  and  $i_1$  and  $i_2$  are the ordered elements of  $I$ .

*Proof of Lemma A.2.* We set  $\mathcal{Q}_k(l) = \{(\mu_i, i = 1, \dots, n) \in \mathbb{R}^n : \#\{i : \mu_i - l < 0\} = k\}$  and  $\mathcal{O}_k(l) = \{(\mu_i, i = 1, \dots, n) \in \mathbb{R}^n : \mu_1 \leq \mu_2 \leq \dots \leq \mu_k \leq l \leq \mu_{k+1} \leq \dots \leq \mu_n\}$ . Using (8) we have for  $p = 1$  or  $2$ :

$$\begin{aligned} \gamma_{n,p}^k(l) &:= \mathbb{E}(\det^p(G_n - lId_n) \mathbf{1}_{\{i(\nabla^2(G_n - lId_n))=k\}}) \\ &= \int_{\mathcal{Q}_k(l)} \prod_{i=1}^n (\mu_i - l)^p f_n(\mu_1, \dots, \mu_n) d\mu_1 \dots d\mu_n \\ &= k_n n! \int_{\mathcal{O}_k(l)} \prod_{i=1}^n (\mu_i - l)^p \prod_{1 \leq i < j \leq n} (\mu_j - \mu_i) \exp\left(-\frac{\sum_{i=1}^n \mu_i^2}{2}\right) d\mu_1 \dots d\mu_n. \end{aligned}$$

We set  $h_{m-1}(\mu_j) = P_{m-1}(\mu_j) \exp(-\mu_j^2/2)$  with  $P_{m-1}(\mu_j) = \mu_j^{m-1}$  and  $P_{m-1}^{(s)}$  denotes the  $s$ th derivative of  $P_{m-1}$ . We construct the  $(n+p) \times (n+p)$  matrix  $\left[ h_{m-1}(\mu_j); P_{m-1}^{(s)}(l) \right]_{\substack{j=1, \dots, n \\ s=0, \dots, p-1 \\ m=1, \dots, n+p}}$  by the concatenation of the  $(n+p) \times n$  matrix  $\left[ h_{m-1}(\mu_j) \right]_{\substack{j=1, \dots, n \\ m=1, \dots, n+p}}$  and the  $(n+p) \times p$  matrix  $\left[ P_{m-1}^{(s)}(l) \right]_{\substack{s=0, \dots, p-1 \\ m=1, \dots, n+p}}$ .

Then calculating  $\det \left[ h_{m-1}(\mu_j); P_{m-1}^{(s)}(l) \right]_{\substack{j=1, \dots, n \\ s=0, \dots, p-1 \\ m=1, \dots, n+p}}$  as a Vandermonde determinant we obtain:

$$\gamma_{n,p}^k(l) = k_n n! \int_{\mathcal{O}_k(l)} (-1)^{np} \det \left[ h_{m-1}(\mu_j); P_{m-1}^{(s)}(l) \right]_{\substack{j=1, \dots, n \\ s=0, \dots, p-1 \\ m=1, \dots, n+p}} d\mu_1 \dots d\mu_n.$$

The integrand is the same on each orthant up to a change of sign. So:

$$\gamma_{n,p}^k(l) = k_n (-1)^{np} \int_{\mathcal{Q}_k(l)} \prod_{1 \leq i < g \leq n} \mathbf{s}(\mu_g - \mu_i) \det \left[ h_{m-1}(\mu_j); P_{m-1}^{(s)}(l) \right]_{\substack{j=1, \dots, n \\ s=0, \dots, p-1 \\ m=1, \dots, n+p}} d\mu_1 \dots d\mu_n.$$

We have for a  $(n+p) \times (n+p)$  matrix  $D$ , for any  $J \in \mathcal{F}_p(n+p)$  fixed:

$$\det D = \sum_{I \in \mathcal{F}_p(n+p)} \epsilon(\sigma_I) \epsilon(\sigma_J) \det[D]_{I,J} \det[D]_{\bar{I}, \bar{J}},$$

where  $\epsilon(\sigma_I)$  is the signature of the permutation  $\sigma_I$  and  $\sigma_I$  is the permutation of  $(1, 2, \dots, n+p)$  such that  $\sigma_I|_{(1, \dots, p)}$  (resp.  $\sigma_I|_{(p+1, \dots, n+p)}$ ) is an increasing one-to-one mapping from  $(1, \dots, p)$  on  $I$  (resp. from  $(p+1, \dots, n+p)$  on  $\bar{I}$ ).

So we have for  $J = \{n+1, \dots, n+p\}$  fixed,  $\epsilon(\sigma_J) = (-1)^{np}$  and:

$$\gamma_{n,p}^k(l) = k_n \int_{\mathcal{Q}_k(l)} \prod_{1 \leq i < g \leq n} \mathbf{s}(\mu_g - \mu_i) \times \left[ \sum_{I \in \mathcal{F}_p(n+p)} \epsilon(\sigma_I) \det [h_{m-1}(\mu_j)]_{\substack{j \in \{1, \dots, n\} \\ m \in I}} \det [P_{m-1}^{(s)}(l)]_{\substack{s=0, \dots, p-1 \\ m \in I}} \right] d\mu_1 \dots \mu_n.$$

We note  $m_1, \dots, m_n$  the ordered elements of  $\bar{I} := \{1, \dots, n+p\} \setminus I$ . Remarking that each term of the  $n!$  terms of  $\det [h_{m-1}(\mu_j)]_{\substack{j \in \{1, \dots, n\} \\ m \in I}}$  contributes equally to the integral, we obtain:

$$\begin{aligned} \gamma_{n,p}^k(l) &= k_n n! \int_{\mathcal{Q}_k(l)} \prod_{1 \leq i < g \leq n} \mathbf{s}(\mu_g - \mu_i) \times \\ &\quad \left[ \sum_{I \in \mathcal{F}_p(n+p)} \epsilon(\sigma_I) \left( \prod_{j=1}^n h_{m_j-1}(\mu_j) \right) \det [P_{m-1}^{(s)}(l)]_{\substack{s=0, \dots, p-1 \\ m \in I}} \right] d\mu_1 \dots \mu_n \\ &= k_n n! \sum_{I \in \mathcal{F}_p(n+p)} \epsilon(\sigma_I) \det [P_{m-1}^{(s)}(l)]_{\substack{s=0, \dots, p-1 \\ m \in I}} \times \\ &\quad \int_{\mathcal{Q}_k(l)} \prod_{1 \leq i < g \leq n} \mathbf{s}(\mu_g - \mu_i) \times \left( \prod_{j=1}^n h_{m_j-1}(\mu_j) \right) d\mu_1 \dots \mu_n \end{aligned}$$

Now we follow the same lines as [14] section 4. Since for  $m$  the integer part of  $n/2$  (see [14]):

$$\prod_{1 \leq i < g \leq n} \mathbf{s}(\mu_g - \mu_i) = \frac{1}{2^m m!} \sum_{j_1=1}^n \dots \sum_{j_n=1}^n \sigma(j_1, \dots, j_n) \mathbf{s}(\mu_{j_2} - \mu_{j_1}) \times \mathbf{s}(\mu_{j_4} - \mu_{j_3}) \times \dots \times \mathbf{s}(\mu_{j_{2m}} - \mu_{j_{2m-1}}),$$

$$\begin{aligned} \gamma_{n,p}^k(l) &= k_n n! \sum_{I \in \mathcal{F}_p(n+p)} \epsilon(\sigma_I) \det [P_{m-1}^{(s)}(l)]_{\substack{s=0, \dots, p-1 \\ m \in I}} \times \\ &\quad \int_{\mathcal{Q}_k(l)} \frac{1}{2^m m!} \sum_{j_1=1}^n \dots \sum_{j_n=1}^n \sigma(j_1, \dots, j_n) \mathbf{s}(\mu_{j_2} - \mu_{j_1}) \times \mathbf{s}(\mu_{j_4} - \mu_{j_3}) \times \dots \times \\ &\quad \mathbf{s}(\mu_{j_{2m}} - \mu_{j_{2m-1}}) \times h_{m_1-1}(\mu_1) \times \dots \times h_{m_n-1}(\mu_n) d\mu_1 \dots \mu_n \\ &= k_n n! \sum_{I \in \mathcal{F}_p(n+p)} \epsilon(\sigma_I) \det [P_{m-1}^{(s)}(l)]_{\substack{s=0, \dots, p-1 \\ m \in I}} \times \\ &\quad \sum_{K \in \mathcal{F}_k(n)} \int \frac{1}{2^m m!} \sum_{j_1=1}^n \dots \sum_{j_n=1}^n \sigma(j_1, \dots, j_n) \mathbf{s}(\mu_{j_2} - \mu_{j_1}) \times \mathbf{s}(\mu_{j_4} - \mu_{j_3}) \times \dots \times \\ &\quad \mathbf{s}(\mu_{j_{2m}} - \mu_{j_{2m-1}}) \times h_{m_1-1}(\mu_1) \mathbb{1}_{\{\mu_1 \in \mathcal{D}_1^K(\mu)\}} \times \dots \times h_{m_n-1}(\mu_n) \mathbb{1}_{\{\mu_n \in \mathcal{D}_n^K(\mu)\}} d\mu_1 \dots \mu_n \end{aligned}$$

Finally, using the same arguments as [14] section 4, we obtain:

$$\gamma_{n,p}^k(l) = k_n n! \sum_{I \in \mathcal{F}_p(n+p)} \epsilon(\sigma_I) \times \sum_{K \in \mathcal{F}_k(n)} \text{Pf} (B_I^K(l)) \times \det [P_{m-1}^{(s)}(l)]_{\substack{s=0, \dots, p-1 \\ m \in I}}. \quad (58)$$

That concludes the proof.  $\square$

## Appendix B: Computation of the conditional variance-covariance matrix of the Hessian and proof of Lemma 5.1

We give the steps to the computation of the conditional variance of  $\nabla^2 X(0)$ ,  $\nabla^2 X(t)$ . Some tedious but easy calculations are not detailed. (see Lemma 8.5 of [7]). We denote  $r(\rho^2)$ ,  $r'(\rho^2)$ ,  $r''(\rho^2)$ ,  $r'''(\rho^2)$  and  $r^{(4)}(\rho^2)$  by  $r$ ,  $r'$ ,  $r''$ ,  $r'''$  and  $r^{(4)}$  for short.  $J_{n,m}$  is the "all-one" matrix of size  $n, m$  :  $(J_{n,m})_{i,j} \equiv 1$ . The elements of the gradient  $\nabla X(t)$  are denoted  $X'_i$ ,  $i = 1, \dots, N$ . Those of the Hessian are denoted  $X''_{ij}$ ,  $i, j = 1, \dots, N$ .

### B.1. Unconditional distribution

#### B.1.1. Gradient

$$\begin{aligned} \text{Var}(\nabla X(t)) &= \text{Var}(\nabla X(0)) = -2r'(0)Id_N, \\ \text{Cov}(X'_1(0), X'_1(t)) &= -2r' - 4\rho^2 r'' \text{ and } \text{Cov}(X'_i(0), X'_i(t)) = -2r' \text{ for } i \neq 1. \end{aligned}$$

Any other covariance is zero.

#### B.1.2. Hessian

We define  $X''_d$  as  $(X''_{11}, \dots, X''_{NN})$  and  $X''_u$  (as upper) as  $\{X''_{i,j}, 1 \leq i < j \leq N\}$ . These two vectors are independent.

$$\begin{aligned} \text{Var}(X''_d(0)) &= 4r''(0)(2Id_N + J_{N,N}), \\ \text{Cov}(X''_d(0), X''_d(t)) &= \begin{pmatrix} 12r'' + 48\rho^2 r''' + 16\rho^4 r^{(4)} & (4r'' + 8\rho^2 r''')J_{1,(N-1)} \\ (4r'' + 8\rho^2 r''')J_{(N-1),1} & 4r''(2Id_{N-1} + J_{(N-1),(N-1)}) \end{pmatrix} \\ \text{Var}(X''_u(0)) &= 4r''(0)Id_{N(N-1)/2}, \\ \text{Cov}(X''_u(0), X''_u(t)) &= \text{diag}((4r'' + 8\rho^2 r''')Id_{N-1}, 4r''Id_{(N-1)(N-2)/2}). \end{aligned}$$

#### B.1.3. Relation between gradient and Hessian

$$\begin{aligned} &E(X''_{11}(0)/X'_1(0), X'_1(t)) \\ &= \frac{1}{4r'(0)^2 - (2r' + 4\rho^2 r'')^2} (0, 12\rho r'' + 8\rho^3 r''') \begin{pmatrix} -2r'(0) & 2r' + 4\rho^2 r'' \\ 2r' + 4\rho^2 r'' & -2r'(0) \end{pmatrix} \begin{pmatrix} X'_1(0) \\ X'_1(t) \end{pmatrix} \\ &= (12\rho r'' + 8\rho^3 r''') \frac{(2r' + 4\rho^2 r'')X'_1(0) - 2r'(0)X'_1(t)}{4r'(0)^2 - (2r' + 4\rho^2 r'')^2} \\ &=: (12\rho r'' + 8\rho^3 r''')K_1(t) \end{aligned}$$

For  $i \neq 1$ :

$$\begin{aligned} E(X''_{1i}(0)/X'_i(0), X'_i(t)) &= \frac{1}{4r'(0)^2 - 4(r')^2} (0, 4\rho r'') \begin{pmatrix} -2r'(0) & 2r' \\ 2r' & -2r'(0) \end{pmatrix} \begin{pmatrix} X'_i(0) \\ X'_i(t) \end{pmatrix} \\ &= 4\rho r'' \frac{2r'X'_i(0) - 2r'(0)X'_i(t)}{4r'(0)^2 - 4(r')^2} =: 4\rho r'' K_i(t) \\ E(X''_{ii}(0)/X'_i(0), X'_i(t)) &= 4\rho r'' K_1(t) \end{aligned}$$

We have equivalent formulas, reversing time, for example:

$$\mathbb{E}(X''_{11}(t)/X'_1(t), X'_1(0)) = (12\rho r'' + 8\rho^3 r''') \frac{-(2r' + 4\rho^2 r'')X'_1(t) + 2r'(0)X'_1(0)}{4(r'(0))^2 - (2r' + 4\rho^2 r'')^2} =: (12\rho r'' + 8\rho^3 r''')\bar{K}_1(t).$$

Any other case corresponds to independence between gradient and Hessian.

### B.2. Conditional distribution

Since the conditional variance-covariance matrix is equal to the unconditional variance-covariance matrix diminished by the variance of the conditional expectation, we compute this last term only.

Let us consider for example the two terms  $\xi_{11}(0), \xi_{11}(t)$ . The  $2 \times 2$  matrix to subtract is  $(12\rho r'' + 8\rho^3 r''')^2 \text{Var}(K_1(t); \bar{K}_1(t))$  with:

$$\text{Var}(K_1(t); \bar{K}_1(t)) = \frac{1}{4(r'(0))^2 - (2r' + 4\rho^2 r'')^2} \begin{pmatrix} -2r'(0) & -(2r' + 4\rho^2 r'') \\ -(2r' + 4\rho^2 r'') & -2r'(0) \end{pmatrix}.$$

In the same way for  $i \neq 1$  :

$$\text{Var}(K_i(t); \bar{K}_i(t)) = \frac{1}{4r'(0)^2 - 4(r')^2} \begin{pmatrix} -2r'(0) & -2r' \\ -2r' & -2r'(0) \end{pmatrix}.$$

Giving the extra term to subtract to get the following lemma

**Lemma B.1.** *We note  $\xi_d(t)$  the vector  $(\xi_{11}(t), \dots, \xi_{NN}(t))$  and  $\xi_u(t)$  the vector  $(\xi_{12}(t), \xi_{13}(t), \dots, \xi_{(N-1)N}(t))$ . The joint distribution of  $(\xi_d(0), \xi_u(0), \xi_d(t), \xi_u(t))$  is centred Gaussian with variance-covariance matrix:*

$$\begin{bmatrix} \Gamma_1 & 0 & \Gamma_3 & 0 \\ 0 & \Gamma_2 & 0 & \Gamma_4 \\ \Gamma_3 & 0 & \Gamma_1 & 0 \\ 0 & \Gamma_4 & 0 & \Gamma_2 \end{bmatrix}.$$

$\Gamma_1$  is the  $N \times N$  matrix:

$$\Gamma_1 = \begin{bmatrix} 12r''(0) & 4r''(0) & \dots & 4r''(0) \\ 4r''(0) & \ddots & 4r''(0) & \vdots \\ \vdots & 4r''(0) & \ddots & 4r''(0) \\ 4r''(0) & \dots & 4r''(0) & 12r''(0) \end{bmatrix} + \frac{\rho^2 r'(0)}{2[r'(0)^2 - (r' + 2r''\rho^2)^2]} \times M,$$

with

$$M = \begin{bmatrix} (12r'' + 8\rho^2 r''')^2 & 4r''(12r'' + 8\rho^2 r''') & \dots & 4r''(12r'' + 8\rho^2 r''') \\ 4r''(12r'' + 8\rho^2 r''') & 16r''^2 & \dots & 16r''^2 \\ \vdots & \vdots & \ddots & \vdots \\ 4r''(12r'' + 8\rho^2 r''') & 16r''^2 & \dots & 16r''^2 \end{bmatrix}.$$

$\Gamma_2$  is the  $\frac{N(N-1)}{2} \times \frac{N(N-1)}{2}$  diagonal matrix:

$$\Gamma_2 = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \text{ with } D_1 = \left( 4r''(0) + \frac{8\rho^2 (r'')^2 r'(0)}{r'(0)^2 - (r')^2} \right) Id_{N-1} \text{ and } D_2 = (4r''(0)) Id_{\frac{(N-1)(N-2)}{2}}.$$

We note  $a := 4r'' + 8\rho^2 r'''$  and  $d := 12r'' + 48\rho^2 r''' + 16\rho^4 r^{(4)}$ .  $\Gamma_3$  is the  $N \times N$  matrix:

$$\Gamma_3 = \begin{bmatrix} d & a & \cdots & \cdots & a \\ a & 12r'' & 4r'' & \cdots & 4r'' \\ \vdots & 4r'' & \ddots & 4r'' & \vdots \\ \vdots & \vdots & 4r'' & \ddots & 4r'' \\ a & 4r'' & \cdots & 4r'' & 12r'' \end{bmatrix} + \frac{\rho^2(r' + 2r''\rho^2)}{2[r'(0)^2 - (r' + 2r''\rho^2)^2]} \times M.$$

$\Gamma_4$  is the  $\frac{N(N-1)}{2} \times \frac{N(N-1)}{2}$  diagonal matrix:

$$\Gamma_4 = \begin{bmatrix} \tilde{D}_1 & 0 \\ 0 & \tilde{D}_2 \end{bmatrix} \text{ with } \tilde{D}_1 = \left( a + \frac{8\rho^2(r'')^2 r'}{r'(0)^2 - r'^2} \right) Id_{N-1} \text{ and } \tilde{D}_2 = 4r'' Id_{\frac{(N-1)(N-2)}{2}}.$$

### B.3. Proof of Lemma 5.1

First note that (30) and (33) are deduced from Lemma B.1 above.

#### B.3.1. Proof of (27)

$$\begin{aligned} \det(\text{Var}(\nabla X(0), \nabla X(t))) &= \det(\text{Var}(X'_1(0), \dots, X'_N(0), X'_1(s), \dots, X'_N(s))) \\ &\simeq \det(\text{Var}(X'_1(0), \dots, X'_N(0), X'_1(0) + \rho X''_{11}(0), \dots, X'_N(0) + \rho X''_{1N}(0))) \\ &\simeq \det(\text{Var}(X'_1(0), \dots, X'_N(0), \rho X''_{11}(0), \dots, \rho X''_{1N}(0))) \\ &\simeq \rho^{2N} \det(\text{Var}(X'_1(0), \dots, X'_N(0), X''_{11}(0), \dots, X''_{1N}(0))). \end{aligned}$$

Using Lemma 5.6 we obtain (27).

#### B.3.2. Proof of (28)

We now consider the case of  $\xi_{11}(t)$ .

$$\begin{aligned} \text{Var}(\xi_{11}(t)) &= \text{Var}(\xi_{11}(0)) = \frac{\det(\text{Var}(X'_1(0), \dots, X'_N(0), X''_{11}(0), X'_1(t), \dots, X'_N(t)))}{\det(\text{Var}(\nabla X(0), \nabla X(t)))} \\ &\simeq \frac{\det(\text{Var}(X'_1(0), \dots, X'_N(0), X''_{11}(0), \rho X''_{11}(0) + \frac{\rho^2}{2} X'''_{111}(0), \rho X''_{12}(0), \dots, \rho X''_{1N}(0)))}{\det(\text{Var}(\nabla X(0), \nabla X(t)))} \\ &\simeq \frac{3^{N-1}}{\lambda_2^N \lambda_4^N} \det(\text{Var}(X'_1(0), \dots, X'_N(0), X''_{11}(0), \frac{\rho}{2} X'''_{111}(0), X''_{12}(0), \dots, X''_{1N}(0))) \\ &\simeq \frac{\rho^2 3^{N-1}}{4\lambda_2^N \lambda_4^N} \det(\text{Var}(X'_1(0), \dots, X'_N(0), X''_{11}(0), \dots, X''_{1N}(0), X'''_{111}(0))). \end{aligned}$$

By Lemma 5.6 we obtain  $\text{Var}(\xi_{11}(t)) \simeq \frac{\rho^2 (\lambda_6 \lambda_2 - \lambda_4^2)}{4 \lambda_2}$ . The other variances are obtained in the same way.



B.3.3. Proof of (29), (31) and (32)

- We consider  $\text{Cov}(\xi_{jj}(t), \xi_{kk}(t))$  for  $j \neq k$  and  $j, k \neq 1$ . Since

$$\text{Cov}(\xi_{jj}(t), \xi_{kk}(t)) = \frac{1}{2} (\text{Var}(\xi_{jj}(t) + \xi_{kk}(t)) - \text{Var}(\xi_{jj}(t)) - \text{Var}(\xi_{kk}(t)))$$

and since, by (28),  $\text{Var}(\xi_{jj}(t)) = \text{Var}(\xi_{kk}(t)) = \frac{8\lambda_4}{9}$ , we just need to calculate  $\text{Var}(\xi_{jj}(t) + \xi_{kk}(t))$ .

$$\begin{aligned} \text{Var}(\xi_{jj}(t) + \xi_{kk}(t)) &= \frac{\det(\text{Var}(X''_{jj}(t) + X''_{kk}(t), X'_1(0), \dots, X'_N(0), X'_1(t), \dots, X'_N(t)))}{\det(\text{Var}(\nabla X(0), \nabla X(t)))} \\ &\simeq \frac{\det(\text{Var}(X''_{jj}(0) + X''_{kk}(0), X'_1(0), \dots, X'_N(0), \rho X''_{11}(0), \rho X''_{12}(0), \dots, \rho X''_{1N}(0)))}{\det(\text{Var}(X'_1(0), \dots, X'_N(0), \rho X''_{11}(0), \rho X''_{12}(0), \dots, \rho X''_{1N}(0)))} \\ &\simeq \frac{\det(\text{Var}(X''_{jj}(0) + X''_{kk}(0), X'_1(0), \dots, X'_N(0), X''_{11}(0), X''_{12}(0), \dots, X''_{1N}(0)))}{\det(\text{Var}(X'_1(0), \dots, X'_N(0), X''_{11}(0), X''_{12}(0), \dots, X''_{1N}(0)))} \end{aligned}$$

By Lemma 5.6 we have

$$\begin{aligned} &\det(\text{Var}(X''_{jj}(0) + X''_{kk}(0), X'_1(0), \dots, X'_N(0), X''_{11}(0), X''_{12}(0), \dots, X''_{1N}(0))) \\ &= \left(\frac{\lambda_4}{3}\right)^{N-1} \lambda_2^N \det(\text{Var}(X''_{11}(0), X''_{jj}(0) + X''_{kk}(0))), \end{aligned}$$

$$\det(\text{Var}(X''_{11}(0), X''_{jj}(0) + X''_{kk}(0))) = \frac{20\lambda_4^2}{9},$$

and

$$\det\left(\text{Var}\left(X'_1(0), \dots, X'_N(0), X''_{11}(0), X''_{12}(0), \dots, X''_{1N}(0)\right)\right) = \frac{\lambda_2^N \lambda_4^N}{3^{N-1}}.$$

Finally we obtain  $\text{Var}(\xi_{jj}(t) + \xi_{kk}(t)) \simeq \frac{20\lambda_4}{9}$ , implying  $\text{Cov}(\xi_{jj}(t), \xi_{kk}(t)) \simeq \frac{2\lambda_4}{9}$ . In the same way we obtain  $\text{Cov}(\xi_{jj}(0), \xi_{kk}(t)) \simeq \frac{2\lambda_4}{9}$  and (31).

- Now let us prove that  $\text{Cov}(\xi_{11}(t), \xi_{jj}(t)) \simeq \rho^2 \frac{11\lambda_2\lambda_6 - 15\lambda_4^2}{180\lambda_2}$  for  $j \neq 1$ . We have, for  $j \neq 1$  (see Section B.1.3):

$$\begin{aligned} \text{Cov}(X''_{jj}(0), X'_i(0)) &= 0, \quad \forall i \in \{1, \dots, N\} \\ \text{Cov}(X''_{jj}(0), X'_i(s)) &= 4\rho r''(\rho^2)\delta_{1i} =: \alpha_2\delta_{1i}. \end{aligned}$$

And with Section B.1.1 we deduce:

$$\begin{aligned} \text{Var}(\xi_{jj}(0)) &= \frac{\det(\text{Var}(X'_1(0), \dots, X'_N(0), X'_1(s), \dots, X'_N(s), X''_{jj}(0)))}{\det(\text{Var}(\nabla X(t), \nabla X(s)))} \\ &= \frac{\det(\text{Var}(\nabla_{-(1)}X(0), \nabla_{-(1)}X(s))) \times \det(\text{Var}(X'_1(0), X'_1(s), X''_{jj}(0)))}{\det(\text{Var}(\nabla X(t), \nabla X(s)))}, \end{aligned}$$

where  $\nabla_{-(1)}X$  denotes the gradient without its first coordinate. Similarly:

$$\text{Var}(\xi_{11}(0)) = \frac{\det(\text{Var}(\nabla_{-(1)}X(0), \nabla_{-(1)}X(s))) \times \det(\text{Var}(X'_1(0), X'_1(s), X''_{11}(0)))}{\det(\text{Var}(\nabla X(t), \nabla X(s)))}$$

and:

$$\text{Var}(\xi_{11}(0) + \xi_{jj}(0)) = \frac{\det(\text{Var}(\nabla_{-(1)}X(0), \nabla_{-(1)}X(s))) \times \det(\text{Var}(X'_1(0), X'_1(s), X''_{11}(0) + X''_{jj}(0)))}{\det(\text{Var}(\nabla X(t), \nabla X(s)))}.$$

It is easy to verify that:

$$\begin{aligned} \det(\text{Var}(X'_1(0), X'_1(s), X''_{11}(0) + X''_{jj}(0))) &= \frac{8}{3}\lambda_4\lambda_2^2 - \lambda_2(\alpha_1 + \alpha_2)^2 - \frac{8}{3}\lambda_4\beta_1^2 \\ \det(\text{Var}(X'_1(0), X'_1(s), X''_{11}(0))) &= \lambda_4\lambda_2^2 - \lambda_2\alpha_1^2 - \lambda_4\beta_1^2 \\ \det(\text{Var}(X'_1(0), X'_1(s), X''_{jj}(0))) &= \lambda_4\lambda_2^2 - \lambda_2\alpha_2^2 - \lambda_4\beta_1^2, \end{aligned}$$

where  $\beta_1 = -2r'(\rho^2) - 4\rho^2r''(\rho^2)$ ,  $\alpha_1 = 12\rho r''(\rho^2) + 8\rho^3r'''(\rho^2)$  and  $\alpha_2 = 4\rho r''(\rho^2)$ . We deduce:

$$\text{Cov}(\xi_{11}(0), \xi_{jj}(0)) = \frac{\det(\text{Var}(\nabla_{-(1)}X(0), \nabla_{-(1)}X(s))) \times \left(\frac{1}{3}\lambda_4\lambda_2^2 - \lambda_2\alpha_1\alpha_2 - \frac{1}{3}\lambda_4\beta_1^2\right)}{\det(\text{Var}(\nabla X(t), \nabla X(s)))}.$$

It is easy to check that:

$$\frac{1}{3}\lambda_4\lambda_2^2 - \lambda_2\alpha_1\alpha_2 - \frac{1}{3}\lambda_4\beta_1^2 \simeq \rho^4 \left( \frac{11}{180}\lambda_2\lambda_4\lambda_6 - \frac{\lambda_4^3}{12} \right).$$

We deduce, using analogous arguments as in Section B.3.1, that  $\text{Cov}(\xi_{11}(0), \xi_{jj}(0)) = \text{Cov}(\xi_{11}(t), \xi_{jj}(t)) \simeq \rho^2 \frac{11\lambda_2\lambda_6 - 15\lambda_4^2}{180\lambda_2}$ .

In the same way we prove  $\text{Cov}(\xi_{11}(0), \xi_{jj}(t)) \simeq \rho^2 \frac{15\lambda_4^2 - \lambda_2\lambda_6}{180\lambda_2}$ .

## References

- [1] ADLER, R. J. (1981). *The Geometry of Random Fields*. Wiley, New York.
- [2] ADLER, R. J. AND TAYLOR, J. (2007). *Random Fields and Geometry*. Springer, New York.
- [3] AUFFINGER, A. AND BEN AROUS, G. (2013). Complexity of random smooth functions on the high-dimensional sphere. *Ann. Probab.*, **41**, 6, 4214–4247.
- [4] AZAÏS, J. M., CIERCO-AYROLLES, C., CROQUETTE, A. (1999). Bounds and asymptotic expansions for the distribution of the maximum of a smooth stationary Gaussian process. *ESAIM: Probab. Statist.*, **3**, 107–129.
- [5] AZAÏS, J. M. AND LEÓN, J. R. (2019). Necessary and sufficient conditions for the finiteness of the second moment of the measure of level sets. *arXiv preprint arXiv:1905.12342*.
- [6] AZAÏS, J. M., LEÓN, J. R., ORTEGA, J. (2005). Geometrical characteristic of Gaussian sea waves. *J. Appl. Probab.*, **42**, 407–425.
- [7] AZAÏS, J. M. AND WSCHBOR, M. (2009). *Level Sets and Extrema of Random Processes and Fields*. John Wiley & Sons, Inc., Hoboken, New Jersey.

- [8] AZAÏS, J. M. AND WSCHEBOR, M. (2008) A general expression for the distribution of the maximum of a Gaussian field and the approximation of the tail. *Stoch. Process. Appl.*, **118**, 1190–1218.
- [9] AZAÏS, J. M. AND WSCHEBOR, M. (2010) Erratum to: A general expression for the distribution of the maximum of a Gaussian field and the approximation of the tail [Stochastic Process. Appl. 118 (7)(2008) 1190–1218]. *Stoch. Process. Appl.*, **120**, 2100–2101.
- [10] BELIAEV, D., CAMMAROTA, V., WIGMAN, I. (2019). Two Point Function for Critical Points of a Random Plane Wave. *International Mathematics Research Notices*, **9**, 2661–2689
- [11] BELYAEV, Y. K. (1966). On the number of intersections of a level by a gaussian stochastic process. *Theory Probab. Appl.*, **11(1)**, 106–113.
- [12] BEN AROUS, G., MEI, S., MONTANARI, A., NICA, M. (2019). The Landscape of the Spiked Tensor Model. *Communications on Pure and Applied Mathematics*, <https://doi.org/10.1002/cpa.21861>
- [13] BERRY, M. V., DENNIS, M. R. (2000). Phase singularities in isotropic random waves. *Proc. R. Soc. Lond. A*, **456**, 2059–2079.
- [14] de BRUIJN, N. G. (1955). On some multiple integrals involving determinants. *J. Indian Math. Soc.*, **19**, 133–151.
- [15] CAMMAROTA, V., MARINUCCI, D., WIGMAN, I. (2016). On the distribution of the critical values of random spherical harmonics. *J. Geom. Anal.*, **26(4)**, 3252–3324.
- [16] CHENG, D., SCHWARTZMAN, A. (2017). Multiple testing of local maxima for detection of peaks in random fields. *Ann. Stat.*, **45**, 529–556.
- [17] CHENG, D., SCHWARTZMAN, A. (2018). Expected number and height distribution of critical points of smooth isotropic Gaussian random fields. *Bernoulli*, **24(4B)**, 3422–3446.
- [18] CHENG, D., XIAO, Y. (2016). The mean Euler characteristic and excursion probability of Gaussian random fields with stationary increments. *Ann. Appl. Probab.*, **26(2)**, 722–759.
- [19] CHEVALIER, C., GINSBOURGER, D. (2013) Fast computation of the multi-points expected improvement with applications in batch selection. *International Conference on Learning and Intelligent Optimization*. Springer, Berlin, Heidelberg.
- [20] ESTRADÉ, A., FOURNIER, J. (2016). Number of critical points of a Gaussian random field: Condition for a finite variance. *Statist. Probab. Lett.*, **118**, 94–99.
- [21] FYODOROV, Y. (2004). Complexity of random energy landscapes, glass transition and absolute value of spectral determinant of random matrices. *Phys. Rev. Lett.*, **92**, 240601. Erratum, *ibid.*, **93**, 149901.
- [22] HOUGH, J. B., KRISHNAPUR, M., PERES, Y., VIRAG, B. (2006). Determinantal processes and independence. *Probab. Surveys*, **3**, 206–229.
- [23] LONGUET-HIGGINS, M. S. (1957). The statistical analysis of a random moving surface. *Phil. Trans. R. Soc. London A*, **249**, 321–387.
- [24] MEHTA, M. L. (2004). *Random Matrices*. 3rd ed. Academic Press, San Diego, CA.
- [25] MEHTA, M. L., NORMAND, J. M. (2001). Moments of the characteristic polynomial in the three ensembles of random matrices. *J. Phys. A: Math. Gen.*, **34**, 4627–4639.
- [26] NICOLAESCU, L. I. (2017). A CLT concerning critical points of random functions on a Euclidean space. *Stoch. Process. Appl.*, **127(10)**, 3412–3446.
- [27] PODGÓRSKI, K., RYCHLIK, I., SJÖ, E. (2000). Statistics for velocities of Gaussian waves. *Internat. J. Offshore Polar Eng.*, **10**, 91–98.
- [28] ROS, V., BEN AROUS, G., BIROLI, G., CAMMAROTA, C. (2019). Complex energy landscapes in spiked-tensor and simple glassy models: Ruggedness, arrangements of local minima, and phase transitions. *Phys. Rev. X*, **9**, 011003.
- [29] SEARLE, S. R. (1971). *Linear Models*. John Wiley & Sons.
- [30] TAYLOR, J. E., WORSLEY, K. J. (2007). Detecting sparse signals in random fields, with an application to brain mapping. *J. Am. Statist. Assoc.*, **102**, 913–928.

- [31] WORSLEY, K. J., MARRETT, S., NEELIN, P., EVANS, A. C. (1996). Searching scale space for activation in PET images. *Human Brain Mapping*, **4**, 74–90.
- [32] WORSLEY, K. J., MARRETT, S., NEELIN, P., VANDAL, A. C., FRISTON, K.J., EVANS, A. C. (1996). A unified statistical approach for determining significant signals in images of cerebral activation. *Human Brain Mapping*, **4**, 58–73.