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# K3 SURFACES WITH MAXIMAL FINITE AUTOMORPHISM GROUPS CONTAINING $M_{20}$

CÉDRIC BONNAFÉ AND ALESSANDRA SARTI

ABSTRACT. It was shown by Mukai that the maximum order of a finite group acting faithfully and symplectically on a K3 surface is 960 and that the group is isomorphic to the group  $M_{20}$ . Then Kondo showed that the maximum order of a finite group acting faithfully on a K3 surface is 3840 and this group contains the Mathieu group  $M_{20}$  with index four. Kondo also showed that there is a unique K3 surface on which this group acts faithfully, which is the Kummer surface  $\text{Km}(E_i \times E_i)$ . In this paper we describe two more K3 surfaces admitting a big finite automorphism group of order 1920, both groups contains  $M_{20}$  as a subgroup of index 2. We show moreover that these two groups and the two K3 surfaces are unique. This result was shown independently by S. Brandhorst and K. Hashimoto in a forthcoming paper, with the aim of classifying all the finite groups acting faithfully on K3 surfaces with maximal symplectic part.

## 1. INTRODUCTION

A K3 surface is a compact complex surface which is simply connected and has trivial canonical bundle. Given a finite group  $\Gamma$  acting on a K3 surface  $X$  we have an exact sequence

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma \longrightarrow \mathbb{Z}/m\mathbb{Z} \longrightarrow 1$$

where the last map is induced by the action on the nowhere vanishing holomorphic 2-form  $\omega_X$ . The group  $\Gamma_0$  is the normal subgroup of maximal order contained in  $\Gamma$  whose automorphisms act trivially on  $\omega_X$ . The automorphisms of  $\Gamma_0$  are called *symplectic*. It was shown by Mukai [9, Theorem 0.3] that a finite group acting faithfully and symplectically on a K3 surface has order bounded by  $960 = |M_{20}|$  and that, if  $G$  is a group of order 960 acting faithfully and symplectically on a K3 surface, then  $G \simeq M_{20}$ . In his paper Mukai gives the example of a K3 surface with such an action, we recall this example in section 4. More generally, it is an interesting question to classify maximal finite groups  $\Gamma$  acting faithfully on a K3 surface. More precisely we say that  $\Gamma$  is a *maximal finite group* acting faithfully on a K3 surface if the following holds: assume  $\Gamma'$  is another finite group acting faithfully on a K3 surface then  $\Gamma$  is not (isomorphic to) a proper subgroup of  $\Gamma'$ .

In Theorem 6.4 we show that there are only three finite groups  $\Gamma$  containing strictly  $\Gamma_0 = M_{20}$  as the normal subgroup of  $\Gamma$  acting faithfully and symplectically and only three K3 surfaces acted on by such a  $\Gamma$ , the main ingredient of the proof is Theorem 2.7. This result is shown also independently in a forthcoming paper of S. Brandhorst and K. Hashimoto [2], where they compute all the finite groups

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acting faithfully on K3 surfaces with maximal symplectic part. In our situation one of the three K3 surfaces mentioned above was constructed by Kondo [7] (this is the only K3 surface acted on faithfully by a finite group of order  $3\,840 = 4 \cdot |M_{20}|$ ), another one was constructed by Mukai [9], and the existence of the last one was showed by Brandhorst-Hashimoto in *loc. cit.*, we give here explicit equations. In the second and in the third case the order of  $\Gamma$  is equal to  $2 \cdot |M_{20}|$ . We denote these three surfaces respectively by  $X_{\text{Ko}}$ ,  $X_{\text{Mu}}$  and  $X_{\text{BH}}$ . In this note, we compute the transcendental lattice of these three K3 surfaces. This was done by Kondo for the surface  $X_{\text{Ko}}$ , we recall it here to have a complete picture, and we compute it for  $X_{\text{Mu}}$  and  $X_{\text{BH}}$ . Accordingly to [4, Section 3] the transcendental lattice of  $X_{\text{Mu}}$  was already known by Mukai, but we could not find explicit computations, so we give it here. We give also equations for the three surfaces. Mukai already provided equations for  $X_{\text{Mu}}$  as a smooth quartic surface in  $\mathbb{P}^3(\mathbb{C})$  (which is the *Maschke surface*, see [4, Section 3]) we compute it here in a different way, but we show that up to a projective transformation of  $\mathbb{P}^3(\mathbb{C})$ , these are equivalent.

The equations for  $X_{\text{Ko}}$  and  $X_{\text{BH}}$  are new. In particular one gets easily a (singular) equation for the first one as a complete intersection of two quartics in weighted projective space  $\mathbb{P}(1, 1, 2, 2, 2)$  by using a result of Inose, [6]. To get the equations for  $X_{\text{BH}}$  one needs a more careful study of the action of  $M_{20}$  on the projective space  $\mathbb{P}^5(\mathbb{C})$ . It turns out that  $X_{\text{BH}}$  is a smooth complete intersection of three quadrics and we give here the equations (this answers a question of S. Brandhorst to the authors). All these three K3 surfaces turn out to be Kummer surfaces of abelian surfaces that are the product of two elliptic curves, see Corollary 2.5. By using results of Shioda and Mitani [15] we compute explicitly the two elliptic curves. We have that

$$X_{\text{Ko}} \cong \text{Km}(E_i \times E_i), \quad X_{\text{Mu}} \cong \text{Km}(E_{i\sqrt{10}} \times E_{i\sqrt{10}}),$$

$$X_{\text{BH}} \cong \text{Km}(E_{\tau_1} \times E_{\tau_2}), \quad \text{with } \tau_1 = \frac{-1+i\sqrt{5}}{2} \text{ and } \tau_2 = 1 + i\sqrt{5}.$$

For the example of  $X_{\text{BH}}$ , we also obtain in Remark 5.12 an explicit set of 16 disjoint smooth rational curves (we are not able to obtain such an explicit set for  $X_{\text{Mu}}$ : see Remark 4.6).

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NOTATION - If  $G$  is a group, we denote by  $G'$  its derived subgroup and by  $Z(G)$  its center. If  $V$  is a vector space, we denote by  $\mathbb{C}[V]$  the algebra of polynomial functions on  $V$  and, if  $k \geq 0$ , we denote by  $\mathbb{C}[V]_k$  its homogeneous component of degree  $k$ . If  $f_1, \dots, f_r \in \mathbb{C}[V]$  are homogeneous, we denote by  $\mathcal{Z}(f_1, \dots, f_r)$  the associated scheme of  $\mathbb{P}(V)$ , defined by  $f_1 = \dots = f_r = 0$ . If  $G$  is a subgroup of  $\mathbf{GL}_{\mathbb{C}}(V)$ , we denote by  $PG$  its image in  $\mathbf{PGL}_{\mathbb{C}}(V)$ . If  $V = \mathbb{C}^n$ , we identify

naturally  $\mathbf{GL}_{\mathbb{C}}(V)$  and  $\mathbf{GL}_n(\mathbb{C})$ . We denote by  $M_{20}$  the Mathieu group of order 960.

If  $\tau \in \mathbb{C}$  has a positive imaginary part, we denote by  $E_{\tau}$  the elliptic curve  $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ . If  $A$  is an abelian surface, we denote by  $\mathrm{Km}(A)$  its associated Kummer surface. We denote by  $\mathbf{L}$  the *K3 lattice*  $E_8(-1) \oplus E_8(-1) \oplus U \oplus U \oplus U$ , where  $U$  is the hyperbolic plane and  $E_8(-1)$  is the lattice  $E_8$  endowed with the opposite quadratic form. If  $X$  is a K3 surface, we denote by  $\mathbf{L}_X$  the lattice  $H^2(X, \mathbb{Z})$  (it turns out that  $\mathbf{L}_X \simeq \mathbf{L}$ ) and by  $\mathbf{T}_X$  its transcendental lattice. Finally, we denote by  $\mathbf{L}_{20}$  the lattice

$$\mathbf{L}_{20} = \begin{pmatrix} 4 & 0 & -2 \\ 0 & 4 & -2 \\ -2 & -2 & 12 \end{pmatrix}.$$

See the Proposition 2.3 below for the reason for this notation.

## 2. K3 SURFACES WITH A FAITHFUL ACTION OF $M_{20}$

We gather in this section some properties of the K3 surfaces admitting a faithful action of the finite group  $M_{20}$  (since  $M_{20}$  is perfect, this is necessarily a symplectic action), and we prove the main result of this paper, namely a classification of K3 surfaces admitting a faithful action of a finite group containing strictly  $M_{20}$ .

If we consider all the K3 surfaces  $X$  that admit a faithful symplectic action of  $M_{20}$ , Xiao [16, Nr. 81, Table 2] proved that the minimal resolution of the quotient of  $X$  by  $M_{20}$  is a K3 surface with Picard number 20. By a result of Inose [6, Corollary 1.2], this means also that  $X$  has Picard number 20. This shows the following, with the same notation as before:

**Proposition 2.1.** *There are at most countably many K3 surfaces with a faithful symplectic action by  $M_{20}$ .*

*Proof.* Since the Picard number is 20, then the moduli space of K3 surfaces with a faithful symplectic  $M_{20}$ -action is 0-dimensional.  $\square$

*Remark 2.2.* Observe that the automorphism group of a K3 surface with Picard number 20 is infinite [14, Theorem 5]. Shioda and Inose show it by exhibiting an elliptic fibration with an infinite order section, this gives an automorphism acting symplectically on the K3 surface with infinite order.  $\blacksquare$

Recall the following result [7, proof of Proposition 2.1]:

**Proposition 2.3.** *Let  $X$  be a K3 surface with a faithful symplectic action by  $M_{20}$ . Then the invariant lattice  $\mathbf{L}_X^{M_{20}}$  is isometric to  $\mathbf{L}_{20}$ .*

*Remark 2.4.* Let

$$\rho_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \rho_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then  $\rho_1$  and  $\rho_2$  belong to the group of isometries of  $\mathbf{L}_{20}$  and it is easily checked that the group of isometries of  $\mathbf{L}_{20}$  is generated by  $\rho_1$ ,  $\rho_2$  and  $-\text{Id}_{\mathbf{L}_{20}}$  (by using for instance the upcoming Lemma 2.8) and has order 16 (see also [7, Proposition 2.1]). ■

**Corollary 2.5.** *If a K3 surface  $X$  admits a faithful action by the group  $M_{20}$  then  $X = \text{Km}(A)$  for a unique abelian surface  $A$ , which is the product of two elliptic curves.*

*Proof.* Let  $(u, v)$  be a  $\mathbb{Z}$ -basis of  $\mathbf{T}_X \subset \mathbf{L}_X^{M_{20}}$ . By Proposition 2.3, we have  $u^2, v^2 \in 4\mathbb{Z}$  and  $u \cdot v \in 2\mathbb{Z}$ . So

$$\mathbf{T}_X \simeq \begin{pmatrix} 4a & 2b \\ 2b & 4c \end{pmatrix}.$$

Hence  $\mathbf{T}_X = \mathbf{T}_A(2)$  where

$$\mathbf{T}_A := \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}.$$

By [15, Section 3], we have  $A \cong E_{\tau_1} \times E_{\tau_2}$  where

$$\tau_1 = \frac{-b + \sqrt{\Delta}}{2a}, \quad \tau_2 = \frac{b + \sqrt{\Delta}}{2}$$

and  $\Delta = b^2 - 4ac$ .

The uniqueness follows from [15, Theorem 5.1]. □

*Remark 2.6.* Let us prove here that  $\mathbf{L}_{20}$  is indecomposable. Assume that it is not indecomposable. Then  $\mathbf{L}_{20} = L_1 \oplus L_2$ , where  $L_1$  has rank 1 and  $L_2$  has rank 2. By the proof of the Corollary 2.5, we have  $L_1 = \langle 4n \rangle$  for some  $n \geq 0$  and

$$L_2 = \begin{pmatrix} 4a & 2b \\ 2b & 4c \end{pmatrix}$$

for some  $a, b, c$ . Then  $160 = \text{disc}(\mathbf{L}_{20}) = \text{disc}(L_1)\text{disc}(L_2) = 16n(4ac - b^2)$ . In other words,  $10 = n(4ac - b^2)$ , which means that  $4ac - b^2 \in \{1, 2, 5, 10\}$ . But  $b^2 \equiv 0$  or  $1 \pmod{4}$ , so  $4ac - b^2 \equiv 3$  or  $4 \pmod{4}$ . This leads to a contradiction. ■

Our main result in this paper is the following:

**Theorem 2.7.** *Assume that  $M_{20}$  acts faithfully on a K3 surface  $X$ , and assume moreover that  $X$  admits a non-symplectic automorphism  $\iota$  acting on it, normalizing  $M_{20}$  and such that  $\iota^2 \in M_{20}$ . We set  $G = \langle \iota \rangle M_{20}$ . Then we have the following three possibilities for the  $G$ -invariant Néron-Severi group of  $X$  and its transcendental lattice:*

- (1)  $\langle 40 \rangle, \quad \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$
- (2)  $\langle 4 \rangle, \quad \begin{pmatrix} 4 & 0 \\ 0 & 40 \end{pmatrix}$
- (3)  $\langle 8 \rangle, \quad \begin{pmatrix} 8 & 4 \\ 4 & 12 \end{pmatrix}$

*All the three cases are possible and are described in the sections 3, 4, 5.*

*Proof.* We only prove here the fact that the Néron-Severi group of  $X$  and its transcendental lattice is necessarily one of the given three forms: the existence of the three examples will be shown in the upcoming sections (and we will add some geometric features of those examples). We first need two technical lemmas:

**Lemma 2.8.** *Up to isometry, there is a unique embedding of the lattice  $\langle 4 \rangle$  (resp.  $\langle 8 \rangle$ , resp.  $\langle 40 \rangle$ ) as a primitive sublattice of  $\mathbf{L}_{20}$ .*

*Proof of Lemma 2.8.* The uniqueness of the embedding of  $\langle 40 \rangle$  is shown in [7, Lemma 3.1]. For the two other cases, let  $(e, f, h)$  denote the canonical basis of the lattice  $\mathbf{L}_{20}$  and let  $L$  be a primitive element of  $\mathbf{L}_{20}$  such that  $L^2 = 4$  (resp. 8). Write  $L = \lambda e + \mu f + \delta h$  with  $\lambda, \mu, \delta \in \mathbb{Z}$ . Then

$$L^2 = (2\lambda - \delta)^2 + (2\mu - \delta)^2 + 10\delta^2,$$

so  $\delta = 0$  and  $\lambda^2 + \mu^2 = 1$  (resp.  $\lambda^2 + \mu^2 = 2$ ). This gives  $(\lambda, \mu) = (\pm 1, 0)$  or  $(0, \pm 1)$  (resp.  $(\pm 1, \pm 1)$ ). So  $L = \pm e$  or  $\pm f$  (resp.  $L = \pm e \pm f$ ), and the four solutions are in the orbit of the group  $\langle -\text{Id}_{\mathbf{L}_{20}}, \rho_1 \rangle$  (resp.  $\langle -\text{Id}_{\mathbf{L}_{20}}, \rho_2 \rangle$ ).  $\square$

We choose an isomorphism between  $\mathbf{L}_{20}$  and  $\mathbf{L}_X^{M_{20}}$ . Then the group  $G/M_{20} = \langle \iota \rangle$  acts on  $\mathbf{L}_{20}$  and  $\iota$  acts by  $-\text{Id}$  on  $\mathbf{T}_X$ . Also, the lattice  $\mathbf{L}_X^G$  has rank 1 because  $\mathbf{T}_X$  has rank 2.

**Lemma 2.9.** *The sublattice  $\mathbf{L}_X^G \oplus \mathbf{T}_X$  has index 2 in  $\mathbf{L}_{20}$ .*

*Proof of Lemma 2.9.* First,  $\mathbf{L}_X^G \oplus \mathbf{T}_X$  is different from  $\mathbf{L}_{20}$  since  $\mathbf{L}_{20}$  is indecomposable (see Remark 2.6). We have

$$\mathbf{L}_X^G = \{L \in \mathbf{L}_{20} \mid \iota(L) = L\},$$

$$\mathbf{T}_X = \{L \in \mathbf{L}_{20} \mid \iota(L) = -L\}.$$

By [11, Section 5], the projection  $\mathbf{L}_{20}/(\mathbf{L}_X^G \oplus \mathbf{T}_X) \rightarrow (\mathbf{L}_X^G)^\vee/\mathbf{L}_X^G$  is a  $\iota$ -invariant monomorphism. This shows in particular that  $\mathbf{L}_{20}/(\mathbf{L}_X^G \oplus \mathbf{T}_X)$  is cyclic. Also, if  $L \in \mathbf{L}_{20}$ , then

$$2L = \underbrace{L + \iota(L)}_{\in \mathbf{L}_X^G} + \underbrace{L - \iota(L)}_{\in \mathbf{T}_X} \in \mathbf{L}_X^G \oplus \mathbf{T}_X.$$

So the sublattice  $\mathbf{L}_X^G \oplus \mathbf{T}_X$  has index 2 in  $\mathbf{L}_{20}$ . This completes the proof of the Lemma.  $\square$

We now come back to the proof of the theorem. We write  $\mathbf{L}_{20}^G = \mathbb{Z}L$ . By the proof of Corollary 2.5, we have  $L^2 = 4n$  (so that  $\mathbf{L}_{20}^G \simeq \langle 4n \rangle$ ) and the transcendental lattice of  $X$  is of the form

$$\mathbf{T}_X = \begin{pmatrix} 4a & 2b \\ 2b & 4c \end{pmatrix}$$

with  $a, b, c$  integers such that  $d := 4ac - b^2 > 0$ ,  $b^2 \leq ac \leq \frac{d}{3}$ ,  $-a \leq b \leq a \leq c$ , see e.g. [14, p. 128]. Hence we have

$$4 = [\mathbf{L}_{20} : \langle 4n \rangle \oplus \mathbf{T}_X]^2 = \frac{16n(4ac - b^2)}{160}.$$

In conclusion

$$n(4ac - b^2) = 2^3 \cdot 5.$$

We discuss two cases.

**Assume that  $b$  is odd.** Then  $4ac - b^2$  is also odd. This means that it is equal to 1 or 5, but then if  $b = 2k + 1$  we get  $4ac - 4k^2 - 4k - 1$  equal to 1 or 5 which is clearly impossible.

**Assume that  $b$  is even.** Then with  $b = 2b'$  we get

$$(ac - b'^2)n = 2 \cdot 5$$

We distinguish four cases:

- (1)  $n = 1, ac - b'^2 = 10,$
- (2)  $n = 2, ac - b'^2 = 5,$
- (3)  $n = 5, ac - b'^2 = 2,$
- (4)  $n = 10, ac - b'^2 = 1.$

By Lemma 2.8, the lattices  $\langle 4 \rangle$ ,  $\langle 8 \rangle$  and  $\langle 40 \rangle$  have a unique primitive embedding in the lattice  $\mathbf{L}_{20}$ :

- (1) If  $n = 1$ , we may assume that  $L = e$ . We now compute the orthogonal complement of  $\mathbb{Z}e$  in the lattice  $\mathbf{L}_{20}$ . This will give us the transcendental lattice. Let now  $\lambda e + \mu f + \delta h$  with  $\lambda, \mu, \delta \in \mathbb{Z}$  be such that

$$\langle \lambda e + \mu f + \delta h, e \rangle = 0$$

This gives  $4\lambda - 2\delta = 0$  so that the orthogonal complement is generated by the elements  $e + 2h$  and  $f$  and considering instead the generators  $e + f + 2h$  and  $f$  we get the lattice given in the theorem.

- (2) If  $n = 2$ , we may assume that  $L = e - f$ . We compute the orthogonal complement of  $e - f$  in  $\mathbf{L}_{20}$  which is generated by  $e + f$  and  $-h$  which are the generators of the rank two lattice whose bilinear form is as given in the theorem.
- (4) If  $n = 10$ , then the orthogonal complement of  $L$  has been computed in [7] and one gets the rank two lattice whose bilinear form is given as in the theorem.

We have respectively  $(a, b, c) = (1, 0, 1)$ ,  $(a, b, c) = (1, 0, 10)$ ,  $(a, b, c) = (2, 2, 3)$ .

We consider now the third case with  $ac - b'^2 = 2$  and we show that it is not possible. The integers  $a, b, c$  satisfy  $-a \leq b \leq a \leq c$ ,  $ac \leq d/3$ ,  $(b')^2 \leq (ac)/4 \leq d/3$ . By the previous computations, we have that  $d = 4(ac - b'^2)$  hence in this case  $d = 8$ , we get that  $b'^2 \leq 2$ . Hence  $b' = 0$  or  $b' = 1$ . In the first case we get  $a = 1$ ,  $c = 2$  which gives the matrix

$$M := \begin{pmatrix} 4 & 0 \\ 0 & 8 \end{pmatrix}.$$

In the second case we get  $a = 1$ ,  $c = 3$  but then  $ac = 3 > 8/3$  so this is not possible. To make the case  $\mathbf{T}_X = M$  possible, we should then find a primitive embedding in  $\mathbf{L}_{20}$  with vectors  $v_1$  and  $v_2$  with  $v_1^2 = 4$ ,  $v_2^2 = 8$ ,  $v_1 \cdot v_2 = 0$  but by the computations in Lemma 2.8 and with the same notations as there we see that we must send  $v_1$  to  $\pm e$  or  $\pm f$  and  $v_2$  to  $\pm e \pm f$ , so these never satisfy the condition  $v_1 \cdot v_2 = 0$ .  $\square$

## 3. KONDO'S EXAMPLE

It was shown by Kondo in [7, Theorem 1] that the maximal order of a finite group acting faithfully on a K3 surface is 3840 and that this bound is reached for a unique K3 surface  $X_{\text{Ko}}$  and a unique faithful action of a unique finite group  $G_{\text{Ko}}$  of order 3840. Kondo shows that  $X_{\text{Ko}} = \text{Km}(E_i \times E_i)$ . Recall that we have an exact sequence

$$(3.1) \quad 1 \longrightarrow M_{20} \longrightarrow G_{\text{Ko}} \longrightarrow \mu_4 \longrightarrow 1,$$

where the last map is induced by the group homomorphism

$$\alpha : G_{\text{Ko}} \longrightarrow \mathbb{C}^*,$$

defined by  $g(\omega_{X_{\text{Ko}}}) = \alpha(g)\omega_X$  and  $\omega_{X_{\text{Ko}}}$  is the holomorphic 2-form that we have fixed on  $X_{\text{Ko}}$ . Recall that  $X_{\text{Ko}} = \text{Km}(E_i \times E_i)$  (see e.g. [7, Proof of Lemma 1.2]) has transcendental lattice

$$\mathbf{T}_{X_{\text{Ko}}} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}.$$

With the previous notation we have:

**Proposition 3.2.** *The invariant Néron–Severi group  $NS(X_{\text{Ko}})^{M_{20}} = \mathbb{Z}L_{40}$  with  $L_{40}^2 = 40$ .*

*Proof.* See [7, Lemma 3.1]. □

*Remark 3.3.* In particular this means that we cannot represent  $X_{\text{Ko}}$  as a quartic surface in  $\mathbb{P}^3(\mathbb{C})$  with a faithful action of  $M_{20}$  by linear transformations of  $\mathbb{P}^3(\mathbb{C})$ . ■

**3.1. A geometric model.** By using a result of Inose [6, Theorem 2] one can view  $X_{\text{Ko}} = \text{Km}(E_i \times E_i)$  as the minimal resolution of a singular surface in  $\mathbb{P}(1, 1, 2, 2, 2)$ . We give here the equation. Inose shows that  $X_{\text{Ko}}$  is the minimal resolution of the quotient of the Fermat quartic surface

$$F : x^4 + y^4 + z^4 + t^4 = 0$$

by the symplectic involution  $\iota : (x : y : z : t) \mapsto (x : y : -z : -t)$ , which has 8 isolated fixed points [10, Section 5]. Since the automorphism is symplectic, the minimal resolution of the quotient  $X_{\text{Ko}} \rightarrow F/\langle \iota \rangle$  is again a K3 surface and the Picard number remains unchanged. Moreover, for the transcendental lattices  $\mathbf{T}_{X_{\text{Ko}}}(2) = \mathbf{T}_F$  holds. The ring of invariant polynomials for the action of  $\iota$  is generated by  $x, y, z^2, t^2, zt$ . We put  $z_0 = x, z_1 = y, z_2 = z^2, z_3 = t^2, z_4 = zt$  and we have then the equations for  $F/\langle \iota \rangle$  in  $\mathbb{P}(1, 1, 2, 2, 2)$ :

$$z_0^4 + z_1^4 + z_2^2 + z_3^2 = 0, \quad z_4^2 = z_2z_3.$$

The eight  $A_1$  singularities are determined as follows. First we have singularities coming from the ambient space. One has to put  $z_0 = z_1 = 0$ , then  $z_2^2 + z_3^2 = 0$  which together with  $z_4^2 = z_2z_3$  gives four  $A_1$  singularities. The others come from the singularities of the cone  $z_4^2 = z_2z_3$ , i.e. with  $z_4 = z_2 = z_3 = 0$  we get the four singularities  $A_1$  with equation  $z_0^4 + z_1^4 = 0$ .

See also [2] for an embedding of  $X_{\text{Ko}}$  in  $\mathbb{P}^{21}(\mathbb{C})$ .



## 4. MUKAI'S EXAMPLE

Let  $G_{\text{Mu}} = \langle s_1, s_2, s_3, s_4 \rangle$ , where

$$s_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad s_2 = \frac{1}{2} \begin{pmatrix} 1 & 1 & i & i \\ 1 & 1 & -i & -i \\ -i & i & 1 & -1 \\ -i & i & -1 & 1 \end{pmatrix},$$

$$s_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad s_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then  $G_{\text{Mu}}$  is the primitive complex reflection group denoted by  $G_{29}$  in Shephard-Todd classification [13]. We denote by  $V$  the vector space  $\mathbb{C}^4$ , and by  $\mathbb{C}[V]$  the algebra of polynomial functions on  $V$ , identified naturally with  $\mathbb{C}[x, y, z, t]$ . If  $m$  is a monomial in  $x, y, z$  and  $t$ , we denote by  $\Sigma(m)$  the sum of all monomials obtained by permutation of the variables.

Note that the derived subgroup  $G'_{\text{Mu}}$  of  $G_{\text{Mu}}$  has index 2, that  $G'_{\text{Mu}} = G_{\text{Mu}} \cap \mathbf{SL}_4(\mathbb{C})$ , so that  $G_{\text{Mu}} = G'_{\text{Mu}} \langle s_1 \rangle$ . Note also that  $Z(G_{\text{Mu}}) \simeq \mu_4 \subset G'_{\text{Mu}}$ . Moreover,  $PG'_{\text{Mu}} \simeq M_{20}$  so that we have a split exact sequence

$$(4.1) \quad 1 \longrightarrow PG'_{\text{Mu}} \simeq M_{20} \longrightarrow PG_{\text{Mu}} \longrightarrow \mu_2 \longrightarrow 1,$$

where the last map is the determinant.

Now, there exists a unique (up to scalar) homogeneous invariant  $f$  of  $G_{\text{Mu}}$  of degree 4: it is given by

$$f = \Sigma(x^4) - 6\Sigma(x^2y^2).$$

We set  $X_{\text{Mu}} = \mathcal{Z}(f)$ . It can easily be checked that  $X_{\text{Mu}}$  is a smooth and irreducible quartic in  $\mathbb{P}^3(\mathbb{C})$ , so that it is a K3 surface, endowed with a faithful symplectic action of  $M_{20}$  and an extra non-symplectic automorphism of order 2, i.e. one can fix it as  $[x : y : z : t] \mapsto [x : y : z : -t]$ , the one induced by  $s_1$ .

In [9, nr. 4 on p. 190] Mukai gives the following equation for some  $M_{20}$ -invariant quartic polynomial

$$\Sigma(x^4) + 12xyzt,$$

and we denote by  $X'_{\text{Mu}}$  the zero set of this polynomial which defines a smooth quartic K3 surface. We have

**Proposition 4.2.** *There exists  $g \in \mathbf{GL}_4(\mathbb{C})$  such that  $g(X_{\text{Mu}}) = X'_{\text{Mu}}$ .*

*Proof.* If one applies to the Mukai's polynomial the change of coordinates:

$$x \mapsto x - y, \quad y \mapsto x + y, \quad z \mapsto z - t, \quad t \mapsto z + t$$

one gets

$$2\Sigma(x^4) + 12x^2y^2 + 12z^2t^2 + 12x^2z^2 - 12x^2t^2 - 12y^2z^2 + 12y^2z^2$$

and by replacing by

$$x \mapsto ix, \quad t \mapsto it, \quad y \mapsto y,$$

and dividing by 2 one finds the polynomial  $f$ . □

Note the following fact:

$$(4.3) \quad \text{If } g \in \mathbf{PGL}_4(\mathbb{C}) \text{ leaves invariant } X_{\text{Mu}} \text{ then } g \in PG_{\text{Mu}}.$$

*Proof.* If  $g \in \mathbf{PGL}_4(\mathbb{C})$  leaves  $X_{\text{Mu}}$  invariant, we may find a representative  $\tilde{g}$  of  $g$  in  $\mathbf{GL}_4(\mathbb{C})$  which leaves  $f$  invariant. Let  $\Gamma = \{\gamma \in \mathbf{GL}_4(\mathbb{C}) \mid \gamma f = f\}$ . We only need to prove that  $\Gamma = G_{\text{Mu}}$ . By [8] or [12, Theorem 2.1],  $\Gamma$  is finite (because  $X_{\text{Mu}}$  is smooth), and contains  $G_{\text{Mu}}$ . Let  $R$  denote the set of reflections in  $G_{\text{Mu}}$  (and recall that  $G_{\text{Mu}} = \langle R \rangle$ ) and let

$$\mathcal{R} = \{\gamma s \gamma^{-1} \mid \gamma \in \Gamma \text{ and } s \in R\},$$

so that  $\mathcal{R}$  is a set of reflections contained in  $\Gamma$ . We set  $\Gamma_{\mathcal{R}} = \langle \mathcal{R} \rangle$ . Then  $\Gamma_{\mathcal{R}}$  is a complex reflection group containing  $G_{\text{Mu}}$ , but it follows from the classification of primitive complex reflection groups that  $\Gamma_{\mathcal{R}} = G_{\text{Mu}}$  or (up to conjugacy) the group denoted by  $G_{31}$  in Shephard-Todd classification [13]. Since  $G_{31}$  has no non-zero invariant of degree 4, this forces  $\Gamma_{\mathcal{R}} = G_{\text{Mu}}$ . In particular,  $G_{\text{Mu}}$  is normal in  $\Gamma$ , and so the result follows from [3, Proposition 3.13] (which says that  $N_{\mathbf{GL}_4(\mathbb{C})}(G_{\text{Mu}}) = G_{\text{Mu}} \cdot \mathbb{C}^\times$ ).  $\square$

The embedding  $X_{\text{Mu}} \hookrightarrow \mathbb{P}^3(\mathbb{C})$  defines an hyperplane section on  $X_{\text{Mu}}$  that we denote by  $L_4$ : then  $L_4^2 = 4$  and  $L_4$  is  $PG_{\text{Mu}}$ -invariant.

**Proposition 4.4.** *With the above notation, we have:*

- (1) *The transcendental lattice of  $X_{\text{Mu}}$  is a rank two lattice given by*

$$\mathbf{T}_{X_{\text{Mu}}} := \begin{pmatrix} 4 & 0 \\ 0 & 40 \end{pmatrix}$$

*and  $NS(X_{\text{Mu}})^{M_{20}} = \mathbb{Z} L_4$  with  $L_4^2 = 4$ .*

- (2) *The quartic  $X_{\text{Mu}}$  is the unique invariant quartic for a faithful action of  $M_{20}$  on  $\mathbb{P}^3$ .*

*Proof.* (1) has been proved in Theorem 2.7, see also [4, Section 3].

(2) Let  $Q \in \mathbb{P}^3(\mathbb{C})$  be a quartic leaved invariant by a faithful action of  $M_{20}$ . This means that there exists a representation of  $M_{20}$  as a subgroup of  $\mathbf{PGL}_4(\mathbb{C})$  which stabilizes  $Q$ . Then  $Q$  is polarized by the lattice  $\langle 4 \rangle$ , so that we have an embedding of  $\langle 4 \rangle$  in the lattice  $\mathbf{L}_Q^{M_{20}}$ . Since this embedding is unique by (1), its orthogonal complement  $\mathbf{T}_Q$  in  $\mathbf{L}_Q^{M_{20}}$  is isometric to  $\mathbf{T}_{X_{\text{Mu}}}$ . So  $Q$  is projectively equivalent to  $X_{\text{Mu}}$ .  $\square$

**Proposition 4.5.** *The quartic  $X_{\text{Mu}}$  is the Kummer surface  $\text{Km}(E_{i\sqrt{10}} \times E_{i\sqrt{10}})$ .*

*Proof.* This follows from Corollary 2.5 and its proof.  $\square$

*Remark 4.6.* As  $X_{\text{Mu}}$  is a Kummer surface, it admits 16 two by two disjoint smooth rational curves (a *Nikulin configuration*). We were not able to find such a set of smooth rational curves, but, using MAGMA, we have at least found 320 conics in

$X_{\text{Mu}}$  (from which it is impossible to extract a Nikulin configuration: we can only extract 12 two by two disjoint conics). Let

$$C_+ = \{[x : y : z : t] \in \mathbb{P}^3(\mathbb{C}) \mid x + y + z = y^2 + yz + z^2 + \frac{3 + \sqrt{10}}{2}t^2 = 0\}$$

and  $C_- = \{[x : y : z : t] \in \mathbb{P}^3(\mathbb{C}) \mid x + y + z = y^2 + yz + z^2 + \frac{3 - \sqrt{10}}{2}t^2 = 0\}$ .

Then  $C_+$  and  $C_-$  are two smooth conics contained in  $X_{\text{Mu}}$  and, if we denote by  $\Omega_{\pm}$  the  $G_{\text{Mu}}$ -orbit of  $C_{\pm}$ , then  $|\Omega_+| \neq |\Omega_-|$ ,  $|\Omega_{\pm}| = 160$ , and all elements of  $\Omega_{\pm}$  are contained in  $X_{\text{Mu}}$ . ■

*Remark 4.7.* Observe that  $PG_{\text{Mu}}$  is a maximal finite subgroup of  $\text{Aut}(X_{\text{Mu}})$ . Indeed, if  $PG_{\text{Mu}} \subsetneq \Gamma \subset \text{Aut}(X_{\text{Mu}})$  with  $\Gamma$  finite, then  $|\Gamma| \geq 2 \cdot |PG_{\text{Mu}}| = 3840$  and so by the result of Kondo in [7] the group  $\Gamma$  would be the group  $G_{\text{Ko}}$  defined in section 3 and  $X_{\text{Mu}}$  would be isomorphic to  $X_{\text{Ko}}$ : this is not the case by Proposition 3.2 and Proposition 4.4. ■

## 5. BRANDHORST-HASHIMOTO'S EXAMPLE

Let  $G_{\text{BH}}$  be the subgroup of  $\mathbf{GL}_6(\mathbb{C})$  generated by

$$t = \text{diag}(-1, 1, 1, 1, 1, 1),$$

$$u = \begin{pmatrix} i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

All the numerical facts about  $G_{\text{BH}}$  stated below can be checked with MAGMA. Then  $|G_{\text{BH}}| = 3840$ ,  $Z(G) = \mu_2$ ,  $|G_{\text{BH}}/G'_{\text{BH}}| = 2$  and there are two exact sequences

$$1 \longrightarrow \mu_2 \longrightarrow G'_{\text{BH}} \longrightarrow M_{20} \longrightarrow 1$$

and

$$(5.1) \quad 1 \longrightarrow M_{20} = PG'_{\text{BH}} \longrightarrow PG_{\text{BH}} \longrightarrow \mu_2 \longrightarrow 1.$$

The second exact sequence splits (for instance by sending the non-trivial element of  $\mu_2$  to  $t$ ) and  $G'_{\text{BH}} = G_{\text{BH}} \cap \mathbf{SL}_6(\mathbb{C})$ . Even though the last exact sequence looks like (4.1),

$$(5.2) \quad \text{The groups } PG_{\text{Mu}} \text{ and } PG_{\text{BH}} \text{ are not isomorphic.}$$

Note that the group  $G'_{\text{BH}}$  is isomorphic to the group denoted by  $2_3.M_{20}$  in the ATLAS of finite groups<sup>1</sup>. We denote by  $V = \mathbb{C}^6$  the natural representation of  $G_{\text{BH}}$  and we identify  $\mathbb{C}[V]$  with  $\mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6]$ . Note that

$$(5.3) \quad G_{\text{BH}} \text{ acts doubly transitively on the set of hyperplanes } \{H_1, \dots, H_6\},$$

where  $H_i$  is defined by  $x_i = 0$ .

<sup>1</sup><http://brauer.maths.qmul.ac.uk/Atlas/v3/group/M20/>

S. Brandhorst and K. Hashimoto [2] proved that there is a unique K3 surface admitting a faithful action of  $PG_{\text{BH}}$  and, in a private communication, he asked the question about the equations of this K3 surface: the aim of this section is to answer the question by exhibiting explicit equations of such a K3 surface.

The group  $G_{\text{BH}}$  contains the group  $N$  of diagonal matrices with coefficients in  $\mu_2$  as a normal subgroup (so  $N \simeq (\mu_2)^6$ ) and we have  $G_{\text{BH}}/N \simeq \mathfrak{A}_5$ . It is easy to see that

$$(5.4) \quad \mathbb{C}[V]^N = \mathbb{C}[x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2].$$

The following facts are checked with MAGMA:

- (a) As a  $G_{\text{BH}}/N$ -module,  $\mathbb{C}[V]_2^N = S_1 \oplus S_2$ , where  $S_1$  and  $S_2$  are the two non-isomorphic irreducible representations of  $G_{\text{BH}}/N \simeq \mathfrak{A}_5$  of dimension 3.
- (b) If we set

$$\begin{cases} q_1 = x_1^2 + x_4^2 - \phi x_5^2 + \phi x_6^2, \\ q_2 = x_2^2 - \phi x_4^2 + x_5^2 - \phi x_6^2, \\ q_3 = x_3^2 + \phi x_4^2 - \phi x_5^2 + x_6^2, \end{cases}$$

then  $(q_1, q_2, q_3)$  is a basis of  $S_1$ .

We then define

$$X_{\text{BH}} = \mathcal{Z}(q_1, q_2, q_3).$$

The next proposition can be proved using MAGMA, but we will provide a proof independent of MAGMA computations.

**Proposition 5.5.** *The scheme  $X_{\text{BH}}$  is smooth, irreducible, of dimension 2.*

The variety  $X_{\text{BH}}$  is then an irreducible smooth complete intersection of three quadrics in  $\mathbb{P}^5(\mathbb{C})$ , so it is a K3 surface. Since the vector space  $S_k$  is stable under the action of  $G_{\text{BH}}$ , the K3 surface  $X_{\text{BH}}$  is endowed with a faithful action of  $PG_{\text{BH}} \simeq \langle t \rangle \rtimes M_{20}$ .

**Corollary 5.6.**  *$X_{\text{BH}}$  is a K3 surface endowed with a faithful action of  $PG_{\text{BH}}$ .*

*Another proof of Proposition 5.5.* First, it follows from (5.4) that

$$(5.7) \quad X_{\text{BH}}/N = \{[y_1 : \cdots : y_6] \in \mathbb{P}^5(\mathbb{C}) \mid \begin{cases} y_1 + y_4 - \phi y_5 + \phi y_6 = 0 \\ y_2 - \phi y_4 + y_5 - \phi y_6 = 0 \\ y_3 + \phi y_4 - \phi y_5 + y_6 = 0 \end{cases} \} \simeq \mathbb{P}^2(\mathbb{C}).$$

Hence  $X_{\text{BH}}/N$  has dimension 2, so  $X_{\text{BH}}$  has dimension 2.

Let us now show that  $X_{\text{BH}}$  is irreducible. Let  $H = N \cap G'_{\text{BH}}$ . Note that

$$\mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6]^H = \mathbb{C}[x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_6^2, x_1 x_2 \cdots x_6],$$

so that  $\mathbb{P}^5(\mathbb{C})/H = \{[y_1 : \cdots : y_6 : z] \in \mathbb{P}(1, \dots, 1, 3) \mid z^2 = \prod_{k=1}^6 y_k\}$ . Therefore,

$$X_{\text{BH}}/H = \{[y_1 : \cdots : y_6 : z] \in \mathbb{P}(1, \dots, 1, 3) \mid z^2 = \prod_{k=1}^6 y_k$$

$$\text{and } \begin{cases} y_1 + y_4 - \phi y_5 + \phi y_6 = 0 \\ y_2 - \phi y_4 + y_5 - \phi y_6 = 0 \\ y_3 + \phi y_4 - \phi y_5 + y_6 = 0 \end{cases} \}$$

Simplifying the equations, one gets

$$X_{\text{BH}}/H = \{[y_4 : y_5 : y_6 : z] \in \mathbb{P}(1, 1, 1, 3) \mid \\ z^2 = y_4 y_5 y_6 (-y_4 + \phi y_5 - \phi y_6)(\phi y_4 - y_5 + \phi y_6)(-\phi y_4 + \phi y_5 - y_6)\}.$$

So  $X_{\text{BH}}/H$  is a K3 surface (with  $A_1$  singularities) which is a double cover of  $\mathbb{P}^2(\mathbb{C})$  ramified on the union of 6 lines in general position. In particular, it is irreducible. This shows that  $H$  acts transitively on the irreducible components of  $X_{\text{BH}}$ . So  $G'_{\text{BH}}$  also acts transitively on the irreducible components. Now, let  $X$  be an irreducible component of  $X_{\text{BH}}$  and let  $K$  denote its stabilizer in  $G'_{\text{BH}}$ . Then  $8 = \deg(X_{\text{BH}}) = \deg(X) \cdot |G'_{\text{BH}}/K|$ . Since  $G'_{\text{BH}}$  has no subgroup of index 2, 4 or 8, we conclude that  $K = G'_{\text{BH}}$ , so that  $X = X_{\text{BH}}$ , as desired.

We now show that  $X_{\text{BH}}$  is smooth. Let  $p = [x_1 : x_2 : x_3 : x_4 : x_5 : x_6] \in X_{\text{BH}}$  and assume that  $p$  is a singular point of  $X_{\text{BH}}$ . Since  $p$  belongs to  $X_{\text{BH}}$ , the equations show that at least two of the  $x_k$ 's are non-zero. By replacing if necessary  $p$  by another point in its  $G_{\text{BH}}$ -orbit, we may assume that  $x_1 x_2 \neq 0$  (thanks to (5.3)). The Jacobian matrix of  $(q_1, q_2, q_3)$  at  $p$  is given by

$$\text{Jac}_p(q_1, q_2, q_3) = \begin{pmatrix} 2x_1 & 0 & 0 & 2x_4 & -2\phi x_5 & 2\phi x_6 \\ 0 & 2x_2 & 0 & -2\phi x_4 & 2x_5 & -2\phi x_6 \\ 0 & 0 & 2x_3 & 2\phi x_4 & -2\phi x_5 & 2x_6 \end{pmatrix}.$$

Then the rank of  $\text{Jac}_p(q_1, q_2, q_3)$  is less than 3, which means that all its minors of size 3 vanish. Therefore,

$$x_{i_1} x_{i_2} x_{i_3} = 0$$

for all  $1 \leq i_1 < i_2 < i_3 \leq 6$ . Since  $x_1 x_2 \neq 0$ , we get  $x_3 = x_4 = x_5 = x_6 = 0$ . But then  $q_1(p) \neq 0$ , which is impossible.  $\square$

*Remark 5.8.* Exchanging  $S_1$  and  $S_2$  (whose characters are Galois conjugate under  $\sqrt{5} \mapsto -\sqrt{5}$ ), one gets another K3 surface  $X'_{\text{BH}}$ , where  $\phi$  is replaced by its Galois conjugate  $\phi' = (1 - \sqrt{5})/2 = 1 - \phi$  in the equations. Let  $\sigma \in \mathbf{GL}_6(\mathbb{C})$  be the matrix

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then  $\sigma$  normalizes  $G_{\text{BH}}$  and  $\sigma(X_{\text{BH}}) = X'_{\text{BH}}$ , so that  $X_{\text{BH}}$  and  $X'_{\text{BH}}$  are isomorphic.  $\blacksquare$

The surface  $X_{\text{BH}}$  is a K3 surface with polarization  $L_8$  satisfying  $L_8^2 = 8$ , and as in section 4 this is invariant by the action of  $M_{20}$ . We have hence an embedding of  $\langle 8 \rangle$  in  $\mathbf{L}_{X_{\text{BH}}}^{M_{20}}$ .

**Proposition 5.9.** *With the above notation, we have:*

- (1) *The transcendental lattice of  $X_{\text{BH}}$  is a rank two lattice given by*

$$\mathbf{T}_{X_{\text{BH}}} = \begin{pmatrix} 8 & 4 \\ 4 & 12 \end{pmatrix}$$

*and  $NS(X_{\text{BH}})^{M_{20}} = \mathbb{Z}L_8$  with  $L_8^2 = 8$ .*

- (2) *The complete intersection  $X_{\text{BH}}$  is the unique K3 surface invariant for a faithful action of  $M_{20}$  in  $\mathbb{P}^5(\mathbb{C})$ .*

*Proof.* (1) has been proved in Theorem 2.7.

(2) follows from the same argument as in Proposition 4.4.  $\square$

*Remark 5.10.* Proposition 5.9 gives another proof that  $X_{\text{BH}} \cong X'_{\text{BH}}$ .  $\blacksquare$

**Proposition 5.11.** *The K3 surface  $X_{\text{BH}}$  is the Kummer surface  $\text{Km}(E_{\tau_1} \times E_{\tau_2})$ , with  $\tau_1 = \frac{-1+i\sqrt{5}}{2}$  and  $\tau_2 = 1 + i\sqrt{5}$ .*

*Proof.* This follows from Corollary 2.5 and its proof.  $\square$

*Remark 5.12* (Smooth rational curves). Using MAGMA, one can find an explicit Nikulin configuration in  $X_{\text{BH}}$  as follows. Let  $C$  denote the conic defined by the equations

$$\begin{cases} x_5 = \sqrt{\phi}x_1, \\ x_4 = \sqrt{\phi}x_2, \\ x_3 = \sqrt{\phi}x_6, \\ x_1^2 - x_2^2 - x_6^2 = 0 \end{cases}$$

and let  $\mathcal{A}$  denote the subgroup of  $G_{\text{BH}}$  generated by

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 0 & -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Then  $C$  is contained in  $X_{\text{BH}}$ . It can be checked with MAGMA that its  $G_{\text{BH}}$ -orbit contains 80 elements, and that its  $\mathcal{A}$ -orbit contains 16 elements which are two by two disjoint (note that  $|\mathcal{A}| = 32$ , that  $\mu_2 \subset \mathcal{A}$  and that  $\mathcal{A}/\mu_2$  is elementary abelian).

Note also that the conic defined by the equations

$$\begin{cases} x_1 + ix_5 - i\phi x_6 = 0, \\ x_3 - i\phi x_5 + i\phi x_6 = 0, \\ x_4 - \phi x_5 + x_6 = 0, \\ x_2^2 - 2\phi x_5^2 + 2(1 + \phi)x_5 x_6 - 2\phi x_6^2 = 0, \end{cases}$$

is contained in  $X_{\text{BH}}$ , and that its  $G_{\text{BH}}$ -orbit contains 96 elements. However, we can only extract subsets of 12 two by two disjoint conics from this orbit.  $\blacksquare$

## 6. FINAL REMARKS

**Proposition 6.1.** *The K3 surfaces  $X_{\text{Mu}}$ ,  $X_{\text{BH}}$  and  $X_{\text{Ko}}$  are two by two non-isomorphic.*

*Proof.* Indeed, they do not have the same transcendental lattice (or equivalently they do not admit polarizations of the same degree).  $\square$

**Proposition 6.2.** *If a K3 surface  $X$  admits a faithful action of  $G_{\text{Ko}}$ ,  $PG_{\text{Mu}}$ , respectively  $PG_{\text{BH}}$  then  $X$  is isomorphic to  $X_{\text{Ko}}$ ,  $X_{\text{Mu}}$ , respectively  $X_{\text{BH}}$ .*

*Proof.* For  $G_{\text{Ko}}$  this is shown in [7, Lemma 3.1]. Before going on, note the following fact, which can easily be checked with MAGMA:

(6.3) *The groups  $PG_{\text{Mu}}$  and  $PG_{\text{BH}}$  are not isomorphic to subgroups of  $G_{\text{Ko}}$ .*

Consider now the group  $G_{\text{Mu}}$ , then  $PG_{\text{Mu}}/M_{20} = \langle \iota \rangle$  and  $\iota$  acts non-symplectically, hence  $X$  is one of the three surfaces of Theorem 2.7 and  $PG_{\text{Mu}}$  leaves invariant the polarization, hence it is realized by linear transformations. We only need to show that  $X_{\text{Ko}}$  and  $X_{\text{BH}}$  do not admit an automorphism group isomorphic to  $PG_{\text{Mu}}$ . Assume it is the case, then  $PG_{\text{Mu}}$  and  $G_{\text{Ko}}$  leaves invariant the polarization of degree  $\langle 40 \rangle$  on  $X_{\text{Ko}}$ , hence by [5, Proposition 5.3.3] the group that they generate together is finite. By the maximality of  $G_{\text{Ko}}$  this means that  $PG_{\text{Mu}}$  is contained in  $G_{\text{Ko}}$  but by (6.3) the group  $G_{\text{Ko}}$  does not contain such a subgroup. With a similar argument if  $PG_{\text{Mu}}$  acts on  $X_{\text{BH}}$  then we conclude that  $PG_{\text{Mu}} \cong PG_{\text{BH}}$  and this is not the case by (5.2). The same argument holds for  $PG_{\text{BH}}$ .  $\square$

**Theorem 6.4.** *Let  $G$  be a maximal finite group with a faithful and non-symplectic action on a K3 surface  $X$  and assume that  $M_{20} \subset G$ . Then  $G$  is isomorphic to  $G_{\text{Ko}}$ ,  $PG_{\text{Mu}}$  or  $PG_{\text{BH}}$ .*

*Proof.* Since  $G$  acts non-symplectically then  $G/M_{20}$  is non-trivial and by [7] it has order at most four. If  $|G/M_{20}| = 4$  then  $G \cong G_{\text{Ko}}$  by [7]. Observe that the group  $G/M_{20}$  acts faithfully on  $\mathbf{L}_{20}$  since it contains  $\mathbf{T}_X$ . By Remark 2.4, the group of isometries of  $\mathbf{L}_{20}$  has order  $2^4$  so it is not possible to have  $|G/M_{20}| = 3$ . We are left with the case  $|G/M_{20}| = 2$ . By Theorem 2.7 the K3 surface  $X$  is isomorphic to  $X_{\text{Ko}}$ ,  $X_{\text{Mu}}$  or  $X_{\text{BH}}$ . By the same argument as in Proposition 6.2 and the maximality of  $G$ , then  $G$  is isomorphic to  $PG_{\text{Mu}}$  or  $PG_{\text{BH}}$ .  $\square$

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