On testing the norm of a random vector in Gaussian noise by tests with uniformly best constant conditional power (UBCCP) on spheres

Dominique Pastor, Laurent Dejean

Collection des rapports de recherche
de Télécom Bretagne

RR-2009002-SC

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Dominique Pastor (Télécom Bretagne, Lab-STICC)
Laurent Déjean (Télécom Bretagne, Lab-STICC)

TELECOM
Bretagne

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Sur les tests à puissance conditionnelle constante sur des sphères et uniformément plus puissants conditionnellement à ces mêmes sphères pour tester la norme d'un vecteur aléatoire dans un bruit Gaussien

Dominique Pastor and Laurent Déjean

Institut TELECOM; TELECOM Bretagne;
UMR CNRS 3192 Lab-STICC
Technopôle Brest-Iroise, CS 83818
29238 Brest Cedex 3, FRANCE

Université Européenne de Bretagne

Research report - Institut TELECOM; TELECOM Bretagne *February 24, 2009*

Abstract

Given $\varrho \in [0, \infty)$, let ϑ be the set of the d-dimensional real random vectors Θ defined on a given probability space $(\Omega, \mathscr{B}, \mathrm{P})$ and such that $\mathrm{P}(\mathscr{H}_0) \in (0, 1)$ with $\mathscr{H}_0 = \big| \|\Theta\| \leqslant \varrho \big|$. Similar to Wald's tests with uniformly best constant power (UBCP) on a family of surfaces, tests with level $\gamma \in (0, 1)$ and uniformly best constant conditional power (UBCCP) on the spheres centred at the origin with radius $\rho \in (\varrho, \infty)$ are defined and established to test $\mathscr{H}_0$ when we observe $Y = \Theta + X$ where $X \sim \mathscr{N}(0, \mathbf{I}_d)$ is independent with Θ. These results extend (A. Wald, 1943, Section 6, Definition III and Proposition III, p. 450).

Keywords

Binary hypothesis testing; statistical decision; uniform best constant power.

Résumé

Soit $\varrho \in [0, \infty)$, considérons l'ensemble ϑ des vecteurs aléatoires réels Θ de dimension d et définis sur un espace probabilisé $(\Omega, \mathscr{B}, \mathrm{P})$ et tels que $\mathrm{P}(\mathscr{H}_0) \in (0, 1)$ où $\mathscr{H}_0 = \big| \|\Theta\| \leqslant \varrho \big|$. On observe $Y = \Theta + X$ où $X \sim \mathscr{N}(0, \mathbf{I}_d)$ est indépendant de Θ. De manière analogue à Wald qui définit les tests uniformément plus puissants sur une famille de surfaces, nous définissons et établissons l'existence de tests de niveau $\gamma \in (0, 1)$, qui ont une puissance constante sur des sphères et qui sont uniformément plus puissants, conditionnellement à ces mêmes sphères, pour tester $\mathscr{H}_0$. Ces résultats étendent (A. Wald, 1943, Section 6, Definition III et Proposition III, p. 450).

Mots-clés

Test binaire d'hypothèses; décision statistique; tests uniformément plus puissants sur des sphères.

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1. Problem statement

The notations introduced in this section are kept with always the same meaning throughout. All the random vectors and variables encountered below are assumed to be defined on the same probability space $(\Omega, \mathscr{B}, \mathrm{P})$. Let ρ be some given non-negative real number and ϑ stand for the set of those d-dimensional real random vectors Θ such that $\mathrm{P}(\mathscr{H}_0) \in (0,1)$ with $\mathscr{H}_0 = \left[\|\Theta\| \leqslant \rho\right]$. Given $\Theta \in \vartheta$, we address the testing of the composite null hypothesis $\mathscr{H}_0$ on the basis of the observation $\Theta + X$ where X is a d-dimensional centred Gaussian real random vector with known definite positive covariance matrix. We assume that Θ and X

are independent. With no loss of generality, we can assume that this covariance matrix is the $d \times d$ identity matrix $\mathbf{I}_d$ and, as usual, we write $X \sim \mathcal{N}(0, \mathbf{I}_d)$. The alternative hypothesis is the measurable subset $\mathscr{H}_\mathrm{A} = \left| \, \|\Theta\| > \varrho \, \right|$ of Ω. We also say that we test the norm of Θ, without mentioning $\mathscr{H}_0$. Below, tests of the null hypothesis $\mathscr{H}_0$ are non-randomized and measurable maps of $\mathbb{R}^d$ into $\{\mathscr{H}_0, \mathscr{H}_\mathrm{A}\}$. The performance of a given test $\mathscr{T}$ for testing $\mathscr{H}_0$ will be evaluated via the following quantities, whose definitions derive from those given in (E.L. Lehmann and J.P. Romano, 2005). First, the size of $\mathscr{T}$ for testing $\mathscr{H}_0$ is defined as the conditional

$$\alpha(\mathscr{T}) \stackrel{\Delta}{=} \sup_{\Theta \in \vartheta} \mathrm{P}\left[\, \mathscr{T}(\Theta + X) = \mathscr{H}_\mathrm{A} \, \middle| \, \mathscr{H}_0 \, \right]. \tag{1}$$

Test $\mathscr{T}$ is then said to have level $\gamma \in [\,0\,,\,1\,]$ if $\alpha(\mathscr{T}) \leqslant \gamma$. Second, the power of $\mathscr{T}$ is defined as the conditional

$$\beta_\Theta(\mathscr{T}) \stackrel{\Delta}{=} \mathrm{P}\left[\, \mathscr{T}(\Theta + X) = \mathscr{H}_\mathrm{A} \, \middle| \, \mathscr{H}_\mathrm{A} \, \right]. \tag{2}$$

Test $\mathscr{T}$ will be said to be more powerful that test $\mathscr{T}'$ for testing $\mathscr{H}_0$ if $\beta_\Theta(\mathscr{T}) \geqslant \beta_\Theta(\mathscr{T}')$. For testing $\mathscr{H}_0$, there is no uniformly most powerful test (UMP) test at level γ, that is, no test with level γ and power larger than or equal to that of any other test with same level. Another criterion for optimality is proposed in the next section to state our main result.

2. Tests with uniformly best constant power (UBCP) on a family of surfaces

These tests are introduced by (A. Wald, 1943, Definition III, Section 6, p. 450) for testing the value of a deterministic but unknown distribution parameter θ when there is no UMP test. They are alternative to tests, such as likelihood ratio tests, whose power is optimal when θ belongs to a certain subset of the parameter space but that might be very inefficient over the complementary of this

subset (M. Fouladirad and I. Nikiforov, 2005, Sec. 3.1, p. 1160). We recall Wald's definition before introducing, in section 3, its adaptation to the problem of testing the norm of Θ.

Let $\{\Upsilon_\rho : \rho \in \mathfrak{I}\}$ be a family of surfaces in a given parameter space ϖ where $\mathfrak{I}$ is some index set. For instance, Υ_ρ may be defined by means of an analytic function (A. Wald, 1943, Section 5, p. 445) and $\mathfrak{I} \subset \mathbb{R}$. Let Y stand for a d-dimensional real random vector whose distribution belongs to some class $\mathscr{P} = \{\mathrm{P}_\theta : \theta \in \varpi\}$. For testing the simple hypothesis $\mathsf{H}_0 : \theta = \theta_0$ against the composite alternative hypothesis $\mathsf{H}_\mathrm{A} : \theta \neq \theta_0$ where $\theta_0 \in \varpi$, the tests considered in this section are, as in section 1, non-randomized and measurable maps of $\mathbb{R}^d$ into $\{\mathsf{H}_0, \mathsf{H}_\mathrm{A}\}$. The power function $\mathrm{P}_\theta\left[\mathscr{T}(Y) = \mathsf{H}_A\right]$ of such a test $\mathscr{T}$ is defined for any $\theta \in \varpi$ as the probability $\mathrm{P}\left[\mathscr{T}(Y) = \mathsf{H}_A\right]$ when the distribution of Y is P_θ.

Definition 1. **[Wald's UBCP criterion]** *For testing* $\mathsf{H}_0 : \theta = \theta_0$ *against* $\mathsf{H}_\mathrm{A} : \theta \neq \theta_0$ *where* $\theta_0 \in \varpi$*, a test* $\mathscr{T}$ *is said to be UBCP on* $\{\Upsilon_\rho : \rho \in \mathfrak{I}\}$ *if it satisfies the following two conditions:*

(a) *Test* $\mathscr{T}$ *has constant power function on every* Υ_ρ*,* $\rho \in \mathfrak{I}$*, in that, given any* $\rho \in \mathfrak{I}$*,* $\mathrm{P}_{\theta_1}\left[\mathscr{T}(Y) = \mathsf{H}_A\right] = \mathrm{P}_{\theta_2}\left[\mathscr{T}(Y) = \mathsf{H}_A\right]$ *for any* $\theta_1, \theta_2 \in \Upsilon_\rho \subset \varpi$*.*

(b) *For any* $\theta \in \varpi$*,* $\mathrm{P}_\theta\left[\mathscr{T}(Y) = \mathsf{H}_A\right] \geqslant \mathrm{P}_\theta\left[\mathscr{T}'(Y) = \mathsf{H}_A\right]$ *for any test* $\mathscr{T}'$ *whose power is constant on every given* Υ_ρ *and such that* $\mathrm{P}_{\theta_0}\left[\mathscr{T}(Y) = \mathsf{H}_A\right] = \mathrm{P}_{\theta_0}\left[\mathscr{T}'(Y) = \mathsf{H}_A\right]$*.*

3. Tests with uniformly best constant conditional power (UBCCP) on spheres

For testing $\mathscr{H}_0 = \left[\|\Theta\| \leqslant \varrho\right]$ against $\mathscr{H}_\mathrm{A} = \left[\|\Theta\| > \varrho\right]$, we transpose Wald's definition as follows. Since we test the norm of Θ and the distribution of X is spherically invariant, we naturally focus on tests that are UBCP, in a certain extended sense to be given, on the family of spheres $\{\rho \mathrm{S}^{d-1} : \rho \in (\varrho, \infty)\}$ where

$\rho \mathrm{S}^{d-1}$ stands for the sphere, in $\mathbb{R}^d$, centred at the origin with radius ρ. For a given Θ and a given test $\mathscr{T} : \mathbb{R}^d \to \{\mathscr{H}_0, \mathscr{H}_\mathrm{A}\}$, the analogue of the power function intervening in definition 1 is the conditional $\mathrm{P}\big[\, \mathscr{T}(\Theta + X) = \mathscr{H}_\mathrm{A} \,\big|\, \|\Theta\| = \rho \,\big]$ defined for any $\rho \in [0, \infty)$ and henceforth called the *conditional power* of $\mathscr{T}$ on $\rho \mathrm{S}^{d-1}$. Test $\mathscr{T}$ is then said to be *conditionally more powerful* than $\mathscr{T}'$ on $\rho \mathrm{S}^{d-1}$ such that $\rho \in (\varrho, \infty)$ if the conditional power $\mathrm{P}\big[\, \mathscr{T}(\Theta + X) = \mathscr{H}_\mathrm{A} \,\big|\, \|\Theta\| = \rho \,\big]$ of $\mathscr{T}$ on $\rho \mathrm{S}^{d-1}$ is larger than or equal to the conditional power $\mathrm{P}\big[\, \mathscr{T}'(\Theta + X) = \mathscr{H}_\mathrm{A} \,\big|\, \|\Theta\| = \rho \,\big]$ of $\mathscr{T}'$ on this sphere. Since

$$\beta_\Theta(\mathscr{T}) = \frac{1}{\mathrm{P}(\mathscr{H}_\mathrm{A})} \int_{(\varrho, \infty)} \mathrm{P}\big[\, \mathscr{T}(\Theta + X) = \mathscr{H}_\mathrm{A} \,\big|\, \|\Theta\| = \rho \,\big] \left(\mathrm{P}\|\Theta\|^{-1}\right)(\mathrm{d}\rho) \tag{3}$$

where $\mathrm{P}\|\Theta\|^{-1}$ stands for the probability distribution of $\|\Theta\|$, a sufficient condition for $\mathscr{T}$ to be more powerful than $\mathscr{T}'$ is that $\mathscr{T}$ be conditionally more powerful than $\mathscr{T}'$ on every $\rho \mathrm{S}^{d-1}$ such that $\rho \in (\varrho, \infty)$. This remark implies that, given some level γ, no test can be conditionally more powerful on every sphere with radius in (ϱ, ∞) than *any* other test with same level: if such a test $\mathscr{T}$ existed, Eq. (3) implies that this test would be UMP at level γ for testing $\mathscr{H}_0$, a contradiction. However, by saying that test $\mathscr{T}$ has *constant conditional power* on $\rho \mathrm{S}^{d-1}$ if $\mathrm{P}\big[\, \mathscr{T}(\Theta + X) = \mathscr{H}_\mathrm{A} \,\big|\, \|\Theta\| = \rho \,\big]$ does not depend on the probability distribution $\mathrm{P}\Theta^{-1}$ of Θ, we transpose Wald's definition and define tests with uniformly best constant conditional power (UBCCP) on $\big\{\rho \mathrm{S}^{d-1} : \rho \in (\varrho, \infty)\big\}$ for testing $\mathscr{H}_0$. Theorem 1 establishes the existence of such tests.

Definition 2. *For testing $\mathscr{H}_0 = \big[\, \|\Theta\| \leqslant \varrho \,\big]$ on the basis of the observation $\Theta + X$ where $\Theta \in \vartheta$, a test $\mathscr{T}^*$ with level $\gamma \in [0, 1]$ is said to be UBCCP on the family of spheres $\big\{\rho \mathrm{S}^{d-1} : \rho \in (\varrho, \infty)\big\}$ if:*

(i) *$\mathscr{T}^*$ has constant conditional power on each of these spheres;*

(ii) *$\mathscr{T}^*$ is conditionally more powerful on each of these spheres than any other test with level γ and constant conditional power on this sphere.*

Lemma 1. *If test $\mathscr{T}^*$ is UBCCP on the family of spheres $\{\rho \mathrm{S}^{d-1} : \rho \in (\varrho, \infty)\}$ for testing $\mathscr{H}_0 = \big[\, \|\Theta\| \leqslant \varrho \,\big]$ on the basis of the observation $\Theta + X$ where $\Theta \in \vartheta$, then $\mathscr{T}^*$ is UMP amongst the tests with same level as $\mathscr{T}^*$ and constant conditional power on each sphere $\rho \mathrm{S}^{d-1}, \rho \in (\varrho, \infty)$.*

PROOF. A straightforward consequence of Eq. (3) and definition 2. □

Remark 1. *For $\Theta = \varepsilon\theta$ where $\theta \in \rho \mathrm{S}^{d-1}$, $\rho \in (0, \infty)$ and ε is Bernoulli, valued in $\{0, 1\}$ and independent with X, $\mathrm{P}\big[\mathscr{T}(\Theta + X) = \mathscr{H}_\mathrm{A} \,\big|\, \|\Theta\| = \rho\big] = \mathrm{P}\big[\mathscr{T}(\theta + X) = \mathscr{H}_\mathrm{A}\big]$. By assuming $\mathrm{P}\big[\,\varepsilon = 1\,\big] \in (0, 1)$, it follows from lemma 1 that definition 2 embraces Wald's definition of UBCP tests on $\{\rho \mathrm{S}^{d-1} : \rho \in (0, \infty)\}$ for testing the mean of $Y \sim \mathscr{N}(\theta, \boldsymbol{I}_d)$.*

4. Theoretical result

For every pair (ρ, λ) of non-negative real numbers, set

$$\mathscr{R}(\rho\,,\,\lambda) \stackrel{\Delta}{=} \frac{e^{-\rho^2/2}}{2^{d/2-1}\Gamma(d/2)} \int_0^\lambda e^{-t^2/2} t^{d-1} {}_0\mathrm{F}_1\left(d/2\,;\, \rho^2 t^2/4\right) \mathrm{d}t \tag{4}$$

where ${}_0\mathrm{F}_1$ is the generalized hypergeometric function (N.N. Lebedev, 1965, p. 275) and Γ the usual gamma function. Given $\rho \in [0, \infty)$, $\mathscr{R}(\rho, \cdot)$ is the distribution function of the square root of any random variable that follows the non-central χ^2 distribution with d degrees of liberty and non-central parameter ρ^2. For any θ in $\mathbb{R}^d$,

$$\mathrm{P}\big[\, \|\theta + X\| \leqslant \lambda \,\big] = \mathscr{R}(\|\theta\|\,,\,\lambda). \tag{5}$$

Given $\lambda \in [0, \infty)$, the *thresholding test* $\mathscr{T}_\lambda$ with threshold height λ is defined for every $y \in \mathbb{R}^d$ by

$$\mathscr{T}_\lambda(y) \stackrel{\Delta}{=} \begin{cases} \mathscr{H}_0 & \text{if} \quad \|y\| \leqslant \lambda, \\ \mathscr{H}_\mathrm{A} & \text{if} \quad \|y\| > \lambda. \end{cases} \tag{6}$$

Theorem 1. *Let $\gamma \in (0, 1)$. For testing $\mathscr{H}_0 = \left[\, \|\Theta\| \leqslant \varrho \, \right]$ given $\Theta + X$ with $\Theta \in \vartheta$ and $X \sim \mathscr{N}(0, \boldsymbol{I}_d)$, the thresholding test $\mathscr{T}_{\lambda^*}$, where λ^* is the unique non-negative real value such that $\mathscr{R}(\varrho\,, \lambda^*) = 1 - \gamma$, is such that*

(i) *$\mathscr{T}_{\lambda^*}$ has size γ and is UBCCP on $\left\{ \rho \mathrm{S}^{d-1} : \rho \in (\varrho\,, \infty) \right\}$; its conditional power is $1 - \mathscr{R}(\rho\,, \lambda^*)$ on any sphere $\rho \mathrm{S}^{d-1}$ with $\rho \in (\varrho\,, \infty)$ and*

$$\beta_\Theta(\mathscr{T}_{\lambda^*}) = \frac{1}{\mathrm{P}(\mathscr{H}_\mathrm{A})} \int_{(\varrho, \infty)} \left(1 - \mathscr{R}(\rho\,, \lambda^*)\right) \left(\mathrm{P}\|\Theta\|^{-1}\right)(\mathrm{d}\rho) \tag{7}$$

(ii) *$\mathscr{T}_{\lambda^*}$ is UMP amongst the tests with level γ and constant conditional power on each sphere $\rho \mathrm{S}^{d-1}$ with $\rho \in (\varrho\,, \infty)$.*

(iii) *$\mathscr{T}_{\lambda^*}$ is unbiased: $\beta_\Theta(\mathscr{T}_{\lambda^*}) \geqslant \gamma$.*

According to remark 1, (A. Wald, 1943, Proposition III, p. 450) derives from theorem 1 above for testing the mean of $Y \sim \mathscr{N}(\theta, \mathbf{I}_d)$.

5. Proof of theorem 1

5.1. Preliminary results

Lemma 2. *For every $\lambda \in [0, \infty)$ and $\left(\mathrm{P}\|\Theta\|^{-1}\right)$ - almost every $\rho \in [0, \infty)$*

$$\mathrm{P}\left[\, \|\Theta + X\| \leqslant \lambda \, \middle| \, \|\Theta\| = \rho \, \right] = \mathscr{R}(\rho\,, \lambda). \tag{8}$$

PROOF. Let B be any Borel set of $\mathbb{R}$ and λ be any element of $[0, \infty)$. From the independence of Θ and X and according to Eq. (5), we have

$$\begin{aligned} \mathrm{P}\left[\, \|\Theta + X\| \leqslant \lambda\,, \, \|\Theta\| \in B \, \middle| \, \Theta = \theta \, \right] &= \mathrm{I}_B(\|\theta\|) \, \mathrm{P}\left[\, \|\theta + X\| \leqslant \lambda \, \right] \\ &= \mathrm{I}_B(\|\theta\|) \, \mathscr{R}(\|\theta\|\,, \lambda) \end{aligned} \tag{9}$$

where, given a set K, I_K is the indicator function of K: $\mathrm{I}_K(x) = 1$ if $x \in K$ and $\mathrm{I}_K(x) = 0$, otherwise. By the standard change-of-variable formula (P. Billingsley, 1995, Theorem 16.13), we now have

$$\int \mathrm{I}_B(\|\theta\|)\,\mathscr{R}(\|\theta\|\,,\lambda)\left(\mathrm{P}\Theta^{-1}\right)(\mathrm{d}\theta) = \int_B \mathscr{R}(\rho\,,\lambda)\left(\mathrm{P}\|\Theta\|^{-1}\right)(\mathrm{d}\rho). \tag{10}$$

Since $\mathrm{P}\left[\,\|\Theta+X\| \leqslant \lambda\,,\,\|\Theta\| \in B\,\right] = \int \mathrm{P}\left[\,\|\Theta+X\| \leqslant \lambda\,,\,\|\Theta\| \in B\,\middle|\,\Theta=\theta\,\right]\left(\mathrm{P}\Theta^{-1}\right)(\mathrm{d}\theta)$, it follows from the second equality in Eq. (9) and Eq. (10) that $\mathrm{P}\left[\,\|\Theta+X\| \leqslant \lambda\,,\,\|\Theta\| \in B\,\right] = \int_B \mathscr{R}(\rho\,,\lambda)\left(\mathrm{P}\|\Theta\|^{-1}\right)(\mathrm{d}\rho)$. The result follows since $\mathrm{P}\left[\,\|\Theta+X\| \leqslant \lambda\,,\,\|\Theta\| \in B\,\right] = \int \mathrm{P}\left[\,\|\Theta+X\| \leqslant \lambda\,,\,\|\Theta\| \in B\,\middle|\,\|\Theta\|=\rho\,\right]\left(\mathrm{P}\|\Theta\|^{-1}\right)(\mathrm{d}\rho)$. □

Lemma 3. *Given any $\lambda \in (0,\infty)$, $\mathscr{R}(\cdot\,,\lambda)$ is decreasing.*

PROOF. This improvement of (Pastor et al, 2002, Lemma IV.2) is proved similarly by refining some arguments. Let ρ and ρ' be two real numbers such that $0 \leqslant \rho < \rho' < \infty$. Let θ and θ' be two colinear vectors of $\mathbb{R}^d$ such that $\|\theta\| = \rho$ and $\|\theta'\| = \rho'$. According to Eq. (5), $\mathscr{R}(\rho\,,\lambda) = \int_{B(\theta,\lambda)} f(x)\,\mathrm{d}x$ and $\mathscr{R}(\rho'\,,\lambda) = \int_{B(\theta',\lambda)} f(x)\,\mathrm{d}x$ where f is the probability density function of X and $B(\theta,\lambda)$ (resp. $B(\theta',\lambda)$) is the closed ball, in $\mathbb{R}^d$, centred at θ (resp. θ') with radius λ. We have $\mathscr{R}(\rho\,,\lambda) - \mathscr{R}(\rho'\,,\lambda) = \int_{B(\theta,\lambda)\setminus B(\theta',\lambda)} \left(f(x) - f(\theta+\theta'-x)\right)\mathrm{d}x$. Let $(e_1, e_2, \ldots, e_d)$ be an orthonormal basis of $\mathbb{R}^d$ such that $\theta = \rho e_1$ and $\theta' = \rho' e_1$. We have $\|\theta+\theta'-x\|^2 - \|x\|^2 = (\rho+\rho')(\rho+\rho'-2x_1)$ for any $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$. If $x \in B(\theta,\lambda)\setminus B(\theta',\lambda)$, then $\|x-\theta'\| > \|x-\theta\|$, which implies that $(\rho'-\rho)(\rho+\rho'-2x_1) > 0$ and, thus, that $\rho+\rho'-2x_1 > 0$ since $\rho' > \rho$. Therefore, $\|\theta+\theta'-x\| > \|x\|$. Since f decreases with the norm of its argument, it follows that $f(x) - f(\theta+\theta'-x) > 0$ so that $\mathscr{R}(\rho\,,\lambda) > \mathscr{R}(\rho'\,,\lambda)$ and the proof is complete. □

Lemma 4. *Given $\gamma \in (0,1]$ and $\rho \in [0,\infty)$, there exists a unique solution $\tau_\gamma(\rho)$ for λ in the equation $1-\mathscr{R}(\rho\,,\lambda) = \gamma$; τ_γ is increasing and continuous everywhere.*

PROOF. **[Existence and unicity]** : Given $\rho \in [0,\infty)$, it follows from Eq. (4) that $\mathscr{R}(\rho\,,\cdot)$ is increasing and continuous and, thus, a one-to-one mapping of $[0,\infty)$ into $[0,1)$. Thence, the existence and the unicity of $\tau_\gamma(\rho)$ for $\gamma \in (0,1]$.

[Increasingness of τ_γ] : Let ρ and ρ' be two non-negative real number such that $\rho < \rho'$. According to lemma 3, $\mathscr{R}(\rho'\,,\tau_\gamma(\rho)) < \mathscr{R}(\rho\,,\tau_\gamma(\rho))$. The right hand side (rhs) in this inequality equals $1-\gamma$ and, thus, $\mathscr{R}(\rho'\,,\tau_\gamma(\rho'))$. The result then follows from the increasingness of $\mathscr{R}(\rho'\,,\cdot)$.

[Continuity of τ_γ] : Given $\rho_0 \in [0,\infty)$, the increasingness of τ_γ implies the existence of a limit $\tau_\gamma(\rho_0^-) \in [0,\infty)$ when ρ tends to ρ_0 from below and the existence of a limit $\tau_\gamma(\rho_0^+) \in [0,\infty)$ when ρ tends to ρ_0 from above. Since $\mathscr{R}$ is continuous in the plane and $\mathscr{R}(\rho\,,\tau_\gamma(\rho)) = \gamma$ for every $\rho \in [0,\infty)$, $\mathscr{R}(\rho_0\,,\tau_\gamma(\rho_0^-)) = \mathscr{R}(\rho_0\,,\tau_\gamma(\rho_0^+)) = \gamma$. Since $\mathscr{R}(\rho_0\,,\cdot)$ is one-to-one, $\tau_\gamma(\rho_0^-) = \tau_\gamma(\rho_0^+) = \tau_\gamma(\rho_0)$ and τ_γ is continuous. □

Lemma 5. $\mathscr{R}(\varrho\,,\lambda) \leqslant \mathrm{P}\left[\, \|\Theta + X\| \leqslant \lambda \,\middle|\, \|\Theta\| \leqslant \varrho \,\right] \leqslant \mathscr{R}(0\,,\lambda)$

PROOF. By Bayes's rule and since $\mathrm{P}\left[\, \|\Theta\| \leqslant \varrho \,\right] \neq 0$, we have $\mathrm{P}\left[\, \|\Theta+X\| \leqslant \lambda \,\middle|\, \|\Theta\| \leqslant \varrho \,\right] = \mathrm{P}\left[\, \|\Theta+X\| \leqslant \lambda\,,\, \|\Theta\| \leqslant \varrho \,\right] / \mathrm{P}\left[\, \|\Theta\| \leqslant \varrho \,\right]$. From the definition of a conditional and lemma 2, $\mathrm{P}\left[\, \|\Theta + X\| \leqslant \lambda\,,\, \|\Theta\| \leqslant \varrho \,\right] = \int_{[0,\varrho]} \mathscr{R}(\rho\,,\lambda)\left(\mathrm{P}\|\Theta\|^{-1}\right)(\mathrm{d}\rho)$. The result then follows from lemma 3. □

Lemma 6. *For every real number $\nu \geqslant 1/2$ and every pair of non-negative real numbers ρ_0 and ρ_1 such that $0 \leqslant \rho_0 < \rho_1$, the map defined for every $x \in [0,\infty)$ by ${}_0\mathrm{F}_1\left(\nu\,;\rho_1^2 x^2/4\right)/{}_0\mathrm{F}_1\left(\nu\,;\rho_0^2 x^2/4\right)$ is increasing.*

PROOF. Set $f(x) = {}_0\mathrm{F}_1\left(\nu\,;\rho_1^2 x^2/4\right)/{}_0\mathrm{F}_1\left(\nu\,;\rho_0^2 x^2/4\right)$ for any $x \in [0,\infty)$. We have $f(0) = 1$ and since ${}_0\mathrm{F}_1(\nu\,;\cdot)$ is increasing, we have $f(x) \geqslant 1$ for any $x \geqslant 0$. For $x \in (0,\infty)$, the derivative of ${}_0\mathrm{F}_1(\nu\,;x)$ with respect to x follows from (N.N. Lebedev, 1965, Sec. 9.14, p.275) and some routine algebra shows that the sign of

$f'(x)$ is that of $q(x) = \frac{\rho_1^2}{\rho_0^2} \frac{{}_0\mathrm{F}_1(\nu+1;\rho_1^2 x^2/4)}{{}_0\mathrm{F}_1(\nu;\rho_1^2 x^2/4)} - \frac{{}_0\mathrm{F}_1(\nu+1;\rho_0^2 x^2/4)}{{}_0\mathrm{F}_1(\nu;\rho_0^2 x 2/4)}$. Put $g(t) = I_\nu(t)/I_{\nu-1}(t)$, $t \in [0,\infty)$, where I_ν is the modified Bessel function (M. Abramowitz and I. Stegun, 1972, Sec. 9.6, p. 374). According to (M. Abramowitz and I. Stegun, 1972, p. 377, Eq. 9.6.47), we have $g(t) = \frac{t}{2\nu} \frac{{}_0\mathrm{F}_1(\nu+1;t^2/4)}{{}_0\mathrm{F}_1(\nu;t^2/4)}$, $t \in [0,\infty)$. Therefore, $q(x) = \frac{2\nu}{\rho_0^2 x}\big(\rho_1 g(\rho_1 x) - \rho_0 g(\rho_0 x)\big)$ whose sign is that of $\rho_1 g(\rho_1 x) - \rho_0 g(\rho_0 x)$. It follows from (Pastor et al, 2002, Lemma B.1, Appendix B, p. 237), g is increasing. Since $\rho_0 < \rho_1$, it follows that $\rho_0 g(\rho_0 x) < \rho_1 g(\rho_0 x) < \rho_1 g(\rho_1 x)$ and the proof is complete. □

Corollary 1. *The family of the non-central χ^2 distributions with d degrees of liberty has monotone likelihood ratio with its non-central parameter.*

Although lemma 6 and its corollary are probably standard, we did not find precise references. So, we have provided a proof of lemma 6.

Proposition 1. *Let $\gamma \in (0,1]$. With the notations of theorem 1, assume that $\Theta = (1-\varepsilon)\rho_0\Theta_0 + \varepsilon\rho_1\Theta_1$ where $\rho_0 < \rho_1$, Θ_0 and Θ_1 are independent and uniformly distributed on S^{d-1}, ε is valued in $\{0,1\}$ and independent of Θ_0 and Θ_1. There exists a UMP test $\mathscr{T}_{\mathrm{UMP}}$ with size γ for testing $h_0 = [\varepsilon = 0]$ against $h_1 = [\varepsilon = 1]$. This test is given for every $y \in \mathbb{R}^d$ by*

$$\mathscr{T}_{\mathrm{UMP}}(y) = \begin{cases} h_0 & \text{if} \quad \|y\| \leqslant \tau_\gamma(\rho_0), \\ h_1 & \text{if} \quad \|y\| > \tau_\gamma(\rho_0) \end{cases} \tag{11}$$

where $\tau_\gamma(\rho_0)$ is the unique solution for λ in the equation $1 - \mathscr{R}(\rho_0, \lambda) = \gamma$ (see lemma 4). The power of this UMP test is $1 - \mathscr{R}(\rho_1, \tau_\gamma(\rho_0))$.

PROOF. Under each hypothesis h_0 and h_1, the probability density function of the observation $\Theta + X$ is given by (Pastor et al, 2002, Proposition V.1, Eq. (18), p.

232). The likelihood ratio for testing h_0 against h_1 is thus

$$\Lambda(y) = e^{-(\rho_1^2-\rho_0^2)/2}\frac{{}_0\mathrm{F}_1\left(d/2\,;\,\rho_1^2\|y\|^2/4\right)}{{}_0\mathrm{F}_1\left(d/2\,;\,\rho_0^2\|y\|^2/4\right)}, y \in \mathbb{R}^d.$$

According to the Neyman-Pearson lemma (E.L. Lehmann and J.P. Romano, 2005, Theorem 3.2.1, Sec. 3.2, p. 60), there exists a UMP test $\mathscr{T}_{\mathrm{UMP}}$ with size γ for testing h_0 against h_1. This test compares Λ to a threshold whose value guarantees that the false alarm probability of $\mathscr{T}_{\mathrm{UMP}}$ is γ. According to lemma 6, $\Lambda(y)$ increases with $\|y\|$. Therefore, given $y \in \mathbb{R}^d$, $\mathscr{T}_{\mathrm{UMP}}(y) = h_0$ if $\|y\| \leqslant \zeta$ and $\mathscr{T}_{\mathrm{UMP}}(y) = h_1$ if $\|y\| > \zeta$ where ζ is determined so as $\mathscr{T}_{\mathrm{UMP}}$ has size γ. From Eq. (4) and (Pastor et al, 2002, Proposition V.1, Eq. (17), p. 232), the size of $\mathscr{T}_{\mathrm{UMP}}$ is $\mathrm{P}\left[\,\|\rho_0\Theta_0 + X\| > \zeta\,\right] = 1 - \mathscr{R}(\rho_0\,,\zeta)$. Thus, according to lemma 4, $\zeta = \tau_\gamma(\rho_0)$. The power $\mathrm{P}\left[\,\|\rho_1\Theta_1 + X\| > \tau_\gamma(\rho_0)\,\right]$ of test $\mathscr{T}_{\mathrm{UMP}}$ now follows from Eq. (4) and (Pastor et al, 2002, Proposition V.1, Eq. (17), p. 232) again. □

5.2. Proof of statement **(i)**

According to lemma 4, $\lambda^* = \tau_\gamma(\varrho)$ is the unique non-negative real number such that $1 - \mathscr{R}(\varrho\,,\lambda^*) = \gamma$. We now prove that $\mathscr{T}_{\lambda^*}$ has size γ. Then, we prove that $\mathscr{T}_{\lambda^*}$ is UBCCP on the appropriate family of spheres.

[Size] : According to lemma 5, $\mathrm{P}\left[\,\|\Theta + X\| > \lambda^* \,\middle|\, \|\Theta\| \leqslant \varrho\,\right] \leqslant 1 - \mathscr{R}(\varrho\,,\lambda^*)$ for any Θ. By definition of λ^*, it follows from Eq. (1) that $\alpha(\mathscr{T}_{\lambda^*}) \leqslant \gamma$. Now, if $\Theta = \theta$ (a-s) with $\theta \in \varrho\mathrm{S}^{d-1}$, Eqs. (1) and (5) imply that $\alpha(\mathscr{T}_{\lambda^*}) \geqslant \mathrm{P}\left[\,\|\Theta + X\| > \lambda^*\,\right] = 1 - \mathscr{R}(\varrho\,,\lambda^*) = \gamma$. Therefore, $\alpha(\mathscr{T}_{\lambda^*}) = \gamma$.

[UBCCP on spheres] : For any $\rho \in (\varrho\,,\infty)$, it follows from Eq. (8) that

$$\mathrm{P}\left[\,\mathscr{T}_{\lambda^*}(\Theta + X) = \mathscr{H}_{\mathrm{A}} \,\middle|\, \|\Theta\| = \rho\,\right] = 1 - \mathscr{R}(\rho\,,\lambda^*), \tag{12}$$

which proves that $\mathscr{T}_{\lambda^*}$ has constant conditional power on $\rho\mathrm{S}^{d-1}$. Eq. (7) is then a direct consequence of the equality above and Eq. (3). To prove that $\mathscr{T}_{\lambda^*}$ is

UBCCP, assume that $\mathscr{T}$ is a test with level γ and constant conditional power on a given sphere $\rho_1 \mathrm{S}^{d-1}$ where $\rho_1 \in (\varrho, \infty)$. The value of $\mathrm{P}\big[\,\mathscr{T}(\Theta+X) = \mathscr{H}_\mathrm{A} \,\big|\, \|\Theta\| = \rho_1\,\big]$ does not depend on Θ. Therefore, we calculate it by considering the particular case $\Theta = (1-\varepsilon)\rho_0\Theta_0 + \varepsilon\rho_1\Theta_1$ where $0 \leqslant \rho_0 \leqslant \varrho$, Θ_0 and Θ_1 are uniformly distributed on S^{d-1}, ε is a random variable valued in $\{0,1\}$ independent with Θ_0 and Θ_1. We have

$$\mathrm{P}\big[\,\mathscr{T}(\Theta+X) = \mathscr{H}_\mathrm{A} \,\big|\, \|\Theta\| = \rho_1\,\big] = \mathrm{P}\big[\,\mathscr{T}(\rho_1\Theta_1 + X) = \mathscr{H}_\mathrm{A}\,\big]. \tag{13}$$

To compare the power of $\mathscr{T}$ to that of $\mathscr{T}_{\lambda^*}$, we test $h_0 = [\,\varepsilon = 0\,]$ against $h_1 = [\,\varepsilon = 1\,]$ given the observation $\Theta + X = (1-\varepsilon)\,\rho_0\,\Theta_0 + \varepsilon\rho_1\Theta_1 + X$. Let $\widetilde{\mathscr{T}}$ stand for the test defined for every $y \in \mathbb{R}^d$ by $\widetilde{\mathscr{T}}(y) = h_1$ if $\mathscr{T}(y) = \mathscr{H}_\mathrm{A}$ and $\widetilde{\mathscr{T}}(y) = h_0$ if $\mathscr{T}(y) = \mathscr{H}_0$. The size of this test is $\alpha(\widetilde{\mathscr{T}}) = \mathrm{P}\big[\,\mathscr{T}(\rho_0\Theta_0 + X) = \mathscr{H}_\mathrm{A}\,\big]$. According to Eq. (1) and since $\alpha(\mathscr{T}) \leqslant \gamma$, we have $\alpha(\widetilde{\mathscr{T}}) \leqslant \gamma$ so that $\widetilde{\mathscr{T}}$ tests h_0 against h_1 at level γ. The power $\beta_\Theta(\widetilde{\mathscr{T}})$ of $\widetilde{\mathscr{T}}$ is now given by

$$\beta_\Theta(\widetilde{\mathscr{T}}) = \mathrm{P}\big[\,\mathscr{T}(\rho_1\Theta_1 + X) = \mathscr{H}_\mathrm{A}\,\big]. \tag{14}$$

According to proposition 1, the UMP test $\mathscr{T}_\mathrm{UMP}$ with size γ for testing h_0 against h_1 is given by Eq. (11) and its power is $\beta_\Theta(\mathscr{T}_\mathrm{UMP}) = 1 - \mathscr{R}(\rho_1, \tau_\gamma(\rho_0))$. Since $\mathscr{T}_\mathrm{UMP}$ is more powerful than any other test for testing h_0 against h_1, we have $\beta_\Theta(\mathscr{T}_\mathrm{UMP}) \geqslant \beta_\Theta(\widetilde{\mathscr{T}})$. It follows from the latter inequality together with Eqs. (13) and (14) that

$$\mathrm{P}\big[\,\mathscr{T}(\Theta+X) = \mathscr{H}_\mathrm{A} \,\big|\, \|\Theta\| = \rho_1\,\big] \leqslant 1 - \mathscr{R}(\rho_1, \tau_\gamma(\rho_0)). \tag{15}$$

Since τ_γ is continuous, $\lim_{\rho_0 \to \varrho} \tau_\gamma(\rho_0) = \tau_\gamma(\varrho) = \lambda^*$. By continuity of $\mathscr{R}(\rho_1, \cdot)$, the rhs in inequality (15) tends to $1 - \mathscr{R}(\rho_1, \lambda^*)$ when ρ_0 tends to ϱ. Therefore, by taking into account Eq. (12), $\mathrm{P}\big[\,\mathscr{T}(\Theta+X) = \mathscr{H}_\mathrm{A} \,\big|\, \|\Theta\| = \rho\,\big] \leqslant \mathrm{P}\big[\,\mathscr{T}_{\lambda^*}(\Theta+X) = \mathscr{H}_\mathrm{A} \,\big|\, \|\Theta\| = \rho\,\big]$. This proves that $\mathscr{T}_{\lambda^*}$ is UBCCP on the family of spheres with radius in (ϱ, ∞).

5.3. Proof of statements **(ii)** *and* **(iii)**

Statement **(ii)** results from lemma 1. Statement **(iii)** derives from Eq. (7) and lemma 3.

Acknowledgement

The authors are very grateful to Lionel Fillatre for the several and profitable discussions on (A. Wald, 1943).

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Campus de Brest
Technopôle Brest-Iroise
CS 83818
29238 Brest Cedex 3
France
Tél. : + 33 (0)2 29 00 11 11
Fax : + 33 (0)2 29 00 10 00

Campus de Rennes
2, rue de la Châtaigneraie
CS 17607
35576 Cesson Sévigné Cedex
France
Tél. : + 33 (0)2 99 12 70 00
Fax : + 33 (0)2 99 12 70 19

Campus de Toulouse
10, avenue Edouard Belin
BP 44004
31028 Toulouse Cedex 04
France
Tél. : +33 (0)5 61 33 83 65
Fax : +33 (0)5 61 33 83 75

Imprimé à Télécom Bretagne
Dépôt légal : Février 2009
ISSN : 1255-2275

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