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On the detection of signals with unknown distributions and priors

Dominique Pastor

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On the detection of signals
with unknown distributions and priors
in white Gaussian noise

Sur la détection de signaux dont les
distributions et les probabilités de présence
sont inconnues dans un bruit blanc Gaussien

Rapport Interne GET / ENST Bretagne

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Abstract

Consider m observations where signals are randomly present or absent in independent and additive white Gaussian noise (AWGN) with standard deviation σ_0 . In various applications, these signals may have unknown distributions and probabilities of presence. If σ_0 is known, their detection can be achieved according to [9] if their norms are larger than or equal to some known lower bound and their probabilities of presence are less than or equal to $1/2$.

The present paper addresses the case where noise has unknown standard deviation. A limit theorem states that σ_0 is the only positive real number that satisfies a specific convergence criterion when m and the minimum amplitude of the signals tend to infinity. A test is then derived from this theorem and [9] for detecting the presence and absence of the signals of interest among the m given observations when σ_0 is unknown. Via a standard Monte-Carlo simulation, the Binary Error Rate (BER) of this test is computed for the non-coherent detection of modulated sinusoidal carriers in independent AWGN, a problem that arises in a number of applications. With a few hundred observations, this BER is sufficiently close to the error probability of the Minimum-Probability-of-Error test to consider that the asymptotic conditions of the limit theorem are not so constraining in practice and can certainly be relaxed.

Keywords

Binary hypothesis testing, Likelihood theory, Multivariate normal distribution, Non-parametric decision, Radar processing, Thresholding test.

Résumé

Considérons m observations où des signaux aléatoires sont présents ou absents dans un bruit blanc Gaussien additif et indépendant d'écart-type σ_0 . Dans de nombreuses applications, ces signaux peuvent avoir des distributions et des probabilités de présence inconnues. Quand σ_0 est connu, ces signaux peuvent être détectés en suivant [9] si leurs normes sont supérieures ou égales à une borne inférieure connue et leurs probabilités de présence inférieures ou égales à $1/2$.

Dans cet article, nous étudions le cas où l'écart-type du bruit n'est pas connu. Un théorème limite énonce que σ_0 est en fait la seule valeur positive réelle qui satisfait un critère de convergence spécifique lorsque m et l'amplitude minimale des signaux tendent vers l'infini. A partir de ce théorème et des résultats de [9], nous dérivons un test pour la détection des signaux à partir de m observations lorsque σ_0 est inconnu. Une simulation de Monte-Carlo permet de calculer le taux d'erreur binaire (TEB) de ce test pour la détection non-cohérente de porteuses sinusodales, un problème que l'on rencontre dans de nombreuses applications. Avec quelques centaines d'observations, le TEB est très proche de la probabilité d'erreur du test du maximum de vraisemblance. Il est donc raisonnable de considérer que les conditions asymptotiques du théorème limite ne sont pas si contraignantes que cela en pratique.

Mots-clés

Décision non paramétrique, Rapport de vraisemblance, Seuils, Test d'hypothèses, Traitement du signal radar, Vecteurs aléatoires Gaussiens.

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1 Introduction

Statistical decision theory plays a crucial role in many signal processing applications. Indeed, the detection of signals of interest can mostly be considered as a standard binary hypothesis testing problem where the null hypothesis is that noise only is present and the alternative hypothesis is that some signal is present in independent and additive noise.

In contrast with the simplicity of this model and even when the noise background is white and Gaussian, the decision can be intricate. Actually, in many applications, very little is known about the observations or about most of their parameters ([4, section I]). For example, the echo received by a radar or an active sonar system from a target basically results from some kind of convolution between a known transmitted pulse and an unknown environment. The situation is even worse in the case of passive sonar systems since the latter receive signals resulting from noise generated by motors and hull vibrations transmitted through a fluctuating environment.

In such situations, the decision cannot be achieved by standard likelihood theory because the usual Bayes, minimax and Neyman-Pearson criteria require full knowledge of the signal distributions. Non-parametric and robust detection ([11, section III.E]), as well as Generalized Likelihood Ratio Tests ([5]), are then alternative formulations making it possible to deal with such cases. Constant False Alarm Rate (CFAR) systems ([8]) standardly used in radar processing for detecting targets with a specified false alarm rate typically derive from such alternative approaches.

In [9], another non-parametric approach is proposed for detecting a signal that has unknown distribution and probability of presence or prior in additive white Gaussian noise (AWGN). In the aforementioned paper, the signal and noise are independent n -dimensional random vectors defined on the same probability space and we investigate how far we can get if the signal is assumed to be less present than absent in the sense that its prior is less than or equal to one half. However, the results established in [9] are workable in practice only if the noise standard deviation is known. The present paper thus addresses the case of an unknown noise standard deviation.

Section 2 presents an easy extension of the results established in [9]. Section 3 states a limit theorem, whose proof is postponed to a dedicated appendix. By combining this theorem and the result of section 2, section 3 also proposes a test for the detection of signals that have unknown probability distributions, amplitudes above or equal to some known lower bound and probabilities of presence less than or equal to $1/2$ in AWGN with unknown standard deviation. Experimental results for the non-coherent detection of modulated sinusoidal carriers are then given in section 4. Some applications

and future extensions are presented in section 5.

2 A thresholding test for detecting signals with unknown distributions and priors in AWGN with known standard deviation

Any random vector or variable encountered in this paper is assumed to be defined on the same probability space (Ω, \mathcal{M}, P) and for every $\omega \in \Omega$ by setting this random vector or variable to 0 on any negligible subset where it could be undefined. As usual, if a property P holds true almost surely, we write P (a-s).

We consider the following binary hypothesis testing problem. The null hypothesis \mathcal{H}_0 is that the observation U is some n -dimensional white Gaussian noise with standard deviation σ_0 in the sense that U has dimension n , is real-valued and Gaussian distributed with a null mean vector and covariance matrix equal to $\sigma_0^2 \mathbf{I}_n$ where \mathbf{I}_n is the $n \times n$ identity matrix. The alternative hypothesis \mathcal{H}_1 is that the observation U is the sum of some n -dimensional random signal Λ and independent white Gaussian noise X with standard deviation σ_0 . The above assumptions can standardly be summarized by

$$\begin{cases} \mathcal{H}_0 & : U \sim \mathcal{N}(0, \sigma_0^2 \mathbf{I}_n), \\ \mathcal{H}_1 & : U = \Lambda + X, X \sim \mathcal{N}(0, \sigma_0^2 \mathbf{I}_n). \end{cases} \quad (1)$$

Following [9], we assign probabilities to the occurrences of hypotheses \mathcal{H}_0 and \mathcal{H}_1 by introducing a random variable $\varepsilon : \Omega \rightarrow \{0, 1\}$ such that Λ , X and ε are mutually independent and

$$U = \varepsilon \Lambda + X. \quad (2)$$

Hence, ε corresponds to the index of the true hypothesis in (1). The probability of presence for Λ is the probability $P(\{\varepsilon = 1\})$ of occurrence of \mathcal{H}_1 .

In what follows, a test \mathcal{T} is any measurable map of \mathbb{R}^n into $\{0, 1\}$. The index of the hypothesis accepted by \mathcal{T} on the basis of the observation U is $\mathcal{T}(U) = \mathcal{T} \circ U$. Therefore, an error occurs when $\mathcal{T}(U)$ and ε differ from each other. The probability $P(\{\mathcal{T} \circ U \neq \varepsilon\})$ is thus the probability of error of test \mathcal{T} for the binary hypothesis testing problem under consideration.

Two tests play a crucial role in the sequel. First, the Minimum-Probability-of-Error (MPE) test is the one that guarantees the smallest probability of error among all possible tests. It is basically a likelihood ratio test and is hereafter denoted by \mathcal{L}_{MPE} . Second, by ‘‘thresholding test with threshold

height $h \in [0, \infty)$ ", we mean the map \mathcal{T}_h defined, for every given $u \in \mathbb{R}^n$, by $\mathcal{T}_h(u) = 1$ if $\|u\| \geq h$ and $\mathcal{T}_h(u) = 0$ otherwise.

Given any n -dimensional random vector Y , let $\|Y\| : \Omega \rightarrow [0, \infty)$ stand for the map that assigns, to any given $\omega \in \Omega$, the standard Euclidean norm of the n -dimensional vector $Y(\omega)$ and define the *minimum amplitude* $\mathbf{a}(Y)$ of Y as the essential infimum of $\|Y\|$. We thus have

$$\mathbf{a}(Y) = \sup\{\alpha \in [0, \infty] : \|Y\| \geq \alpha \text{ (a.s.)}\}. \quad (3)$$

In the following statement, which derives from [9, Theorem VII.1], V stands for the function defined for every non-negative real number ρ by

$$\begin{aligned} V(\rho) = & \frac{e^{-\rho^2/2}}{2^{n/2}\Gamma(n/2)} \int_0^{\xi(\rho)} e^{-t^2/2} t^{n-1} {}_0F_1(n/2; \rho^2 t^2/4) dt \\ & + \frac{1}{2} \left[1 - \frac{1}{2^{n/2-1}\Gamma(n/2)} \int_0^{\xi(\rho)} e^{-t^2/2} t^{n-1} dt \right], \end{aligned} \quad (4)$$

where ${}_0F_1$ is the generalized hypergeometric function ([7, p. 275]) and $\xi(\rho)$ is the unique positive solution for x in the equation

$${}_0F_1(n/2; \rho^2 x^2/4) = e^{\rho^2/2}. \quad (5)$$

Proposition 2.1 *Everything being as above,*

- (i) *if Λ has a probability of presence $P(\{\varepsilon = 1\})$ less than or equal to one half, $V(\mathbf{a}(\Lambda)/\sigma_0)$ is an upper bound for the probability of error of both the likelihood ratio test \mathcal{L}_{MPE} and the thresholding test $\mathcal{T}_{\sigma_0\xi(\mathbf{a}(\Lambda)/\sigma_0)}$ with threshold height $\sigma_0\xi(\mathbf{a}(\Lambda)/\sigma_0)$.*
- (ii) *This bound is attained by both \mathcal{L}_{MPE} and $\mathcal{T}_{\sigma_0\xi(\mathbf{a}(\Lambda)/\sigma_0)}$ if Λ is uniformly distributed on the sphere centred at the origin with radius $\mathbf{a}(\Lambda)$ and has a probability of presence equal to one half.*

Hence, given some non-negative real value A , the thresholding test $\mathcal{T}_{\sigma_0\xi(A/\sigma_0)}$ is non-parametric for detecting any signal less present than absent and **relatively large** in the sense that its norm is larger than or equal to A . The performance measurement guaranteed by this test over the whole class of signals under consideration is then $V(A/\sigma_0)$. This bound is sharp because attained in the least favourable case of a signal with prior equal to 1/2 and uniform distribution on the sphere centred at the origin with radius A .

As a straightforward application of proposition 2.1, the non-coherent detection of a modulated sinusoidal carrier in AWGN is particularly relevant for telecommunication, radar and sonar applications.

The problem is to detect a sinusoidal carrier in a background of independent AWGN when the carrier has known amplitude A , phase uniformly distributed on $[0, 2\pi]$ and prior equal to one half ([11, Example III.B.5, p. 65]). The in-phase and quadrature components of the observation can then be interpreted as noisy measurements of the coordinates of a point in the plane, this point being either at the origin or uniformly distributed on the circle with radius A centred at the origin.

The MPE solution for this binary hypothesis testing problem is known to be the thresholding test with threshold height $\sigma_0^2 I_0^{-1}(e^{A^2/2\sigma_0^2})/A$ ([11, Example II.E.1, p. 33]) where I_0 is the zeroth-order modified Bessel function of the first kind. This result can be derived from statement **(ii)** of proposition 2.1, equation (5) with $n = 2$ and the fact that $I_0(x) = {}_0F_1(1; x^2/4)$ ([1, Eq. 9.6.47, p. 377]). Proposition 2.1 extends the usefulness of this standard result because it states that the same thresholding test can be used when the probability of presence might be less than one half, the amplitude might be above A and the distribution of the phase is not necessarily uniform.

3 A limit theorem and a test for detecting relatively large signals in AWGN with unknown standard deviation

In this section, we address the case of an unknown standard deviation because the results of the previous section are workable in practice only if the noise standard deviation is known, which is seldom the case in many applications. For instance, the standard deviation of a radar system's thermal noise is often unknown and estimating it so as to perform the radar target detection with some guaranteed performance measurement is a standard problem ([8]).

Our approach concerns the model described in the next subsection. This model basically extends that of subsection 2 to sequences of observations. On the basis of this model, after introducing some additional notations, subsection 3.3 states a limit theorem that can be used to estimate the noise standard deviation. Subsection 3.4 then proposes a detection test that combines proposition 2.1 and the limit theorem established below.

3.1 The model

In order to estimate the noise standard deviation, we consider a sequence of observations obeying the following model.

To begin with, given some positive real number σ_0 and some natural number n , a sequence $X = (X_k)_{k \in \mathbb{N}}$ of n -dimensional real random vectors is called an *n-dimensional white Gaussian noise (WGN) with standard deviation σ_0* if the random vectors X_k , $k \in \mathbb{N}$, are mutually independent and identically Gaussian distributed with null mean vector and covariance matrix $\sigma_0^2 \mathbf{I}_n$.

Let $\varepsilon = (\varepsilon_k)_{k \in \mathbb{N}}$ be a sequence of random variables valued in $\{0, 1\}$, $\Lambda = (\Lambda_k)_{k \in \mathbb{N}}$ stand for some sequence of n -dimensional real random vectors and $X = (X_k)_{k \in \mathbb{N}}$ be some n -dimensional WGN with standard deviation σ_0 . Suppose that, for every $k \in \mathbb{N}$, Λ_k , X_k and ε_k are mutually independent. We consider the sequence $U = (U_k)_{k \in \mathbb{N}}$ of observations defined by $U = \varepsilon \Lambda + X$, that is the sequence such that, for every given natural number k , $U_k = \varepsilon_k \Lambda_k + X_k$. The sequence U models a sequence of observations where random signals are either present or absent in independent AWGN: the background of independent AWGN is modelled by the sequence X ; given $k \in \mathbb{N}$, Λ_k stands for some possible signal and ε_k models the possible occurrence of Λ_k . As in section 2, the two possible values for ε_k form a hypothesis pair according to which U_k is distributed; the null hypothesis is that $\varepsilon_k = 0$ and the alternative one is $\varepsilon_k = 1$.

3.2 Some further notations

Let \mathfrak{S} henceforth stand for the set of all the sequences of n -dimensional real random vectors. We define the *minimum amplitude of an element $\Lambda = (\Lambda_k)_{k \in \mathbb{N}}$ of \mathfrak{S}* as the supremum $\mathbf{a}(\Lambda)$ of the set of those $\alpha \in [0, \infty]$ such that, for every natural number k , $\|\Lambda_k\|$ is larger than or equal to α (a-s):

$$\mathbf{a}(\Lambda) = \sup \{ \alpha \in [0, \infty] : \forall k \in \mathbb{N}, \|\Lambda_k\| \geq \alpha \text{ (a-s)} \}. \quad (6)$$

This definition extends that given in (3) since the minimum amplitude of any given random vector Y is the minimum amplitude of the sequence $(Y_k)_{k \in \mathbb{N}}$ where $Y_k = Y$, $k = 1, 2, \dots$

For every given $\Lambda = (\Lambda_k)_{k \in \mathbb{N}}$, the reader will easily verify that $\mathbf{a}(\Lambda)$ is finite, that $\|\Lambda_k\| \geq \mathbf{a}(\Lambda)$ for every $k \in \mathbb{N}$ and that, given $\alpha \in [0, \infty]$, $\mathbf{a}(\Lambda) \geq \alpha$ if and only if, for every $k \in \mathbb{N}$, $\|\Lambda_k\| \geq \alpha$ (a-s).

If f is some map of \mathfrak{S} into \mathbb{R} , we will say that the limit of f is $\ell \in \mathbb{R}$ when $\mathbf{a}(\Lambda)$ tends to ∞ and write that $\lim_{\mathbf{a}(\Lambda) \rightarrow \infty} f(\Lambda) = \ell$ if, for any positive real value η , there exists some $\alpha_0 \in (0, \infty)$ such that, for every $\alpha \geq \alpha_0$ and every $\Lambda \in \mathfrak{S}$ such that $\mathbf{a}(\Lambda) \geq \alpha$, we have that $|f(\Lambda) - \ell| \leq \eta$.

We now introduce the two subsets of \mathfrak{S} that are used below.

Given some non-negative real number a , let $L^a(\Omega, \mathbb{R}^n)$ stand for the set of those n -dimensional real random vectors $Y : \Omega \rightarrow \mathbb{R}^n$ for which $E[\|Y\|^a] < \infty$. We will hereafter deal with the set of those elements $\Lambda = (\Lambda_k)_{k \in \mathbb{N}}$ of \mathfrak{S} such that $\Lambda_k \in L^a(\Omega, \mathbb{R}^n)$ for every $k \in \mathbb{N}$ and $\sup_{k \in \mathbb{N}} E[\|\Lambda_k\|^a]$ is finite. We denote this subset of \mathfrak{S} by $\ell^\infty(\mathbb{N}, L^a(\Omega, \mathbb{R}^n))$. This standard notation is used because, even though $\ell^\infty(\mathbb{N}, L^0(\Omega, \mathbb{R}^n))$ is trivially \mathfrak{S} and $L^a(\Omega, \mathbb{R}^n)$ is a Banach space only if $1 \leq a < \infty$, $L^a(\Omega, \mathbb{R}^n)$ remains a complete metric space even when $0 < a < 1$ ([6]).

Now, let $L^\infty(\Omega, \mathbb{R}^n)$ be the Banach space whose elements are the real random vectors $Y : \Omega \rightarrow \mathbb{R}^n$ whose essential supremum norm $\|Y\|_\infty$ is finite. We then define $\ell^\infty(\mathbb{N}, L^\infty(\Omega, \mathbb{R}^n))$ as the set of those sequences $\Lambda = (\Lambda_k)_{k \in \mathbb{N}}$ of \mathfrak{S} such that $\Lambda_k \in L^\infty(\Omega, \mathbb{R}^n)$ for every $k \in \mathbb{N}$ and $\sup_{k \in \mathbb{N}} \|\Lambda_k\|_\infty$ is finite. The set $\ell^\infty(\mathbb{N}, L^\infty(\Omega, \mathbb{R}^n))$ is a Banach space once it is endowed with the norm $\|\cdot\|_\infty$ defined for any element $\Lambda = (\Lambda_k)_{k \in \mathbb{N}}$ of $\ell^\infty(\mathbb{N}, L^\infty(\Omega, \mathbb{R}^n))$ by

$$\|\Lambda\|_\infty = \sup_{k \in \mathbb{N}} \|\Lambda_k\|_\infty. \quad (7)$$

For every given $k \in \mathbb{N}$, $\|\Lambda_k\|$ is then less than or equal to $\|\Lambda\|_\infty$ (a-s). If $\alpha \in [0, \infty]$, $\|\Lambda\|_\infty \leq \alpha$ if and only if, for every $k \in \mathbb{N}$, $\|\Lambda_k\| \leq \alpha$ (a-s). For every $k \in \mathbb{N}$, $\mathbf{a}(\Lambda) \leq \|\Lambda_k\| \leq \|\Lambda\|_\infty$ (a-s).

If f is some map of $\ell^\infty(\mathbb{N}, L^\infty(\Omega, \mathbb{R}^n))$ into \mathbb{R} , we will say that the limit of f is $\ell \in \mathbb{R}$ when $\|\Lambda\|_\infty$ tends to 0 and write that $\lim_{\|\Lambda\|_\infty \rightarrow 0} f(\Lambda) = \ell$ if, given any positive real value η , there exists some $\alpha_0 \in (0, \infty)$ such that, for every real value $\alpha \leq \alpha_0$ and every $\Lambda \in \ell^\infty(\mathbb{N}, L^\infty(\Omega, \mathbb{R}^n))$ such that $\|\Lambda\|_\infty \leq \alpha$, we have that $|f(\Lambda) - \ell| \leq \eta$.

The reader will easily verify that, given two non-negative (possibly extended) real numbers a and b such that $0 \leq a \leq b \leq \infty$,

$$\ell^\infty(\mathbb{N}, L^b(\Omega, \mathbb{R}^n)) \subset \ell^\infty(\mathbb{N}, L^a(\Omega, \mathbb{R}^n)). \quad (8)$$

3.3 A limit theorem for estimating the noise standard deviation

Let a **thresholding function** be any non-decreasing continuous and positive real function $\theta : [0, \infty) \rightarrow (0, \infty)$ such that $\theta(x) = Cx + \gamma(x)$ where $0 < C < 1$, γ is positive and $\lim_{x \rightarrow \infty} \gamma(x) = 0$. Given any $q \in [0, \infty)$, let Υ_q be the map of $[0, \infty)$ into $[0, \infty)$ defined for every $x \in [0, \infty)$ by

$$\Upsilon_q(x) = \int_0^x t^{q+n-1} e^{-t^2/2} dt. \quad (9)$$

We can now state the following theorem whose proof is given in appendix. In this theorem, neither the priors nor the signals are required to be identically distributed. The convergence criterion derives from a corollary of Kolmogorov's classical strong limit theorem and not from usual generalizations of the central limit theorem such as the Lindeberg and the Lyapounov theorems. For every given random vector $Y : \Omega \rightarrow \mathbb{R}^n$ and any $\tau \in \mathbb{R}$, the notation $\mathcal{I}(\|Y\| \leq \tau)$ henceforth stands for the indicator function of the event $\{\|Y\| \leq \tau\}$.

Theorem 3.1 *Let $U = (U_k)_{k \in \mathbb{N}}$ be some element of \mathfrak{S} such that $U = \varepsilon \Lambda + X$ where $\Lambda = (\Lambda_k)_{k \in \mathbb{N}}$, $X = (X_k)_{k \in \mathbb{N}}$ and $\varepsilon = (\varepsilon_k)_{k \in \mathbb{N}}$ are an element of \mathfrak{S} , some n -dimensional WGN with standard deviation σ_0 and a sequence of random variables valued in $\{0, 1\}$ respectively.*

Assume that

- (H1) *for every $k \in \mathbb{N}$, Λ_k , X_k and ε_k are mutually independent;*
- (H2) *the random vectors U_k , $k \in \mathbb{N}$, are mutually independent;*
- (H3) *the set of priors $\{P(\{\varepsilon_k = 1\}) : k \in \mathbb{N}\}$ has an upper bound p in $(0, 1)$ and the random variables ε_k , $k \in \mathbb{N}$, are mutually independent;*
- (H4) *there exists some $\nu \in (0, \infty]$ such that $\Lambda \in \ell^\infty(\mathbb{N}, L^\nu(\Omega, \mathbb{R}^n))$.*

Let r and s be any two non-negative real numbers such that $0 \leq s < r \leq \nu/2$. Given some natural number m and any pair (σ, T) of positive real numbers, define the random variable $\Delta_m(\sigma, T)$ by

$$\Delta_m(\sigma, T) = \left| \frac{\sum_{k=1}^m \|U_k\|^r \mathcal{I}(\|U_k\| \leq \sigma T)}{\sum_{k=1}^m \|U_k\|^s \mathcal{I}(\|U_k\| \leq \sigma T)} - \sigma^{r-s} \frac{\Upsilon_r(T)}{\Upsilon_s(T)} \right|. \quad (10)$$

Then, given any thresholding function θ ,

- (i) σ_0 *is the unique positive real number σ such that, for every $\beta_0 \in (0, 1]$,*

$$\lim_{\mathbf{a}(\Lambda) \rightarrow \infty} \left\| \overline{\lim}_m \Delta_m(\sigma, \beta \theta(\mathbf{a}(\Lambda)/\sigma)) \right\|_\infty = 0 \quad (11)$$

uniformly in $\beta \in [\beta_0, 1]$;

(ii) if $\nu = \infty$, σ_0 is in addition such that, for every $\beta_0 \in (0, 1]$,

$$\lim_{\|\Lambda\|_\infty \rightarrow 0} \left\| \overline{\lim}_m \Delta_m(\sigma_0, \beta\theta(\|\Lambda\|_\infty/\sigma_0)) \right\|_\infty = 0 \quad (12)$$

uniformly in $\beta \in [\beta_0, 1]$.

Typical values for p and ν are $p = 1/2$ and $\nu = 2$. Actually, in many practical cases, signals are less present than absent, have finite second order moments and the sequence of these second-order moments is finitely upper-bounded.

For $\tau \in [0, \infty)$, the ratio $\sum_{k=1}^m \|U_k\|^r \mathcal{I}(\|U_k\| \leq \tau) / \sum_{k=1}^m \|U_k\|^s \mathcal{I}(\|U_k\| \leq \tau)$ is defined everywhere for the following reason. Let x_1, x_2, \dots, x_m be m real numbers. If there exists at least one natural number $k \in \{1, \dots, m\}$ such that $x_k \neq 0$, the finiteness of the ratio $\sum_{k=1}^m |x_k|^r / \sum_{k=1}^m |x_k|^s$ is trivial. Since $r > s$, the definition of this ratio is then extended by continuity by setting $\sum_{k=1}^m |x_k|^r / \sum_{k=1}^m |x_k|^s = 0$ if $(x_1, \dots, x_m) = (0, \dots, 0)$.

Theorem 3.1 concerns positive solutions of (11) only because $\sigma = 0$ trivially satisfies (11) regardless of the specific convergence involved. Straightforwardly, (11) is also satisfied for all $\sigma \in [0, \infty)$ when $r = s \geq 0$. This explains why it is assumed that $r > s \geq 0$. Finally, statement (i) is of more interest than statement (ii). First, the former specifies the uniqueness of the solution in σ to (11) whereas this uniqueness is not stated by the latter and remains an open question; second, the assumption $\nu = \infty$ is very strong for, according to (8), $\ell^\infty(\mathbb{N}, L^\infty(\Omega, \mathbb{R}^n))$ is a subset of every $\ell^\infty(\mathbb{N}, L^\nu(\Omega, \mathbb{R}^n))$, $\nu \in [0, \infty)$, and, as mentioned above, (H4) is satisfied in many practical cases with $\nu = 2$.

3.4 The Essential Supremum Test

With the same notations as those used above and on the basis of proposition 2.1 and theorem 3.1, this section introduces a test. The aim of this test is to detect signals in independent AWGN with unknown standard deviation when the signals have norms larger than or equal to some known $A \in [0, \infty)$ and when hypotheses (H1-H4) of theorem 3.1 are satisfied with $p = 1/2$ and $\nu = 2$. Hence, $\mathbf{a}(\Lambda) \geq A$ and we consider the typical case already mentioned above of signals less present than absent and whose norms have finite second-order moments that form a bounded sequence. We must choose r and s such that $0 \leq s < r \leq 1$. We fix $r = 1$ and $s = 0$.

We start by noticing that the map ξ introduced in proposition 2.1 is basically a thresholding function with $C = 1/2$. This follows from the three following facts. First, the increasingness of the generalized hypergeometric function ${}_0F_1(n/2; \cdot)$ and that of the function of ρ in the right hand side

of (5) imply the increasingness of ξ ; second, ξ is a positive map because $\xi(0) = \sqrt{n}$; third, $\xi(\rho) = (\rho/2) + (\log 2/\rho)(1 + \psi_1(\rho))$ if $n = 1$ and $\xi(\rho) = (\rho/2) + (n-1)(\log \rho/\rho)(1 + \psi_n(\rho))$ otherwise, with $\lim_{\rho \rightarrow +\infty} \psi_n(\rho) = 0$ for all $n \geq 1$ ([9, Lemma VI.1]). Therefore, one single map, that is ξ , can serve both for estimating σ_0 and making the decision. The analysis of other choices for the thresholding function has not yet been addressed.

Now, let $L \in \mathbb{N}$ and set $\beta_\ell = \ell/L$ for every $\ell \in \{1, \dots, L\}$. Given U_1, \dots, U_m , theorem 3.1 suggests estimating σ_0 by a possibly local minimum $\hat{\sigma}_0(m)$ of

$$\sup_{\ell \in \{1, \dots, L\}} \left\{ \frac{\sum_{k=1}^m \|U_k\| \mathcal{I}(\|U_k\| \leq \beta_\ell \sigma \xi(A/\sigma))}{\sum_{k=1}^m \mathcal{I}(\|U_k\| \leq \beta_\ell \sigma \xi(A/\sigma))} - \sigma \frac{\int_0^{\beta_\ell \xi(A/\sigma)} t^n e^{-t^2/2} dt}{\int_0^{\beta_\ell \xi(A/\sigma)} t^{n-1} e^{-t^2/2} dt} \right\} \quad (13)$$

and proposition 2.1 to use the test $\mathcal{I}(\|\cdot\| \geq \hat{\sigma}_0(m)\xi(A/\hat{\sigma}_0(m)))$ on every observation U_k , $k = 1, \dots, m$, for deciding about the value of ε_k . More specifically, the algorithm we propound is the following one. It is called the *Essential Supremum Test* (EST) for amplitude A and sample size m because the essential supremum norm plays a crucial role in theorem 3.1 and m observations are needed to detect signals that have norms above or equal to A . This test is denoted $\text{EST}(A, m)$.

The EST(A, m).

Inputs:

- The m observations U_k , $k = 1, \dots, m$, to process. In what follows, $U_{[k]}$, $k = 1, \dots, m$, stands for the sequence of observations U_1, \dots, U_m , sorted by increasing norm.
- The lower bound A for the signal norms ($A \leq \mathbf{a}(\Lambda)$).
- A probability value Q close to 1 but less than or equal to $1 - \frac{m}{4(m/2-1)^2}$. A typical choice is $Q = 0.95$, provided that $m \geq 24$. According to Bienaymé-Chebyshev's inequality, the probability that the number of observations due to noise alone be above $m/2 - hm$ is larger than or equal to Q if $h = 1/\sqrt{4m(1-Q)}$. The test is thus asked to accept the null hypothesis at least $k_{\min} = m/2 - hm$ times given the m observations to process.

Outputs: The sets \mathcal{D}_0 and \mathcal{D}_1 where \mathcal{D}_0 (resp. \mathcal{D}_1) is the set of those $k \in \{1, \dots, m\}$ for which the null hypothesis, that is $\varepsilon_k = 0$, is accepted (resp. rejected).

Initialization

SET $\sigma_{\min} = \|U_{[k_{\min}]}\|/\sqrt{n}$. This value is the left endpoint of the search interval for minimizing the discrete cost (13). It guarantees a minimum of $m/2 - hm$ acceptances of the null hypothesis.

SET $\sigma_{\max} = \|U_{[m]}\|/\sqrt{n}$. This value is the right endpoint of the search interval for minimizing (13). The reason for this choice is the following one. An estimate $\hat{\sigma}_0(m)$ larger than σ_{\max} would imply that $\|U_k\| \leq \hat{\sigma}_0(m)\xi(A/\hat{\sigma}_0(m))$ for every $k = 1, 2, \dots, m$, since $\xi(\rho) \geq \xi(0)$ for all $\rho \in [0, \infty[$ and $\xi(0) = \sqrt{n}$ ([9]). The outcome of the $\text{EST}(A, m)$ could then be that no signal is present whereas the full absence of signals amongst m observations is hardly probable when m is large, provided that the probabilities of presence are not too small.

Estimation of the noise standard deviation and decision-making

COMPUTE a minimum $\hat{\sigma}_0(m)$ of (13) within the search interval $[\sigma_{\min}, \sigma_{\max}]$.

Any minimization routine for scalar bounded non-linear functions is suitable. For instance, the experimental results presented in the next section were obtained with the MATLAB routine `fminbnd.m`, based on the parabolic interpolation ([12]).

SET $\mathcal{D}_0 = \{k \in \{1, \dots, m\} : \|U_k\| < \hat{\sigma}_0(m)\xi(A/\hat{\sigma}_0(m))\}$.

SET $\mathcal{D}_1 = \{1, \dots, m\} \setminus \mathcal{D}_0$.

4 Experimental results

With the notations and assumptions of theorem 3.1, for any given $k \in \mathbb{N}$, the probability of error $P(\{\mathcal{I}(\|U_k\| \geq \hat{\sigma}_0(m)\xi(A/\hat{\sigma}_0(m))) \neq \varepsilon_k\})$ can be expected not to significantly exceed $V(A/\sigma_0)$ when A and m are large enough. This claim remains intuitive because, unlike proposition 2.1, theorem 3.1 and the construction of the $\text{EST}(A, m)$ pinpoint no performance measurement guaranteed over the whole class of those signals that are less present than absent and whose norms are above or equal to A . Therefore, in this section, we restrict attention to some experimental results concerning the non-coherent detection of modulated sinusoidal carriers in AWGN. For every given $k \in \mathbb{N}$,

we thus assume that U_k , Λ_k and X_k are two-dimensional random vectors and that each Λ_k is uniformly distributed on the circle centred at the origin with radius A . In order to guarantee the mutual independence of the observations, the random vectors Λ_k , $k \in \mathbb{N}$, are assumed to be mutually independent. We also assume that the priors $P(\{\varepsilon_k = 1\})$, $k = 1, \dots, m$, all equal some $p \leq 1/2$.

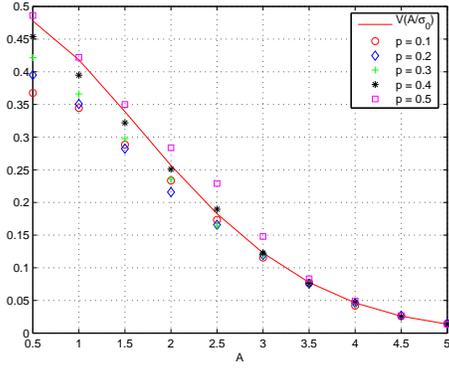
When σ_0 is known, the error probability of the MPE test for deciding about the value of every given ε_k is $V(A/\sigma_0)$ if $p = 1/2$ and is less than or equal to this upper bound if $p \leq 1/2$ (see subsection 2). Given σ_0 , A , m and p , we achieve a Monte-Carlo simulation and compute the Binary Error Rate (BER) of the EST(A, m) as follows. Independent trials of m observations each are carried out until two conditions are fulfilled.

First, a minimum number M of trials must be achieved. This number is specified beforehand. Inasmuch as the decision about the presence or the absence of signals is made on the observations used for estimating σ_0 , the accuracy of the estimate affects m decisions at one go. This effect is then reduced by performing a minimum number of trials.

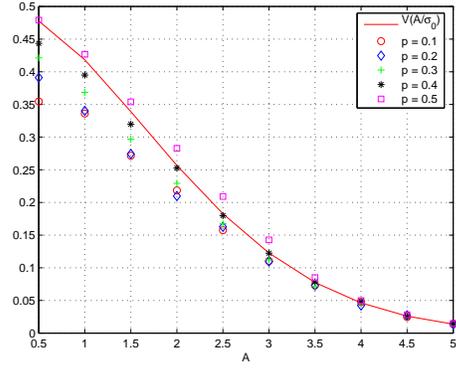
Second, the total number N_e of errors committed by the EST(A, m) for detecting the presence or the absence of signals must be above or equal to some specified number N . If j is the first trial number larger than or equal to M and for which the total number of errors N_e becomes larger than or equal to N , the BER of the EST(A, m) is then defined as the ratio $N_e/(j \times m)$.

The simulation is performed with $\sigma_0 = 1$. The pre-specified number of errors is fixed to $N = 400$ and the minimum number of trials is $M = 100$. As far as the parameters of the EST(A, m) are concerned, we take $L = m$ and $Q = 0.95$ according to a few preliminary trials. The results obtained for various values of A , m and p are those of figure 1. As expected, when m and A increase, the BER of the EST(A, m) becomes still closer to the probability of error of the MPE test.

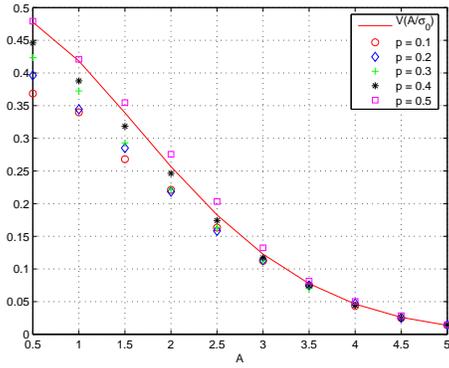
Still on the basis of theorem 3.1, it can be expected that, given $k \in \mathbb{N}$, the error probability $P(\{\mathcal{I}(\|U_k\| \geq \hat{\sigma}_0(m)\xi(A/\hat{\sigma}_0(m))) \neq \varepsilon_k\})$ remains even less than or equal to $V(A/\sigma_0)$ if the ratio $\mathbf{a}(\Lambda)/A$ is large enough. To check this claim experimentally, Monte-Carlo simulations were carried out when the EST(A, m) was asked to process observations where the signals had their norms equal to $1.1220 \times A$ instead of A . In other words, the actual amplitudes of the signals were 1 dB larger than expected. The BERs thus obtained are those of figure 2. Roughly speaking, we can say that the loss incurred by the use of the EST(A, m) instead of the MPE test is 1 dB for all the amplitudes, probabilities of presence and values of m that were tested.



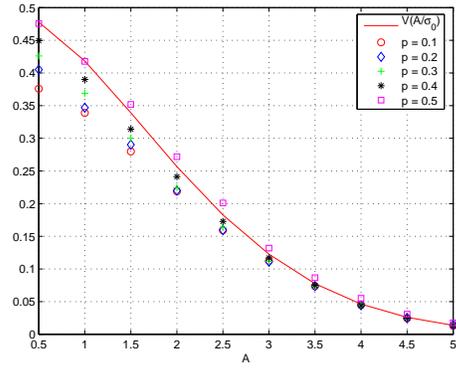
(a) $m = 100$



(b) $m = 200$

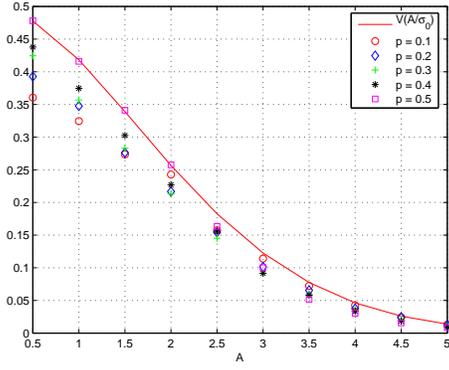


(c) $m = 400$

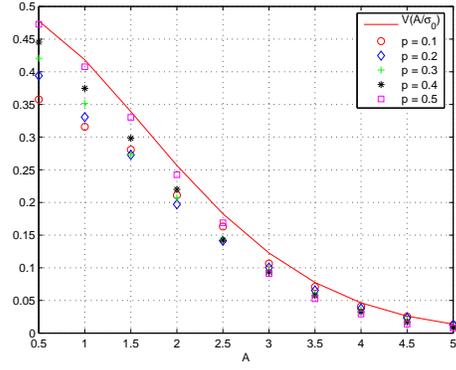


(d) $m = 800$

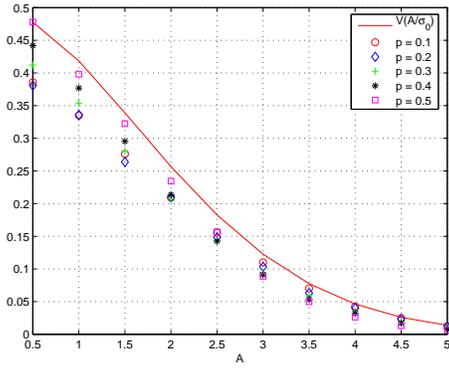
Figure 1: BER of the $EST(A, m)$ for different values for m in comparison with $V(A/\sigma_0)$



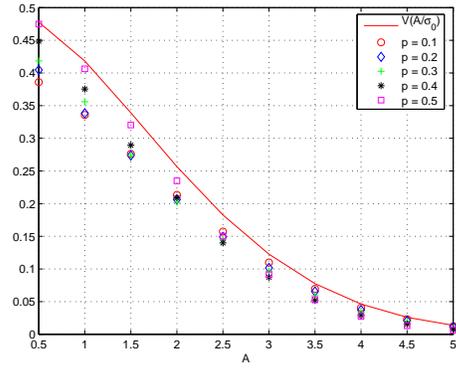
(a) $m = 100$



(b) $m = 200$



(c) $m = 400$



(d) $m = 800$

Figure 2: BER of the $EST(A, m)$ for different values for m in comparison with $V(A/\sigma_0)$

5 Perspectives and extensions

This paper has presented a limit theorem and a new test for the detection of signals whose amplitudes have a known lower bound but whose distributions and priors less than or equal to one half are unknown in AWGN with unknown standard deviation.

In forthcoming work, we wish to analyse the influence of L , r , s and the thresholding function on the performance of the EST. We will also address the possible links between the approach followed above and that proposed in [3]. It seems also particularly relevant to analyse to what extent the asymptotic conditions of theorem 3.1 can actually be relaxed since experimental results suggest that these conditions are not so constraining in practice.

A natural application of the approach presented in this paper is the design of Constant False Alarm Rate (CFAR) systems for the detection of radar targets. Our intention is to study to what extent proposition 2.1, theorem 3.1 and the EST are complementary to standard results and algorithms such as those described in [8]. In this respect, [10] introduces an estimator of the noise standard deviation when only non signal-free observations are available; this estimator derives from the theoretical and experimental results presented above.

Another illustrative application of the foregoing is described in [2]. Given speech signals corrupted by independent and additive white Gaussian noise, the estimator proposed in [10] and derived from the contents of the present paper is used to estimate the noise standard deviation, and thus, to adjust a speech-enhancement algorithm based on a perceptual model of the speech signals. The performance measurements yielded by this approach are very promising.

6 Acknowledgement.

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Appendix

Proof of theorem 3.1

After stating some preliminary results in the next subsection, our first purpose is to prove that the noise standard deviation satisfies (11) and (12). This will be achieved in three steps, namely subsections B, C and D. Then, in subsection E, we complete the proof of statement (i) by showing that the noise standard deviation is the sole solution in σ to (11). From now on, we assume that $\sigma_0 = 1$ and carry out the proof in this case. The reader will easily verify that this simplification infers no loss of generality.

A Preliminary results

For every $k \in \mathbb{N}$, X_k and Λ_k are assumed to be independent and the distribution of X_k is absolutely continuous with respect to Lebesgue's measure in \mathbb{R}^n . Therefore, given $k \in \mathbb{N}$, the distribution of the random vector $\Lambda_k + X_k$ is absolutely continuous with respect to Lebesgue's measure in \mathbb{R}^n . Its density, denoted hereafter by $f_{\Lambda_k + X_k}$, is given by

$$f_{\Lambda_k + X_k}(u) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{\|u-\lambda\|^2}{2}} dP_{\Lambda_k}(\lambda), u \in \mathbb{R}^n \quad (14)$$

where the distribution P_{Λ_k} of Λ_k is, by definition, the positive measure such that $P_{\Lambda_k}(E) = P(\Lambda_k^{-1}(E))$ for any Borel set E of \mathbb{R}^n .

For every $k \in \mathbb{N}$, the distribution of $\|\Lambda_k + X_k\|$ is absolutely continuous with respect to Lebesgue's measure. The density $f_{\|\Lambda_k + X_k\|}$ of this distribution is given, for any non-negative real number x , by

$$f_{\|\Lambda_k + X_k\|}(x) = \frac{2\pi^{n/2}}{\Gamma(n/2)} x^{n-1} \int_{S^{n-1}} f_{\Lambda_k + X_k}(xu) d\Sigma(u), \quad (15)$$

where Σ stands for the rotation-invariant positive Borel measure on the sphere $S^{n-1} = \{u \in \mathbb{R}^n : \|u\| = 1\}$ for which $\Sigma(S^{n-1}) = 1$ and Γ is the usual Gamma function. It then follows from (14), (15) and Fubini's theorem, that, for any non-negative real number x ,

$$f_{\|\Lambda_k + X_k\|}(x) = \frac{e^{-x^2/2} x^{n-1}}{2^{n/2-1} \Gamma(n/2)} \int_{\mathbb{R}^n} e^{-\|\lambda\|^2/2} \left(\int_{S^{n-1}} e^{x(\lambda|u)} d\Sigma(u) \right) dP_{\Lambda_k}(\lambda), \quad (16)$$

where $(u|v) = \sum_{i=1}^n u_i v_i$, $u, v \in \mathbf{R}^n$, is the usual scalar product in \mathbf{R}^n . It is then known ([9, Lemma A.1]) that $\int_{S^{n-1}} e^{x(\lambda|u)} d\Sigma(u) = {}_0F_1(n/2; x^2 \|\lambda\|^2/4)$,

where ${}_0F_1$ is the generalized hypergeometric function ([7, p. 275]). We thus derive from (16) that

$$f_{\|\Lambda_k + X_k\|}(x) = \frac{e^{-x^2/2} x^{n-1}}{2^{n/2-1} \Gamma(n/2)} \int_{\mathbb{R}^n} e^{-\|\lambda\|^2/2} {}_0F_1(n/2; x^2 \|\lambda\|^2/4) dP_{\Lambda_k}(\lambda). \quad (17)$$

For any non-negative real numbers q and T , the following equalities are trivial but will prove very useful in the sequel.

On the one hand, we have that

$$E [\|\Lambda_k + X_k\|^q \mathcal{I}(\|\Lambda_k + X_k\| \leq T)] = \int_0^T x^q f_{\|\Lambda_k + X_k\|}(x) dx. \quad (18)$$

On the other hand, we can write that

$$E [\|U_k\|^q \mathcal{I}(\|U_k\| \leq T)] = p_k E [\|\Lambda_k + X_k\|^q \mathcal{I}(\|\Lambda_k + X_k\| \leq T)] + q_k \int_0^T x^q f_n(x) dx \quad (19)$$

where

$$f_n(x) = \frac{1}{2^{\frac{n}{2}-1} \Gamma(\frac{n}{2})} e^{-\frac{x^2}{2}} x^{n-1}, x \geq 0, \quad (20)$$

is the probability density function of the square root of some centred chi-2 distribution with n degrees of freedom and where, from now on, p_k and q_k stand for $P(\{\varepsilon_k = 1\})$ and $1 - p_k$ respectively.

B The case of large sample sizes

The behaviour of $\sum_{k=1}^m \|U_k\|^r \mathcal{I}(\|U_k\| \leq T) / \sum_{k=1}^m \|U_k\|^s \mathcal{I}(\|U_k\| \leq T)$ is given in lemma B.2 when m is large and T is any positive real value. The analysis derives from a corollary of Kolmogorov's classical strong limit theorem. This corollary is given in ([13, chapter 5, p 108, corollary 3]). We recall it after correcting some minor typing errors.

Lemma [Rao] B.1 *If Y_1, Y_2, \dots is a sequence of mutually independent random variables on (Ω, \mathcal{M}, P) with non-negative means $\alpha_1, \alpha_2, \dots$, and variances $\sigma_1^2, \sigma_2^2, \dots$, such that $\sum_{m=1}^{+\infty} \alpha_m = +\infty$ (i) and $\sum_{m=1}^{+\infty} \sigma_m^2 / a_m^2 < +\infty$ (ii), where $a_m = \sum_{k=1}^m \alpha_k$, then $\lim_m (1/a_m) \sum_{k=1}^m Y_m = 1$ (a-s).*

Proposition B.2 *Under hypotheses (H2), (H3) and (H4), for every non-negative real number q less than or equal to $\nu/2$ and every positive real number T ,*

$$\lim_{m \rightarrow +\infty} \sum_{k=1}^m \|U_k\|^q \mathcal{I}(\|U_k\| \leq T) / E \left[\sum_{k=1}^m \|U_k\|^q \mathcal{I}(\|U_k\| \leq T) \right] = 1 \text{ (a-s)}.$$

PROOF: Lemma B.1 applies to the sequence of mutually independent variables $(\|U_k\|^q \mathcal{I}(\|U_k\| \leq T))_{k \in \mathbb{N}}$ if we prove that the sequences

$$(Var(\|U_k\|^q \mathcal{I}(\|U_k\| \leq T)))_{k \in \mathbb{N}} \text{ and } (E[\|U_k\|^q \mathcal{I}(\|U_k\| \leq T)])_{k \in \mathbb{N}}$$

are upper-bounded **(i')** and bounded away from zero **(ii')** respectively. Actually, the two conditions **(i')** and **(ii')** are sufficient to fulfil conditions **(i)** and **(ii)** of lemma B.1.

Clearly, $X \in \ell^\infty(\mathbb{N}, L^{2q}(\Omega, \mathbb{R}^n))$; on the other hand, it follows from (8) that $\ell^\infty(\mathbb{N}, L^\nu(\Omega, \mathbb{R}^n))$ is a subset of $\ell^\infty(\mathbb{N}, L^{2q}(\Omega, \mathbb{R}^n))$. Thereby, $\Lambda + X$ is an element of $\ell^\infty(\mathbb{N}, L^{2q}(\Omega, \mathbb{R}^n))$. Condition **(i')** then holds true as a consequence of the inequality

$$Var(\|U_k\|^q \mathcal{I}(\|U_k\| \leq T)) \leq E[\|\Lambda_k + X_k\|^{2q}] + E[\|X_k\|^{2q}],$$

which straightforwardly follows from (19).

Condition **(ii')** derives from the non-vanishing behaviour of the probabilities of absence q_k , $k \in \mathbb{N}$, the fact that T is positive and the easy inequality

$$E[\|U_k\|^q \mathcal{I}(\|U_k\| \leq T)] \geq (1-p) \int_0^T x^q f_n(x) dx,$$

which also follows from (19). It then suffices to apply lemma B.1 to conclude. ■

The subsequent result straightforwardly derives from the foregoing.

Corollary B.3 *Under hypotheses **(H2)**, **(H3)** and **(H4)**, for every pair (r, s) of real numbers such that $0 \leq s < r \leq \nu/2$ and every positive real number T ,*

$$\lim_{m \rightarrow \infty} \left(\frac{\sum_{k=1}^m \|U_k\|^r \mathcal{I}(\|U_k\| \leq T)}{\sum_{k=1}^m \|U_k\|^s \mathcal{I}(\|U_k\| \leq T)} \times \frac{E\left[\sum_{k=1}^m \|U_k\|^s \mathcal{I}(\|U_k\| \leq T)\right]}{E\left[\sum_{k=1}^m \|U_k\|^r \mathcal{I}(\|U_k\| \leq T)\right]} \right) = 1 \quad (a-s).$$

C The case of large and small amplitudes

In contrast with the previous result, the thresholding function θ is of most importance in the present section where, given $\beta \in [0, 1]$, we derive the

behaviour of $\frac{E[\sum_{k=1}^m \|U_k\|^r \mathcal{I}(\|U_k\| \leq \beta\theta(\mathbf{a}(\Lambda)))]}{E[\sum_{k=1}^m \|U_k\|^s \mathcal{I}(\|U_k\| \leq \beta\theta(\mathbf{a}(\Lambda)))]}$ when $\mathbf{a}(\Lambda)$ tends to ∞ and that of $\frac{E[\sum_{k=1}^m \|U_k\|^r \mathcal{I}(\|U_k\| \leq \beta\theta(\|\Lambda\|_\infty))]}{E[\sum_{k=1}^m \|U_k\|^s \mathcal{I}(\|U_k\| \leq \beta\theta(\|\Lambda\|_\infty))]}$ when $\|\Lambda\|_\infty$ tends to 0. The starting point of the analysis is the following result.

Lemma C.1 *Under hypothesis (H1), for every given $q \in [0, \infty)$,*

(i) $\lim_{\mathbf{a}(\Lambda) \rightarrow \infty} E[\|\Lambda_k + X_k\|^q \mathcal{I}(\|\Lambda_k + X_k\| \leq \beta\theta(\mathbf{a}(\Lambda)))] = 0$, uniformly in k and $\beta \in [0, 1]$.

(ii) *If $\Lambda \in \ell^\infty(\mathbb{N}, L^\infty(\Omega, \mathbb{R}^n))$, then, in addition to statement (i),*

$$\lim_{\|\Lambda\|_\infty \rightarrow 0} \left(E[\|\Lambda_k + X_k\|^q \mathcal{I}(\|\Lambda_k + X_k\| \leq \beta\theta(\|\Lambda\|_\infty))] - E[\|X_k\|^q \mathcal{I}(\|X_k\| \leq \beta\theta(\|\Lambda\|_\infty))] \right) = 0,$$

uniformly in k and $\beta \in [0, 1]$.

PROOF: We start by proving statement (i). For the sake of simplifying notations, we put $\rho = \mathbf{a}(\Lambda)$. Given any $\beta \in [0, 1]$, we easily derive from (18) that $E[\|\Lambda_k + X_k\|^q \mathcal{I}(\|\Lambda_k + X_k\| \leq \beta\theta(\rho))]$ is less than or equal to $\theta(\rho)^q P(\{\|\Lambda_k + X_k\| \leq \theta(\rho)\})$. Our proof consists in showing that the latter quantity tends to 0 uniformly with k when ρ tends to ∞ .

We derive from the properties of θ that, for ρ large enough, $C\rho < \theta(\rho) \leq \rho$. According to (14) and since, for any natural number k , $\|\Lambda_k\| \geq \rho$ (a-s), the probability that $\Lambda_k + X_k$ lies inside the ball of radius $\theta(\rho)$ can be written as follows:

$$\begin{aligned} & P(\{\|\Lambda_k + X_k\| \leq \theta(\rho)\}) \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\|x\| \leq C\rho} \int_{\|\lambda\| \geq \rho} e^{-\|x-\lambda\|^2/2} dP_{\Lambda_k}(\lambda) dx, \\ & \quad + \frac{1}{(2\pi)^{n/2}} \int_{C\rho < \|x\| \leq \theta(\rho)} \int_{\|\lambda\| \geq \rho} e^{-\|x-\lambda\|^2/2} dP_{\Lambda_k}(\lambda) dx. \end{aligned} \quad (21)$$

Because $\|x - \lambda\| \geq \rho(1 - C)$ for $\|x\| \leq C\rho$ and $\|\lambda\| \geq \rho$, the first term in the right-hand side of (21) is bounded by a constant multiplied with $\rho^n e^{-\frac{(1-C)^2}{2}\rho^2}$.

For $C\rho < \|x\| \leq \theta(\rho)$ and $\|\lambda\| \geq \rho$, $\|x - \lambda\|$ is larger than or equal to $\rho(1 - C) - \gamma(\rho)$. Hence, by taking into account the volume of the shell $\{x \in \mathbb{R}^n : C\rho < \|x\| \leq \theta(\rho)\}$, the second term in the right hand side of (21) is less than a constant times $\left(1 + \frac{\gamma(\rho)}{C\rho}\right)^n - 1 \rho^n e^{-((1-C)-\gamma(\rho)/\rho)^2 (\rho^2/2)}$.

In addition, $\theta(\rho) \sim C\rho$ when ρ tends to $+\infty$. It thus follows from the foregoing that, for every non-negative real number q , $\theta(\rho)^q P(\{\|\Lambda_k + X_k\| < \theta(\rho)\})$ tends to 0 uniformly with k when $\rho = \mathbf{a}(\Lambda)$ tends to ∞ , which completes the proof of statement **(i)**.

We now turn our attention to the proof of statement **(ii)**. Still for notational sake, we set $\rho = \|\Lambda\|_\infty$. An easy consequence of (18) is that, for any $\beta \in [0, 1]$, the absolute value of the difference

$$E\left[\|\Lambda_k + X_k\|^q \mathcal{I}(\|\Lambda_k + X_k\| \leq \beta\theta(\rho))\right] - E\left[\|X_k\|^q \mathcal{I}(\|X_k\| \leq \beta\theta(\rho))\right]$$

is less than or equal to $\theta(\rho)^q \int_0^{\theta(\rho)} |f_{\|\Lambda_k + X_k\|}(x) - f_n(x)| dx$. Our intention is then to prove that the latter quantity tends to 0 uniformly in k when ρ tends to 0. The hypergeometric function ${}_0F_1(n/2; \cdot)$ is continuous, increasing and ${}_0F_1(n/2; 0) = 1$. Hence, for $\|\lambda\| \leq \rho$ and $x \in [0, \theta(\rho)]$,

$$1 \leq {}_0F_1\left(n/2; x^2\|\lambda\|^2/4\right) \leq {}_0F_1\left(n/2; \rho^2\theta(\rho)^2/4\right).$$

From the inequality above, (17), (20), and the fact that for every $k \in \mathbb{N}$, $\|\Lambda_k\|$ is less than or equal to ρ (a-s), it follows that, for all $x \in [0, \theta(\rho)]$,

$$\frac{e^{-x^2/2}x^{n-1}}{2^{n/2-1}\Gamma(n/2)} \left(e^{-\rho^2/2} - 1\right) \leq f_{\|\Lambda_k + X_k\|}(x) - f_n(x)$$

and

$$f_{\|\Lambda_k + X_k\|}(x) - f_n(x) \leq \frac{e^{-x^2/2}x^{n-1}}{2^{n/2-1}\Gamma(n/2)} \left({}_0F_1\left(n/2; \rho^2\theta(\rho)^2/4\right) - 1\right).$$

The two functions of ρ that bracket $f_{\|\Lambda_k + X_k\|}(x) - f_n(x)$ in the inequalities above both trivially tend to 0 independently of k when ρ tends to 0. Since θ is continuous at the origin, we have that $\lim_{\rho \rightarrow 0} \mathcal{I}_{[0, \theta(\rho)]}(x)\theta(\rho)^q |f_{\|\Lambda_k + X_k\|}(x) - f_n(x)| = 0$, uniformly in k . It follows from Lebesgue's dominated convergence theorem that $\theta(\rho)^q \int_0^{\theta(\rho)} |f_{\|\Lambda_k + X_k\|}(x) - f_n(x)| dx$ tends to 0 uniformly in k when ρ tends to 0, which completes the proof. \blacksquare

Thanks to lemma C.1, the following proposition concludes this subsection.

Proposition C.2 *Under hypotheses **(H1)**, **(H2)** and **(H3)**, for every given $\beta_0 \in (0, 1]$ and every given pair (r, s) of non-negative real numbers such that $0 \leq s < r$,*

$$\lim_{\mathbf{a}(\Lambda) \rightarrow \infty} \left(\frac{E \left[\sum_{k=1}^m \|U_k\|^r \mathcal{I}(\|U_k\| \leq \beta\theta(\mathbf{a}(\Lambda))) \right]}{E \left[\sum_{k=1}^m \|U_k\|^s \mathcal{I}(\|U_k\| \leq \beta\theta(\mathbf{a}(\Lambda))) \right]} - \frac{\Upsilon_r(\beta\theta(\mathbf{a}(\Lambda)))}{\Upsilon_s(\beta\theta(\mathbf{a}(\Lambda)))} \right) = 0 \quad (22)$$

uniformly in m and $\beta \in [\beta_0, 1]$.

Furthermore, if Λ belongs to $\ell^\infty(\mathbb{N}, L^\infty(\Omega, \mathbb{R}^n))$,

$$\lim_{\|\Lambda\|_\infty \rightarrow 0} \left(\frac{E \left[\sum_{k=1}^m \|U_k\|^r \mathcal{I}(\|U_k\| \leq \beta\theta(\|\Lambda\|_\infty)) \right]}{E \left[\sum_{k=1}^m \|U_k\|^s \mathcal{I}(\|U_k\| \leq \beta\theta(\|\Lambda\|_\infty)) \right]} - \frac{\Upsilon_r(\beta\theta(\|\Lambda\|_\infty))}{\Upsilon_s(\beta\theta(\|\Lambda\|_\infty))} \right) = 0 \quad (23)$$

uniformly in m and $\beta \in [\beta_0, 1]$.

PROOF: We prove (22) and (23) together. As above, let ρ stand for either $\mathbf{a}(\Lambda)$ or $\|\Lambda\|_\infty$. In the latter case, it is implicitly assumed that $\Lambda \in \ell^\infty(\mathbb{N}, L^\infty(\Omega, \mathbb{R}^n))$.

Let q be any non-negative real value, β_0 any element of $(0, 1]$ and β any element of $[\beta_0, 1]$. It follows from (19) that

$$\begin{aligned} & E \left[\sum_{k=1}^m \|U_k\|^q \mathcal{I}(\|U_k\| \leq \beta\theta(\rho)) \right] \\ &= \sum_{k=1}^m p_k \left(E [\|\Lambda_k + X_k\|^q \mathcal{I}(\|\Lambda_k + X_k\| \leq \beta\theta(\rho))] \right. \\ &\quad \left. - \delta_{\ell,0} E [\|X_k\|^q \mathcal{I}(\|X_k\| \leq \beta\theta(\rho))] \right) \\ &\quad + \left(\sum_{k=1}^m q_k + \delta_{\ell,0} \sum_{k=1}^m p_k \right) \int_0^{\beta\theta(\rho)} x^q f_n(x) dx, \end{aligned} \quad (24)$$

where $\ell = \infty$ if $\rho = \mathbf{a}(\Lambda)$ and $\ell = 0$ otherwise and $\delta_{a,b}$, $a, b \in \mathbb{R}$, stands for the usual Kronecker symbol: $\delta_{a,b} = 1$ if $a = b$ and $\delta_{a,b} = 0$ otherwise.

According to lemma C.1, if $\rho = \mathbf{a}(\Lambda)$ (resp. $\rho = \|\Lambda\|_\infty$), there exists some positive real number α_0 such that, for every $\alpha \geq \alpha_0$ (resp. $\alpha \leq \alpha_0$), every Λ such that $\mathbf{a}(\Lambda) \geq \alpha$ (resp. $\|\Lambda\|_\infty \leq \alpha$) and every $\beta \in [\beta_0, 1]$, each factor of p_k in the right hand side of (24) is less than or equal to η . By taking into account that each p_k is smaller than or equal to $p < 1$ and that $\sum_{k=1}^m q_k + \delta_{\ell,0} \sum_{k=1}^m p_k$ remains positive, it follows that

$$\left| \frac{E \left[\sum_{k=1}^m \|U_k\|^q \mathcal{I}(\|U_k\| \leq \beta\theta(\rho)) \right]}{\sum_{k=1}^m q_k + \delta_{\ell,0} \sum_{k=1}^m p_k} - \int_0^{\beta\theta(\rho)} x^q f_n(x) dx \right| \leq \frac{p}{1-p} \eta$$

for ρ large enough (resp. small enough). The right hand side of the latter inequality does not depend on k . Moreover,

$$\lim_{\rho \rightarrow \ell} \int_0^{\beta\theta(\rho)} x^q f_n(x) dx = \int_0^{\beta\theta(\ell)} x^q f_n(x) dx,$$

whether $\ell = 0$ or $\ell = \infty$ since $\theta(0) > 0$, $\theta(\infty) = \infty$ and $\beta \geq \beta_0 > 0$. Thereby, we get that

$$\lim_{\rho \rightarrow \ell} \frac{E \left[\sum_{k=1}^m \|U_k\|^q \mathcal{I}(\|U_k\| \leq \beta\theta(\rho)) \right]}{\sum_{k=1}^m q_k + \delta_{\ell,0} \sum_{k=1}^m p_k} = \int_0^{\beta\theta(\ell)} x^q f_n(x) dx \in (0, \infty),$$

uniformly in m and $\beta \in [\beta_0, 1]$. Given two real numbers $r > s \geq 0$, it straightforwardly follows that

$$\lim_{\rho \rightarrow \ell} \frac{E \left[\sum_{k=1}^m \|U_k\|^r \mathcal{I}(\|U_k\| \leq \beta\theta(\rho)) \right]}{E \left[\sum_{k=1}^m \|U_k\|^s \mathcal{I}(\|U_k\| \leq \beta\theta(\rho)) \right]} = \frac{\int_0^{\beta\theta(\ell)} x^r f_n(x) dx}{\int_0^{\beta\theta(\ell)} x^s f_n(x) dx},$$

which concludes the proof. ■

D Combining large sample sizes and large or small amplitudes

Corollary B.3 holds true for any threshold T and real values r and s such that $0 \leq s < r \leq \nu/2$. Proposition C.2 is valid for all non-negative real values r and s such that $s < r$ and thresholds deriving from the thresholding function θ . Hence, the following lemma D.1 particularizes corollary B.3 to such thresholds. The rest of the section then combines proposition C.2 and lemma D.1.

Lemma D.1 *Under hypotheses (H1), (H2), (H3) and (H4), let r and s be two real numbers such that $0 \leq s < r \leq \nu/2$. Then, for any $\beta_0 \in (0, 1]$, there exists some positive real number α_0 (resp. α_1) such that, for any α in $[\alpha_0, \infty)$ (resp. $(0, \alpha_1]$), any $\Lambda \in \ell^\infty(\mathbb{N}, L^\nu(\Omega, \mathbb{R}^n))$ (resp. $\Lambda \in \ell^\infty(\mathbb{N}, L^\infty(\Omega, \mathbb{R}^n))$) such*

that $\mathbf{a}(\Lambda) \geq \alpha$ (resp. $\|\Lambda\|_\infty \leq \alpha$) and any $\beta \in [\beta_0, 1]$,

$$\lim_{m \rightarrow +\infty} \left(\frac{\sum_{k=1}^m \|U_k\|^r \mathcal{I}(\|U\| \leq \beta\theta(\rho))}{\sum_{k=1}^m \|U_k\|^s \mathcal{I}(\|U\| \leq \beta\theta(\rho))} - \frac{E \left[\sum_{k=1}^m \|U_k\|^r \mathcal{I}(\|U_k\| \leq \beta\theta(\rho)) \right]}{E \left[\sum_{k=1}^m \|U_k\|^s \mathcal{I}(\|U_k\| \leq \beta\theta(\rho)) \right]} \right) = 0 \text{ (a-s)} \quad (25)$$

with $\rho = \mathbf{a}(\Lambda)$ (resp. $\rho = \|\Lambda\|_\infty$).

PROOF: By differentiating $\Upsilon_r(x)/\Upsilon_s(x)$ with respect to $x \geq 0$, it follows that Υ_r/Υ_s is a non-decreasing map of $[0, \infty]$ into $[0, \Upsilon_r(\infty)/\Upsilon_s(\infty)]$ with $\Upsilon_r(\infty)/\Upsilon_s(\infty) < \infty$.

On the other hand, since $\theta(x) \geq \theta(0) > 0$ for every non-negative real number x , a consequence of (18) and (19) is that

$$\frac{E \left[\sum_{k=1}^m \|U_k\|^r \mathcal{I}(\|U_k\| \leq \beta\theta(\mathbf{a}(\Lambda))) \right]}{E \left[\sum_{k=1}^m \|U_k\|^s \mathcal{I}(\|U_k\| \leq \beta\theta(\mathbf{a}(\Lambda))) \right]} > 0$$

for all $\beta > 0$. Let β_0 be an element of $(0, 1]$ and η be some positive real number. It follows from proposition C.2 and the remarks above that there exists some positive real number α_0 such that, for any real number α larger than or equal to α_0 , any $\Lambda \in \ell^\infty(\mathbb{N}, L^\nu(\Omega, \mathbb{R}^n))$ such that $\mathbf{a}(\Lambda) \geq \alpha$, any $\beta \in [\beta_0, 1]$ and any $m \in \mathbb{N}$,

$$0 < \frac{E \left[\sum_{k=1}^m \|U_k\|^r \mathcal{I}(\|U_k\| \leq \beta\theta(\mathbf{a}(\Lambda))) \right]}{E \left[\sum_{k=1}^m \|U_k\|^s \mathcal{I}(\|U_k\| \leq \beta\theta(\mathbf{a}(\Lambda))) \right]} \leq \frac{\Upsilon_r(\infty)}{\Upsilon_s(\infty)} + \eta.$$

If $\alpha \in [\alpha_0, \infty)$ and $\beta \in [\beta_0, 1]$, it follows from the latter inequality that, for every $\omega \in \Omega$ and every $m \in \mathbb{N}$,

$$\begin{aligned} & \left| \frac{\sum_{k=1}^m \|U_k(\omega)\|^r \mathcal{I}_{[0, \beta\theta(\mathbf{a}(\Lambda))]}(\|U_k(\omega)\|)}{\sum_{k=1}^m \|U_k(\omega)\|^s \mathcal{I}_{[0, \beta\theta(\mathbf{a}(\Lambda))]}(\|U_k(\omega)\|)} - \frac{E \left[\sum_{k=1}^m \|U_k\|^r \mathcal{I}(\|U_k\| \leq \beta\theta(\mathbf{a}(\Lambda))) \right]}{E \left[\sum_{k=1}^m \|U_k\|^s \mathcal{I}(\|U_k\| \leq \beta\theta(\mathbf{a}(\Lambda))) \right]} \right| \\ & \leq \left(\frac{\Upsilon_r(\infty)}{\Upsilon_s(\infty)} + \eta \right) \\ & \times \left| \frac{\sum_{k=1}^m \|U_k(\omega)\|^r \mathcal{I}_{[0, \beta\theta(\mathbf{a}(\Lambda))]}(\|U_k(\omega)\|) E \left[\sum_{k=1}^m \|U_k\|^s \mathcal{I}(\|U_k\| \leq \beta\theta(\mathbf{a}(\Lambda))) \right]}{\sum_{k=1}^m \|U_k(\omega)\|^s \mathcal{I}_{[0, \beta\theta(\mathbf{a}(\Lambda))]}(\|U_k(\omega)\|) E \left[\sum_{k=1}^m \|U_k\|^r \mathcal{I}(\|U_k\| \leq \beta\theta(\mathbf{a}(\Lambda))) \right]} - 1 \right|, \end{aligned}$$

where \mathcal{I}_K stands for the indicator function of a given set K . According to corollary B.3, for every ω in some measurable subset with measure 1 of Ω , the second factor in the right hand side of the inequality above can be made arbitrarily small for m large enough, which completes the proof for $\mathbf{a}(\cdot)$. The reader will then easily obtain the same result for $\|\cdot\|_\infty$ by mimicking the foregoing. \blacksquare

We now combine the almost everywhere convergence established by the latter lemma with the uniform convergence stated in proposition C.2 so as to complete the proof that $\sigma = 1$ satisfies (11) and (12). As a matter of fact, we prove (11) only, because the proof of (12) can be achieved in exactly the same way as below by replacing $\mathbf{a}(\cdot)$ with $\|\cdot\|_\infty$.

Let $\beta_0 \in (0, 1]$ and η be any positive real number. Let α_1 be a positive real value such that, for all $\alpha \in [\alpha_1, \infty)$, all $\Lambda \in \ell^\infty(\mathbb{N}, L^\nu(\Omega, \mathbb{R}^n))$ such that $\mathbf{a}(\Lambda) \geq \alpha$ and every $\beta \in [\beta_0, 1]$, (25) holds true with $\rho = \mathbf{a}(\Lambda)$.

We derive from proposition C.2 the existence of another positive real number α_2 such that, for any $\alpha \in [\alpha_2, \infty)$, any $\Lambda \in \ell^\infty(\mathbb{N}, L^\nu(\Omega, \mathbb{R}^n))$ with $\mathbf{a}(\Lambda) \geq \alpha$, any $\beta \in [\beta_0, 1]$ and any natural number m ,

$$\left| \frac{E \left[\sum_{k=1}^m \|U_k\|^r \mathcal{I}(\|U_k\| \leq \beta\theta(\mathbf{a}(\Lambda))) \right]}{E \left[\sum_{k=1}^m \|U_k\|^s \mathcal{I}(\|U_k\| \leq \beta\theta(\mathbf{a}(\Lambda))) \right]} - \frac{\Upsilon_r(\beta\theta(\mathbf{a}(\Lambda)))}{\Upsilon_s(\beta\theta(\mathbf{a}(\Lambda)))} \right| \leq \eta. \quad (26)$$

For any α larger than or equal to $\alpha_0 = \max(\alpha_1, \alpha_2)$, any Λ in $\ell^\infty(\mathbb{N}, L^\nu(\Omega, \mathbb{R}^n))$ such that $\mathbf{a}(\Lambda) \geq \alpha$ and any $\beta \in [\beta_0, 1]$, it then follows from (25) and (26)

that

$$\overline{\lim}_m \left| \frac{\sum_{k=1}^m \|U_k\|^r \mathcal{I}(\|U_k\| \leq \beta\theta(\mathbf{a}(\Lambda)))}{\sum_{k=1}^m \|U_k\|^s \mathcal{I}(\|U_k\| \leq \beta\theta(\mathbf{a}(\Lambda)))} - \frac{\Upsilon_r(\beta\theta(\mathbf{a}(\Lambda)))}{\Upsilon_s(\beta\theta(\mathbf{a}(\Lambda)))} \right| \leq \eta \text{ (a-s),}$$

or equivalently, that

$$\left\| \overline{\lim}_m \left| \frac{\sum_{k=1}^m \|U_k\|^r \mathcal{I}(\|U_k\| \leq \beta\theta(\mathbf{a}(\Lambda)))}{\sum_{k=1}^m \|U_k\|^s \mathcal{I}(\|U_k\| \leq \beta\theta(\mathbf{a}(\Lambda)))} - \frac{\Upsilon_r(\beta\theta(\mathbf{a}(\Lambda)))}{\Upsilon_s(\beta\theta(\mathbf{a}(\Lambda)))} \right| \right\|_{\infty} \leq \eta. \quad (27)$$

Summarizing, for any given real number $\beta_0 \in (0, 1]$ and any positive real number η , we have found a positive real number α_0 such that, for any $\alpha \in [\alpha_0, \infty)$, any $\Lambda \in \ell^\infty(\mathbb{N}, L^p(\Omega, \mathbb{R}^n))$ that satisfies $\mathbf{a}(\Lambda) \geq \alpha$ and any $\beta \in [\beta_0, 1]$, (27) holds true. This means that $\sigma = 1$ satisfies (11).

E Uniqueness of the solution

Suppose the existence of two positive real numbers $\sigma_1 \geq \sigma_2 > 0$ that both satisfy (11). For notational sake, put $\rho_i = \mathbf{a}(\Lambda)/\sigma_i$, $i \in \{1, 2\}$ and set $\beta_i = \theta(\rho_i)/\rho_i$, $i = 1, 2$. For every positive real number x , $0 < \Upsilon_r(x)/\Upsilon_s(x) < \infty$. Therefore, for any given pair $(m, \omega) \in \mathbb{N} \times \Omega$, we can write that

$$\begin{aligned} & \left| \sigma_1^{r-s} \frac{\Upsilon_r(\beta_2\theta(\rho_1))}{\Upsilon_s(\beta_2\theta(\rho_1))} - \sigma_2^{r-s} \frac{\Upsilon_r(\beta_1\theta(\rho_2))}{\Upsilon_s(\beta_1\theta(\rho_2))} \right| \leq \\ & \left| \frac{\sum_{k=1}^m \|U_k\|^r \mathcal{I}_{[0, \beta_2\sigma_1\theta(\rho_1)]}(\|U_k(\omega)\|)}{\sum_{k=1}^m \|U_k\|^s \mathcal{I}_{[0, \beta_2\sigma_1\theta(\rho_1)]}(\|U_k(\omega)\|)} - \sigma_1^{r-s} \frac{\Upsilon_r(\beta_2\theta(\rho_1))}{\Upsilon_s(\beta_2\theta(\rho_1))} \right| \\ & + \left| \frac{\sum_{k=1}^m \|U_k\|^r \mathcal{I}_{[0, \beta_2\sigma_1\theta(\rho_1)]}(\|U_k(\omega)\|)}{\sum_{k=1}^m \|U_k\|^s \mathcal{I}_{[0, \beta_2\sigma_1\theta(\rho_1)]}(\|U_k(\omega)\|)} - \sigma_2^{r-s} \frac{\Upsilon_r(\beta_1\theta(\rho_2))}{\Upsilon_s(\beta_1\theta(\rho_2))} \right|. \end{aligned}$$

We derive from this inequality that

$$\begin{aligned} & \left| \sigma_1^{r-s} \frac{\Upsilon_r(\beta_2\theta(\rho_1))}{\Upsilon_s(\beta_2\theta(\rho_1))} - \sigma_2^{r-s} \frac{\Upsilon_r(\beta_1\theta(\rho_2))}{\Upsilon_s(\beta_1\theta(\rho_2))} \right| \leq \\ & \left\| \left\| \frac{\overline{\lim}_m \left| \frac{\sum_{k=1}^m \|U_k\|^r \mathcal{I}(\|U_k\| \leq \beta_2\sigma_1\theta(\rho_1))}{\sum_{k=1}^m \|U_k\|^s \mathcal{I}(\|U_k\| \leq \beta_2\sigma_1\theta(\rho_1))} - \sigma_1^{r-s} \frac{\Upsilon_r(\beta_2\theta(\rho_1))}{\Upsilon_s(\beta_2\theta(\rho_1))} \right| \right\|_\infty \right. \\ & \left. + \left\| \frac{\overline{\lim}_m \left| \frac{\sum_{k=1}^m \|U_k\|^r \mathcal{I}(\|U_k\| \leq \beta_2\sigma_1\theta(\rho_1))}{\sum_{k=1}^m \|U_k\|^s \mathcal{I}(\|U_k\| \leq \beta_2\sigma_1\theta(\rho_1))} - \sigma_2^{r-s} \frac{\Upsilon_r(\beta_1\theta(\rho_2))}{\Upsilon_s(\beta_1\theta(\rho_2))} \right| \right\|_\infty \right\|. \end{aligned}$$

By taking into account that $\beta_1\sigma_2\theta(\rho_2) = \beta_2\sigma_1\theta(\rho_1)$, it follows from the inequality above and the definition of Δ_m given by (10) that

$$\begin{aligned} & \left| \sigma_1^{r-s} \frac{\Upsilon_r(\beta_2\theta(\rho_1))}{\Upsilon_s(\beta_2\theta(\rho_1))} - \sigma_2^{r-s} \frac{\Upsilon_r(\beta_1\theta(\rho_2))}{\Upsilon_s(\beta_1\theta(\rho_2))} \right| \\ & \leq \left\| \overline{\lim}_m \Delta_m(\sigma_1, \beta_2\theta(\rho_1)) \right\|_\infty + \left\| \overline{\lim}_m \Delta_m(\sigma_2, \beta_1\theta(\rho_2)) \right\|_\infty. \quad (28) \end{aligned}$$

Since σ_1 and σ_2 are both assumed to satisfy statement **(i)** of theorem 3.1, then, for every given positive real number η , there exists some $\alpha_0 \in (0, \infty)$ such that, for every real number $\alpha \geq \alpha_0$ and every $\Lambda \in \ell^\infty(\mathbb{N}, L^\nu(\Omega, \mathbb{R}^n))$ that satisfies $\mathbf{a}(\Lambda) \geq \alpha$,

$$\left\| \overline{\lim}_m \Delta_m(\sigma_1, \beta_2\theta(\rho_1)) \right\|_\infty \leq \eta \text{ and } \left\| \overline{\lim}_m \Delta_m(\sigma_2, \beta_1\theta(\rho_2)) \right\|_\infty \leq \eta. \quad (29)$$

Indeed, according to the definition of a thresholding function, $\mathbf{a}(\Lambda)$ can be chosen large enough so that β_1 and β_2 are both elements of $(C, 1]$. It then follows from (28) and (29) that

$$\lim_{a \rightarrow \infty} \left(\sigma_1^{r-s} \frac{\Upsilon_r(\theta(a/\sigma_1)\theta(a/\sigma_2)/(a/\sigma_2))}{\Upsilon_s(\theta(a/\sigma_1)\theta(a/\sigma_2)/(a/\sigma_2))} - \sigma_2^{r-s} \frac{\Upsilon_r(\theta(a/\sigma_1)\theta(a/\sigma_2)/(a/\sigma_1))}{\Upsilon_s(\theta(a/\sigma_1)\theta(a/\sigma_2)/(a/\sigma_1))} \right) = 0.$$

According to the asymptotic behaviour of θ , the limits of the respective coefficients of σ_1^{r-s} and σ_2^{r-s} in the equality above are both equal to $\Upsilon_r(\infty)/\Upsilon_s(\infty)$ when a tends to ∞ . Since this common limit is positive and finite, it follows that $\sigma_1 = \sigma_2$ since $r > s$.

References

- [1] M. Abramowitz, I. Stegun, *Handbook of Mathematical Functions, Ninth printing*, Dover Publications Inc., New York, 1972.
- [2] A. Amehraye, D. Pastor, S. Ben Jebara, On the Application of Recent Results in Statistical Decision and Estimation Theory to Perceptual Filtering of Noisy Speech Signals, *Proceedings of the second IEEE-EURASIP International Symposium on Control, Communications, and Signal Processing*, ISCCSP'06, Marrakech, Morocco, 2006.
- [3] D. Donoho, I. Johnstone, Ideal Spatial Adaptation by Wavelet Shrinkage, *Biometrika*, 81, 425-455, 1994.
- [4] T. Kailath, H. V. Poor, Detection of Stochastic Processes, *IEEE Transactions on Information Theory*, 44, 2230-2259, 1998.
- [5] S. M. Kay, *Fundamentals of Statistical Signal Processing, Volume II: Detection Theory*, Prentice Hall, Upper Saddle River, 1998.
- [6] G. Köthe, *Topological Vector Spaces-I*, Springer-Verlag (Grundlehren der Mathematischen Wissenschaften, 159), Berlin, 1969.
- [7] N. N. Lebedev, *Special Functions and their Applications*, Prentice-Hall, Englewood Cliffs, 1965.
- [8] G. Minkler, J. Minkler, *The Principles of Automatic Radar Detection in Clutter, CFAR*, Magellan Book Company, Baltimore, 1990.
- [9] D. Pastor, R. Gay, B. Groenenboom, A Sharp Upper-Bound for the Probability of Error of the Likelihood Ratio Test for Detecting Signals in White Gaussian Noise, *IEEE Transactions on Information Theory*, 48, pp 228-238, 2002.
- [10] D. Pastor, Estimating the standard deviation of some additive white Gaussian noise on the basis of non signal-free observations, *Proceedings of the 2006 International Conference on Acoustics, Speech and Signal Processing*, ICASSP'06, Toulouse, France, 2006.
- [11] H. V. Poor, *An Introduction to Signal Detection and Estimation, 2nd Edition*, Springer-Verlag, New York, 1994.
- [12] W. H. Press, S. A. Teukolsky, W. T. Vetterling, B. P. Flannery, *Numerical recipes in C, The Art of Scientific Computing, second edition*, University Press, Cambridge, 1992.

- [13] M. M. Rao, *Conditional Measures and Applications*, Marcel Dekker, New York, 1993.