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Abstract

In the Colonel Blotto game, two players with a fixed budget simultaneously allocate their resources across n battlefields to maximize the aggregate value gained from the battlefields where they have the higher allocation. Despite its long-standing history and important applicability, the Colonel Blotto game still lacks a complete Nash equilibrium characterization in its most general form—the non-constant-sum version with asymmetric players and heterogeneous battlefields. In this work, we propose a simply-constructed class of strategies—the independently uniform strategies—and we prove them to be approximate equilibria of the non-constant-sum Colonel Blotto game; moreover, we also characterize the approximation error according to the game's parameters. We also introduce an extension called the Lottery Blotto game, with stochastic winner-determination rules allowing more flexibility in modeling practical contexts. We prove that the proposed strategies are also approximate equilibria of the Lottery Blotto game.

Keywords: resource allocation games, epsilon-equilibrium, Colonel Blotto game, Lottery Blotto game, contest success function

1. INTRODUCTION

The *Colonel Blotto game* (henceforth, CB game) is one of the most well-known resource allocation games. Its description is very simple: two players, each having a fixed amount of resources (called budget), compete over a finite number of battlefields. Each battlefield is evaluated by the players with a certain value. Players simultaneously allocate their resources toward the battlefields and each player's payoff is her aggregate gains from all the battlefields. In each battlefield, the winner, who is simply the one that has the higher allocation, gains the corresponding value and the loser gains zero—this is called the winner-takes-all rule—; in battlefields with tie allocations, the value is shared between the players with a predetermined tie-breaking rule, e.g., sharing equally between them. Throughout its long-standing history since its first introduction by Borel (1921), the CB game has attracted interest from different research communities for its potential to elegantly model a large range of practical situations. One of its original applications is military logistics, see e.g., Gross (1950); Gross & Wagner (1950); but it is also used to model problems in politics (where political parties distribute their budgets to compete over voters), see e.g., Myerson (1993); Kovenock & Roberson (2012); Roberson (2006); in cybersecurity (where effort is distributed to attack/defend targets), see e.g., Chia (2012); Schwartz et al. (2014); in online advertising (where marketing campaigns allocate the time to broadcast ads to attract web users), see e.g., Masucci & Silva (2014, 2015); in telecommunication (where network service providers distribute and lease their spectrum to the users), see e.g., Hajimirsaadeghi & Mandayam (2017).

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In this paper, we consider the most general version of the *non-constant-sum CB game*, where the evaluations of the battlefields' values can be heterogeneous across battlefields and different between the two players; moreover, players' budgets can be asymmetric. Despite the long-standing history of the CB game, the characterization of the Nash equilibrium in this most general version remains an open question—even the existence of an equilibrium has not been proved or disproved. Our study also examines the Nash equilibrium but we take a different angle: instead of looking for an exact equilibrium, *our first contribution* is to propose a class of approximate equilibria of the non-constant-sum CB game, called the IU^{γ^*} strategies.[1] Importantly, we characterize the approximation error of this solution according to the games' parameters and show that it is negligible when the number of battlefields is sufficiently large (it quickly decreases as the number of battlefields increases). Note also that it is simple and efficient to construct IU^{γ^*} strategies even in large-scale problems. Our work extends the state-of-the-art where the only known results regarding the equilibria of the non-constant-sum CB game[2] are given by Kovenock & Roberson (2015). They provide a set of univariate marginal distributions (one per battlefield) that are the equilibrium marginals if they can be achieved with the budget constraints. They then indicate a sufficient condition for this to hold[3]—which is identical to that of Schwartz et al. (2014) for the constant-sum case—, that only covers a restricted range of games; and they also show a necessary condition where there is no equilibrium satisfying such a set of marginals.

The constant-sum CB game, where both players assign the same value to each battlefield, has been studied profoundly in the literature; however, even in this simpler version the equilibrium characterization is still not completely solved. When players have symmetric budgets, the equilibria are constructed by Borel & Ville (1991) in the game involving three battlefields and by Gross & Wagner (1950); Gross (1950) in the game containing any number of battlefields (see also Laslier (2002); Thomas (2017) for a modern presentation of this solution). For the constant-sum CB game with asymmetric budgets, equilibria characterization remains an open question in general; the exceptions are the following restricted cases: the games with only two battlefields (Macdonell & Mastronardi (2015)), the games with any number of battlefields but homogeneous values (Roberson (2006)), and the games where there exists a sufficient number of battlefields of each possible value (Schwartz et al. (2014)). In our model of the non-constant-sum CB game, we make no assumption on the players' symmetry nor on the battlefields homogeneity; therefore, our results for the non-constant-sum CB game can be trivially adapted to the constant-sum game with the most general configuration of parameters, i.e., the IU^{γ^*} strategy is also an approximate equilibrium. Moreover, we show an additional result that the IU^{γ^*} strategy is also an approximate max-min strategy of the constant-sum CB game (with the same approximation error). It is also worth mentioning that a few works have considered extensions to non-constant-sum of the CB game, though with a significantly different flavor. In particular, Hortala-Vallve & Llorente-Saguer (2012) consider the discrete CB game and identify conditions under which a pure Nash equilibrium exists; and Kvasov (2007); Roberson & Kvasov (2012) consider the relaxation of the use-it-or-lose-it rule that changes the payoffs.

In practice, there exists situations where the winner-takes-all rule of the CB game is too restrictive. In order to model these situations with more flexibility, in this work, we introduce and study an extension of the non-constant-sum CB game, called the *Lottery Blotto game* (henceforth, LB game),[4] where each player only gains a part of her value in each battlefield. Alternatively, one can interpret the LB game as a version of the CB game in which each player wins a battlefields' value with a certain probability depending on players' allocations on that battlefield and this probability can be non-zero even for the player with smaller allocation. Some examples where the LB game model may prove to be useful are online advertising competitions, political contests for voters' attention, research and development activities, radio-wave transmission with noises, etc. We formulate the LB game by presenting the players' payoffs based

[1] We explain the name in Section 4.1.

[2] It is called the generalized Colonel Blotto game by Kovenock & Roberson (2015).

[3] The set of battlefields are partitioned such that two battlefields are in the same partition if they have the same (normalized) values; the sufficient condition on the attainability of equilibria requires a sufficient number of battlefields in each partition.

[4] Note that the LB game is also a non-constant-sum game; however, to lighten the notation, hereinafter, we do not highlight this and only call it the LB game in places with no ambiguity.

on the concept of contest success function (henceforth, CSF). CSFs, studied profoundly in the rent-seeking literature—see e.g., Skaperdas (1996); Corchón (2007)—, are functions that take the players' allocations as inputs and output the probability of winning a battlefield. The definition of CSF that we adopt (see Section 2.2) also includes the winner-takes-all rule as a special case so that the CB game is a particular case of the LB game. Similar to the non-constant-sum CB game, the equilibrium characterization of the LB game is an open question, except for several particular instances. *Our second main contribution* is to prove that the IU^{γ^*} strategy is also an approximate equilibrium of the LB game with an approximation error that decreases quickly as the number of battlefields increases and the corresponding CSFs converge pointwise to that of the CB game.

The LB game that we propose as an extension of the CB game with a general CSF, has not been formally defined in previous works; but several particular instances have been considered. Friedman (1958) investigated the pure equilibrium of the (constant-sum) LB game where players' gains in each battlefield follows the Tullock function (termed by Tullock (2001)).[5][6] Osório (2013) studied an extension of this model with a generalization of the Tullock function (coincidentally, it is also called there by the Lottery Blotto game); however, only numerically computed approximate-results of the equilibrium are proposed and no tractable close-form solution is provided in the general cases where battlefields' values are asymmetric across players. The CSFs considered in these works belong to one specific class that we call the ratio-form CSFs (see Section 2.2 for a formal definition). Note that our result works for any LB game with any general CSF; therefore, they can be applied to the LB game with ratio-form CSFs. As an illustration, we analyze our IU^{γ^*} strategy in the LB games with two of the most well-known cases of ratio-form CSFs, the power form and the logit form (see Section 2.2 for more details)—in this case we obtain more precise results on the convergence of the error.

We note finally that a strategy construction similar to the IU^{γ^*} strategy can be found in Vu et al. (2018) for the (constant-sum) discrete CB game (i.e., where the budgets and every allocation are required to be integers) with asymmetric budgets and heterogeneous battlefields. Due to the discrete condition, their analysis has essential differences to our work; particularly, their asymptotic results involve a double limits of the number of battlefields and the ratio of players' budgets; moreover, the convergence of the players' payoffs in their work does not have the difficulties of continuous allocation encountered in our work. Finally, they do not consider the non-constant-sum CB game and the extension to the LB game.

The remainder of this paper is organized as follows. Section 2 introduces the formulations of the non-constant-sum CB game and the LB game. Although the LB game model is essentially more general; we first focus on the CB game due to the fact that the CB game is a more classical game and our analysis for the LB game also depends on our results for the CB game. After providing some preliminary results for the CB game in Section 3, we propose the IU^{γ^*} strategy in Section 4 and state the result that any IU^{γ^*} strategy is an approximate equilibrium of the non-constant-sum CB game. In Section 5, we claim the results that the IU^{γ^*} strategy is also an approximate equilibrium of the LB game. Finally, the detailed proofs of all lemmas and theorems are given in Appendix.

Throughout the paper, we use bold symbols (e.g., $\boldsymbol{x}$) to denote vectors and subscript indices to denote its elements (e.g., $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$). The notation $[n]$ denotes the set $\{1, 2, \dots, n\}$, for any $n \in \mathbb{N} \backslash \{0\}$. We often use the letter p to denote a player and use $-p$ to indicate her opponent in the games; $R^n_{\geq 0}$ denotes the set of all n-tuples whose elements are non-negative ($R_{\geq 0} := R^1_{\geq 0}$). We denote the Euler's number by e. For any random variable X, we use F_X to denote its corresponding cumulative density function (abbreviated by CDF). Finally, we use $\mathbb{P}(E)$ to denote the probability that an event E happens and $\mathbb{E}X$ to denote the expectation of a random variable X. A table of notations that are used in this work (Table A.2) is given in Appendix A.

[5] That is the LB game where two players, called A and B, commonly evaluate each battlefield i with a value w_i; if players allocates x^A_i, x^B_i to battlefield i then player A gains $x^A_i w_i/(x^A_i + x^B_i)$ and player B gains $x^B_i w_i/(x^A_i + x^B_i)$ from this battlefield.

[6] A similar function to define the winning probability is also used by Rinott et al. (2012) to study a variant of the CB game involving sequential tournaments.

2. GAMES FORMULATION

In this section, we define the two games that are our main focus: in Section 2.1, we introduce the non-constant-sum Colonel Blotto game; in Section 2.2, we present the Lottery Blotto game, as an extension of the Colonel Blotto game.

2.1. The Colonel Blotto game

We consider the following one-shot, complete information game between two players A and B. Each player has a fixed amount of resources (called the *budgets*), denoted X_A and X_B, respectively. Without loss of generality, we assume that $0 < X_A \leq X_B$. Players simultaneously allocate their resources across n *battlefields* ($n \geq 3$). Each battlefield $i \in [n]$ is embedded with two parameters $w_i^A, w_i^B > 0$, corresponding to the *values* at which player A and player B respectively assess this battlefield. A *pure strategy* of player $p \in \{A, B\}$ is a vector $\boldsymbol{x}^p = (x_i^p)_{i \in [n]} \in \mathbb{R}^n_{\geq 0}$ that satisfies the budget constraint $\sum_{i=1}^n x_i^p \leq X_p$. In each battlefield i, when player p allocates strictly more than her opponent, she gains completely her embedded values w_i^p while the opponent gains 0. In case of a tie, i.e., if $x_i^A = x_i^B$, then player A receives αw_i^A and player B receives $(1-\alpha) w_i^B$, where $\alpha \in [0,1]$ is a fixed parameter. Each player's payoff is the summation of values she gains from all battlefields; formally, for any pure strategy profile $(\boldsymbol{x}^A, \boldsymbol{x}^B)$, the payoffs of players A and B are $\Pi^A(\boldsymbol{x}^A, \boldsymbol{x}^B) = \sum_{i=1}^n w_i^A \cdot \beta_A\left(x_i^A, x_i^B\right)$ and $\Pi^B(\boldsymbol{x}^A, \boldsymbol{x}^B) = \sum_{i=1}^n w_i^B \cdot \beta_B\left(x_i^A, x_i^B\right)$ respectively; here, β_A and β_B (henceforth, we called them the Blotto functions) are functions defined as follows:

$$\beta_A(x,y) = \begin{cases} 1 \text{ , if } x > y \\ \alpha \text{ , if } x = y \\ 0 \text{ , if } x < y \end{cases} \quad \text{and} \quad \beta_B(x,y) = \begin{cases} 1 \text{ , if } y > x \\ 1-\alpha \text{ , if } y = x \\ 0 \text{ , if } y < x \end{cases} \text{, for all } x, y \in \mathbb{R}_{\geq 0}. \tag{2.1}$$

Definition 2.1. ***A non-constant-sum Colonel Blotto game****, denoted by $\mathcal{CB}_n$, is the game defined above; in particular, the action set of player $p \in \{A, B\}$ is $\{\boldsymbol{x}^p \in \mathbb{R}^n_{\geq 0} : \sum_{i=1}^n x_i^p \leq X_p\}$ and her payoff is $\Pi^p(\boldsymbol{x}^A, \boldsymbol{x}^B)$ when players A and B play the pure strategies $\boldsymbol{x}^A$ and $\boldsymbol{x}^B$ respectively.*

To lighten the notation, we only include the subscript n—the number of battlefields—in the notation $\mathcal{CB}_n$ and omit the other parameters; in particular the values X_A, X_B, α and w_i^A, w_i^B for $i \in [n]$. Hereinafter, in places with no ambiguity, we drop the term non-constant-sum and simply address the game $\mathcal{CB}_n$ as the Colonel Blotto game. In this game, a *mixed strategy* is a joint distribution on the allocations of all battlefields, such that any drawn pure strategy of a player is an n-tuple that satisfies her budget constraint. We reuse the notations $\Pi^A(s_A, s_B)$ and $\Pi^B(s_A, s_B)$ to denote the payoffs of players A and B when they play the mixed strategies s_A and s_B, respectively. Note that the definition of $\mathcal{CB}_n$ above allows asymmetry in players' budgets and heterogeneity in battlefields values; moreover, it allows battlefield values to differ between the two players. Furthermore, the defined payoff functions can be understood as if we randomly break the tie (if it happens) such that player A wins battlefield i with probability α while player B wins it with probability $(1-\alpha)$. This includes all the classical tie-breaking rules considered in the literature; for instance, the rule of giving the whole value to player B used by Roberson (2006); Schwartz et al. (2014) corresponds to $\alpha = 0$; the 50-50 rule used by Kovenock & Roberson (2015); Ahmadinejad et al. (2016); Behnezhad et al. (2017) corresponds to $\alpha = 1/2$.

In this paper, we also often work with the *normalized values* of the battlefields defined as $v_i^A := w_i^A / W_A$ and $v_i^B := w_i^B / W_B$, where $W_A := \sum_{j=1}^n w_j^A$ and $W_B := \sum_{j=1}^n w_j^B$ for $i \in [n]$. We trivially observe that $v_i^p \in [0,1]$ for all i and that $\sum_{j=1}^n v_j^p = 1$. Most of our analysis relies on an additional assumption that the battlefields' values are bounded away from zero and infinity (see the Assumption ($A0$) below). This is a fairly mild assumption that is satisfied in most of (if not all) practical applications.

$$(A0) \qquad \exists \underline{w}, \bar{w} > 0 : \underline{w} \leq w_i^p \leq \bar{w}, \forall i \in [n], \forall p \in \{A, B\}$$

As a direct consequence, the normalized values satisfy

$$\frac{\underline{w}}{n\bar{w}} \leq v_i^p \leq \frac{\bar{w}}{n\underline{w}}, \quad \forall i \in [n], \forall p \in \{A, B\}. \tag{2.2}$$

Finally, we note that most works in the literature (the only exception, in our knowledge, being the work of Kovenock & Roberson (2015)) focus only on the *constant-sum* Colonel Blotto game where players have the same evaluations on battlefields' values. The non-constant-sum game $\mathcal{CB}_n$ given in Definition 2.1 is more general; hence all our results for $\mathcal{CB}_n$ can be straightforwardly applied to this constant-sum version as well. However, for the purpose of comparing with the literature and because we can show stronger results in this special case, it is useful to also formally define the constant-sum game variant as follows.

Definition 2.2. ***A constant-sum Colonel Blotto game****, denoted by $\mathcal{CB}^c_n$, is a game that has the same formulation as the game $\mathcal{CB}_n$ but with the additional condition that $w^A_i = w^B_i, \forall i \in [n]$.*

As a trivial corollary of this additional condition, in $\mathcal{CB}^c_n$, players also have common normalized valuation on battlefields, i.e., $v^A_i = v^B_i$ for all $i \in [n]$ and the players' maximum payoffs are equal, i.e., $W_A = W_B$.

2.2. The contest success functions and the Lottery Blotto game

In this section, we present a new game, the Lottery Blotto game, that extends the model of the Colonel Blotto game. This new game is based on the notion of contest success functions (CSFs), that we introduce below before defining the game model.

Contest success functions (CSFs) are functions that quantify the winning probability in *contests* (also called *rent-seeking* competitions) where several players compete for a single prize by exerting resources/efforts. CSFs can be defined for any number of players (see e.g., a general definition by Skaperdas (1996)), but in this work, we focus only on the case of two players.

Definition 2.3. *$\zeta_A : \mathbb{R}^2_{\geq 0} \rightarrow \mathbb{R}$ and $\zeta_B : \mathbb{R}^2_{\geq 0} \rightarrow \mathbb{R}$ is a pair of contest success functions (**CSFs**) if and only if the following two conditions are satisfied:*

($C1$) *$\zeta_A(x, y), \zeta_B(x, y) \geq 0$ and $\zeta_A(x, y) + \zeta_B(x, y) = 1$, $\forall x, y \geq 0$.*

($C2$) *$\zeta_A(x, y)$ (resp. $\zeta_B(x, y)$) is non-decreasing in x (resp. in y) and non-increasing in y (resp. in x).*

Intuitively, the function ζ_A (resp. ζ_B) maps any pair of players' invested resources to the probability that player A (resp. player B) wins the prize. Condition ($C1$) indicates that the outputs of any pair of the CSFs always satisfy the condition of a probability distribution. On the other hand, Condition ($C2$) states that a player's winning probability increases (or at least stays the same) when she increases her effort and decreases (or at least stays the same) when her opponent increases her effort. We note that Definition 2.3 allows a more general definitions of the CSFs (in two-player cases) compared to the definition given by Skaperdas (1996); Hirshleifer (1989); Clark & Riis (1998) that contains other assumptions.[7] While many of the CSFs considered in the literature are continuous functions, we do not include continuity in Definition 2.3 to keep the generality. Importantly, the Blotto functions β_A, β_B of the game $\mathcal{CB}_n$ (i.e., the winner-takes-all rule) satisfy Conditions ($C1$) and ($C2$), hence β_A, β_B are CSFs. Besides these functions, some examples of other CSFs considered in the literature are:

(a) $\zeta_A(x, y) = x/(x + y)$ and $\zeta_B(x, y) = y/(x + y)$, proposed by Tullock (2001);
(b) $\zeta_A(x, y) = \max\left\{\min\left\{\frac{1}{2} + C(x-y), 1\right\}, 0\right\}$ and $\zeta_B(x, y) = 1 - \zeta_A(x, y)$, proposed by Che & Gale (2000), where $C > 0$ is a fixed parameter;
(c) $\zeta_A(x, y) = \frac{1}{2} - \frac{y-x}{2y}$ if $x \leq y$ and $\zeta_A(x, y) = \frac{1}{2} + \frac{x-y}{2x}$ if $x \geq y$; and $\zeta_B(x, y) = 1 - \zeta_A(x, y)$, proposed by Alcalde & Dahm (2007).

Building on the notion of CSFs and the Colonel Blotto game, we now define a new game model based on the following idea: in a game $\mathcal{CB}_n$, we view each battlefield as a contest between players where the prize is the battlefield's value and players' effort correspond to their allocations; by doing this, each pair of CSFs defines an instance of a new game where the probability of winning a battlefield follows them accordingly.

[7] For example, Skaperdas (1996) defines ζ_A, ζ_B with an axiom of anonymity; they also require that any player who puts a strictly positive amount of resources has a strictly positive probability of winning the prize; Clark & Riis (1998) considers the CSFs additionally satisfying the Choice Axiom. These are technical conditions needed for proving their results and we omit them here lest they unnecessarily limit our scope of study.

Definition 2.4. *Let $\zeta = (\zeta_A, \zeta_B)$ be a pair of CSFs.* ***A Lottery Blotto game*** *with n battlefields, denoted $\mathcal{LB}_n(\zeta)$, is the game with the same players A and B and the same strategy sets as in $\mathcal{CB}_n$; but where payoffs are given, for any pure strategy profile $(\boldsymbol{x}^A, \boldsymbol{x}^B)$, by*

$$\Pi^A_\zeta(\boldsymbol{x}^A, \boldsymbol{x}^B) = \sum\nolimits_{i=1}^{n} w_i^A \cdot \zeta_A\left(x_i^A, x_i^B\right) \qquad and \qquad \Pi^B_\zeta(\boldsymbol{x}^A, \boldsymbol{x}^B) = \sum\nolimits_{i=1}^{n} w_i^B \cdot \zeta_B\left(x_i^A, x_i^B\right).$$

The Lottery Blotto game model is more flexible than that of the Colonel Blotto game, as it allows choosing the CSFs that define the players' payoffs for each specific practical situation. Intuitively, the players' payoffs in a Lottery Blotto game can be seen as the expected payoffs in the Colonel Blotto game with respect to the following random process determining the winner in any battlefield i: player A wins with probability $\zeta_A(x_i^A, x_i^B)$ and player B wins with probability $\zeta_B(x_i^A, x_i^B)$ if they allocate x_i^A and x_i^B respectively. Similar to the game $\mathcal{CB}_n$, players' payoffs in the $\mathcal{LB}_n$ game are also monotonic with respect to the allocations in a battlefield (due to Condition $(C2)$).

Besides the Lottery Blotto game with the generally defined CSFs, we additionally consider the games corresponding to the CSFs that belong to a special class called the *ratio-form* CSFs. These are the CSFs that are studied the most profoundly in the literature. We will use the games with these ratio-form CSFs to illustrate the results obtained in the Lottery Blotto game.

Definition 2.5. *CSFs $\zeta_A, \zeta_B : \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$ are called* ***ratio-form CSFs*** *if they have the form:*

$$\zeta_A(x, y) = \frac{\eta(x)}{\eta(x) + \kappa(y)} \quad and \quad \zeta_B(x, y) = \frac{\kappa(y)}{\eta(x) + \kappa(y)},$$

where $\eta, \kappa : \mathbb{R}_{\geq 0} \to \mathbb{R}$ are non-negative functions such that ζ_A and ζ_B satisfy Conditions $(C1)$ and $(C2)$.

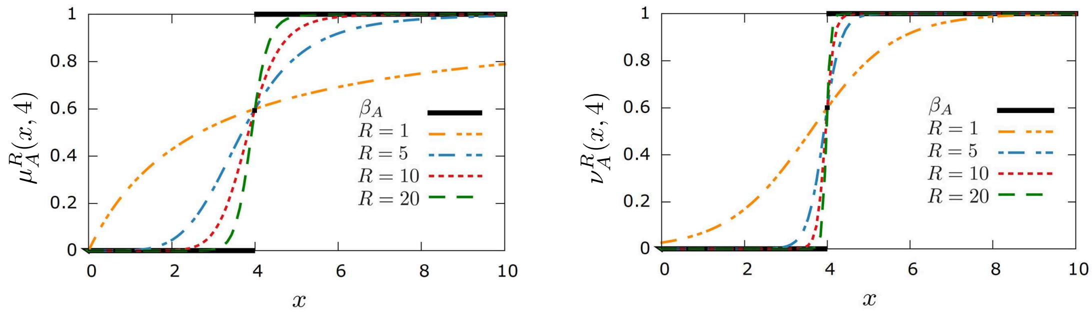
Two classical ratio-form CSFs in the literature (see e.g., Hillman & Riley (1989); Corchón & Dahm (2010)) are the power form where $\eta(z) = \kappa(z) = z^R, \forall z \geq 0$ and the logit form where $\eta(z) = \kappa(z) = e^{Rz}, \forall z \geq 0$, where $R > 0$ is a parameter chosen a priori. These functions yield the sharing 50-50 tie-breaking rule, i.e., $\zeta_A(x, y) = \zeta_B(x, y) = 1/2$ if $x = y$. We define in Table 1 the generalized versions of these ratio-form CSFs using the parameter $\alpha \in (0, 1)$ that leads to the general tie-breaking rule as in the Colonel Blotto game $\mathcal{CB}_n$.[8] Henceforth, we use the terms power and logit form to indicate the CSFs μ^R and ν^R with this generalization. It is trivial to verify that both pairs (μ^R_A, μ^R_B) and (ν^R_A, ν^R_B) satisfy the Conditions $(C1)$ and $(C2)$. An important remark is that both the power and logit form CSFs converge pointwise toward the Blotto functions β_A, β_B as R tends to infinity (see Section 5.2 for more details). This convergence can be observed in Figure 1 that illustrates several instances of the ratio-form CSFs in comparison with the Blotto functions.

Table 1: Power and logit form CSFs with generalized tie-breaking rule ($\alpha \in (0, 1)$).

Name	Notation	If $x^2 + y^2 > 0$	If $x = y = 0$
Power form	$\mu^R := (\mu^R_A, \mu^R_B)$	$\mu^R_A(x, y) = \frac{\alpha x^R}{\alpha x^R + (1-\alpha) y^R}$; $\mu^R_B(x, y) = \frac{(1-\alpha) y^R}{\alpha x^R + (1-\alpha) y^R}$	$\mu^R_A(x, y) = \alpha$ $\mu^R_B(x, y) = 1 - \alpha$
Logit form	$\nu^R := (\nu^R_A, \nu^R_B)$	$\nu^R_A(x, y) = \frac{\alpha e^{xR}}{\alpha e^{xR} + (1-\alpha) e^{yR}}$; $\nu^R_B(x, y) = \frac{(1-\alpha) e^{yR}}{\alpha e^{xR} + (1-\alpha) e^{yR}}$	$\nu^R_A(x, y) = \alpha$ $\nu^R_B(x, y) = 1 - \alpha$

Throughout the paper, to refer to a Colonel Blotto game $\mathcal{CB}_n$ that has the same parameters n, X_A, X_B, w_i^A, $w_i^B, \forall i \in [n]$ as a Lottery Blotto game $\mathcal{LB}_n$, we call $\mathcal{CB}_n$ the *corresponding game* of $\mathcal{LB}_n$ and vice versa. Note that, to derive our results for $\mathcal{LB}_n$, we will also use Assumption $(A0)$ introduced above.

[8]When $\alpha = 1/2$, the CSFs μ^R and ν^R match the classical power form and logit form CSFs. Note that we exclude the cases where $\alpha = 0$ or $\alpha = 1$ since these are the trivial cases: in the corresponding Lottery Blotto game, a player, say $p \in \{A, B\}$, always has the payoff W_p while player $-p$'s payoff is always zero regardless how they allocate their resources.



(a) Power form CSF, $X_A = 10, \alpha = 0.6$.

(b) Logit CSF, $X_A = 10, \alpha = 0.6$.

Figure 1: Examples of power form and logit form CSFs in comparison with the Blotto functions.

3. PRELIMINARIES

In this section, we briefly review some results from the literature that are useful for our analyses of the Colonel Blotto games and the Lottery Blotto games; and we show new bounds on the involved parameters, based on Assumption $(A0)$, that are essential for the asymptotic analysis in the next sections.

The Nash equilibrium characterization of the non-constant-sum Colonel Blotto game still remains an open question. However, under certain assumptions, the set of *univariate marginal distributions* of players in an equilibrium of the game $\mathcal{CB}_n$ is well-known. To see this, observe that we can break down the problem of finding the best-response of a player against a fixed strategy of her opponent into solving n all-pay auctions involving the Lagrange multipliers corresponding to the budget constraints (see e.g., Kovenock & Roberson (2015); Roberson (2006); Schwartz et al. (2014)). The equilibrium of two-player all-pay auctions is well-known and can be expressed as uniform-type distributions (see e.g., Hillman & Riley (1989); Baye et al. (1996)). In other words, these uniform-type distributions are the set of Nash equilibrium univariate marginals of the Colonel Blotto game. We have a Nash equilibrium if we can construct a joint distribution with these univariate marginals such that its realizations always satisfy the budget constraints. However, as mentioned in Section 1, the existence of such a construction is known only for some special cases and remains unknown in the general setting of $\mathcal{CB}_n$. Note that if we consider a relaxation of the game that only requires the budget constraints to be hold in expectation (this relaxation is called the General Lotto game by Kovenock & Roberson (2015)), an equilibrium is to independently draw allocations from the uniform-type distributions.

Although in this work we do not attempt to solve the open question of the equilibria characterization of the non-constant-sum Colonel Blotto game, we still use several preliminary results from this approach to construct an approximate equilibrium of the games. We present these results below, using a notation similar to Kovenock & Roberson (2015).

For each instance of the game $\mathcal{CB}_n$ (and of the game $\mathcal{LB}_n$), for any $\gamma \in (0, \infty)$, we define

$$\Omega_A(\gamma) := \left\{ i \in [n] : v_i^A / v_i^B > \gamma \right\},$$

and consider the following equation with the variable γ (other coefficients are the parameters of $\mathcal{CB}_n$ and $\mathcal{LB}_n$):

$$\frac{X_B \gamma}{X_A} = \frac{\gamma^2 \sum_{i \in \Omega_A(\gamma)} \frac{(v_i^B)^2}{v_i^A} + \sum_{i \notin \Omega_A(\gamma)} v_i^A}{\sum_{i \in \Omega_A(\gamma)} v_i^B + \frac{1}{\gamma^2} \sum_{i \notin \Omega_A(\gamma)} \frac{(v_i^A)^2}{v_i^B}}. \tag{3.1}$$

Let us denote by $\mathcal{S}_n$ *the set containing all positive solutions* of Equation (3.1) corresponding to the game $\mathcal{CB}_n$ (or $\mathcal{LB}_n$).[9] Based on Brouwer's fixed-point theorem, the following lemma is proved by Kovenock & Roberson (2015).

[9]Note that (3.1) and $\mathcal{S}_n$ also depend on other parameters of the game $\mathcal{CB}_n$ but we use the notation with only the subscript n and omit other parameters to lighten the notation.

Lemma 3.1. *For any game $\mathcal{CB}_n$ (or $\mathcal{LB}_n$), Equation* (3.1) *has at least one positive solution; i.e., $\mathcal{S}_n \neq \emptyset$.*

Equation (3.1) may have more than one solution and it can be solved in $\mathcal{O}(n\ln(n))$ time.[10] Now, corresponding to each positive solution $\gamma^* \in \mathcal{S}_n$, we define two constants,[11] namely λ^*_A and λ^*_B as follows:

$$\lambda^*_A := \frac{(\gamma^*)^2}{2X_B} \sum_{i\in\Omega_A(\gamma^*)} \frac{\left(v_i^B\right)^2}{v_i^A} + \frac{1}{2X_B} \sum_{i\notin\Omega_A(\gamma^*)} v_i^A, \tag{3.2}$$

$$\lambda^*_B := \frac{1}{2X_A} \sum_{i\in\Omega_A(\gamma^*)} v_i^B + \frac{1}{2(\gamma^*)^2 X_A} \sum_{i\notin\Omega_A(\gamma^*)} \frac{\left(v_i^A\right)^2}{v_i^B}. \tag{3.3}$$

Note importantly that we have $\gamma^* = \lambda^*_A/\lambda^*_B$ (see Lemma A1 in Appendix A for a proof). We now use these constants λ^*_A and λ^*_B to define several important distributions.

Definition 3.2. *Given a game $\mathcal{CB}_n$ (or $\mathcal{LB}_n$), for any $\gamma^* \in \mathcal{S}_n$ and the corresponding constants λ^*_A, λ^*_B, we define the following random variables and distributions,*[12] *for each $i \in [n]$:*

(a) *If $i \in \Omega_A(\gamma^*)$ (i.e., $\frac{v_i^A}{\lambda^*_A} > \frac{v_i^B}{\lambda^*_B}$), we define $A^S_{\gamma^*,i}$ and $B^W_{\gamma^*,i}$ as the random variables whose distributions are*

$$F_{A^S_{\gamma^*,i}}(x) := \frac{x\lambda^*_B}{v_i^B}, \forall x \in \left[0, \frac{v_i^B}{\lambda^*_B}\right], \tag{3.4}$$

$$F_{B^W_{\gamma^*,i}}(x) := \frac{\frac{v_i^A}{\lambda^*_A} - \frac{v_i^B}{\lambda^*_B}}{\frac{v_i^A}{\lambda^*_A}} + \frac{x\lambda^*_A}{v_i^A}, \forall x \in \left[0, \frac{v_i^B}{\lambda^*_B}\right]. \tag{3.5}$$

(b) *If $i \notin \Omega_A(\gamma^*)$ (i.e., $\frac{v_i^A}{\lambda^*_A} \leq \frac{v_i^B}{\lambda^*_B}$), we define $A^W_{\gamma^*,i}$ and $B^S_{\gamma^*,i}$ as the random variables whose distributions are*

$$F_{A^W_{\gamma^*,i}}(x) := \frac{\frac{v_i^B}{\lambda^*_B} - \frac{v_i^A}{\lambda^*_A}}{\frac{v_i^B}{\lambda^*_B}} + \frac{x\lambda^*_B}{v_i^B}, \forall x \in \left[0, \frac{v_i^A}{\lambda^*_A}\right], \tag{3.6}$$

$$F_{B^S_{\gamma^*,i}}(x) := \frac{x\lambda^*_A}{v_i^A}, \forall x \in \left[0, \frac{v_i^A}{\lambda^*_A}\right]. \tag{3.7}$$

To lighten the notation, hereinafter, we often commonly denote these random variables as follows (the corresponding distributions are denoted by $F_{A^*_i}$ and $F_{B^*_i}$):

$$A^*_i := \begin{cases} A^S_{\gamma^*,i} & \text{if } i \in \Omega_A(\gamma^*) \\ A^W_{\gamma^*,i} & \text{if } i \notin \Omega_A(\gamma^*) \end{cases} \quad \text{and} \quad B^*_i := \begin{cases} B^S_{\gamma^*,i} & \text{if } i \notin \Omega_A(\gamma^*) \\ B^W_{\gamma^*,i} & \text{if } i \in \Omega_A(\gamma^*) \end{cases}. \tag{3.8}$$

We term these distributions the *uniform-type distributions*: $F_{A^S_{\gamma^*,i}}(x)$ is the continuous uniform distribution on $\left[0, v_i^B/\lambda^*_B\right]$ and $F_{B^W_{\gamma^*,i}}(x)$ is the distribution placing a positive mass $\left(\frac{v_i^A}{\lambda^*_A} - \frac{v_i^B}{\lambda^*_B}\right)\Big/\frac{v_i^A}{\lambda^*_A}$ at 0 and

[10] To solve this equation, we first sort out all ratios v_i^A/v_i^B in a non-decreasing order (which can be done in $\mathcal{O}(n\ln(n))$), then there are three possible cases: $\gamma^* < \min\{v_i^A/v_i^B, i \in [n]\}$ or $\gamma^* \geq \max\{v_i^A/v_i^B, i \in [n]\}$ or $\exists j : \gamma^* \in \left[v_j^A/v_j^B, v_{j+1}^A/v_{j+1}^B\right)$. In all of these cases, Equation (3.1) becomes a cubic equation; therefore, it can be solved algebraically.

[11] These constants are the Lagrange multipliers corresponding to the budget constraints in finding players' best-response; see Kovenock & Roberson (2015) for more details.

[12] Here, the superscripts S and W, standing for strong and weak, are used to emphasize the intuition on players' incentive to play according to these distributions in the CB games: if $i \in \Omega_A(\gamma^*) := \left\{i : v_i^A/\lambda^*_A > v_i^B/\lambda^*_B\right\}$, player A has a "stronger" incentive to win battlefield i and player B has a "weaker" incentive; if $i \notin \Omega_A(\gamma^*)$, the roles of players are exchanged.

uniformly distributing the remaining mass on $\left(0, v_i^B/\lambda_B^*\right]$; similarly, $F_{B^S_{\gamma^*,i}}$ is the uniform distribution on $[0, v_i^A/\lambda_A^*]$ and $F_{B^W_{\gamma^*,i}}$ is uniform on $(0, v_i^A/\lambda_A^*]$ with a positive mass at 0.

If player A can construct and plays a mixed strategy such that her sampled allocation to any battlefield $i \in [n]$ follows the distribution $F_{A_i^*}$, it is optimal for player B to play such that her allocation to i follows $F_{B_i^*}$ (if it is possible) and vice versa. We will revisit this result (with more details) in Section 4 and in Lemma B3 in Appendix B. Therefore, under the condition that player A and player B can respectively construct joint distributions of $F_{A_i^*}, \forall i \in [n]$ and $F_{B_i^*}, \forall i \in [n]$ such that their sampled allocations satisfy the budget constraint, these mixed strategies yield an equilibrium of the game $\mathcal{CB}_n$. However, in general, that condition does not always hold. For instance, although A_i^* and B_i^* have finite upper-bounds,[13] we note that among these random variables, some may (with strictly positive probability) exceed the budgets X_A, X_B for certain parameters' configuration of the game; therefore, allocating according to $F_{A_i^*}, F_{B_i^*}$ may violate the budget constraints and it is then trivial that there exists no equilibrium yielding $F_{A_i^*}, F_{B_i^*}, \forall i \in [n]$ as marginals. On the other hand, given fixed X_A, X_B, if n is large enough, we can guarantee that A_i^*, B_i^* do not exceed the budgets for each i; however, even in this case, we still do not have guarantees on the summation of allocations sampled from all $A_i^*, B_i^*, i \in [n]$, i.e., it is still unknown if there exists an equilibrium yielding $F_{A_i^*}, F_{B_i^*}, i \in [n]$ as marginals. Note importantly that the budget-constraints violation of A_i^*, B_i^* does not affect our work and our results hold for any parameters' configuration of the games.

Finally, under Assumption $(A0)$, we obtain a novel result, presented below as Proposition 3.3, stating that the parameters γ^*, λ_A^* and λ_B^* are all bounded. The main results of this work are based on asymptotic analyses in terms of the number of battlefields of the games; therefore, it is noteworthy that the bounds of these parameters do not depend on n. The proof of this proposition is given in Appendix A. From the proof of Proposition 3.3, we observe that as the ratios $\bar{w}/\underline{w}$ and (or) X_B/X_A increase, the ranges in which γ^* and λ_A^*, λ_B^* belong to also become larger (i.e., the ratios $\bar{\gamma}/\underline{\gamma}$ and $\bar{\lambda}/\underline{\lambda}$ also increase).

Proposition 3.3. *Under Assumption $(A0)$, for any game $\mathcal{CB}_n$ (or $\mathcal{LB}_n$), there exist constants $\underline{\gamma}, \bar{\gamma}, \underline{\lambda}, \bar{\lambda} > 0$, that do not depend on n, such that for any $\gamma^* \in \mathcal{S}_n$ and its corresponding λ_A^*, λ_B^*, we have $\underline{\gamma} \leq \gamma^* \leq \bar{\gamma}$ and $\underline{\lambda} \leq \lambda_A^*, \lambda_B^* \leq \bar{\lambda}$.*

4. APPROXIMATE EQUILIBRIA OF THE COLONEL BLOTTO GAME

In this section, we propose a class of strategies in the Colonel Blotto game $\mathcal{CB}_n$, called the independently uniform strategies, and we show that it is an approximate Nash equilibrium (and an approximate min-max strategy in the constant-sum case). Note that the independently uniform strategies are also approximate equilibria of the Lottery Blotto game $\mathcal{LB}_n$, we analyze that in Section 5.

We begin by recalling the concept of approximate Nash equilibria (see e.g., Myerson (1991); Nisan et al. (2007)) in the context of our games: *for any $\varepsilon \geq 0$, an* ε-equilibrium *of the game $\mathcal{CB}_n$ is any strategy profile (s^*, t^*) such that $\Pi^A(s, t^*) \leq \Pi^A(s^*, t^*) + \varepsilon$ and $\Pi^B(s^*, t) \leq \Pi^B(s^*, t^*) + \varepsilon$ for any strategy s and t of players A and B.* Replacing Π^A and Π^B by Π^A_ζ and Π^B_ζ, we have the definition of ε-equilibria of the Lottery Blotto games $\mathcal{LB}_n(\zeta)$. Hereinafter, we use the generic term *approximate equilibrium* whenever the approximation error ε need not be emphasized.

4.1. The Independently Uniform strategies

Given a game $\mathcal{CB}_n$ (or $\mathcal{LB}_n$), consider the corresponding Equation (3.1) and set $\mathcal{S}_n$. For any $\gamma^* \in \mathcal{S}_n$, we define in Definition 4.1 a mixed strategy via an algorithm, called Algorithm 1. We term this strategy as the *independently uniform* strategy (or IU^{γ^*} strategy), parameterized by γ^*. Intuitively, this strategy is constructed by a simple procedure: players start by *independently* drawing n numbers from the *uniform-type* distributions defined in Definition 3.2, then they re-scale these numbers to guarantee the budget constraints.

[13] Trivially from Proposition 3.3, the random variables $A_i^*, B_i^*, \forall n, \forall i \in [n]$ are upper-bounded by $\bar{w}/(\underline{w} n \underline{\lambda})$. In the remainders of the paper, we sometimes need an upper-bound of these random variables that does not depend on n: we can prove that they are bounded by $2X_B$ (see Lemma A1 in Appendix A).

Definition 4.1 (The independently uniform strategy). *For any $\gamma^* \in \mathcal{S}_n$ and any player $p \in \{A, B\}$, $\mathrm{IU}_p^{\gamma^*}$ is the* ***mixed*** *strategy of player p where her allocation $\boldsymbol{x}^p$ is randomly generated from Algorithm 1.*

Algorithm 1: IU^{γ^*} strategy-generation algorithm.

Input: $n \in \mathbb{N}$, $w_i^A, w_i^B \in [\underline{w}, \bar{w}], \forall i \in [n]$, budgets X_A, X_B, $\gamma^* \in \mathcal{S}_n$
Output: $\boldsymbol{x}^A, \boldsymbol{x}^B \in \mathbb{R}^n_{\geq 0}$

1 Draw $a_i \sim F_{A_i^*}, b_i \sim F_{B_i^*}, \forall i \in [n]$ independently
2 **if** $\sum_{j\in[n]} a_j = 0$ **then**
3 $\quad x_i^A := 0, \forall i \in [n]$
4 **else**
5 $\quad x_i^A := \frac{a_i}{\sum_{j\in[n]} a_j} X_A, \forall i \in [n]$
6 **if** $\sum_{j\in[n]} b_j = 0$ **then**
7 $\quad x_i^B := 0, \forall i \in [n]$
8 **else**
9 $\quad x_i^B := \frac{b_i}{\sum_{j\in[n]} b_j} X_B, \forall i \in [n]$

Henceforth, we use the term IU^{γ^*} strategy to denote the strategy profile $(\mathrm{IU}_A^{\gamma^*}, \mathrm{IU}_B^{\gamma^*})$. We also simply use the notation IU^{γ^*} in some places to commonly address either $\mathrm{IU}_A^{\gamma^*}$ or $\mathrm{IU}_B^{\gamma^*}$ strategy in case the name of the player need not be specified. Observe that for any player $p \in \{A, B\}$, any output $\boldsymbol{x}^p$ from Algorithm 1 is an n-tuple that satisfies her budget constraint. In other words, $\mathrm{IU}_p^{\gamma^*}$ is a mixed strategy that is implicitly defined by Algorithm 1 and each run of this algorithm outputs a feasible pure strategy sampled from $\mathrm{IU}_p^{\gamma^*}$. Note that the marginals of the IU^{γ^*} strategy are *not* the uniform-type distributions $F_{A_i^*}, F_{B_i^*}, i \in [n]$ defined in Section 3. In terms of computational complexity, Algorithm 1 terminates in $\mathcal{O}(n)$ time. Below we discuss the specificity of the outputs of Algorithm 1 in the cases where $\sum_{j\in[n]} a_j = 0$ or $\sum_{j\in[n]} b_j = 0$.

Remark 4.2. *If $\sum_{j\in[n]} a_j = 0$ or $\sum_{j\in[n]} b_j = 0$, the $\mathrm{IU}_p^{\gamma^*}$ strategy allocates zero resource to all battlefields for the corresponding player (line 3 and line 7 of Algorithm 1). It may seem more natural that, if $\sum_{j\in[n]} a_j = 0$, player A allocates equally on all battlefields, i.e., set $x_i^A := X_A/n, \forall i \in [n]$ in line 3 of Algorithm 1 (and similarly for player B). In reality though, these assignments can be chosen to be any arbitrary n-tuple $\boldsymbol{x}^p$ in $\mathbb{R}^n_{\geq 0}$ as long as $\sum_{i\in[n]} x_i^p \leq X_p$ without affecting the results in our work. This comes from the fact that in most cases, the conditions in line 2 and 6 hold with probability zero. They can happen with a positive probability only when one player is the "weak player" and the other is the "strong player" on all of the battlefields (i.e., either $\Omega_A(\gamma^*) = \emptyset$ or $\Omega_A(\gamma^*) = [n]$), e.g., in a constant-sum game $\mathcal{CB}_n^c$. Yet, even in this case, this probability decreases exponentially as the number of battlefields increases (see (B.18) in Appendix B). The asymptotic order of the approximation error in all of our results is larger than this probability; therefore, it does not matter which assignment we choose in lines 3 and 7 of Algorithm 1. Here, we choose to assign $x_i^A = 0, \forall i$ and $x_i^B = 0, \forall i$ to ease the notation in the proofs of the results in the following sections; in particular, it avoids creating a discontinuity outside 0 in the CDF of the effective allocation in each battlefield (see also Lemma B1 in Appendix B).*

4.2. Approximate equilibria of the non-constant-sum Colonel Blotto game $\mathcal{CB}_n$

We now present our main result stating that the IU^{γ^*} strategy is an approximate equilibrium with an error that is only a negligible fraction of the maximum payoffs that the players can achieve, quickly decreasing as n increases. In the following results, note that since we focus on the setting of games with a large number of battlefields, we now focus on characterizing the approximation error according to n and treat other parameters of the $\mathcal{CB}_n$ games, including $X_A, X_B, \underline{w}, \bar{w}$ and α, as constants (but not the values

$w_i^p, v_i^p, \forall i \in [n], p \in \{A, B\}$). Using the notation $\tilde{\mathcal{O}}$—a variant of the big-$\mathcal{O}$ notation that ignores the logarithmic factors—, we have the first result as follows.

Theorem 4.3.

(i) In any game $\mathcal{CB}_n$, there exists a positive number $\varepsilon = \tilde{\mathcal{O}}(n^{-1/2})$ such that for any $\gamma^* \in \mathcal{S}_n$, the following inequalities hold for any pure strategy $\boldsymbol{x}^A$ and $\boldsymbol{x}^B$ of players A and B:

$$\Pi^A(\boldsymbol{x}^A, \mathrm{IU}_B^{\gamma^*}) \leq \Pi^A(\mathrm{IU}_A^{\gamma^*}, \mathrm{IU}_B^{\gamma^*}) + \varepsilon W_A, \tag{4.1}$$

$$\Pi^B(\mathrm{IU}_A^{\gamma^*}, \boldsymbol{x}^B) \leq \Pi^B(\mathrm{IU}_A^{\gamma^*}, \mathrm{IU}_B^{\gamma^*}) + \varepsilon W_B. \tag{4.2}$$

(ii) For any $\varepsilon \in (0, 1]$, there exists $C^* > 0$ (that does not depend on ε) such that in any game $\mathcal{CB}_n$ with $n \geq C^* \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$, (4.1) and (4.2) hold for any $\gamma^* \in \mathcal{S}_n$, any pure strategy $\boldsymbol{x}^A$, $\boldsymbol{x}^B$ of players A and B.

A proof of this theorem is presented in Appendix B. The two results stated in Theorem 4.3 are two equivalent statements that can be interpreted from different perspectives as follows. Result (i) states that given a priori a game $\mathcal{CB}_n$, there exists no unilateral deviation from the IU^{γ^*} strategy that can profit any player $p \in \{A, B\}$ more than a small portion of her maximum payoff W_p. As a trivial corollary, the IU^{γ^*} strategy is an approximate equilibrium of the game $\mathcal{CB}_n$ with a bounded approximation error (depending on n); this is formally stated as follows:

Corollary 4.4. *In any game $\mathcal{CB}_n$, there exists a positive number $\varepsilon = \tilde{\mathcal{O}}\left(n^{-1/2}\right)$ such that for any $\gamma^* \in \mathcal{S}_n$, the IU^{γ^*} strategy is an εW-equilibrium where $W := \max\{W_A, W_B\}$.*

The bound $\tilde{\mathcal{O}}(n^{-1/2})$ tells us the order of and how fast the level of error ε decreases if we consider games with larger and larger numbers of battlefields. Moreover, note that this upper-bound on ε also depends on other parameters of the game $\mathcal{CB}_n$, including $X_A, X_B, \underline{w}, \bar{w}$ and α.[14] We can extract from the proof of Theorem 4.3 that as $\bar{w}/\underline{w}$ and/or X_B/X_A increases, ε also increases, i.e., for games with higher heterogeneity of the battlefields values and/or higher asymmetry in players' budgets, the IU^{γ^*} strategy yields higher errors. Additionally, we note that to keep the generality, Result (i) is presented such that the approximation error ε is commonly addressed for any IU^{γ^*} strategy with any $\gamma^* \in \mathcal{S}_n$. For each specific solution γ^* of Equation (3.1) (implying λ_A^* and λ_B^*), the corresponding IU^{γ^*} strategy is an approximate equilibrium of $\mathcal{CB}_n$ with an approximation error that is at most (and it might be strictly smaller than) ε.

On the other hand, Result (ii) of Theorem 4.3 indicates the number of battlefields that a Colonel Blotto game needs to contain in order to guarantee a desired level of the approximation error by using the IU^{γ^*} strategy as an approximate equilibrium. Hence, in practical situations involving large instances of the Colonel Blotto game, the IU^{γ^*} strategy (simply and efficiently constructed by Algorithm 1) can be used as a safe replacement for any Nash equilibrium whose construction may be unknown or too complicated. Now, let us introduce an important notation:

Definition 4.5. *Corresponding to the players' allocations toward each battlefield $i \in [n]$, let $F_{A_i^n}$ and $F_{B_i^n}$ denote the univariate marginal distributions of the $\mathrm{IU}_A^{\gamma^*}$ and $\mathrm{IU}_B^{\gamma^*}$ strategies (see (B.1) and (B.2) in Appendix B for a more explicit formulation of $F_{A_i^n}$ and $F_{B_i^n}$).*

Intuitively, Result (ii) can be proved by showing the two following results: *(a)* when player B's allocation to the battlefield $i \in [n]$ follows $F_{B_i^*}$, the best response of player A is to play such that her allocation to i follows the distribution $F_{A_i^*}$ (and vice versa); *(b)* as n—the number of battlefields—increases, $F_{A_i^n}$ and $F_{B_i^n}$ uniformly converge toward the distributions $F_{A_i^*}$ and $F_{B_i^*}$, i.e., the marginal distributions of the IU^{γ^*} strategy approximate the distributions $F_{A_i^*}$ and $F_{B_i^*}$. This convergence can be proved by applying concentration

[14]This dependency is implicitly presented in the asymptotic notation $\tilde{\mathcal{O}}$ in Result (i) and the constant C^* in Result (ii).

inequalities on the random variables $\sum_{j\in[n]} A_j^*$ and $\sum_{j\in[n]} B_j^*$ (see Lemma B4 in Appendix B); moreover, the relation between ε and n in the results of Theorem 4.3 depends directly on the rate of this convergence. In this work, we use the Hoeffding's inequality (Hoeffding (1963)) that yields a better convergence rate than working with other types of concentration inequalities (e.g., Chebyshev's inequality). To complete the proof of Result (*ii*), we finally show that as n increases, when player $-p \in \{A, B\}$ plays the $\mathrm{IU}_{-p}^{\gamma^*}$ strategy, the $\mathrm{IU}_p^{\gamma^*}$'s payoff of player p converges toward her best-response payoff. Note that these payoffs can be written as expectations with respect to different measures (see (B.3), (B.4) and Lemma B2 in Appendix B). To prove the convergence of payoffs, we use a variant of the portmanteau theorem (see Lemma B5 in Appendix B) regarding the equivalent definitions of the weak convergence of a sequence of measures. Note importantly that a direct application of the portmanteau theorem leads to a slow convergence rate (notably, (4.1) and (4.2) only hold when $n = \Omega(\varepsilon^{-4})$). This is due to the fact that the players' payoffs involve the bounded Lipschitz functions $F_{A_i^*}$ and $F_{B_i^*}$ and that these functions depend on n, particularly, their Lipschitz constants (that are either λ_A^*/v_i^A or λ_B^*/v_i^B) increase as n increases. In order to obtain the convergence rate as indicated in Theorem 4.3, we exploit the special relation between $F_{A_i^n}$ and $F_{A_i^*}$, and between $F_{B_i^n}$ and $F_{B_i^*}$; then we apply a telescoping-sum trick allowing us to avoid the need of using the Lipschitz properties (for more details, see the proof of Lemma B5 in Appendix B.5).

4.3. Approximate equilibria of the constant-sum Colonel Blotto game $\mathcal{CB}_n^c$

In this section, we discuss the constant-sum variant $\mathcal{CB}_n^c$ of the Colonel Blotto game, defined in Definition 2.2. As an instance of the non-constant-sum game $\mathcal{CB}_n$, the game $\mathcal{CB}_n^c$ satisfies all results presented in Sections 4.1 and 4.2. Additionally, we show that any IU^{γ^*} strategy is an approximate max-min strategy of the game $\mathcal{CB}_n^c$.

In any game $\mathcal{CB}_n^c$, Equation (3.1) has a unique solution $\gamma^* = X_B/X_A \geq 1$; this γ^* uniquely induces $\lambda_A^* = 1/(2X_B)$ and $\lambda_B^* = X_A/(2{X_B}^2)$. Moreover, in $\mathcal{CB}_n^c$, $v_i^A/v_i^B = 1 \leq X_B/X_A = \gamma_A^*/\gamma_B^*, \forall i \in [n]$; therefore, we have $\Omega_A(\gamma^*) = \emptyset$; intuitively, player A is the "weak player" (and B the "strong player") in *all* battlefields. Recall the notation $W := \max\{W_A, W_B\}$, in the constant-sum game $\mathcal{CB}_n^n$, we have $W = W_A = W_B$. Applying Theorem 4.3, we obtain the following result.

Corollary 4.6. *In any game $\mathcal{CB}_n^c$, there exists a positive number $\varepsilon \leq \tilde{O}(n^{-1/2})$ such that the IU^{γ^*} strategy is an εW-equilibrium with $\gamma^* \in \mathcal{S}_n = \{X_B/X_A\}$.*

Note that if a Nash equilibrium exists in $\mathcal{CB}_n^c$, then the set of equilibrium univariate marginal distributions is unique (see e.g., Corollary 1 of Kovenock & Roberson (2015)) and they correspond to the distributions $F_{A^W_{\gamma^*,i}}$ and $F_{B^S_{\gamma^*,i}}$, defined in (3.6) and (3.7), where λ_A^* and λ_B^* are respectively replaced by $1/(2X_B)$ and $X_A/(2{X_B}^2)$. The marginals of the IU^{γ^*} strategy with $\gamma^* = X_B/X_A$ converge toward these unique equilibrium marginals.

Finally, we also deduce that the IU^{γ^*} strategy is an approximate max-min strategy of the game $\mathcal{CB}_n^c$; formally stated as follows.

Corollary 4.7. *In any game $\mathcal{CB}_n^c$, there exists a positive number $\varepsilon \leq \tilde{O}(n^{-1/2})$ such that the following inequalities hold for $\gamma^* \in \mathcal{S}_n = \{X_B/X_A\}$ and any strategy $\tilde{s}$ and $\tilde{t}$ of players A and B:*

$$\min_t \Pi^A(\tilde{s}, t) \leq \min_t \Pi^A(\mathrm{IU}_A^{\gamma^*}, t) + \varepsilon W, \tag{4.3}$$

$$\min_s \Pi^B(s, \tilde{t}) \leq \min_s \Pi^B(s, \mathrm{IU}_B^{\gamma^*}) + \varepsilon W. \tag{4.4}$$

Intuitively, if player $p \in \{A, B\}$ plays the $\mathrm{IU}_p^{\gamma^*}$ strategy, she guarantees a near-optimal payoff even in the worst-case scenario when her opponent $-p$ plays strategies that minimize p's payoff (no matters how it affects $-p$'s payoff). The proofs of Corollary 4.6 and Corollary 4.7 can be trivially deduced by applying specifically Theorem 4.3 to the constant-sum Colonel Blotto games and thus are omitted in this work.

5. APPROXIMATE EQUILIBRIA OF THE LOTTERY BLOTTO GAME

In this section, we present the results regarding the IU^{γ^*} strategy in the Lottery Blotto games. In Section 5.1, we analyze the game $\mathcal{LB}_n(\zeta)$ with an arbitrary pair of CSFs $\zeta = (\zeta_A, \zeta_B)$ and show that the IU^{γ^*} strategy is an approximate equilibrium of $\mathcal{LB}_n(\zeta)$ with an error depending on the number of battlefields as well as the dissimilarity between ζ_A and β_A (and between ζ_B and β_B). In Section 5.2, we illustrate this result in two particular instances, the games $\mathcal{LB}_n(\mu^R)$ and $\mathcal{LB}_n(\nu^R)$, belonging to the class of ratio-form Lottery Blotto games. We characterize the approximation error of the IU^{γ^*} strategy in these games according to n and the parameter R of these CSFs.

5.1. Approximate equilibria of Lottery Blotto games $\mathcal{LB}_n(\zeta)$ with general CSFs

We start by defining a parameter that expresses the dissimilarity between a given pair of CSFs $\zeta = (\zeta_A, \zeta_B)$ and the Blotto functions β_A, β_B (defined in (2.1)). First recall that for any n and $i \in [n]$, the random variables A_i^*, B_i^* are upper-bounded by $2X_B$ (see Lemma A1 in Appendix A) and by definition, the variables A_i^n, B_i^n are trivially upper-bounded by X_A, X_B (and thus by $2X_B$). Then, given any $\varepsilon > 0$, for any $x^* \in [0, 2X_B]$ and $y^* \in [0, 2X_B]$ (i.e., any number that can be sampled from $F_{A_i^*}, F_{B_i^*}, F_{A_i^n}$ or $F_{B_i^n}$), we introduce the following sets:

$$\mathcal{X}_\zeta(y^*, \varepsilon) := \{x \in [0, 2X_B] : |\zeta_A(x, y^*) - \beta_A(x, y^*)| \geq \varepsilon\}, \tag{5.1}$$
$$\mathcal{Y}_\zeta(x^*, \varepsilon) := \{y \in [0, 2X_B] : |\zeta_B(x^*, y) - \beta_B(x^*, y)| \geq \varepsilon\}. \tag{5.2}$$

Definition 5.1. *For any pair of CSFs $\zeta = (\zeta_A, \zeta_B)$, $\varepsilon > 0$ and $\gamma^* \in \mathcal{S}_n$, we define the following set*[15]

$$\Delta_{\gamma^*}(\zeta, \varepsilon) := \left\{ \delta \in [0,1] : \max_{i \in [n]} \max_{y^* \in [0, 2X_B]} \int_{\mathcal{X}_\zeta(y^*, \varepsilon)} \mathrm{d}F_{A_i^*}(x) \leq \delta, \quad \textit{and} \quad \max_{i \in [n]} \max_{x^* \in [0, 2X_B]} \int_{\mathcal{Y}_\zeta(x^*, \varepsilon)} \mathrm{d}F_{B_i^*}(y) \leq \delta \right\}.$$

Intuitively, the set $\Delta_{\gamma^*}(\zeta, \varepsilon)$ contains all numbers $\delta \in [0,1]$ such that for any allocation y^* of player B toward an arbitrary battlefield i, if player A draws an allocation x from the distribution $F_{A_i^*}$, it only happens with probability at most δ that the value of the CSF ζ_A at (x, y^*) is significantly different (i.e., ε-away) from that of the Blotto function β_A; and we have a similar statement for the distribution $F_{B_i^*}$ of player B and any allocation x^* of player A. Note that the set $\Delta_{\gamma^*}(\zeta, \varepsilon)$ depends on $F_{A_i^*}$ and $F_{B_i^*}$, thus it depends on γ^*. We can trivially see that $\Delta_{\gamma^*}(\zeta, \varepsilon)$ is an interval with the form $[\delta_0, 1]$ since if $\delta_0 \in \Delta_{\gamma^*}(\zeta, \varepsilon)$ then $\delta \in \Delta_{\gamma^*}(\zeta, \varepsilon)$ for any $\delta \geq \delta_0$.

Based on the convergence of $F_{A_i^n}$ and $F_{B_i^n}$ toward $F_{A_i^*}$ and $F_{B_i^*}$ (see Lemma B4 in Appendix B), we can prove the following lemma (a formal proof is given in Appendix C.1):

Lemma 5.2. *For any $\varepsilon \in (0,1]$, there exists a constant $L_0 > 0$ (that does not depend on ε), such that for any $n \geq L_0 \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$, for any game $\mathcal{LB}_n(\zeta)$, $\gamma^* \in \mathcal{S}_n$, $\delta \in \Delta_{\gamma^*}(\zeta, \varepsilon)$ and $i \in [n]$, we have:*

$$\max\left\{ \sup_{y^* \in [0, 2X_B]} \int_{\mathcal{X}_\zeta(y^*, \varepsilon)} \mathrm{d}F_{A_i^n}(x), \sup_{x^* \in [0, 2X_B]} \int_{\mathcal{Y}_\zeta(x^*, \varepsilon)} \mathrm{d}F_{B_i^n}(y) \right\} \leq \delta + \varepsilon. \tag{5.3}$$

Intuitively, this lemma provides an upper-bound for the probability of the value of the CSFs ζ being ε-away from the Blotto functions when player A (resp. player B) plays such that her allocation to battlefields i follows $F_{A_i^n}$ (resp. $F_{B_i^n}$), i.e., when she plays the IU^{γ^*} strategy.

Based on the definition of $\Delta_{\gamma^*}(\zeta, \varepsilon)$ and Lemma 5.2, we can now show the following result regarding the IU^{γ^*} strategy in Lottery Blotto games.

Theorem 5.3. ***(Approximate equilibria of the Lottery Blotto game).***

[15] Note that $F_{A_i^*}, F_{B_i^*}$ are continuous, bounded functions on $[0, 2X_B]$; therefore, they attain a maximum on this interval.

(i) In any game $\mathcal{LB}_n(\zeta)$, there exists a positive number $\varepsilon \leq \tilde{\mathcal{O}}(n^{-1/2})$ such that for any $\gamma^* \in \mathcal{S}_n$ and $\delta \in \Delta_{\gamma^*}(\zeta, \varepsilon)$, the following inequalities hold for any pure strategy $\boldsymbol{x}^A$ and $\boldsymbol{x}^B$ of players A and B:

$$\Pi^A_\zeta(\boldsymbol{x}^A, \mathrm{IU}^{\gamma^*}_B) \leq \Pi^A_\zeta(\mathrm{IU}^{\gamma^*}_A, \mathrm{IU}^{\gamma^*}_B) + (8\delta + 13\varepsilon)\, W_A, \tag{5.4}$$

$$\Pi^B_\zeta(\mathrm{IU}^{\gamma^*}_A, \boldsymbol{x}^B) \leq \Pi^B_\zeta(\mathrm{IU}^{\gamma^*}_A, \mathrm{IU}^{\gamma^*}_B) + (8\delta + 13\varepsilon)\, W_B. \tag{5.5}$$

(ii) For any $\varepsilon \in (0,1]$, there exists a constant $L^* > 0$ (that does not depend on ε) such that in any game $\mathcal{LB}_n(\zeta)$ where $n \geq L^* \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$, (5.4) and (5.5) hold for any $\gamma^* \in \mathcal{S}_n$, $\delta \in \Delta_{\gamma^*}(\zeta, \varepsilon)$ and any pure strategy $\boldsymbol{x}^A, \boldsymbol{x}^B$ of players A and B.

The proof of this theorem is given in Appendix C.2. The main idea to prove these results is that we can approximate the players' payoffs in the game $\mathcal{LB}_n(\zeta)$ when they play the IU^{γ^*} strategies by that in the corresponding game $\mathcal{CB}_n$ (the difference between these payoffs is controlled by the parameter $\delta \in \Delta_{\gamma^*}(\zeta, \varepsilon)$); and then use the results from Section 4 for the game $\mathcal{CB}_n$ (involving the error ε) to prove (5.4) and (5.5). The coefficients (8 and 13) in front of the parameters δ and ε come from the application of several triangle inequalities to connect these approximate results. Note that if the CSFs ζ_A and ζ_B are Lipschitz continuous on $[0, 2X_B] \times [0, 2X_B]$, we can avoid the need to approximate several terms involved in the analysis of using the IU^{γ^*} strategy in the game $\mathcal{LB}_n(\zeta)$ via the corresponding terms in the game $\mathcal{CB}_n$; thus, we can improve the results in Theorem 5.3 to obtain an approximation error of $2\delta + 5\varepsilon$ instead of $8\delta + 13\varepsilon$ (see Remark C3 in Appendix C.5 for more details). Here, to keep the generality, we do not include the continuity assumption of the CSFs in Theorem 5.3 (recall that our definition of a CSF allows for discontinuity).

Intuitively, Result (i) of Theorem 5.3 determines the order of the approximattion error while using IU^{γ^*} in any given game $\mathcal{LB}_n(\zeta)$. Straightforwardly, we can deduce that the IU^{γ^*} strategy is an approximate equilibrium of the game $\mathcal{LB}_n(\zeta)$, formally stated as follows.

Corollary 5.4. *In any game $\mathcal{LB}_n(\zeta)$, there exists a positive number $\varepsilon \leq \tilde{\mathcal{O}}(n^{-1/2})$ such that for any $\gamma^* \in \mathcal{S}_n$ and $\delta \in \Delta_{\gamma^*}(\zeta, \varepsilon)$, the IU^{γ^*} strategy is an $(8\delta + 13\varepsilon)\, W$-equilibrium where $W := \max\{W_A, W_B\}$.*

We observe that the error bound in Theorem 5.3 (and in Corollary 5.4) is valid for any δ of the set $\Delta_{\gamma^*}(\zeta, \varepsilon)$. Naturally, it is the tightest for $\delta_0 = \min\{\delta : \delta \in \Delta_{\gamma^*}(\zeta, \varepsilon)\}$; but this quantity is not always easy to compute; for instances, in the Lottery Blotto games with the power and logit form CSFs (i.e., μ^R and ν^R). Still, in Section 5.2, we show that there exists an element of $\Delta_{\gamma^*}(\mu^R, \varepsilon)$ and $\Delta_{\gamma^*}(\nu^R, \varepsilon)$ that is negligibly small, given appropriate parameter's configurations of the games; in other words, we can still obtain a good error's upper-bound for the IU^{γ^*} strategy in these games. Note that, on the other hand, the Colonel Blotto game $\mathcal{CB}_n$ can be considered as an instance of the game $\mathcal{LB}_n(\zeta)$ where the CSFs are $\zeta_A = \beta_A$ and $\zeta_B = \beta_B$; therefore, it also satisfies Theorem 5.3. In $\mathcal{CB}_n$, we trivially have $\mathcal{X}_\zeta(y^*, \varepsilon) = \mathcal{Y}_\zeta(x^*, \varepsilon) = \emptyset$ for any x^*, y^*; thus $\Delta_{\gamma^*}(\zeta, \varepsilon) = [0, 1]$ for any $\varepsilon > 0$ and $\min\{\delta : \delta \in \Delta_{\gamma^*}(\zeta, \varepsilon)\} = 0$.[16] This is consistent with results obtained in Theorem 4.3 in Section 4.

In Theorem 5.3, Result (ii) is an equivalent statement of Result (i). It indicates the number of battlefields needed to guarantee a certain level of approximation error when using the IU^{γ^*} strategy in the game $\mathcal{LB}_n(\zeta)$. For instance, to obtain an approximate equilibrium of the game $\mathcal{LB}_n(\zeta)$ where the level of error is less than a certain number $\bar{\varepsilon}$, one needs $\varepsilon \leq \bar{\varepsilon}$ such that we can find a $\delta \in \Delta_{\gamma^*}(\zeta, \varepsilon)$ satisfying $8\delta + 13\varepsilon \leq \bar{\varepsilon}$; from these parameters, by Result (ii), one can deduce the sufficient number of battlefields needed for an $\mathcal{LB}_n$ game to yield that desired level of error.

Finally, in the constant-sum variant of the Lottery Blotto game denoted by $\mathcal{LB}^{\mathrm{c}}_n(\zeta)$ (i.e., when $w^A_i = w^B_i$, $\forall i \in [n]$), we can easily deduce from Theorem 5.3 that the IU^{γ^*} strategy is also an approximate max-min strategy:

[16] Note also that for the case of the game $\mathcal{CB}_n$, the left-hand side in (5.3) equals zero for any n and $i \in [n]$.

Corollary 5.5. *In any game $\mathcal{LB}^c_n(\zeta)$, there exists $\varepsilon \leq \tilde{\mathcal{O}}(n^{-1/2})$ such that for any $\gamma^* \in \mathcal{S}_n = \{X_B/X_A\}$ and $\delta \in \Delta_{\gamma^*}(\zeta, \varepsilon)$, the following inequalities hold for any strategy $\tilde{s}$ and $\tilde{t}$ of players A and B:*[17]

$$\min_t \Pi^A_\zeta(\tilde{s}, t) \leq \min_t \Pi^A_\zeta(\mathrm{IU}^{\gamma^*}_A, t) + (8\delta + 13\varepsilon)W,$$
$$\min_s \Pi^B_\zeta(s, \tilde{t}) \leq \min_s \Pi^B_\zeta(s, \mathrm{IU}^{\gamma^*}_B) + (8\delta + 13\varepsilon)W.$$

5.2. Approximate equilibria of the ratio-form Lottery Blotto games $\mathcal{LB}_n(\mu^R)$ and $\mathcal{LB}_n(\nu^R)$

We now consider the ratio-form Lottery Blotto games $\mathcal{LB}_n(\mu^R)$ and $\mathcal{LB}_n(\nu^R)$. Recall that the corresponding CSFs are defined in Table 1 and that for those CSFs, we do not consider the degenerate cases where $\alpha = 0$ or $\alpha = 1$ in which trivial equilibria exist. The games $\mathcal{LB}_n(\mu^R)$ and $\mathcal{LB}_n(\nu^R)$ are instances of the game $\mathcal{LB}_n(\zeta)$ studied in Section 5.1; therefore, by Theorem 5.3 (and Corollary 5.4), the IU^{γ^*} strategy is also an approximate equilibrium of them. In this section, we focus on characterizing the approximation error of the IU^{γ^*} strategy in these games according to n (the number of battlefields) and R (the corresponding parameter of the CSFs). We will show that this error quickly tends to zero as n and R increase under appropriate conditions. To do this, we first notice that although it is non-trivial to analyze the closed form of the sets $\Delta_{\gamma^*}(\mu^R, \varepsilon)$ and $\Delta_{\gamma^*}(\nu^R, \varepsilon)$ and find their minimum, we can find small elements of theses sets.

Lemma 5.6. *Fix $n \geq 2$, $R > 0$ and $\alpha \in (0, 1)$, for any $\varepsilon < \min\{\alpha, 1 - \alpha\}$, we have:*[18]

(i) In any game $\mathcal{LB}_n(\mu^R)$ with α as the tie-breaking parameter, there exists $\delta_\mu = \min\{1, \mathcal{O}\big(n(\varepsilon^{-1/R} - 1)\big)\}$ such that $\delta_\mu \in \Delta_{\gamma^*}(\mu^R, \varepsilon)$ for any $\gamma^* \in \mathcal{S}_n$.

(ii) In any game $\mathcal{LB}_n(\nu^R)$ with α as the tie-breaking parameter, there exists $\delta_\nu = \min\{1, \mathcal{O}\big(nR^{-1}\ln(\varepsilon^{-1})\big)\}$ such that $\delta_\nu \in \Delta_{\gamma^*}(\nu^R, \varepsilon)$ for any $\gamma^* \in \mathcal{S}_n$.

The proof of Lemma 5.6 is given in Appendix D.1. Note that for the sake of generality, the parameters δ_μ and δ_ν are indicated in this lemma in such a way that they do not depend on γ^*, but for each $\gamma^* \in \mathcal{S}_n$, we can find smaller elements of the corresponding sets $\Delta_{\gamma^*}(\mu^R, \varepsilon)$ and $\Delta_{\gamma^*}(\nu^R, \varepsilon)$. More importantly, for a fixed n, the numbers δ_μ and δ_ν decrease as R increases; but δ_μ and δ_ν increase as ε decreases. While the lemma is valid for any parameter values, since 1 is a trivial element of $\Delta_{\gamma^*}(\mu^R, \varepsilon)$ and $\Delta_{\gamma^*}(\nu^R, \varepsilon)$, it is useful only if $\delta_\mu, \delta_\nu < 1$; this is guaranteed whenever $R \geq \mathcal{O}\left(n \ln(\varepsilon^{-1})\right)$. Note finally that the condition $\varepsilon < \min\{\alpha, 1-\alpha\}$ in the statement of Lemma 5.6 does not limit its use since our goal is to obtain asymptotic results on the IU^{γ^*} strategy when ε tends to 0. Moreover, in the games $\mathcal{LB}_n(\mu^R)$ and $\mathcal{LB}_n(\nu^R)$ where α is either very close to 0 or 1, one player has a very high advantage and always obtains large gains from all battlefields (where her allocation is strictly positive) while her opponent gains very little regardless of her allocations; therefore, there exists (many) trivial approximate equilibria with small errors.

Combining the results of Corollary 5.4 and Lemma 5.6, we can deduce directly that in any game $\mathcal{LB}_n(\mu^R)$ (resp. $\mathcal{LB}_n(\nu^R)$), there exists $\varepsilon \leq \tilde{\mathcal{O}}(n^{-1/2})$ such that for any $\gamma^* \in \mathcal{S}_n$, the IU^{γ^*} strategy is an $(8\varepsilon + 13\delta_\mu)W$-equilibrium (resp. $(8\varepsilon + 13\delta_\nu)W$-equilibrium). Next, we look for the asymptotic relation between these error terms and the parameters n, R of the games. First, as n increases, the error level ε decreases; on the other hand, from Lemma 5.6, the number δ_μ (and δ_ν) decreases if R increases with a faster rate than $\tilde{\mathcal{O}}(n)$. However, there is a trade-off between ε and δ_μ (or δ_ν): as ε decreases, δ_μ (and δ_ν) increases and vice versa. To handle this trade-off between δ_μ and ε (resp. δ_ν and ε), we can first find a condition on n that generates a small error ε, and then find a condition on R (with respect to n) such that the error δ_μ (resp. δ_ν) is of the same order as ε. Formally, we state the result that the IU^{γ^*} strategy yields an approximate equilibrium of the games $\mathcal{LB}_n(\mu^R)$ and $\mathcal{LB}_n(\nu^R)$ with any arbitrary small error in the next theorem.

[17] Recall that in the constant-sum variant, $W := \max\{W_A, W_B\} = W_A = W_B$.
[18] The asymptotic notations are taken w.r.t. when $\varepsilon \to 0$.

Theorem 5.7. ***(Approximate equilibria of the ratio-form Lottery Blotto games)*** *For any $\bar{\varepsilon} > 0$ and $\alpha \in (0,1)$ such that $\bar{\varepsilon} < \min\{\alpha, 1-\alpha\}$, there exists $\tilde{L} > 0$ such that for any $n \geq \tilde{L}\bar{\varepsilon}^{-2} \ln\left(\frac{1}{\min\{\bar{\varepsilon}, 1/e\}}\right)$, $R \geq \mathcal{O}\left(\frac{n}{\bar{\varepsilon}} \ln\left(\frac{1}{\bar{\varepsilon}}\right)\right)$ and $\gamma^* \in \mathcal{S}_n$, the IU^{γ^*} strategy is an $\bar{\varepsilon}W$-equilibrium of any game $\mathcal{LB}_n(\mu^R)$ and $\mathcal{LB}_n(\nu^R)$ having α as the tie-breaking-rule parameter.*

The proof of this theorem is based on Theorem 5.3 and Lemma 5.6 (see Appendix D.2 for more details). Theorem 5.7 involves a double limit in R and n. Intuitively, if n and R increase but R increases with a slower rate, then ε decreases but the corresponding δ_μ and δ_ν do not decrease; thus, the total error is not guaranteed to decrease.

6. CONCLUSION

In this work, we consider the most general variant of the Colonel Blotto game—the non-constant-sum variant with heterogeneous battlefields and asymmetric players. While most of (if not all) works in the literature attempt (but do not completely succeed) to construct an exact equilibrium of the CB game by looking for a joint distribution with the uniform-type marginals that satisfies the budget constraints, we take a different angle. We propose a class of strategies called the IU^{γ^*} strategies that is simply constructed by an efficient algorithm; the IU^{γ^*} strategies guarantee the budget constraints but their marginals are not the uniform-type distributions. Yet, we prove the IU^{γ^*} strategy to be an approximate equilibrium of the CB games. We also define an extended game called the Lottery Blotto game and obtain similar results. We characterize the approximate error in our results in terms of the number of battlefields of the games. Our work extends the scope of applications of the CB games and its variants.

Throughout the paper, we emphasized the dependence of the approximation error on the number of battlefields n. Yet, although the dependence on other parameters of the games is not explicitly emphasized, it can be extracted from our analysis and the proofs of the stated results. It is also interesting to note that although the notion of approximate equilibrium is defined in terms of payoffs (the payoffs when players play the IU^{γ^*} strategy are close to optimal), the IU^{γ^*} strategy also approximates the equilibrium marginals (if an equilibrium exists)—that is, it is also an approximate equilibrium in terms of strategies.

Our approximation results are valid even in the case where no equilibrium exists (and we do not include the assumption that requires the existence of the equilibrium). Particularly in the cases of the CB game where it is known that there exists no equilibrium yielding the uniform-type marginals, the IU^{γ^*} strategy is still an approximate equilibrium, yet we suspect that in those cases the approximation error might be large. On the other hand, our work does not solve the question of the existence of an exact Nash equilibrium. In particular, we leave as future work the investigation of possible conditions under which a Nash equilibrium exists, for instance for a large-enough number of battlefields. We also finally note that, in the non-constant-sum version, the existence of multiple solutions γ^* of Equation (3.1) leads to problems of equilibrium selection (in practical contexts involving a social welfare measurement) among the IU^{γ^*} strategies with different $\gamma^* \in \mathcal{S}_n$, which we also leave as future work.

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REFERENCES

Ahmadinejad, A. M., Dehghani, S., Hajiaghayi, M. T., Lucier, B., Mahini, H., & Seddighin, S. (2016). From duels to battlefields: Computing equilibria of blotto and other games. In *Proceedings of the Thirtieth AAAI Conference on Artificial Intelligence (AAAI)* (pp. 369–375).

Alcalde, J., & Dahm, M. (2007). Tullock and hirshleifer: a meeting of the minds. *Review of Economic Design*, *11*, 101–124.

Baye, M. R., Kovenock, D., & de Vries, C. G. (1996). The all-pay auction with complete information. *Economic Theory*, *8*, 291–305.

Behnezhad, S., Dehghani, S., Derakhshan, M., Aghayi, M. T. H., & Seddighin, S. (2017). Faster and simpler algorithm for optimal strategies of blotto game. In *Proceedings of the 31st AAAI Conference on Artificial Intelligence (AAAI)* (pp. 369–375).

Borel, E. (1921). La théorie du jeu et les équations intégrales à noyau symétrique. *Comptes rendus de l'Académie des Sciences*, *173*, 58.

Borel, E., & Ville, J. (1991). *Application de la théorie des probabilités aux jeux de hasard*. Original edition by Gauthier-Villars, Paris, 1938; reprinted at the end of Théorie mathématique du bridge à la portée de tous, by E. Borel & A. Chéron, Editions Jacques Gabay, Paris.

Che, Y.-K., & Gale, I. (2000). Difference-form contests and the robustness of all-pay auctions. *Games and Economic Behavior*, *30*, 22 – 43.

Chia, P. H. (2012). Colonel Blotto in web security. In *The Eleventh Workshop on Economics and Information Security, WEIS Rump Session* (pp. 141–150).

Clark, D. J., & Riis, C. (1998). Contest success functions: an extension. *Economic Theory*, *11*, 201–204.

Corchón, L., & Dahm, M. (2010). Foundations for contest success functions. *Economic Theory*, *43*, 81–98.

Corchón, L. C. (2007). The theory of contests: a survey. *Review of Economic Design*, *11*, 69–100.

Friedman, L. (1958). Game-theory models in the allocation of advertising expenditures. *Operations Research*, *6*, 699–709.

Gross, O., & Wagner, R. (1950). *A continuous Colonel Blotto game*. Technical Report RAND project air force Santa Monica CA.

Gross, O. A. (1950). *The symmetric Blotto game*. Technical Report. US Air Force Project RAND Research Memorandum.

Hajimirsaadeghi, M., & Mandayam, N. B. (2017). A dynamic colonel Blotto game model for spectrum sharing in wireless networks. In *Proceedings of the 55th Annual Allerton Conference on Communication, Control, and Computing (Allerton)* (pp. 287–294).

Hillman, A. L., & Riley, J. G. (1989). Politically contestable rents and transfers. *Economics & Politics*, *1*, 17–39.

Hirshleifer, J. (1989). Conflict and rent-seeking success functions: Ratio vs. difference models of relative success. *Public choice*, *63*, 101–112.

Hoeffding, W. (1963). Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association,*, *58*, 13–30.

Hortala-Vallve, R., & Llorente-Saguer, A. (2012). Pure strategy Nash equilibria in non-zero sum colonel Blotto games. *International Journal of Game Theory*, *41*, 331–343.

Kovenock, D., & Roberson, B. (2012). Coalitional Colonel Blotto games with application to the economics of alliances. *Journal of Public Economic Theory*, *14*, 653–676.

Kovenock, D., & Roberson, B. (2015). Generalizations of the General Lotto and Colonel Blotto games. CESifo Working Paper No. 5291.

Kvasov, D. (2007). Contests with limited resources. *Journal of Economic Theory*, *136*, 738 – 748.

Laslier, J.-F. (2002). How two-party competition treats minorities. *Review of Economic Design*, *7*, 297–307.

Macdonell, S. T., & Mastronardi, N. (2015). Waging simple wars: a complete characterization of two-battlefield blotto equilibria. *Economic Theory*, *58*, 183–216.

Masucci, A. M., & Silva, A. (2014). Strategic resource allocation for competitive influence in social networks. In *Proceedings of the 52nd Annual Allerton Conference on Communication, Control, and Computing (Allerton)* (pp. 951–958).

Masucci, A. M., & Silva, A. (2015). Defensive resource allocation in social networks. In *Proceedings of the 54th IEEE Conference on Decision and Control (CDC)* (pp. 2927–2932).

Myerson, R. (1991). *Game Theory: Analysis of Conflict*. Harvard University Press.

Myerson, R. B. (1993). Incentives to cultivate favored minorities under alternative electoral systems. *American Political Science Review*, *87*, 856–869.

Nisan, N., Roughgarden, T., Tardos, E., & Vazirani, V. V. (2007). *Algorithmic Game Theory*. New York, NY, USA: Cambridge University Press.

Osório, A. (2013). The lottery Blotto game. *Economics Letters*, *120*, 164–166.

Rinott, Y., Scarsini, M., & Yu, Y. (2012). A Colonel Blotto gladiator game. *Mathematics of Operations Research*, *37*, 574–590.

Roberson, B. (2006). The Colonel Blotto game. *Economic Theory*, *29*, 1–24.

Roberson, B., & Kvasov, D. (2012). The non-constant-sum Colonel Blotto game. *Economic Theory*, *51*, 397–433.

Schwartz, G., Loiseau, P., & Sastry, S. S. (2014). The heterogeneous Colonel Blotto game. In *Proceedings of the 7th International Conference on Network Games, Control and Optimization (NetGCoop)* (pp. 232–238).

Skaperdas, S. (1996). Contest success functions. *Economic theory*, *7*, 283–290.

Thomas, C. (2017). N-dimensional Blotto game with heterogeneous battlefield values. *Economic Theory*, (pp. 1–36).

Tullock, G. (2001). Efficient rent seeking. In *: Lockard A.A., Tullock G. (eds) Efficient Rent Seeking Chronicle of an Intellectual Quagmire. Springer, Boston, MA*.

Van der Vaart, A. W. (2000). *Asymptotic statistics* volume 3. Cambridge university press.

Vu, D. Q., Loiseau, P., & Silva, A. (2018). Efficient computation of approximate equilibria in discrete colonel blotto games. In *Proceedings of the 27th International Joint Conference on Artificial Intelligence and the 23rd European Conference on Artificial Intelligence (IJCAI-ECAI)* (pp. 519–526).

Appendix A. NOMENCLATURES AND PRELIMINARIES

Table A.2: Table of Notation

		Abbreviation
CB ($\mathcal{CB}_n$)	$\triangleq$	non-constant-sum Colonel Blotto game (with n battlefields)
LB ($\mathcal{LB}_n$)	$\triangleq$	non-constant-sum Lottery Blotto game (with n battlefields)
$\mathcal{CB}_n^c, \mathcal{LB}_n^c$	$\triangleq$	the constant-sum versions of $\mathcal{CB}_n$ and $\mathcal{LB}_n$ games
CSF	$\triangleq$	contest success function.
IU^{γ^*} $(= (\mathrm{IU}_A^{\gamma^*}, \mathrm{IU}_B^{\gamma^*}))$	$\triangleq$	independent uniform strategy (corresponding to γ^*)
		Games' Parameters
X_A, X_B	$\triangleq$	budgets of player A and B respectively
n	$\triangleq$	number of battlefields
w_i^A, w_i^B	$\triangleq$	values of battlefield i assessed by player A and B respectively
$\underline{w}, \bar{w}$	$\triangleq$	lower and upper bounds of battlefields' values
W_A, W_B	$\triangleq$	sums of battlefields' values, $W_A := \sum_{i=1}^n w_i^A$, $W_B := \sum_{i=1}^n w_i^B$
W	$\triangleq$	$\max\{W_A, W_B\}$
v_i^A, v_i^B	$\triangleq$	normalized values of battlefield i assessed by player A and B
x_i^A, x_i^B	$\triangleq$	the allocation to battlefield i of player A and B respectively
$\Pi^A(s,t), \Pi^B(s,t)$	$\triangleq$	players' payoffs in CB games when playing the strategies s and t
α	$\triangleq$	the tie-breaking parameter
β_A, β_B	$\triangleq$	Blotto functions (see (2.1))
ζ	$\triangleq$	(ζ_A, ζ_B)—the generic CSFs
$\mathcal{LB}_n(\zeta)$	$\triangleq$	the LB game with CSFs ζ_A, ζ_B
μ^R	$\triangleq$	(μ_A^R, μ_B^R)—the power form CSFs with parameter R (see Table 1)
ν^R	$\triangleq$	(ν_A^R, ν_B^R)—the logit form CSFs with parameter R (see Table 1)
$\Pi_\zeta^A(s,t), \Pi_\zeta^B(s,t)$	$\triangleq$	players' payoffs in $\mathcal{LB}_n(\zeta)$ games when playing the strategies s, t
$\mathcal{X}_\zeta(y^*, \varepsilon), \mathcal{Y}_\zeta(x^*, \varepsilon)$	$\triangleq$	the sets characterizing the dissimilarity between (β_A, β_B) and (ζ_A, ζ_B) (see (5.1), (5.2))
$\Delta_{\gamma^*}(\zeta, \varepsilon)$	$\triangleq$	see Definition 5.1
		IU^{γ^*} Strategies
γ^*	$\triangleq$	a positive solution of Equation (3.1)
$\mathcal{S}_n$	$\triangleq$	the set of positive solutions of Equation (3.1) w.r.t. $\mathcal{CB}_n$ (or $\mathcal{LB}_n$)
$\underline{\gamma}, \bar{\gamma}$	$\triangleq$	lower and upper bounds of any $\gamma^* \in \mathcal{S}_n$ (see Proposition 3.3)
$\Omega_A(\gamma^*)$	$\triangleq$	$\{i \in [n] : v_i^A / v_i^B > \gamma^*\}$
λ_A^*, λ_B^*	$\triangleq$	Lagrange multipliers corresponding to γ^* (see (3.2), (3.3))
$\underline{\lambda}, \bar{\lambda}$	$\triangleq$	lower and upper bounds of λ_A^*, λ_B^* (see Proposition 3.3)
$F_{A_i^*}, F_{B_i^*}$	$\triangleq$	uniform-type distributions (see (3.4)-(3.8))
$A_{\gamma^*,i}^S, A_{\gamma^*,i}^W, A_i^n$ $B_{\gamma^*,i}^S, B_{\gamma^*,i}^W, B_i^n$	$\triangleq$	random variables defined in (3.4)-(3.8)
$F_{A_i^n}, F_{B_i^n}$	$\triangleq$	the marginals corresponding to battlefield i of the IU^{γ^*} strategy
$A_{=0}, A_{>0}$	$\triangleq$	the events $\{\sum_{i\in[n]} A_i^* = 0\}$ and $\{\sum_{i\in[n]} A_i^* > 0\}$, respectively
$B_{=0}, B_{>0}$	$\triangleq$	the events $\{\sum_{i\in[n]} B_i^* = 0\}$ and $\{\sum_{i\in[n]} B_i^* > 0\}$, respectively

Lemma A1. *Given a game $\mathcal{CB}_n$ (or $\mathcal{LB}_n$), for any $\gamma^* \in \mathcal{S}_n$, we have:*

(i) $\lambda_A^*, \lambda_B^* > 0$ *and* $\gamma^* = \lambda_A^* / \lambda_B^*$.

(ii) For any $i \in [n]$, $\mathbb{E}[A_{\gamma^*,i}^S] = \frac{1}{2}\frac{v_i^B}{\lambda_B^*}$, $\mathbb{E}[A_{\gamma^*,i}^W] = \left(\frac{v_i^A}{\lambda_A^*}\right)^2 \frac{\lambda_B^*}{2v_i^B}$, $\mathbb{E}[B_{\gamma^*,i}^S] = \frac{1}{2}\frac{v_i^A}{\lambda_A^*}$ *and* $\mathbb{E}[B_{\gamma^*,i}^W] = \left(\frac{v_i^B}{\lambda_B^*}\right)^2 \frac{\lambda_A^*}{2v_i^A}$.

(iii) $X_A = \sum_{i\in[n]} \mathbb{E}[A_i^*]$ *and* $X_B = \sum_{i\in[n]} \mathbb{E}[B_i^*]$.

(iv) *For any* $i \in [n]$, A_i^* *and* B_i^* *have a constant upper-bound; particularly,* $\mathbb{P}\left(A_i^* \le 2X_B\right) = \mathbb{P}\left(B_i^* \le 2X_B\right) = 1$.

Proof.

(i) The positivity of λ_A^* and λ_B^* follows from the positivity of γ^* and the definitions of λ_A^* and λ_B^* in (3.2) and (3.3). By dividing (3.2) by (3.3) and combining with (3.1), we trivially have that $\gamma^* = \lambda_A^*/\lambda_B^*$.

(ii) These results come directly from the definitions of the distributions $F_{A^S_{\gamma^*,i}}$, $F_{A^W_{\gamma^*,i}}$ $F_{B^S_{\gamma^*,i}}$ and $F_{B^W_{\gamma^*,i}}$.

(iii) We multiply both sides of (3.3) by X_A/λ_B^* and both sides of (3.2) by X_B/λ_A^* then using the fact that $\gamma^* = \lambda_A^*/\lambda_B^*$ to obtain the following:

$$X_A = \sum_{j\in\Omega_A(\gamma^*)} \frac{1}{2}\frac{v_j^B}{\lambda_B^*} + \sum_{j\notin\Omega_A(\gamma^*)} \left(\frac{v_j^A}{\lambda_A^*}\right)^2 \frac{\lambda_B^*}{2v_j^B}, \tag{A.1}$$

$$X_B = \sum_{j\in\Omega_A(\gamma^*)} \left(\frac{v_j^B}{\lambda_B^*}\right)^2 \frac{\lambda_A^*}{2v_j^A} + \sum_{j\notin\Omega_A(\gamma^*)} \frac{1}{2}\frac{v_j^A}{\lambda_A^*}. \tag{A.2}$$

Combining with *(ii)*, we deduce that $X_A = \sum_{i\in[n]} \mathbb{E}[A_i^*]$ and $X_B = \sum_{i\in[n]} \mathbb{E}[B_i^*]$.

(iv) If $i \in \Omega_A(\gamma^*)$, we have $A_i^* = A^S_{\gamma^*,i}$ and $B_i^* = B^W_{\gamma^*,i}$. Recalling Definition 3.2, we have that $\mathbb{P}\left(A_i^S \le v_i^B/\lambda_B^*\right) = 1$ and $\mathbb{P}\left(B_i^W \le v_i^B/\lambda_B^*\right) = 1$. On the other hand, from (A.1), we deduce

$$X_B \ge X_A \ge \sum_{j\in\Omega_A(\gamma^*)} \frac{v_j^B}{2\lambda_B^*} \ge \frac{v_i^B}{2\lambda_B^*}.$$

Therefore, $\mathbb{P}(A_i^S \le 2X_B) \ge \mathbb{P}\left(A_i^S \le v_i^B/\lambda_B^*\right) = 1$ and $\mathbb{P}(B_i^W \le 2X_B) \ge \mathbb{P}(B_i^W \le v_i^B/\lambda_B^*) = 1$. We conclude that for any $i \in \Omega_A(\gamma^*)$, A_i^*, B_i^* are bounded by $2X_B$.

If $i \notin \Omega_A(\gamma^*)$, we have $A_i^* = A^W_{\gamma^*,i}$ and $B_i^* = B^S_{\gamma^*,i}$. Recalling Definition 3.2, we have that $\mathbb{P}\left(A_i^W \le v_i^A/\lambda_A^*\right) = 1$ and $\mathbb{P}\left(B_i^S \le v_i^A/\lambda_A^*\right) = 1$. On the other hand, from (A.2), we deduce

$$X_B \ge \sum_{j\notin\Omega_A(\gamma^*)} \frac{v_j^A}{2\lambda_A^*} \ge \frac{v_i^A}{2\lambda_A^*}.$$

Therefore, $\mathbb{P}(A_i^W \le 2X_B) \ge \mathbb{P}\left(A_i^W \le v_i^A/\lambda_A^*\right) = 1$ and $\mathbb{P}(B_i^S \le 2X_B) \ge \mathbb{P}(B_i^S \le v_i^A/\lambda_A^*) = 1$. We conclude that for $i \notin \Omega_A(\gamma^*)$, A_i^*, B_i^* are also bounded by $2X_B$.

Proposition 3.3. *Under Assumption (A0), for any game* $\mathcal{CB}_n$ *(or* $\mathcal{LB}_n$*), there exist constants* $\underline{\gamma}, \bar{\gamma}, \underline{\lambda}, \bar{\lambda} > 0$, *that do not depend on n, such that for any* $\gamma^* \in \mathcal{S}_n$ *and its corresponding* λ_A^*, λ_B^*, *we have* $\underline{\gamma} \le \gamma^* \le \bar{\gamma}$ *and* $\underline{\lambda} \le \lambda_A^*, \lambda_B^* \le \bar{\lambda}$.

Proof. Let $\gamma^* \in \mathcal{S}_n$, we consider the following cases:

Case 1: If $0 < \gamma^* < \min\limits_{i\in[n]} \left\{\frac{v_i^A}{v_i^B}\right\}$. In this case, $\Omega_A(\gamma^*) = [n]$, and since γ^* is a solution of (3.1), we deduce:

$$\gamma^* = \frac{X_B}{X_A} \frac{\sum_{i=1}^n v_i^B}{\sum_{i=1}^n \frac{(v_i^B)^2}{v_i^A}} \ge \frac{X_B}{X_A} \frac{n\frac{\underline{w}}{n\bar{w}}}{n\frac{\left(\frac{\bar{w}}{n\underline{w}}\right)^2}{\frac{\underline{w}}{n\bar{w}}}} = \frac{X_B}{X_A}\left(\frac{\underline{w}}{\bar{w}}\right)^4.$$

Here, the inequality comes directly from (2.2).

Case 2: If $\gamma^* \geq \max\limits_{i\in[n]}\left\{\frac{v_i^A}{v_i^B}\right\}$. In this case, $\Omega_A(\gamma^*) = \emptyset$, and since γ^* is a solution of (3.1), we deduce:

$$\gamma^* = \frac{X_B}{X_A}\frac{\sum_{i=1}^n \frac{(v_i^A)^2}{v_i^B}}{\sum_{i=1}^n v_i^A} \leq \frac{X_B}{X_A}\left(\frac{\bar{w}}{\underline{w}}\right)^4.$$

Case 3: If $\exists i,j : \frac{v_i^A}{v_i^B} \leq \gamma^* < \frac{v_j^A}{v_j^B}$. In this case, trivially from (2.2), we have $\gamma^* \in \left[\left(\frac{\underline{w}}{\bar{w}}\right)^2, \left(\frac{\bar{w}}{\underline{w}}\right)^2\right]$.

In conclusion, by denoting $\underline{\gamma} := \min\left\{\frac{X_B}{X_A}\left(\frac{\underline{w}}{\bar{w}}\right)^4, \left(\frac{\underline{w}}{\bar{w}}\right)^2\right\}$ and $\bar{\gamma} := \max\left\{\frac{X_B}{X_A}\left(\frac{\bar{w}}{\underline{w}}\right)^4, \left(\frac{\bar{w}}{\underline{w}}\right)^2\right\} = \frac{X_B}{X_A}\left(\frac{\bar{w}}{\underline{w}}\right)^4$, we have the conclusion on the bounds of γ^*.

On the other hand, from the definition of λ_A^* in (3.2), we deduce

$$\begin{aligned}
\lambda_A^* &\geq \frac{(\gamma^*)^2}{2X_B}\sum_{i\in\Omega_A(\gamma^*)}\left(\frac{\underline{w}}{n\bar{w}}\right)^2\frac{1}{\frac{\bar{w}}{n\underline{w}}} + \frac{1}{2X_B}\sum_{i\notin\Omega_A(\gamma^*)}\frac{\underline{w}}{n\bar{w}} \\
&\geq \min\left\{\frac{(\gamma^*)^2}{2X_B}, \frac{1}{2X_B}\right\}\cdot\sum_{i\in[n]}\frac{1}{n}\left(\frac{\underline{w}}{\bar{w}}\right)^3 \\
&\geq \min\left\{\frac{(\gamma^*)^2}{2X_B}, \frac{1}{2X_B}\right\}\cdot\left(\frac{\underline{w}}{\bar{w}}\right)^3.
\end{aligned}$$

Similarly, we have the upper-bound

$$\lambda_A^* \leq \max\left\{\frac{(\gamma^*)^2}{2X_B}, \frac{1}{2X_B}\right\}\cdot\left[\sum_{i\in\Omega_A(\gamma^*)}\frac{1}{n}\left(\frac{\bar{w}}{\underline{w}}\right)^3 + \sum_{i\notin\Omega_A(\gamma^*)}\frac{1}{n}\left(\frac{\bar{w}}{\underline{w}}\right)^3\right] = \max\left\{\frac{(\gamma^*)^2}{2X_B}, \frac{1}{2X_B}\right\}\cdot\left(\frac{\bar{w}}{\underline{w}}\right)^3.$$

Similarly, we can prove that $\min\left\{\frac{1}{2X_A}, \frac{1}{2\gamma^{*2}X_A}\right\}\left(\frac{\underline{w}}{\bar{w}}\right)^3 \leq \lambda_B^* \leq \max\left\{\frac{1}{2X_A}, \frac{1}{2\gamma^{*2}X_A}\right\}\left(\frac{\bar{w}}{\underline{w}}\right)^3$; therefore,

$$\min\left\{\frac{\gamma^{*2}}{2X_B}, \frac{1}{2X_B}, \frac{1}{2X_A}, \frac{1}{2(\gamma^*)^2X_A}\right\}\left(\frac{\underline{w}}{\bar{w}}\right)^3 \leq \lambda_A^*, \lambda_B^* \leq \max\left\{\frac{\gamma^{*2}}{2X_B}, \frac{1}{2X_B}, \frac{1}{2X_A}, \frac{1}{2(\gamma^*)^2X_A}\right\}\left(\frac{\bar{w}}{\underline{w}}\right)^3.$$

Since $\gamma^* \in [\underline{\gamma}, \bar{\gamma}]$, λ_A^* and λ_B^* are bounded in $[\underline{\lambda}, \bar{\lambda}]$, where

$$\begin{aligned}
\underline{\lambda} &:= \min\left\{\frac{\underline{\gamma}^2}{2X_B}, \frac{1}{2X_B}, \frac{1}{2X_A}, \frac{1}{2\bar{\gamma}^2X_A}\right\}\left(\frac{\underline{w}}{\bar{w}}\right)^3, \\
\bar{\lambda} &:= \max\left\{\frac{\bar{\gamma}^2}{2X_B}, \frac{1}{2X_B}, \frac{1}{2X_A}, \frac{1}{2\underline{\gamma}^2X_A}\right\}\left(\frac{\bar{w}}{\underline{w}}\right)^3.
\end{aligned}$$

□

Finally, we prove a trivial result that will be used quite often in the remainder of this work.

Lemma A2. *For any* $\hat{\varepsilon} > 0$ *and* $\hat{C} \geq 1$, *we have that* $(\ln(\hat{C})+1)\ln\left(\frac{1}{\min\{\hat{\varepsilon},1/e\}}\right) \geq \ln\left(\frac{\hat{C}}{\hat{\varepsilon}}\right)$.

PROOF. <u>Case 1:</u> If $\hat{\varepsilon} < 1/e$. In this case, we have $\ln(1/\hat{\varepsilon}) > 1$; therefore,

$$(\ln(\hat{C})+1)\ln\left(\frac{1}{\min\{\hat{\varepsilon},1/e\}}\right) = (\ln(\hat{C})+1)\ln\left(\frac{1}{\hat{\varepsilon}}\right) = \ln(\hat{C})\ln\left(\frac{1}{\hat{\varepsilon}}\right) + \ln\left(\frac{1}{\hat{\varepsilon}}\right) > \ln(\hat{C}) + \ln\left(\frac{1}{\hat{\varepsilon}}\right) = \ln\left(\frac{\hat{C}}{\hat{\varepsilon}}\right).$$

<u>Case 2:</u> If $\hat{\varepsilon} \geq 1/e$. We have $\ln(1/\hat{\varepsilon}) \leq 1$; therefore,

$$(\ln(\hat{C})+1)\ln\left(\frac{1}{\min\{\hat{\varepsilon},1/e\}}\right) = (\ln(\hat{C})+1)\ln\left(\frac{1}{1/e}\right) = \ln(\hat{C}) + 1 \geq \ln(\hat{C}) + \ln\left(\frac{1}{\hat{\varepsilon}}\right) = \ln\left(\frac{\hat{C}}{\hat{\varepsilon}}\right).$$

□

Appendix B. PROOF OF THEOREM 4.3

First note that in the remainders of the paper, for any bounded, non-negative random variable Z (i.e., $\exists C > 0 : \mathbb{P}(Z \in [0, C]) = 1$), any measurable function g on $\mathbb{R}$, we write $\int_0^\infty g(x)\mathrm{d}F_Z(x)$ instead of $\int_0^C g(x)\mathrm{d}F_Z(x)$ if there is no need to emphasize the bounds of Z. For the sake of notation, we also denote by $A_{=0}$ the event $\left\{\sum_{j\in[n]} A_j^* = 0\right\}$ and by $A_{>0}$ its complement event, that is $\left\{\sum_{j\in[n]} A_j^* > 0\right\}$. Similarly, we denote by $B_{=0}$ the event $\left\{\sum_{j\in[n]} B_j^* = 0\right\}$ and by $B_{>0}$ the event $\left\{\sum_{j\in[n]} B_j^* > 0\right\}$.

Recall the notation $F_{A_i^n}$ and $F_{B_i^n}$ as the univariate marginal distributions corresponding to battlefield $i \in [n]$ of the $\mathrm{IU}_A^{\gamma^*}$ and $\mathrm{IU}_B^{\gamma^*}$ strategies (the corresponding random variables are denoted A_i^n and B_i^n). Due to the definition of the IU^{γ^*} strategy (via Algorithm 1), for any $x \geq 0$ and $i \in [n]$, we have:

$$\begin{aligned} F_{A_i^n}(x) &= \mathbb{P}\left(\{A_i^n \leq x\} \bigcap A_{=0}\right) + \mathbb{P}\left(\{A_i^n \leq x\} \bigcap A_{>0}\right) \\ &= \mathbb{P}\left(A_{=0}\right) + \mathbb{P}\left(\left\{\frac{A_i^* \cdot X_A}{\sum_{j\in[n]} A_j^*} \leq x\right\} \bigcap A_{>0}\right). \end{aligned} \tag{B.1}$$

Here, we have used the fact that if $\sum_{j\in[n]} A_j^* = 0$ (i.e., when $A_{=0}$ happens), then $A_i^n = 0$ by definition and thus, $\mathbb{P}(A_i^n \leq x) = 1$ and $\mathbb{P}\left(\{A_i^n \leq x\} \bigcap A_{=0}\right) = \mathbb{P}(A_{=0})$. Similarly to (B.1), for any $x \geq 0$ and $i \in [n]$,

$$F_{B_i^n}(x) = \mathbb{P}\left(B_{=0}\right) + \mathbb{P}\left(\left\{\frac{B_i^* \cdot X_B}{\sum_{j\in[n]} B_j^*} \leq x\right\} \bigcap B_{>0}\right). \tag{B.2}$$

Regarding the random variables A_i^n and B_i^n ($i \in [n]$), we prepare a lemma stating several useful results as follows (its proof is given in Appendix B.1).

Lemma B1. *For any n and $i \in [n]$, we have*

(i) $\mathbb{P}(A_i^n = 0) = \mathbb{P}(A_i^* = 0)$ *and* $\mathbb{P}(B_i^n = 0) = \mathbb{P}(B_i^* = 0)$.

(ii) $\mathbb{P}(A_i^n = x) = \mathbb{P}(B_i^n = y) = 0$ *for any* $x \in (0, \infty)\backslash\{X_A\}$ *and* $y \in (0, \infty)\backslash\{X_B\}$.

(iii) $\mathbb{P}(A_i^n = X_A) \leq \left(1 - \frac{\lambda}{\bar{\lambda}} \frac{\underline{w}^2}{\bar{w}^2}\right)^{n-1}$ *and* $\mathbb{P}(B_i^n = X_B) \leq \left(1 - \frac{\lambda}{\bar{\lambda}} \frac{\underline{w}^2}{\bar{w}^2}\right)^{n-1}$.

Intuitively, Result *(ii)* states that the function $F_{A_i^n}$ (resp. $F_{B_i^n}$) is continuous on $(0, X_A)$ (resp. $(0, X_B)$). The discontinuity of $F_{A_i^n}$ (resp. $F_{B_i^n}$) at X_A (resp. at X_B) is due to the normalization step involved in the definition of the IU^{γ^*} strategy; note that the probability that $A_i^n = X_A$ (resp. $B_i^n = X_B$) quickly tends to zero when n increases as has been shown in Result *(iii)*. Finally, Result *(i)* shows that in some cases, $F_{A_i^n}$ and $F_{B_i^n}$ may be discontinuous at 0. This is due to the fact that the functions $F_{A_i^*}$ and $F_{B_i^*}$ may be discontinuous at 0. Moreover, recall that we chose the assignments of the outputs in line 3 and 7 of Algorithm 1 to be allocating zero to every battlefield, i.e., the mass at 0 of $F_{A_i^n}$ and $F_{B_i^n}$ is added by a (negligibly small) positive probability. While other assignments do not affect our results, they make $F_{A_i^n}$ (resp. $F_{B_i^n}$) be discontinuous at some points differing from 0 and X_A (resp. X_B), e.g., if in line 3 of Algorithm 1, we assign $x_i^A = X_A/n$, the distribution $F_{A_i^n}$ would also be discontinuous at the point X_A/n. Our choice of assignments provides more convenience in our analysis since we have to consider their discontinuity at 0 in any case.

Finally, with all the preparation steps mentioned above, we are ready to prove Theorem 4.3.

Theorem 4.3.

(i) In any game $\mathcal{CB}_n$, there exists a positive number $\varepsilon = \tilde{\mathcal{O}}(n^{-1/2})$ such that for any $\gamma^* \in \mathcal{S}_n$, the following inequalities hold for any pure strategy $\boldsymbol{x}^A$ and $\boldsymbol{x}^B$ of players A and B:

$$\Pi^A(\boldsymbol{x}^A, \mathrm{IU}_B^{\gamma^*}) \leq \Pi^A(\mathrm{IU}_A^{\gamma^*}, \mathrm{IU}_B^{\gamma^*}) + \varepsilon W_A, \tag{4.1}$$

$$\Pi^B(\mathrm{IU}_A^{\gamma^*}, \boldsymbol{x}^B) \leq \Pi^B(\mathrm{IU}_A^{\gamma^*}, \mathrm{IU}_B^{\gamma^*}) + \varepsilon W_B. \tag{4.2}$$

(ii) For any $\varepsilon \in (0,1]$, there exists $C^* > 0$ (that does not depend on ε) such that in any game $\mathcal{CB}_n$ with $n \geq C^* \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$, (4.1) and (4.2) hold for any $\gamma^* \in \mathcal{S}_n$, any pure strategy $\boldsymbol{x}^A$, $\boldsymbol{x}^B$ of players A and B.

Proof. In this section, we first give a proof of Result (*ii*) of Theorem 4.3. Result (*i*) will be deduced from (*ii*). We first look for the condition on n such that (4.1) holds for any pure strategy $\boldsymbol{x}^A$ of player A. The proof that (4.2) holds for any pure strategy of player B under the same condition can be done similarly and thus is omitted.

First, we write explicitly the payoffs of player A when player B plays the $\mathrm{IU}_B^{\gamma^*}$ strategy and player A plays either the pure strategy $\boldsymbol{x}^A$ or the $\mathrm{IU}_A^{\gamma^*}$ strategy:

$$\Pi^A(\boldsymbol{x}^A, \mathrm{IU}_B^{\gamma^*}) = \alpha \sum_{i=1}^n w_i^A \mathbb{P}(B_i^n = x_i^A) + \sum_{i=1}^n w_i^A \mathbb{P}(B_i^n < x_i^A), \tag{B.3}$$

$$\begin{aligned}\Pi^A(\mathrm{IU}_A^{\gamma^*}, \mathrm{IU}_B^{\gamma^*}) &= \alpha \sum_{i=1}^n w_i^A \mathbb{P}(B_i^n = A_i^n) + \sum_{i=1}^n w_i^A \mathbb{P}(B_i^n < A_i^n) \\ &= \alpha \sum_{i=1}^n \int_0^\infty w_i^A \mathbb{P}(B_i^n = x) \mathrm{d}F_{A_i^n}(x) + \sum_{i=1}^n \int_0^\infty w_i^A \mathbb{P}(B_i^n < x) \mathrm{d}F_{A_i^n}(x). \end{aligned} \tag{B.4}$$

We then prepare a useful lemma, its proof is given in Appendix B.2. Intuitively, this lemma shows that as n is large enough, we can prove (4.1) without the need of analyzing separately the case where players get tie allocations (that is our results hold regardless of the tie-breaking-rule parameter α).

Lemma B2. *Given $\varepsilon \in (0,1]$, there exists a constant $C_0^* > 0$ (that does not depend on ε) such that for any $n \geq C_0^* \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$, for any game $\mathcal{CB}_n$ and $\gamma^* \in \mathcal{S}_n$ the following inequality is a sufficient condition of (4.1):*

$$\sum_{i=1}^n v_i^A F_{B_i^n}\left(x_i^A\right) \leq \sum_{i=1}^n \int_0^\infty v_i^A F_{B_i^n}(x) \mathrm{d}F_{A_i^n}(x) + \frac{\varepsilon}{2}. \tag{B.5}$$

In the remainders of the proof, we focus on (B.5) and look for the condition of n such that it holds; this will be done in the following five steps. After that, from Lemma B2, we can conclude that (4.1) also holds with the corresponding condition on n.

<u>Step 1:</u> Prove that $\{F_{A_i^*}\}_i$ is optimal against $\{F_{B_i^*}\}_i$.

Lemma B3. *In any game $\mathcal{CB}_n$, for any pure strategy $\boldsymbol{x}^A$ of player A and $\gamma^* \in \mathcal{S}_n$, we have*

$$\sum_{i=1}^n v_i^A F_{B_i^*}\left(x_i^A\right) \leq \sum_{i=1}^n \int_0^\infty v_i^A F_{B_i^*}(x) \mathrm{d}F_{A_i^*}(x). \tag{B.6}$$

The proof of Lemma B3 is given in Appendix B.3. This lemma can be interpreted as follows: if the allocation of player B to battlefield i follows the distribution $F_{B_i^*}$, then it is optimal for player A to play such that her allocation at this battlefield follows $F_{A_i^*}$ (we do not know if it is possible to construct a mixed strategy such that player A's allocation at battlefield i follows $F_{A_i^*}$ for all $i \in [n]$; however, this does not affect our results in this work). Using this lemma, we will analyze the validity of (B.5) by proving that, as $n \to \infty$, the terms in (B.5) respectively converge toward the terms in (B.6). To do this, we consider the next step.

<u>Step 2:</u> Prove that $F_{A_i^n}$ and $F_{B_i^n}$ uniformly converge toward $F_{A_i^*}$ and $F_{B_i^*}$ as n increases.

Lemma B4. *For any $\varepsilon_1 \in (0,1]$, there exists $C_1 > 0$ (that does not depend on ε_1) such that for any $n \geq C_1 \varepsilon_1^{-2} \ln\left(\frac{1}{\min\{\varepsilon_1, 1/e\}}\right)$ and $i \in [n]$,*

$$\sup_{x \in [0,\infty)} \left| F_{A_i^n}(x) - F_{A_i^*}(x) \right| \leq \varepsilon_1 \quad \text{and} \quad \sup_{x \in [0,\infty)} \left| F_{B_i^n}(x) - F_{B_i^*}(x) \right| \leq \varepsilon_1. \tag{B.7}$$

A proof of this lemma is given in Appendix B.4. The main intuition of this result comes from the fact that A_i^n (resp. B_i^n) is the normalization of $A_i^*, i \in [n]$ (except for the special cases of the events $A_{=0}$ and $B_{=0}$) and the use of concentration inequalities on the random variables $\sum_{j\in[n]} A_j^*$ (and $\sum_{j\in[n]} B_j^*$). In this work, we apply the Hoeffding's inequality (Theorem 2, Hoeffding (1963)) to obtain the rate of convergence indicated here in Lemma B4.

<u>Step 3:</u> *Prove that the left-hand-side of* (B.5) *converges toward the left-hand-side of* (B.6). Take C_1 as indicated in Lemma B4, we define $C_1^* := 16C_1(\ln(4)+1)$ and deduce that $\frac{C_1^*}{\varepsilon^2}\ln\left(\frac{1}{\min\{\varepsilon,1/e\}}\right) \geq C_1\left(\frac{4}{\varepsilon}\right)^2 \ln\left(\frac{1}{\min\{\frac{\varepsilon}{4},\frac{1}{e}\}}\right)$.[19] Therefore, take $\varepsilon_1 := \varepsilon/4$, for any $n \geq C_1^*\varepsilon^{-2}\ln\left(\frac{1}{\min\{\varepsilon,1/e\}}\right)$, we have $n \geq C_1\varepsilon_1^{-2}\ln\left(\frac{1}{\min\{\varepsilon_1,1/e\}}\right)$; apply Lemma B4, for any pure strategy $\boldsymbol{x}^A$ of player A, we have

$$\begin{aligned}\left|\sum_{i=1}^n v_i^A F_{B_i^n}\left(x_i^A\right) - \sum_{i=1}^n v_i^A F_{B_i^*}\left(x_i^A\right)\right| &\leq \sum_{i=1}^n v_i^A \sup_{x\in[0,\infty)} \left|F_{B_i^n}(x) - F_{B_i^*}(x)\right| \\ &\leq \sum_{i=1}^n v_i^A \frac{\varepsilon}{4} = \frac{\varepsilon}{4}. \end{aligned} \tag{B.8}$$

<u>Step 4:</u> *Prove that the right-hand-side of* (B.5) *converges toward the right-hand-side of* (B.6). We consider the difference of the involved terms as follows.

$$\begin{aligned}&\left|\sum_{i=1}^n \int_0^\infty v_i^A F_{B_i^n}(x)\mathrm{d}F_{A_i^n}(x) - \sum_{i=1}^n \int_0^\infty v_i^A F_{B_i^*}(x)\mathrm{d}F_{A_i^*}(x)\right| \\ \leq &\sum_{i=1}^n v_i^A \int_0^\infty \left|F_{B_i^n}(x) - F_{B_i^*}(x)\right| \mathrm{d}F_{A_i^n}(x) + \sum_{i=1}^n v_i^A \left|\int_0^\infty F_{B_i^*}(x)\mathrm{d}F_{A_i^n}(x) - \int_0^\infty F_{B_i^*}(x)\mathrm{d}F_{A_i^*}(x)\right|. \end{aligned}\tag{B.9}$$

Let us define $C_2^* := C_1 \cdot 64(\ln(8)+1)$ (again, C_1 is the constant indicated in Lemma B4), we have that $C_2^*\varepsilon^{-2}\ln\left(\frac{1}{\min\{\varepsilon,1/e\}}\right) \geq C_1\left(\frac{8}{\varepsilon}\right)^2\ln\left(\frac{1}{\min\{\frac{\varepsilon}{8},\frac{1}{e}\}}\right)$.[20] Therefore, take $\varepsilon_1 := \varepsilon/8$, for any $n \geq C_2^*\varepsilon^{-2}\ln\left(\frac{1}{\min\{\varepsilon,1/e\}}\right)$, we have $n \geq \varepsilon_1^{-2}\ln\left(\frac{1}{\min\{\varepsilon_1,1/e\}}\right)$ and by Lemma B4, we have

$$\sum_{i=1}^n v_i^A \int_0^\infty \left|F_{B_i^n}(x) - F_{B_i^*}(x)\right| \mathrm{d}F_{A_i^n}(x) \leq \sum_{i=1}^n v_i^A \int_0^\infty \frac{\varepsilon}{8}\mathrm{d}F_{A_i^n}(x) = \sum_{i=1}^n v_i^A \frac{\varepsilon}{8}. \tag{B.10}$$

Now, we need to find an upper-bound of the second term in the right-hand-side of(B.9). To do this, we present a lemma, called Lemma B5 (stated below), that is based on the portmanteau lemma (see, e.g., Van der Vaart (2000)) regarding the weak convergence of a sequence of measures. Note importantly that by a direct application of the portmanteau lemma (since $F_{B_i^*}$ is Lipschitz continuous and from Lemma B4, $F_{A_i^n}$ uniformly converges to $F_{A_i^*}$), we can prove that $\int_0^\infty F_{B_i^*}(x)\mathrm{d}F_{A_i^n}(x)$ converges toward $\int_0^\infty F_{B_i^*}(x)\mathrm{d}F_{A_i^*}(x)$ as $n\to\infty$; however, note that the convergence rate obtained by doing this is large due to the fact that the Lipschitz constant of $F_{B_i^*}$ (that is λ_A^*/v_i^A) increases as n increases. To obtain a better convergence rate as indicated in Lemma B5, we exploit the properties of the involved functions that allow us to use the telescoping sum trick (see Appendix B.5 for more details).

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[19]This is due to $C_1^*\cdot\varepsilon^{-2}\ln\left(\frac{1}{\min\{\varepsilon,1/e\}}\right) = C_1\left(\frac{4}{\varepsilon}\right)^2\cdot(\ln(4)+1)\ln\left(\frac{1}{\min\{\varepsilon,1/e\}}\right) \geq C_1\cdot\left(\frac{4}{\varepsilon}\right)^2\ln\left(\frac{4}{\varepsilon}\right)$; here, we have applied Lemma A2 with $\hat{\varepsilon} := \varepsilon$ and $\hat{C} := 4$; moreover, $\frac{\varepsilon}{4} = \min\{\frac{\varepsilon}{4},\frac{1}{e}\}$ since $\varepsilon \leq 1$; thus, we can rewrite $\ln\left(\frac{4}{\varepsilon}\right) = \ln\left(\frac{1}{\min\{\varepsilon/4,1/e\}}\right)$.

[20]This is due to $C_2^*\cdot\varepsilon^{-2}\ln\left(\frac{1}{\min\{\varepsilon,1/e\}}\right) = C_1\left(\frac{8}{\varepsilon}\right)^2\cdot(\ln(8)+1)\ln\left(\frac{1}{\min\{\varepsilon,1/e\}}\right) \geq C_1\cdot\left(\frac{8}{\varepsilon}\right)^2\ln\left(\frac{8}{\varepsilon}\right)$; here, we have applied Lemma A2 with $\hat{\varepsilon} := \varepsilon$ and $\hat{C} := 8$; moreover, $\frac{\varepsilon}{8} = \min\{\frac{\varepsilon}{8},\frac{1}{e}\}$ since $\varepsilon \leq 1$; thus, we can rewrite $\ln\left(\frac{8}{\varepsilon}\right) = \ln\left(\frac{1}{\min\{\varepsilon/8,1/e\}}\right)$.

Lemma B5. *For any $\varepsilon_2 \in (0,1]$, there exists a constant $C_2 > 0$ (that does not depend on ε_2) such that for any $n \geq C_2 \cdot \varepsilon_2^{-2} \ln\left(\frac{1}{\min\{\varepsilon_2, 1/e\}}\right)$ and $i \in [n]$, we have*

$$\left| \int_0^\infty F_{B_i^*}(x) \mathrm{d}F_{A_i^n}(x) - \int_0^\infty F_{B_i^*}(x) \mathrm{d}F_{A_i^*}(x) \right| \leq \varepsilon_2. \tag{B.11}$$

The proof of Lemma B5 is given in Appendix B.5. Based on this constant C_2, we define $C_3^* := 8^2 C_2(\ln 8 + 1)$. Now, take $\varepsilon_2 := \varepsilon/8$, we have that $C_3^* \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right) \geq C_2 \varepsilon_2^{-2} \ln\left(\frac{1}{\min\{\varepsilon_2, 1/e\}}\right)$;[21] therefore, for any $n \geq C_3^* \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$, we have $n \geq C_2 \varepsilon_2^{-2} \ln\left(\frac{1}{\min\{\varepsilon_2, 1/e\}}\right)$ and by Lemma B5, we deduce

$$\left| \int_0^\infty F_{B_i^*}(x) \mathrm{d}F_{A_i^n}(x) - \int_0^\infty F_{B_i^*}(x) \mathrm{d}F_{A_i^*}(x) \right| \leq \varepsilon/8.$$

Combine this with (B.9) and (B.10), for any $n = \max\{C_2^*, C_3^*\} \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$, we have

$$\left| \sum_{i=1}^n \int_0^\infty v_i^A F_{B_i^n}(x) \mathrm{d}F_{A_i^n}(x) - \sum_{i=1}^n \int_0^\infty v_i^A F_{B_i^*}(x) \mathrm{d}F_{A_i^*}(x) \right| \leq \sum_{i=1}^n v_i^A \varepsilon/8 + \sum_{i=1}^n v_i^A \varepsilon/8 = \frac{\varepsilon}{4}. \tag{B.12}$$

<u>*Step 5:*</u> *Conclusion.* For any $n \geq \max\{C_1^*, C_2^*, C_3^*\} \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$ and any pure strategy $\boldsymbol{x}^A$ of player A, we conclude that

$$\begin{aligned}
\sum_{i=1}^n v_i^A F_{B_i^n}\left(x_i^A\right) &\leq \sum_{i=1}^n v_i^A F_{B_i^*}\left(x_i^A\right) + \frac{\varepsilon}{4} && \text{(from (B.8))} \\
&\leq \sum_{i=1}^n \int_0^\infty v_i^A F_{B_i^*}(x) \mathrm{d}F_{A_i^*}(x) + \frac{\varepsilon}{4} && \text{(from (B.6))} \\
&\leq \sum_{i=1}^n \int_0^\infty v_i^A F_{B_i^n}(x) \mathrm{d}F_{A_i^n}(x) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} && \text{(from (B.12))} \\
&= \sum_{i=1}^n \int_0^\infty v_i^A F_{B_i^n}(x) \mathrm{d}F_{A_i^n}(x) + \frac{\varepsilon}{2}.
\end{aligned}$$

This is exactly (B.5); therefore, applying Lemma B2 (involving C_0^*), denote $C_{(4.1)}^* := \max\{C_0^*, C_1^*, C_2^*, C_3^*\}$, we have proved that (4.1) holds for any $n \geq C_{(4.1)}^* \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$. Similarly, we can prove that there exists a constant $C_{(4.2)}^*$ such that (4.2) holds for any $n \geq C_{(4.2)}^* \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$. Finally, define $C^* := \max\{C_{(4.1)}^*, C_{(4.2)}^*\}$, we conclude the proof for Result (ii).

Now, to obtain Result (i), we prove that Result (ii) implies Result (i). Note that the constant C^* found in the Result (ii) does not depend on neither n nor ε. Moreover, the function

$$\begin{aligned}
\xi \colon (0, \infty) &\to (0, \infty) \\
\tilde{\varepsilon} &\mapsto C^* \tilde{\varepsilon}^{-2} \ln\left(\frac{1}{\min\{\tilde{\varepsilon}, 1/e\}}\right).
\end{aligned}$$

[21] Once again, apply Lemma A2, $C_3^* \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right) = C_2 \left(\frac{8}{\varepsilon}\right)^2 (\ln(8)+1) \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right) \geq C_2 \left(\frac{8}{\varepsilon}\right)^2 \ln\left(\frac{8}{\varepsilon}\right)$; moreover, we have $\varepsilon_2 := \frac{\varepsilon}{8} = \min\left\{\frac{\varepsilon}{8}, \frac{1}{e}\right\}$.

is continuous and increases to infinity when ε tends to zero. Therefore, for any $n \geq 1$, there exists an $\varepsilon > 0$ such that $n = C^*\varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$. Now, apply Result (ii), (4.1) and (4.2) hold in the game $\mathcal{CB}_n$ for any $\gamma^* \in \mathcal{S}_n$ and pure strategies $\boldsymbol{x}^A, \boldsymbol{x}^B$. We conclude the proof by notice that if $\varepsilon \geq 1/e$, we have $n = C^*\varepsilon^{-2}$ and thus $\varepsilon = \sqrt{n/C^*} = \mathcal{O}(n^{-1/2})$; on the other hand, if $\varepsilon < \frac{1}{e}$, we have $\ln\left(\frac{1}{\varepsilon}\right) > 1$ that induces $n = C^*\varepsilon^{-2} \ln\left(\frac{1}{\varepsilon}\right) \geq C^*\varepsilon^{-2} \geq \frac{C^*}{\varepsilon}$, thus, $\frac{1}{\varepsilon} \leq \frac{n}{C^*}$. We deduce that $\varepsilon = \sqrt{\frac{C^*}{n} \ln\left(\frac{1}{\varepsilon}\right)} \leq \sqrt{\frac{C^*}{n} \ln\left(\frac{n}{C^*}\right)} = \tilde{\mathcal{O}}(n^{-1/2})$.

□

Appendix B.1. Proof of Lemma B1

(i) Assuming $A_i^* = 0$, if $\sum_{j \neq i} A_j^* = 0$ then $A_i^n = 0$ (due to line 3 of Algorithm 1) and if $\sum_{j \neq i} A_j^* > 0$ then $A_i^n = A_i^* / \sum_{j \in [n]} A_j^* = 0$. Reversely, assuming $A_i^n = 0$, then regardless whether $\sum_{j \in [n]} A_j^* = 0$ or $\sum_{j \in [n]} A_j^* > 0$, we have $A_i^* = 0$. Therefore, $A_i^n = 0 \Leftrightarrow A_i^* = 0$ for any n and $i \in [n]$. Similarly, we can prove that $B_i^n = 0 \Leftrightarrow B_i^* = 0$.

(ii) The results are trivial in cases where $x > X_A$ and $y > X_B$ due to the definition of A_i^n and B_i^n (that guarantees that with probability 1, $A_i^n \leq X_A$ and $B_i^n \leq X_B$). In the following, we consider the case where $x \in (0, X_A)$. For any n, $i \in [n]$, we denote $Z_i := \sum_{j \neq i} A_j^*$ and obtain:

$$
\begin{aligned}
&\mathbb{P}(A_i^n = x) \\
=&\mathbb{P}\left(\{A_i^n = x\} \bigcap A_{>0}\right) && \text{(since } x > 0\text{)} \\
=&\mathbb{P}\left(\left\{A_i^* = \frac{x}{X_A} \sum\nolimits_{j \in [n]} A_j^*\right\} \bigcap A_{>0}\right) \\
=&\mathbb{P}\left(\left\{A_i^* \left(1 - \frac{x}{X_A}\right) = \frac{x}{X_A} \sum\nolimits_{j \neq i} A_j^*\right\} \bigcap A_{>0}\right) \\
=&\mathbb{P}\left(\left\{A_i^* = \frac{Z_i \cdot x}{X_A - x}\right\} \bigcap A_{>0}\right) && \text{(note that } X_A - x > 0\text{)} \\
\leq&\mathbb{P}(\{A_i^* = Z_i = 0\} \cap A_{>0}) + \int_{z>0} \mathbb{P}\left(A_i^* = \frac{z \cdot x}{X_A - x}\right) \mathrm{d}F_{Z_i}(z) \\
\leq&\mathbb{P}(A_{=0} \cap A_{>0}) + \int_{z>0} 0 \ \mathrm{d}F_{Z_i}(z) \\
=&0.
\end{aligned}
$$

Here, the second-to-last inequality comes from the fact that $\frac{zx}{X_A - x} > 0, \forall z > 0, \forall x \in (0, X_A)$ and $\mathbb{P}(A_i^* = a) = 0$ for any $a > 0$. Similarly, we can prove that $\mathbb{P}(B_i^n = y) = 0$ for any $y \in (0, X_B)$.

(iii) We have

$$
\begin{aligned}
\mathbb{P}(A_i^n = X_A) &= \mathbb{P}\left(\left\{A_i^* = \sum_{j \in [n]} A_j^*\right\} \bigcap A_{>0}\right) \\
&\leq \mathbb{P}\left(\sum_{j \neq i} A_j^* = 0\right) \\
&= \prod_{j \neq i} \mathbb{P}\left(A_j^* = 0\right) && \text{(since } A_j^*, j \in [n] \text{ are non-negative and independent).}
\end{aligned}
$$

Now, if there exists $j \neq i$ such that $j \in \Omega_A(\gamma^*)$, then $\mathbb{P}(A_j^* = 0) = 0$ due to the fact that $A_j^* = A_{\gamma^*,j}^S$ and the definition of $A_{\gamma^*,j}^S$ (see (3.4)). In this case, $\prod_{j \neq i} \mathbb{P}\left(A_j^* = 0\right) = 0$. On the other hand, if

$j \notin \Omega_A(\gamma^*)$ for any $j \neq i$, then $A_j^* = A_{\gamma^*,j}^W$ for $j \neq i$; therefore,

$$\prod_{j\neq i} \mathbb{P}\left(A_j^* = 0\right) = \prod_{j\neq i}\left[\left(\frac{v_j^B}{\lambda_B^*} - \frac{v_j^A}{\lambda_A^*}\right)\bigg/\frac{v_j^B}{\lambda_B^*}\right] = \prod_{j\neq i}\left(1 - \frac{v_j^A}{v_j^B}\frac{\lambda_B^*}{\lambda_A^*}\right) \leq \left(1 - \frac{\underline{\lambda}}{\overline{\lambda}}\frac{\frac{\underline{w}}{n\bar{w}}}{\frac{\bar{w}}{n\underline{w}}}\right)^{n-1} = \left(1 - \frac{\underline{\lambda}}{\overline{\lambda}}\frac{\underline{w}^2}{\bar{w}^2}\right)^{n-1}.$$

Here, to obtain the last equality, we use (2.2) for the bounds of v_j^A, v_j^B and Proposition 3.3 for the bounds of λ_A^*, λ_B^*.

Similarly, we can obtain $\mathbb{P}(B_i^n = X_B) \leq \left(1 - \frac{\underline{\lambda}}{\overline{\lambda}}\frac{\underline{w}^2}{\bar{w}^2}\right)^{n-1}$.

Appendix B.2. Proof of Lemma B2

Fix $\varepsilon \in (0, 1]$ and assume that (B.5) is satisfied, we prove that (4.1) also holds by comparing the terms in (B.5) with the terms in (4.1). First, due to the fact that $\alpha \leq 1$, we can find a lower bound of the left-hand side of (B.5) as follows:

$$\begin{aligned}\sum_{i=1}^n v_i^A F_{B_i^n}\left(x_i^A\right) &= \sum_{i=1}^n v_i^A \mathbb{P}(B_i^n = x_i^A) + \sum_{i=1}^n v_i^A \mathbb{P}(B_i^n < x_i^A)\\ &\geq \alpha \sum_{i=1}^n v_i^A \mathbb{P}(B_i^n = x_i^A) + \sum_{i=1}^n v_i^A \mathbb{P}(B_i^n < x_i^A)\\ &= \Pi^A(\boldsymbol{x}^A, \mathrm{IU}_B^{\gamma^*})/W_A. \end{aligned} \tag{B.13}$$

Now, we turn our focus to the right-hand-side of (B.5), we can rewrite the involved term as follows.

$$\sum_{i=1}^n \int_0^\infty v_i^A F_{B_i^n}(x)\mathrm{d}F_{A_i^n}(x) = \sum_{i=1}^n \int_0^\infty v_i^A \mathbb{P}(B_i^n = x)\mathrm{d}F_{A_i^n}(x) + \sum_{i=1}^n \int_0^\infty v_i^A \mathbb{P}(B_i^n < x)\mathrm{d}F_{A_i^n}(x).$$

We observe that $\sum_{i=1}^n \int_0^\infty v_i^A F_{B_i^n}(x)\mathrm{d}F_{A_i^n}(x)$ is very similar to the expression of $\Pi^A(\boldsymbol{x}^A, \mathrm{IU}_B^{\gamma^*})$ stated in (B.4). The main difference lies at the coefficient of the term related to the tie cases that is the tie-breaking parameter α. Therefore, we consider the following two cases of α:

Case 1: $\alpha = 1$. For any n, divide two sides of (B.4) (with $\alpha = 1$) by W_A and recall that $v_i^A := w_i^A/W_A, \forall i$, we trivially have $\sum_{i=1}^n \int_0^\infty v_i^A F_{B_i^n}(x)\mathrm{d}F_{A_i^n}(x) = \Pi^A(\mathrm{IU}_A^{\gamma^*}, \mathrm{IU}_B^{\gamma^*})/W_A$.

Case 2: $\alpha < 1$. Due to Results *(ii)* and *(iii)* of Lemma B1, for any $x > 0$, we have $\mathbb{P}(B_i^n = x) \leq D^{n-1}$ where we define $D := \left(1 - \frac{\underline{\lambda}}{\overline{\lambda}}\frac{\underline{w}^2}{\bar{w}^2}\right) < 1$. We consider two cases of α as follows.

- If $2(1-\alpha) \leq 1$, define $\hat{C}_1 := \frac{1}{\ln(1/D)} + 1 > 0$, we have that[22] $\hat{C}_1 \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right) \geq \log_D \varepsilon + 1$; therefore, for any $n \geq \hat{C}_1 \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$, we obtain $n - 1 \geq \log_D \varepsilon$ and

$$D^{n-1} \leq D^{\log_D \varepsilon} = \varepsilon \leq \frac{\varepsilon}{2(1-\alpha)} \qquad \text{(note that } D < 1 \text{ and in this case } 2(1-\alpha) \leq 1).$$

[22] If $\varepsilon < 1/e$, then $\ln(1/\varepsilon) > 1$ and $\hat{C}_1 \ln\left(\frac{1}{\min\{\varepsilon,1/e\}}\right) = \frac{\ln(1/\varepsilon)}{\ln(1/D)} + \ln(1/\varepsilon) > \log_D \varepsilon + 1$; otherwise, if $\varepsilon \geq 1/e$, we have $\ln(1/\varepsilon) \leq 1$ and $\hat{C}_1 \ln\left(\frac{1}{\min\{\varepsilon,1/e\}}\right) = \frac{1}{\ln(1/D)} + 1 \geq \frac{\ln(1/\varepsilon)}{\ln(1/D)} + 1 = \log_D \varepsilon + 1$ (note that $\ln(1/D) > 0$ since $D < 1$).

- If $2(1-\alpha)>1$, define $\hat{C}_2 := \frac{1}{\ln(1/D)} + \frac{\ln(2-2\alpha)}{\ln(1/D)} + 1 > 0$; we have $\hat{C}_2 \ln\left(\frac{1}{\min\{\varepsilon,1/e\}}\right) \geq \log_D \frac{\varepsilon}{2(1-\alpha)} + 1$.[23] We conclude that for any $n \geq \hat{C}_2 \ln\left(\frac{1}{\min\{\varepsilon,1/e\}}\right)$, we obtain $n-1 \geq \log_D\left(\frac{\varepsilon}{2(1-\alpha)}\right)$ and

$$D^{n-1} \leq D^{\log_D \frac{\varepsilon}{2(1-\alpha)}} = \frac{\varepsilon}{2(1-\alpha)}.$$

Let us define $C_0^* = \max\{\hat{C}_1, \hat{C}_2\} > 0$, we conclude that for any $\alpha < 1$, $n \geq C_0^* \ln\left(\frac{1}{\min\{\varepsilon,1/e\}}\right)$, $i \in [n]$ and $x > 0$, we have

$$\mathbb{P}(B_i^n = x) \leq D^{n-1} \leq \frac{\varepsilon}{2(1-\alpha)}. \tag{B.14}$$

Note also that $\mathbb{P}(A_i^n = B_i^n = 0) = \mathbb{P}(A_i^n = 0)\mathbb{P}(B_i^n = 0) = \mathbb{P}(A_i^* = 0)\mathbb{P}(B_i^* = 0) = 0, \forall i,$[24] we conclude that when $\alpha < 1$, for any $n \geq C_0^* \ln\left(\frac{1}{\min\{\varepsilon,1/e\}}\right)$, we have

$$\begin{aligned}
&\sum_{i=1}^n \int_0^\infty v_i^A F_{B_i^n}(x) \mathrm{d}F_{A_i^n}(x)\\
=&\left[\sum_{i=1}^n \int_0^\infty v_i^A \mathbb{P}(B_i^n < x) \mathrm{d}F_{A_i^n}(x) + \alpha \sum_{i=1}^n \int_0^\infty v_i^A \mathbb{P}(B_i^n = x) \mathrm{d}F_{A_i^n}(x)\right] + (1-\alpha)\sum_{i=1}^n \int_0^\infty v_i^A \mathbb{P}(B_i^n = x) \mathrm{d}F_{A_i^n}(x)\\
=&\Pi^A(\mathrm{IU}_A^{\gamma^*}, \mathrm{IU}_B^{\gamma^*})/W_A + (1-\alpha)\sum_{i=1}^n v_i^A \int_{(0,\infty)} \frac{\varepsilon}{2(1-\alpha)} \mathrm{d}F_{A_i^n}(x) + (1-\alpha)\sum_{i=1}^n v_i^A \mathbb{P}(A_i^n = B_i^n = 0)\\
=&\Pi^A(\mathrm{IU}_A^{\gamma^*}, \mathrm{IU}_B^{\gamma^*})/W_A + (1-\alpha)\sum_{i=1}^n v_i^A \int_{(0,\infty)} \frac{\varepsilon}{2(1-\alpha)} \mathrm{d}F_{A_i^n}(x) + 0\\
\leq&\Pi^A(\mathrm{IU}_A^{\gamma^*}, \mathrm{IU}_B^{\gamma^*})/W_A + (1-\alpha)\frac{\varepsilon}{2(1-\alpha)}\\
=&\Pi^A(\mathrm{IU}_A^{\gamma^*}, \mathrm{IU}_B^{\gamma^*})/W_A + \varepsilon/2.
\end{aligned}$$

In conclusion, regardless of the value of α, for any $n \geq C_0^* \ln\left(\frac{1}{\min\{\varepsilon,1/e\}}\right)$, we have

$$\sum_{i=1}^n \int_0^\infty v_i^A F_{B_i^n}(x) \mathrm{d}F_{A_i^n}(x) \leq \Pi^A(\mathrm{IU}_A^{\gamma^*}, \mathrm{IU}_B^{\gamma^*})/W_A + \varepsilon/2. \tag{B.15}$$

Combine (B.13), (B.15) and the assumption that (B.5) holds, for any $n \geq C_0^* \ln\left(\frac{1}{\min\{\varepsilon,1/e\}}\right)$, we have

$$\frac{\Pi^A(\boldsymbol{x}^A, \mathrm{IU}_B^{\gamma^*})}{W_A} \overset{(B.13)}{\leq} \sum_{i=1}^n v_i^A F_{B_i^n}\left(x_i^A\right) \overset{(B.5)}{\leq} \sum_{i=1}^n \int_0^\infty v_i^A F_{B_i^n}(x) \mathrm{d}F_{A_i^n}(x) + \varepsilon/2 \overset{(B.15)}{\leq} \frac{\Pi^A(\mathrm{IU}_A^{\gamma^*}, \mathrm{IU}_B^{\gamma^*})}{W_A} + \varepsilon.$$

By multiplying both sides of this inequality by W_A, we obtain (4.1). □

[23] If $\varepsilon < 1/e$, we have $\hat{C}_2 \ln\left(\frac{1}{\min\{\varepsilon,1/e\}}\right) = \log_D \varepsilon + \left(\log_{1/D}(2-2\alpha) + 1\right) \ln\left(\frac{1}{\varepsilon}\right) > \log_D \varepsilon - \log_D(2-2\alpha) + 1$; otherwise, if $\varepsilon \geq 1/e$, we have $\hat{C}_2 \ln\left(\frac{1}{\min\{\varepsilon,1/e\}}\right) = \hat{C}_2 \geq \frac{\ln(1/\varepsilon)}{\ln(1/D)} + \frac{\ln(2-2\alpha)}{\ln(1/D)} + 1 \geq \log_D \varepsilon - \log_D(2-2\alpha) + 1$ (since $1 \geq \ln\left(\frac{1}{\varepsilon}\right)$).

[24] Note that if $i \in \Omega_A(\gamma^*)$ then $\mathbb{P}(A_i^* = 0) = 0$, if $i \notin \Omega_A(\gamma^*)$ then $\mathbb{P}(B_i^* = 0) = 0$ (see (3.4)-(3.8)); therefore, $\mathbb{P}(A_i^* = 0)\mathbb{P}(A_i^* = 0) = 0, \forall i$.

Appendix B.3. Proof of Lemma B3

We compute the right-hand-side of (B.6) based on the definition of $F_{A_i^*}$ and $F_{B_i^*}$ (see Definition 3.2).

$$\begin{aligned}\sum_{i=1}^{n}\int_0^\infty v_i^A F_{B_i^*}(x)\mathrm{d}F_{A_i^*}(x) &= \sum_{i\in\Omega_A(\gamma^*)}\int_0^\infty v_i^A F_{B^W_{\gamma^*,i}}(x)\mathrm{d}F_{A^S_{\gamma^*,i}}(x) + \sum_{i\notin\Omega_A(\gamma^*)}\int_0^\infty v_i^A F_{B^S_{\gamma^*,i}}(x)\mathrm{d}F_{A^W_{\gamma^*,i}}(x)\\ &= \sum_{i\in\Omega_A(\gamma^*)}\int_0^{\frac{v_i^B}{\lambda_B^*}} v_i^A\left(\frac{\frac{v_i^A}{\lambda_A^*}-\frac{v_i^B}{\lambda_B^*}}{\frac{v_i^A}{\lambda_A^*}}+\frac{x}{\frac{v_i^A}{\lambda_A^*}}\right)\frac{1}{\frac{v_i^B}{\lambda_B^*}}\mathrm{d}x + \sum_{i\notin\Omega_A(\gamma^*)}\int_0^{\frac{v_i^A}{\lambda_A^*}} v_i^A\frac{x}{\frac{v_i^A}{\lambda_A^*}}\frac{1}{\frac{v_i^B}{\lambda_B^*}}\mathrm{d}x\\ &= \sum_{i\in\Omega_A(\gamma^*)} v_i^A\left(1-\frac{v_i^B\gamma^*}{2v_i^A}\right)+\sum_{i\notin\Omega_A(\gamma^*)}(v_i^A)^2\frac{1}{2\gamma^* v_i^B}. \qquad \text{(B.16)}\end{aligned}$$

On the other hand, for any pure strategy $\boldsymbol{x}^A$ of player A, we have:

$$\begin{aligned}\sum_{i=1}^{n} v_i^A F_{B_i^*}\left(x_i^A\right) &= \sum_{i\in\Omega_A(\gamma^*)} v_i^A F_{B^W_{\gamma^*,i}}\left(x_i^A\right)+\sum_{i\notin\Omega_A(\gamma^*)} v_i^A F_{B^S_{\gamma^*,i}}\left(x_i^A\right)\\ &\le \sum_{i\in\Omega_A(\gamma^*)} v_i^A\left(\frac{\frac{v_i^A}{\lambda_A^*}-\frac{v_i^B}{\lambda_B^*}}{\frac{v_i^A}{\lambda_A^*}}+\frac{x_i^A\lambda_A^*}{v_i^A}\right)+\sum_{i\notin\Omega_A(\gamma^*)} v_i^A\left(\frac{x_i^A\lambda_A^*}{v_i^A}\right)\\ &\le \sum_{i\in\Omega_A(\gamma^*)}\left(\frac{v_i^A}{\lambda_A^*}-\frac{v_i^B}{\lambda_B^*}\right)\lambda_A^*+\lambda_A^* X_A \qquad \left(\text{since } \textstyle\sum_{i=1}^n x_i^A\le X_A\right)\\ &= \sum_{i\in\Omega_A(\gamma^*)} v_i^A\left(1-\frac{v_i^B\gamma^*}{2v_i^A}\right)+\sum_{i\notin\Omega_A(\gamma^*)}(v_i^A)^2\frac{1}{2\gamma^* v_i^B}. \qquad \text{(B.17)}\end{aligned}$$

Here, to obtain the last equality, we use (A.1) to rewrite X_A. Finally, from (B.16) and (B.17), we conclude that (B.6) holds for any $\boldsymbol{x}^A$ and γ^*. □

Appendix B.4. Proof of Lemma B4

Since the definition of $F_{A_i^n}$ involves $\mathbb{P}(A_{=0})$ (see (B.1)), we first look for an upper-bound of $\mathbb{P}(A_{=0})$. For any n and $\gamma^*\in\mathcal{S}_n$, if $\Omega_A(\gamma^*)\neq\emptyset$, i.e., there exists i such that $A_i^* = A^S_{\gamma^*,i}$, then $\mathbb{P}(A_i^*=0)=0$ due to the definition of $A^S_{\gamma^*,i}$ (see (3.4)); in this case, $\mathbb{P}(A_{=0})=\prod_{j\in[n]}\mathbb{P}\left(A_j^*=0\right)=0$. On the other hand, if $\Omega_A(\gamma^*)=\emptyset$, then $A_j^*=A^W_{\gamma^*,j}$ for any $j\in[n]$; therefore,

$$\mathbb{P}(A_{=0})=\prod_{j\in[n]}\mathbb{P}\left(A_j^*=0\right)=\prod_{j\in[n]}\left[\left(\frac{v_j^B}{\lambda_B^*}-\frac{v_j^A}{\lambda_A^*}\right)\Big/\frac{v_j^B}{\lambda_B^*}\right]=\prod_{j\in[n]}\left(1-\frac{v_j^A}{v_j^B}\frac{\lambda_B^*}{\lambda_A^*}\right)\le\left(1-\frac{\underline{\lambda}}{\overline{\lambda}}\frac{\underline{w}^2}{\overline{w}^2}\right)^n. \qquad \text{(B.18)}$$

Here, the last inequality comes directly from (2.2) and Proposition 3.3. Recall the notation $D:=\left(1-\frac{\underline{\lambda}}{\overline{\lambda}}\frac{\underline{w}^2}{\overline{w}^2}\right)$ and define $\tilde{C}_0:=\frac{\ln(4)+1}{\ln(1/D)}>0$, we have $\tilde{C}_0\ln\left(\frac{1}{\min\{\varepsilon_1,1/e\}}\right)\ge\log_D\left(\frac{\varepsilon_1}{4}\right)$.[25] Therefore, for any $n\ge\tilde{C}_0\ln\left(\frac{1}{\min\{\varepsilon_1,1/e\}}\right)$ we have $n\ge\log_D(\varepsilon_1/4)$ and since $D<1$ we have:

$$\mathbb{P}(A_{=0})\le D^n\le D^{\log_D(\varepsilon_1/4)}=\varepsilon_1/4. \qquad \text{(B.19)}$$

[25]This is due to the fact that $\tilde{C}_0\cdot\ln\left(\frac{1}{\min\{\varepsilon_1,1/e\}}\right)=\frac{1}{\ln(1/D)}(\ln(4)+1)\ln\left(\frac{1}{\min\{\varepsilon_1,1/e\}}\right)\ge\frac{\ln(4/\varepsilon_1)}{\ln(1/D)}=\log_D\left(\frac{\varepsilon_1}{4}\right)$; here, we have applied Lemma A2 (see Appendix A) for $\hat{\varepsilon}:=\varepsilon_1$ and $\hat{C}:=4$.

Now, we look for an upper-bound of $\mathbb{P}(A_{>0})$. For any n, define the constants $\epsilon_n := \frac{\varepsilon_1}{4}\frac{\underline{w}}{n\bar{w}\bar{\lambda}}$ and $\tau := \frac{1}{X_A}\left(\frac{\bar{w}}{n\underline{w}\underline{\lambda}}\frac{1}{\epsilon_n}+1\right) = \frac{1}{X_A}\left[\frac{4\bar{\lambda}}{\varepsilon_1\underline{\lambda}}\left(\frac{\bar{w}}{\underline{w}}\right)^2+1\right]$, we consider the following term for any $i \in [n]$:

$$
\begin{aligned}
&\mathbb{P}\left(\left\{A_i^* - \frac{A_i^*}{\sum_{j\in[n]} A_j^*}X_A > \epsilon_n\right\}\bigcap A_{>0}\right) \\
&\leq \mathbb{P}\left(\left\{\left|A_i^* - \frac{A_i^*}{\sum_{j\in[n]} A_j^*}X_A\right| > \epsilon_n\right\}\bigcap A_{>0}\right) \\
&\leq \mathbb{P}\left(A_i^*\left|\sum\nolimits_{j\in[n]} A_j^* - X_A\right| > \epsilon_n \sum\nolimits_{j\in[n]} A_j^*\right) \\
&= \mathbb{P}\left(A_i^*\left|\sum\nolimits_{j\in[n]} A_j^* - X_A\right| > \epsilon_n X_A - \epsilon_n\left(X_A - \sum\nolimits_{j\in[n]} A_j^*\right)\right) \\
&\leq \mathbb{P}\left(A_i^*\left|\sum\nolimits_{j\in[n]} A_j^* - X_A\right| > \epsilon_n X_A - \epsilon_n\left|\sum\nolimits_{j\in[n]} A_j^* - X_A\right|\right) \\
&= \mathbb{P}\left(\left|\sum\nolimits_{j\in[n]} A_j^* - X_A\right| > \frac{\epsilon_n X_A}{A_i^* + \epsilon_n}\right) \\
&\leq \mathbb{P}\left(\left|\sum\nolimits_{j\in[n]} A_j^* - X_A\right| > \frac{\epsilon_n X_A}{\frac{\bar{w}}{n\underline{w}\underline{\lambda}} + \epsilon_n}\right) \\
&= \mathbb{P}\left(\left|\sum\nolimits_{j\in[n]} A_j^* - X_A\right| > \frac{1}{\tau}\right).
\end{aligned}
\tag{B.20}
$$

Here, the second-to-last inequality comes from the fact that for any $i \in [n]$, A_i^* is upper-bounded by either v_i^A/λ_A^* or v_i^B/λ_B^* (see (3.4) and (3.6)), thus, it is bounded by $\bar{w}/(n\underline{w}\underline{\lambda})$ (due to (2.2) and Proposition 3.3).

Recall that $X_A = \mathbb{E}\left[\sum_{j=1}^{n} A_j^*\right]$ (see Lemma A1-(*iii*)), we use the Hoeffding's inequality (see e.g., Theorem 2, Hoeffding (1963)) on the random variables $A_i^*, i \in [n]$ (bounded in $[0, \bar{w}/(n\underline{w}\underline{\lambda})]$) to obtain

$$
\begin{aligned}
P\left(\left|\sum_{j\in[n]} A_j^* - X_A\right| > \frac{1}{\tau}\right) &\leq 2\exp\left(\frac{-2\frac{1}{\tau^2}}{\sum\limits_{j\in[n]}\left(\frac{\bar{w}}{n\underline{w}\underline{\lambda}}\right)^2}\right) \\
&= 2\exp\left[\frac{-2n}{\tau^2}\left(\frac{\underline{\lambda}\underline{w}}{\bar{w}}\right)^2\right].
\end{aligned}
\tag{B.21}
$$

Now, we define $\tilde{C}_1 := \frac{1}{2}\left(\frac{4}{X_A}\frac{\bar{\lambda}}{\underline{\lambda}}\frac{\bar{w}^2}{\underline{w}^2} + \frac{1}{X_A}\right)^2(\ln 8 + 1)\left(\frac{\bar{w}}{\underline{w}\underline{\lambda}}\right)^2$; due to the definition of τ, we have that[26] $\tilde{C}_1 \cdot \frac{1}{\varepsilon_1^2}\ln\left(\frac{1}{\min\{\varepsilon_1, 1/e\}}\right) \geq \frac{\tau^2}{2}\ln\left(\frac{8}{\varepsilon_1}\right)\left(\frac{\bar{w}}{\underline{w}\underline{\lambda}}\right)^2$; therefore, for any $n \geq \tilde{C}_1\varepsilon_1^{-2}\ln\left(\frac{1}{\min\{\varepsilon_1, 1/e\}}\right)$, we can deduce that $\frac{2n}{\tau^2}\left(\frac{\underline{w}\underline{\lambda}}{\bar{w}}\right)^2 \geq \ln\left(\frac{8}{\varepsilon_1}\right)$ and thus,

$$
2\exp\left[\frac{-2n}{\tau^2}\left(\frac{\underline{\lambda}\underline{w}}{\bar{w}}\right)^2\right] \leq 2\exp\left[-\ln\left(\frac{8}{\varepsilon_1}\right)\right] = \frac{\varepsilon_1}{4}.
\tag{B.22}
$$

[26] This is due to $\tilde{C}_1 \cdot \frac{1}{\varepsilon_1^2}\ln\left(\frac{1}{\min\{\varepsilon_1, 1/e\}}\right) = \frac{1}{2}\left[\frac{1}{X_A}\left(\frac{4}{\varepsilon_1}\frac{\bar{\lambda}}{\underline{\lambda}}\frac{\bar{w}^2}{\underline{w}^2} + \frac{1}{\varepsilon_1}\right)\right]^2 \cdot (\ln(8)+1)\ln\left(\frac{1}{\min\{\varepsilon_1, 1/e\}}\right) \cdot \left(\frac{\bar{w}}{\underline{w}\underline{\lambda}}\right)^2 \geq \frac{\tau^2}{2} \cdot \ln\left(\frac{8}{\varepsilon_1}\right) \cdot \left(\frac{\bar{w}}{\underline{w}\underline{\lambda}}\right)^2$; here, we have used Lemma A2 with $\hat{\varepsilon} := \varepsilon_1$ and $\hat{C} := 8$ and the fact that $1/\varepsilon \geq 1$.

Combining (B.20), (B.21) and (B.22), we deduce that

$$\mathbb{P}\left(\left\{A_i^* - \frac{A_i^*}{\sum_{j\in[n]} A_j^*} X_A > \epsilon_n\right\}\bigcap A_{>0}\right) \le \frac{\varepsilon_1}{4}, \forall n \ge \tilde{C}_1 \varepsilon_1^{-2} \ln\left(\frac{1}{\min\{\varepsilon_1, 1/e\}}\right). \tag{B.23}$$

Finally, note that for any n, $i \in [n]$ and $x \ge 0$, we also have

$$\begin{aligned}
&\mathbb{P}\left(\left\{\frac{A_i^* \cdot X_A}{\sum_{j\in[n]} A_j^*} \le x\right\}\bigcap A_{>0}\right)\\
=&\mathbb{P}\left(\left\{\frac{A_i^* X_A}{\sum\limits_{j\in[n]} A_j^*} \le x\right\}\bigcap\left\{A_i^* - \frac{A_i^* X_A}{\sum\limits_{j\in[n]} A_j^*} \le \epsilon_n\right\}\bigcap A_{>0}\right) + \mathbb{P}\left(\left\{\frac{A_i^* \cdot X_A}{\sum\limits_{j\in[n]} A_j^*} \le x\right\}\bigcap\left\{A_i^* - \frac{A_i^* X_A}{\sum\limits_{j\in[n]} A_j^*} > \epsilon_n\right\}\bigcap A_{>0}\right)\\
\le&\mathbb{P}\left(\{A_i^* \le x + \epsilon_n\}\right) + \mathbb{P}\left(\left\{A_i^* - \frac{A_i^* X_A}{\sum\limits_{j\in[n]} A_j^*} > \epsilon_n\right\}\bigcap A_{>0}\right).
\end{aligned} \tag{B.24}$$

Therefore, define $C_1 := \max\{\tilde{C}_0, \tilde{C}_1\}$, for any $n \ge C_1 \varepsilon_1^{-2} \ln\left(\frac{1}{\min\{\varepsilon_1, 1/e\}}\right)$, from (B.1), we have

$$\begin{aligned}
&F_{A_i^n}(x) - F_{A_i^*}(x)\\
=&\mathbb{P}(A_{=0}) + \mathbb{P}\left(\left\{\frac{A_i^* \cdot X_A}{\sum_{j\in[n]} A_j^*} \le x\right\}\bigcap A_{>0}\right) - F_{A_i^*}(x)\\
\le&\frac{\varepsilon_1}{4} + \mathbb{P}\left(\{A_i^* \le x + \epsilon_n\}\right) + \mathbb{P}\left(\left\{A_i^* - \frac{A_i^* X_A}{\sum\limits_{j\in[n]} A_j^*} > \epsilon_n\right\}\bigcap A_{>0}\right) - F_{A_i^*}(x) \quad \text{(due to (B.19) and (B.24))}\\
\le&\frac{\varepsilon_1}{4} + F_{A_i^*}(x + \epsilon_n) + \frac{\varepsilon_1}{4} - F_{A_i^*}(x) \quad \text{(due to (B.23))}.
\end{aligned} \tag{B.25}$$

The final step is to bound the term $F_{A_i^*}(x + \epsilon_n) - F_{A_i^*}(x)$; we present this as the following lemma.

Lemma B6. *For any $\epsilon > 0$, $n > 0$, $i \in [n]$ and $x \in [0, \infty)$, we have $F_{A_i^*}(x + \epsilon) - F_{A_i^*}(x) \le \frac{\epsilon \lambda_B^*}{v_i^B}$.*

Proof. If $i \in \Omega_A(\gamma^*)$, then $A_i^* = A_{\gamma^*,i}^S$ and

$$F_{A_{\gamma^*,i}^S}(x+\epsilon) - F_{A_{\gamma^*,i}^S}(x) = \begin{cases} \frac{(x+\epsilon)\lambda_B^*}{v_i^B} - \frac{x\lambda_B^*}{v_i^B} = \frac{\epsilon\lambda_B^*}{v_i^B}, & \text{if } 0 \le x < \frac{v_i^B}{\lambda_B^*} - \epsilon \\ 1 - \frac{x\lambda_B^*}{v_i^B} \le \frac{\epsilon\lambda_B^*}{v_i^B}, & \text{if } \frac{v_i^B}{\lambda_B^*} - \epsilon \le x \le \frac{v_i^B}{\lambda_B^*} \\ 1 - 1 \le \frac{\epsilon v_i^B}{\lambda_B^*}, & \text{if } x > \frac{v_i^B}{\lambda_B^*} \end{cases}. \tag{B.26}$$

On the other hand, if $i \notin \Omega_A(\gamma^*)$, then $A_i^* = A_{\gamma^*,i}^W$ and we have

$$F_{A_{\gamma^*,i}^W}(x+\epsilon) - F_{A_{\gamma^*,i}^W}(x) = \begin{cases} \frac{(x+\epsilon)\lambda_B^*}{v_i^B} - \frac{x\lambda_B^*}{v_i^B} = \frac{\epsilon\lambda_B^*}{v_i^B}, & \text{if } 0 \le x < \frac{v_i^A}{\lambda_A^*} - \epsilon \\ 1 - \frac{\frac{v_i^B}{\lambda_B^*} - \frac{v_i^A}{\lambda_A^*}}{\frac{v_i^B}{\lambda_B^*}} - \frac{x\lambda_B^*}{v_i^B} \le \frac{\epsilon\lambda_B^*}{v_i^B}, & \text{if } \frac{v_i^A}{\lambda_A^*} - \epsilon \le x \le \frac{v_i^A}{\lambda_A^*} \\ 1 - 1 \le \frac{\epsilon v_i^B}{\lambda_B^*}, & \text{if } x > \frac{v_i^A}{\lambda_A^*} \end{cases}. \tag{B.27}$$

□

Combine this lemma with (B.25) and recall the definition of ϵ_n (which induces that $\epsilon_n\lambda_B^*/v_i^B \le \varepsilon_1/2$), we conclude that $F_{A_i^n}(x)–F_{A_i^*}(x) \le \varepsilon_1$ for any $n \ge C_1\varepsilon_1^{-2}\ln\left(\frac{1}{\min\{\varepsilon_1,1/e\}}\right)$. Similarly, for $n \ge C_1\varepsilon_1^{-2}\ln\left(\frac{1}{\min\{\varepsilon_1,1/e\}}\right)$ and $i \in [n]$, we can deduce that $F_{A_i^n}(x)-F_{A_i^*}(x) \ge -\varepsilon_1$ for any $x \in [0,\infty)$. We conclude that for any $n \ge C_1\varepsilon_1^{-2}\ln\left(\frac{1}{\min\{\varepsilon_1,1/e\}}\right)$, $\sup\limits_{x\in[0,\infty)}\left|F_{A_i^n}(x) - F_{A_i^*}(x)\right| \le \varepsilon_1$. The inequality $\sup\limits_{x\in[0,\infty)}\left|F_{B_i^n}(x)-F_{B_i^*}(x)\right| \le \varepsilon_1$ can be proved in a similar way. □

Appendix B.5. Proof of Lemma B5

In this proof, we will use the notation $\mathbb{E}f(X) := \int_0^\infty f(z)\mathrm{d}F_Z(x)$ and $\mathbb{E}_{\mathcal{I}}f(X) := \int_{\mathcal{I}} f(z)\mathrm{d}F_Z(x)$ for any function f, random variable Z and interval $\mathcal{I}$. To simplify the notation, let us define $M := \frac{\bar{\lambda}}{\underline{\lambda}}\frac{\bar{w}^2}{\underline{w}^2}$ and we denote by $\mathcal{I}_i$ the interval $\left[0, v_i^B/\lambda_B^*\right]$. For any $\varepsilon_2 \in (0,1]$, we define $\delta_2 := \frac{\varepsilon_2}{6+2M}$. We first consider the case where $i \in \Omega_A(\gamma^*)$, i.e., $B_i^* = B_{\gamma^*,i}^W$. Note that $F_{A_i^n}(x) = F_{A_i^*}(x) = 1$ for any $x \ge 2X_B$ (see Lemma A1-(iv)); the left-hand-side of (B.11) can be rewritten as follows.

$$\begin{aligned}
&\left|\mathbb{E}F_{B_i^*}(A_i^n) - \mathbb{E}F_{B_i^*}(A_i^*)\right| \\
=&\left|\int_{[0,2X_B]} F_{B_i^*}(x)\mathrm{d}F_{A_i^n}(x) - \int_{[0,2X_B]} F_{B_i^*}(x)\mathrm{d}F_{A_i^*}(x)\right| \\
\le&\left|\mathbb{E}_{[0,v_i^B/\lambda_B^*]}F_{B_{\gamma^*,i}^W}(A_i^n) - \mathbb{E}_{[0,v_i^B/\lambda_B^*]}F_{B_{\gamma^*,i}^W}(A_i^*)\right| + \left|\int\limits_{\left(v_i^B/\lambda_B^*,2X_B\right]} \mathrm{d}F_{A_i^n}(x) - \int\limits_{\left(v_i^B/\lambda_B^*,2X_B\right]} \mathrm{d}F_{A_i^*}(x)\right| \\
=&\left|\mathbb{E}_{\mathcal{I}_i}F_{B_{\gamma^*,i}^W}(A_i^n) - \mathbb{E}_{\mathcal{I}_i}F_{B_{\gamma^*,i}^W}(A_i^*)\right| + \left|F_{A_i^n}(2X_B) - F_{A_i^n}(v_i^B/\lambda_B^*) - F_{A_i^*}(2X_B) + F_{A_i^*}(v_i^B/\lambda_B^*)\right| \\
\le&\left|\mathbb{E}_{\mathcal{I}_i}F_{B_{\gamma^*,i}^W}(A_i^n) - \mathbb{E}_{\mathcal{I}_i}F_{B_{\gamma^*,i}^W}(A_i^*)\right| + 2\sup_{x\in[0,\infty)}\left|F_{A_i^n}(x) - F_{A_i^*}(x)\right|.
\end{aligned} \tag{B.28}$$

We now focus on bounding the first term in (B.28). Let us define $K := \left\lceil \frac{M}{\delta_2}\right\rceil$ and $K+1$ points x_j such that $x_0 := 0$ and $x_j := x_{j-1} + \frac{v_i^B}{\lambda_B^* K}, \forall j \in [K]$. In other words, we have the partitions $\mathcal{I}_i = \bigcup_{j=1}^K P_j$ where we denote by P_1 the interval $[x_0, x_1]$ and by P_j the interval $(x_{j-1}, x_j]$ for $j = 2, \ldots, K$. For any $x, x' \in P_j, \forall j \in [K]$, from the definition of $B_{\gamma^*,i}^W$, we have

$$|F_{B_{\gamma^*,i}^W}(x) - F_{B_{\gamma^*,i}^W}(x')| = \frac{\lambda_A^*}{v_i^A}|x - x'| \le \frac{\lambda_A^*}{v_i^A}\cdot\frac{v_i^B}{\lambda_B^*}\cdot\frac{1}{K} \le \frac{\bar{\lambda}n\bar{w}}{\underline{w}}\cdot\frac{\bar{w}}{n\underline{w}\underline{\lambda}}\cdot\frac{1}{K} = \frac{M}{K} \le \delta_2. \tag{B.29}$$

Now, we define the function $g(x) := \sum\limits_{j=1}^K F_{B_{\gamma^*,i}^W}(x_j)\mathbf{1}_{P_j}(x)$. Here, $\mathbf{1}_{P_j}$ is the indicator function of the set P_j. From this definition and Inequality (B.29), we trivially have $|F_{B_{\gamma^*,i}^W}(x) - g(x)| \le \delta_2$, $\forall x \in \mathcal{I}_i$. Therefore,

$$\left|\mathbb{E}_{\mathcal{I}_i}F_{B_{\gamma^*,i}^W}(A_i^n) - \mathbb{E}_{\mathcal{I}_i}g(A_i^n)\right| \le \int_{I_i}\left|F_{B_{\gamma^*,i}^W}(x) - g(x)\right|\mathrm{d}F_{A_i^n}(x) \le \int_{\mathcal{I}_i}\delta_2\mathrm{d}F_{A_i^n}(x) \le \delta_2, \tag{B.30}$$

$$\left|\mathbb{E}_{\mathcal{I}_i}F_{B_{\gamma^*,i}^W}(A_i^*) - \mathbb{E}_{\mathcal{I}_i}g(A_i^*)\right| \le \int_{I_i}\left|F_{B_{\gamma^*,i}^W}(x) - g(x)\right|\mathrm{d}F_{A_i^*}(x) \le \int_{\mathcal{I}_i}\delta_2\mathrm{d}F_{A_i^*}(x) \le \delta_2. \tag{B.31}$$

Now, we note that for any $j \in [K]$, $F_{B_{\gamma^*,i}^W}(x_j) = \sum\limits_{m=0}^j\left[F_{B_{\gamma^*,i}^W}(x_m) - F_{B_{\gamma^*,i}^W}(x_{m-1})\right]$; here, for the sake of notation, we denote by x_{-1} an arbitrary negative number (that is $F_{B_{\gamma^*,i}^W}(x_{-1}) = 0$). Using this, we have:

$$\left|\mathbb{E}_{\mathcal{I}_i}g\left(A_i^n\right) - \mathbb{E}_{\mathcal{I}_i}g\left(A_i^*\right)\right|$$

$$
\begin{aligned}
&= \left| \sum_{j=1}^{K} F_{B^{W}_{\gamma^*,i}}(x_j) \left[\mathbb{E}_{\mathcal{I}_i} \mathbf{1}_{P_j}(A_i^n) - \mathbb{E}_{\mathcal{I}_i} \mathbf{1}_{P_j}(A_i^*) \right] \right| \\
&= \left| \sum_{j=1}^{K} F_{B^{W}_{\gamma^*,i}}(x_j) \left[\mathbb{P}(A_i^n \in P_j) - \mathbb{P}(A_i^* \in P_j) \right] \right| \\
&= \left| \sum_{j=1}^{K} \left(\sum_{m=0}^{j} \left[F_{B^{W}_{\gamma^*,i}}(x_m) - F_{B^{W}_{\gamma^*,i}}(x_{m-1}) \right] \left[\mathbb{P}(A_i^n \in P_j) - \mathbb{P}(A_i^* \in P_j) \right] \right) \right| \\
&\le \left| \left[F_{B^{W}_{\gamma^*,i}}(x_0) - F_{B^{W}_{\gamma^*,i}}(x_{-1}) \right] \sum_{j=1}^{K} \left[\mathbb{P}(A_i^n \in P_j) - \mathbb{P}(A_i^* \in P_j) \right] \right| \\
&\qquad + \left| \sum_{m=1}^{K} \left(\left[F_{B^{W}_{\gamma^*,i}}(x_m) - F_{B^{W}_{\gamma^*,i}}(x_{m-1}) \right] \sum_{j=m}^{K} \left[\mathbb{P}(A_i^n \in P_j) - \mathbb{P}(A_i^* \in P_j) \right] \right) \right|. \qquad \text{(B.32)}
\end{aligned}
$$

Note that $\mathbb{P}(A_i^n \in P_j) - \mathbb{P}(A_i^* \in P_j) = F_{A_i^n}(x_j) - F_{A_i^n}(x_{j-1}) - F_{A_i^*}(x_j) + F_{A_i^*}(x_{j-1})$.[27] Moreover, due to the fact that $x_0 = 0$ and $F_{B^{W}_{\gamma^*,i}}(x_{-1}) = 0$, we can rewrite the first term in (B.32) as follows:

$$
\begin{aligned}
&\left| \left[F_{B^{W}_{\gamma^*,i}}(x_0) - F_{B^{W}_{\gamma^*,i}}(x_{-1}) \right] \sum_{j=1}^{K} \left[\mathbb{P}(A_i^n \in P_j) - \mathbb{P}(A_i^* \in P_j) \right] \right| \\
=& \left| F_{B^{W}_{\gamma^*,i}}(0) \cdot \left[\sum_{j=1}^{K} \left(F_{A_i^n}(x_j) - F_{A_i^n}(x_{j-1}) - F_{A_i^*}(x_j) + F_{A_i^*}(x_{j-1}) \right) \right] \right| \\
=& \left| F_{B^{W}_{\gamma^*,i}}(0) \cdot \left[F_{A_i^n}(x_K) - F_{A_i^n}(x_0) - F_{A_i^*}(x_K) + F_{A_i^*}(x_0) \right] \right| \\
\le& F_{B^{W}_{\gamma^*,i}}(0) \cdot 2 \sup_{x \in [0,\infty)} \left| F_{A_i^n}(x) - F_{A_i^*}(x) \right| \\
\le& 2 \sup_{x \in [0,\infty)} \left| F_{A_i^n}(x) - F_{A_i^*}(x) \right|. \qquad \text{(B.33)}
\end{aligned}
$$

Now, recall that $x_m = x_{m-1} + v_i^B/(\lambda_B^* \cdot K), \forall m \in [K]$, by the definition of $F_{B^{W}_{\gamma^*,i}}$, we deduce that $F_{B^{W}_{\gamma^*,i}}(x_m) - F_{B^{W}_{\gamma^*,i}}(x_{m-1}) = \frac{v_i^B}{\lambda_B^* K} \frac{\lambda_A^*}{v_i^A} \le \frac{\bar{\lambda}}{\underline{\lambda}} \frac{\bar{w}^2}{\underline{w}^2} \frac{1}{K} = \frac{M}{K}, \forall m \in [K]$. Therefore, the second term in (B.32) is

$$
\begin{aligned}
&\left| \sum_{m=1}^{K} \left(\left[F_{B^{W}_{\gamma^*,i}}(x_m) - F_{B^{W}_{\gamma^*,i}}(x_{m-1}) \right] \sum_{j=m}^{K} \left[\mathbb{P}(A_i^n \in P_j) - \mathbb{P}(A_i^* \in P_j) \right] \right) \right| \\
=& \left| \sum_{m=1}^{K} \left(\left[F_{B^{W}_{\gamma^*,i}}(x_m) - F_{B^{W}_{\gamma^*,i}}(x_{m-1}) \right] \sum_{j=m}^{K} \left[F_{A_i^n}(x_j) - F_{A_i^n}(x_{j-1}) - F_{A_i^*}(x_j) + F_{A_i^*}(x_{j-1}) \right] \right) \right| \\
=& \left| \sum_{m=1}^{K} \left(\left[F_{B^{W}_{\gamma^*,i}}(x_m) - F_{B^{W}_{\gamma^*,i}}(x_{m-1}) \right] \left[F_{A_i^n}(x_K) - F_{A_i^n}(x_{m-1}) - F_{A_i^*}(x_K) + F_{A_i^*}(x_{m-1}) \right] \right) \right| \\
\le& \sum_{m=1}^{K} \left(F_{B^{W}_{\gamma^*,i}}(x_m) - F_{B^{W}_{\gamma^*,i}}(x_{m-1}) \right) \cdot 2 \sup_{x \in [0,\infty)} \left| F_{A_i^n}(x) - F_{A_i^*}(x) \right|
\end{aligned}
$$

[27] For any $j \ge 2$, this is trivially since $P_j := (x_{j-1}, x_j]$. For $P_1 = [0, x_1]$, we have that $\mathbb{P}(A_i^n \in P_1) - \mathbb{P}(A_i^* \in P_1) = \mathbb{P}(A_i^n \in (0, x_1]) - \mathbb{P}(A_i^* \in (0, x_1]) + \mathbb{P}(A_i^n = 0) - \mathbb{P}(A_i^* = 0)$; moreover, due to Lemma B1-(i), we also note that $\mathbb{P}(A_i^n = 0) = \mathbb{P}(A_i^* = 0)$.

$$\leq \sum_{m=1}^{K} \frac{M}{K} \cdot 2 \sup_{x\in[0,\infty)} \left|F_{A_i^n}(x) - F_{A_i^*}(x)\right|$$
$$=2M \sup_{x\in[0,\infty)} \left|F_{A_i^n}(x) - F_{A_i^*}(x)\right|. \tag{B.34}$$

Inject (B.33) and (B.34) into (B.32), we obtain that

$$|\mathbb{E}_{\mathcal{I}_i} g\left(A_i^n\right) - \mathbb{E}_{\mathcal{I}_i} g\left(A_i^*\right)| \leq (2+2M) \sup_{x\in[0,\infty)} \left|F_{A_i^n}(x) - F_{A_i^*}(x)\right|. \tag{B.35}$$

Apply the triangle inequality and combine (B.30), (B.31), (B.35), we have that:

$$\left|\mathbb{E}_{\mathcal{I}_i} F_{B^W_{\gamma^*,i}}(A_i^n) - \mathbb{E}_{\mathcal{I}_i} F_{B^W_{\gamma^*,i}}(A_i^*)\right| \leq 2\delta_2 + (2+2M) \sup_{x\in[0,\infty)} \left|F_{A_i^n}(x) - F_{A_i^*}(x)\right|.$$

From this and (B.28), we obtain that

$$|\mathbb{E}F_{B_i^*}(A_i^n) - \mathbb{E}F_{B_i^*}(A_i^*)| \leq 2\delta_2 + (4+2M) \sup_{x\in[0,\infty)} \left|F_{A_i^n}(x) - F_{A_i^*}(x)\right|. \tag{B.36}$$

Recall the constant C_1 indicated in Lemma B4, we define $C_2 := C_1 \cdot (6+2M)^2 \left[\ln(6+2M)+1\right]$ (note that C_2 does not depend on n nor ε_2) and deduce that $C_2\varepsilon_2^{-2}\ln\left(\frac{1}{\min\{\varepsilon_2,1/e\}}\right) \geq C_1\delta_2^{-2}\ln\left(\frac{1}{\min\{\delta_2,1/e\}}\right)$.[28] Take $\varepsilon_1 := \delta_2$, for any $n \geq C_2\varepsilon_2^{-2}\ln\left(\frac{1}{\min\{\varepsilon_2,1/e\}}\right)$, we have $n \geq C_1\varepsilon_1^{-2}\ln\left(\frac{1}{\min\{\varepsilon_1,1/e\}}\right)$ and by applying Lemma B4, we obtain that $\sup_{x\in[0,\infty)} \left|F_{A_i^n}(x) - F_{A_i^*}(x)\right| \leq \varepsilon_1 = \delta_2$ and thus by (B.36), we have:

$$|\mathbb{E}F_{B_i^*}(A_i^n) - \mathbb{E}F_{B_i^*}(A_i^*)| \leq 2\delta_2 + (4+2M)\,\delta_2 = (6+2M)\delta_2 = \varepsilon_2.$$

This is exactly (B.11). We can have a similar result in the case where $i \notin \Omega_A(\gamma^*)$ (its proof is omitted here) and we conclude the proof of this lemma. ☐

Appendix C. PROOF OF RESULTS IN SECTION 5

Appendix C.1. Proof of Lemma 5.2

Lemma 5.2. *For any $\varepsilon \in (0,1]$, there exists a constant $L_0 > 0$ (that does not depend on ε), such that for any $n \geq L_0\varepsilon^{-2}\ln\left(\frac{1}{\min\{\varepsilon,1/e\}}\right)$, for any game $\mathcal{LB}_n(\zeta)$, $\gamma^* \in \mathcal{S}_n$, $\delta \in \Delta_{\gamma^*}(\zeta,\varepsilon)$ and $i \in [n]$, we have:*

$$\max\left\{\sup_{y^*\in[0,2X_B]} \int_{\mathcal{X}_\zeta(y^*,\varepsilon)} \mathrm{d}F_{A_i^n}(x), \sup_{x^*\in[0,2X_B]} \int_{\mathcal{Y}_\zeta(x^*,\varepsilon)} \mathrm{d}F_{B_i^n}(y)\right\} \leq \delta + \varepsilon. \tag{5.3}$$

Fix $y^* \in [0,2X_B]$, we look for the condition on n such that $\int_{\mathcal{X}_\zeta(y^*,\varepsilon)} \mathrm{d}F_{A_i^n}(x) \leq \delta+\varepsilon$ holds. The condition corresponding to the inequality $\int_{\mathcal{Y}_\zeta(x^*,\varepsilon)} \mathrm{d}F_{B_i^n}(x) \leq \varepsilon+\delta$ with $x^* \in [0,2X_B]$ can be proved similarly and thus is omitted in this section.

First, we note that if $\mathcal{X}_\zeta(y^*,\varepsilon)$ is empty, $\int_{\mathcal{X}_\zeta(y^*,\varepsilon)} \mathrm{d}F_{A_i^n}(x) = 0$ and the result trivially holds. Now, let us assume that $\mathcal{X}_\zeta(y^*,\varepsilon) \neq \emptyset$, we can write $\mathcal{X}_\zeta(y^*,\varepsilon) = I_1 \bigcup I_2 \bigcup I_3$ with[29]

$$I_1 := \{x \in [0,2X_B] : x = y^*, |\zeta_A(x,y^*) - \alpha| \geq \varepsilon\},$$

---

[28] Apply Lemma A2, we have $C_2\varepsilon_2^{-2}\ln\left(\frac{1}{\min\{\varepsilon_2,1/e\}}\right) = C_1\left(\frac{6+2M}{\varepsilon_2}\right)^2\left[\ln(6+2M)+1\right]\ln\left(\frac{1}{\min\{\varepsilon_2,1/e\}}\right) \geq C_1\left(\frac{6+2M}{\varepsilon_2}\right)^2\ln\left(\frac{6+2M}{\varepsilon_2}\right)$. Moreover, since $\frac{\varepsilon_2}{6+2M} = \min\left\{\frac{\varepsilon_2}{6+2M}, \frac{1}{e}\right\} = \min\left\{\delta_2, \frac{1}{e}\right\}$ (due to the fact that $\delta_2 = \varepsilon_2/(6+2M) < 1/e$). Therefore, we have $C_2\varepsilon_2^{-2}\ln\left(\frac{1}{\min\{\varepsilon_2,1/e\}}\right) \geq C_1\delta_2^{-2}\ln\left(\frac{1}{\min\{\delta_2,1/e\}}\right)$.

[29] Recall that by definition, $\beta_A(x,y^*) = \alpha$ if $x = y^*$, $\beta_A(x,y^*) = 0$ if $x < y^*$ and $\beta_A(x,y^*) = 1$ if $x > y^*$

$$I_2 := \{x \in [0, 2X_B] : x < y^*, \zeta_A(x, y^*) \geq \varepsilon\},$$
$$I_3 := \{x \in [0, 2X_B] : x > y^*, 1 - \zeta_A(x, y^*) \geq \varepsilon\}.$$

It is trivial that I_1 is either an empty set or a singleton; on the other hand, due to the monotonicity of the CSF ζ_A (see $(C2)$, Definition 2.3), I_2 and I_3 are either empty sets or half intervals. Moreover, for any arbitrary distribution F, we have that

$$\int_{x \in I'} \mathrm{d}F(x) = \begin{cases} 0 \text{ , if } I' = \emptyset, \\ F(a) \text{ , if } I' = \{a\}, \text{i.e., } I' \text{ is a singleton,} \\ F(b) - F(a) \text{ , if } I' = (a, b], \text{i.e., } I' \text{ is a half interval.} \end{cases}$$

Therefore, we can deduce that

$$\int_{\mathcal{X}_\zeta(y^*,\varepsilon)} \mathrm{d}F_{A_i^n}(x) - \int_{\mathcal{X}_\zeta(y^*,\varepsilon)} \mathrm{d}F_{A_i^*}(x) = \sum_{j=1}^{3} \left(\int_{I_j} \mathrm{d}F_{A_i^n}(x) - \int_{I_j} \mathrm{d}F_{A_i^*}(x) \right) \leq 5 \sup_{x \in [0,\infty)} |F_{A_i^n}(x) - F_{A_i^*}(x)|$$

Recall the constant C_1 indicated in Lemma B4, we define $L_0 := C_1 5^2(\ln(5) + 1)$. Note that L_0 does not depend on the choice of y^*. Take $\varepsilon_1 := \varepsilon/5$, we can deduce that $L_0 \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right) \geq C_1 \varepsilon_1^2 \ln\left(\frac{1}{\min\{\varepsilon_1, 1/e\}}\right)$.[30] Therefore, for any $n \geq L_0 \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$, we have $n \geq C_1 \varepsilon_1^2 \ln\left(\frac{1}{\min\{\varepsilon_1, 1/e\}}\right)$ and by Lemma B4, $\sup_{x \in [0,\infty)} |F_{A_i^n}(x) - F_{A_i^*}(x)| \leq \varepsilon_1 = \varepsilon/5$. Hence, for any $n \geq L_0 \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$ and $\delta \in \Delta_{\gamma^*}(\zeta, \varepsilon)$,

$$\int_{\mathcal{X}_\zeta(y^*,\varepsilon)} \mathrm{d}F_{A_i^n}(x) \leq \int_{\mathcal{X}_\zeta(y^*,\varepsilon)} \mathrm{d}F_{A_i^*}(x) + 5 \cdot \varepsilon/5 \leq \delta + \varepsilon.$$

$\square$

Appendix C.2. Proof of Theorem 5.3

Theorem 5.3. ***(Approximate equilibria of the Lottery Blotto game).***

(i) In any game $\mathcal{LB}_n(\zeta)$, there exists a positive number $\varepsilon \leq \tilde{\mathcal{O}}(n^{-1/2})$ such that for any $\gamma^* \in \mathcal{S}_n$ and $\delta \in \Delta_{\gamma^*}(\zeta, \varepsilon)$, the following inequalities hold for any pure strategy $\boldsymbol{x}^A$ and $\boldsymbol{x}^B$ of players A and B:

$$\Pi_\zeta^A(\boldsymbol{x}^A, \mathrm{IU}_B^{\gamma^*}) \leq \Pi_\zeta^A(\mathrm{IU}_A^{\gamma^*}, \mathrm{IU}_B^{\gamma^*}) + (8\delta + 13\varepsilon)\, W_A, \tag{5.4}$$
$$\Pi_\zeta^B(\mathrm{IU}_A^{\gamma^*}, \boldsymbol{x}^B) \leq \Pi_\zeta^B(\mathrm{IU}_A^{\gamma^*}, \mathrm{IU}_B^{\gamma^*}) + (8\delta + 13\varepsilon)\, W_B. \tag{5.5}$$

(ii) For any $\varepsilon \in (0, 1]$, there exists a constant $L^* > 0$ (that does not depend on ε) such that in any game $\mathcal{LB}_n(\zeta)$ where $n \geq L^* \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$, (5.4) and (5.5) hold for any $\gamma^* \in \mathcal{S}_n$, $\delta \in \Delta_{\gamma^*}(\zeta, \varepsilon)$ and any pure strategy $\boldsymbol{x}^A, \boldsymbol{x}^B$ of players A and B.

Proof. We first give the proof of Result *(ii)*. For the sake of brevity, we only focus on (5.4). The proof that (5.5) holds under the same condition can be done similarly and thus is omitted. Note that in this proof, we often use the Fubini's Theorem to exchange the order of the double integrals.

[30] Note that $\varepsilon_1 = \frac{\varepsilon}{5}$ and apply Lemma A2, $L_0 \cdot \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right) = C_1 \left(\frac{5}{\varepsilon}\right)^2 \cdot (\ln(5) + 1) \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right) \geq C_1 \cdot \left(\frac{5}{\varepsilon}\right)^2 \ln\left(\frac{5}{\varepsilon}\right)$; moreover, $\frac{\varepsilon}{5} = \min\{\frac{\varepsilon}{5}, \frac{1}{e}\}$ since $\varepsilon \leq 1$; thus, we can rewrite $\ln\left(\frac{5}{\varepsilon}\right) = \ln\left(\frac{1}{\min\{\varepsilon/5, 1/e\}}\right) = \ln\left(\frac{1}{\min\{\varepsilon_1, 1/e\}}\right)$.

Recall that $\boldsymbol{x}^A = (x_i^A)_{i\in[n]}$, by the definition of the payoff functions in $\mathcal{LB}_n(\zeta)$, (5.4) can be rewritten as

$$\sum_{i=1}^{n}\left(v_i^A \int_0^\infty \zeta_A\left(x_i^A, y\right) \mathrm{d}F_{B_i^n}(y)\right) - \sum_{i=1}^{n}\left(v_i^A \int_0^\infty\int_0^\infty \zeta_A(x,y)\,\mathrm{d}F_{A_i^n}(x)\,\mathrm{d}F_{B_i^n}(y)\right) \leq 8\delta + 13\varepsilon. \tag{C.1}$$

We now prove that (C.1) holds under appropriate parameters values. To do this, we prepare two useful lemmas as follows.

Lemma C1. *For any pair of CSFs $\zeta = (\zeta_A, \zeta_B)$, any $\varepsilon \in (0,1]$ and $x^* \in [0, 2X_B]$, the following results hold:*

(*i*) *For any n, $i \in [n]$ and $\delta \in \Delta_{\gamma^*}(\zeta, \varepsilon)$,*

$$\left|\int_0^\infty \zeta_A\left(x^*, y\right) \mathrm{d}F_{B_i^*}(y) - \int_0^\infty \beta_A\left(x^*, y\right) \mathrm{d}F_{B_i^*}(y)\right| \leq \delta + \varepsilon. \tag{C.2}$$

(*ii*) *There exists a constant $L_1 > 0$ such that for any $n \geq L_1 \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$, $i \in [n]$ and $\delta \in \Delta_{\gamma^*}(\zeta, \varepsilon)$,*

$$\left|\int_0^\infty \zeta_A\left(x^*, y\right) \mathrm{d}F_{B_i^n}(y) - \int_0^\infty \beta_A\left(x^*, y\right) \mathrm{d}F_{B_i^n}(y)\right| \leq \delta + 2\varepsilon. \tag{C.3}$$

Lemma C2. *For any $\varepsilon \in (0,1]$, there exists a constant $L_2 > 0$ such that for any $n \geq L_2 \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$, any game $\mathcal{LB}_n(\zeta)$, any $\delta \in \Delta_{\gamma^*}(\zeta, \varepsilon)$ and $i \in [n]$, we have:*

$$\left|\int_0^\infty \zeta_A(x,y)\,\mathrm{d}F_{B_i^n}(y) - \int_0^\infty \zeta_A(x,y)\,\mathrm{d}F_{B_i^*}(y)\right| \leq 2\delta + 4\varepsilon, \forall x \geq 0, \tag{C.4}$$

$$\left|\int_0^\infty\int_0^\infty \zeta_A(x,y)\,\mathrm{d}F_{A_i^*}(x)\mathrm{d}F_{B_i^*}(y) - \int_0^\infty\int_0^\infty \zeta_A(x,y)\,\mathrm{d}F_{A_i^*}(x)\mathrm{d}F_{B_i^n}(y)\right| \leq 2\delta + 3\varepsilon, \tag{C.5}$$

$$\left|\int_0^\infty\int_0^\infty \zeta_A(x,y)\,\mathrm{d}F_{B_i^n}(y)\mathrm{d}F_{A_i^*}(x) - \int_0^\infty\int_0^\infty \zeta_A(x,y)\,\mathrm{d}F_{B_i^n}(y)\mathrm{d}F_{A_i^n}(x)\right| \leq 2\delta + 4\varepsilon. \tag{C.6}$$

Lemma C1 states the relation between the first term appearing in the left-hand-side of (C.1) and the corresponding terms when we replace the CSF ζ by the Blotto functions β and replace $F_{B_i^n}$ by $F_{B_i^*}$. These relations are useful to connect the statement we want to prove and the results obtained in Section 4. A proof of Lemma C1 is given in Appendix C.3. On the other hand, Lemma C2 indicates several useful inequalities involving the players' payoffs in the game $\mathcal{LB}_n$ (when they play according to the IU^{γ^*} strategy or playing such that the marginals are $F_{A_i^*}, F_{B_i^*}$). Its proof is given in Appendix C.4 that is based on Lemma C1 and the convergence of the distributions $F_{A_i^n}, F_{B_i^n}$ toward $F_{A_i^*}, F_{B_i^*}$ (i.e., Lemma B4).

We have another remark: for any n and $i \in [n]$,

$$\mathbb{P}(A_i^* = B_i^* = x) = 0, \forall x \geq 0. \tag{C.7}$$

This can be trivially proved as follows: first, $\mathbb{P}(A_i^* = B_i^* = x) = \mathbb{P}(A_i^* = x)\mathbb{P}(B_i^* = x)$ since they are independent; now, if $x > 0$, both $F_{A_i^*}$ and $F_{B_i^*}$ are continuous at x and thus $\mathbb{P}(A_i^* = x) = \mathbb{P}(B_i^* = x) = 0$; on the other hand, if $x = 0$, in the case where $i \in \Omega_A(\gamma^*)$, since $A_i^* = A_{\gamma^*,i}^S$, we have $\mathbb{P}(A_i^* = x) = 0$, in the case where $i \notin \Omega_A(\gamma^*)$, since $B_i^* = B_{\gamma^*,i}^S$, we have $\mathbb{P}(B_i^* = x) = 0$.

Finally, use Lemma C1 and C2 and take $L^* = \max\{L_1, L_2\}$, for any $n \geq L^* \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$, $\delta \in \Delta_{\gamma^*}(\zeta, \varepsilon)$ and any pure strategy $\boldsymbol{x}^A$ of player A, we have:

$$\begin{aligned}
&\sum_{i=1}^{n}\left(v_i^A \int_0^\infty \zeta_A\left(x_i^A, y\right) \mathrm{d}F_{B_i^n}(y)\right) \\
\leq &\sum_{i=1}^{n}\left(v_i^A \int_0^\infty \zeta_A\left(x_i^A, y\right) \mathrm{d}F_{B_i^*}(y)\right) + \sum_{i=1}^{n} v_i^A (2\delta + 4\varepsilon) && \text{(due to (C.4))} \\
= &\sum_{i=1}^{n}\left(v_i^A \int_0^\infty \zeta_A\left(x_i^A, y\right) \mathrm{d}F_{B_i^*}(y)\right) + 2\delta + 4\varepsilon && \left(\text{note that } \textstyle\sum_{i=1}^{n} v_i^A = 1\right) \\
\leq &\sum_{i=1}^{n}\left(v_i^A \int_0^\infty \beta_A\left(x_i^A, y\right) \mathrm{d}F_{B_i^*}(y)\right) + 3\delta + 5\varepsilon && \text{(due to (C.2))} \\
= &\sum_{i=1}^{n}\left[v_i^A \left(\alpha \mathbb{P}(B_i^* = x_i^A) + \mathbb{P}(B_i^* < x_i^A)\right)\right] + 3\delta + 5\varepsilon \\
\leq &\sum_{i=1}^{n} v_i^A F_{B_i^*}\left(x_i^A\right) + 3\delta + 5\varepsilon && (\text{since } \alpha \leq 1) \\
\leq &\sum_{i=1}^{n}\left(v_i^A \int_0^\infty F_{B_i^*}(x) \mathrm{d}F_{A_i^*}(x)\right) + 3\delta + 5\varepsilon && \text{(due to Lemma B3)} \\
= &\sum_{i=1}^{n}\left(v_i^A \int_0^\infty \mathbb{P}(B_i^* < x) \mathrm{d}F_{A_i^*}(x)\right) + 3\delta + 5\varepsilon && \text{(due to (C.7))} \\
\leq &\sum_{i=1}^{n}\left(v_i^A \int_0^\infty \int_0^\infty \beta_A(x, y) \mathrm{d}F_{B_i^*}(y) \mathrm{d}F_{A_i^*}(x)\right) + 3\delta + 5\varepsilon \\
\leq &\sum_{i=1}^{n}\left(v_i^A \int_0^\infty \int_0^\infty \zeta_A(x, y) \mathrm{d}F_{B_i^*}(y) \mathrm{d}F_{A_i^*}(x)\right) + 4\delta + 6\varepsilon && \text{(due to (C.2))} \\
\leq &\sum_{i=1}^{n}\left(v_i^A \int_0^\infty \int_0^\infty \zeta_A(x, y) \mathrm{d}F_{A_i^*}(x) \mathrm{d}F_{B_i^n}(y)\right) + 6\delta + 9\varepsilon && \text{(due to (C.5))} \\
\leq &\sum_{i=1}^{n}\left(v_i^A \int_0^\infty \int_0^\infty \zeta_A(x, y) \mathrm{d}F_{B_i^n}(y) \mathrm{d}F_{A_i^n}(x)\right) + 8\delta + 13\varepsilon && \text{(due to (C.6)).}
\end{aligned}$$

Hence, we conclude that for $n \geq L^* \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$ (C.1) holds and thus, (5.4) also holds.

To prove that Result (ii) implies Result (i), we can proceed similarly to the proof that Theorem 4.3-(ii) implies Theorem 4.3-(i) (see Appendix B). We conclude this proof.

$\square$

Appendix C.3. Proof of Lemma C1

First, we prove (C.2). Note that $F_{B_i^*}(y) = 1, \forall y > 2X_B$ (see Lemma A1-(iv)), for any n, $i \in [n]$ and $\delta \in \Delta_{\gamma^*}(\zeta, \varepsilon)$, we have

$$
\begin{aligned}
&\left|\int\limits_0^\infty \zeta_A\left(x^*, y\right) \mathrm{d}F_{B_i^*}\left(y\right) - \int\limits_0^\infty \beta_A\left(x^*, y\right) \mathrm{d}F_{B_i^*}\left(y\right)\right| \\
\leq &\int_{\mathcal{Y}_\zeta(x^*,\varepsilon)} |\zeta_A(x^*,y) - \beta_A(x^*,y)|\, \mathrm{d}F_{B_i^*}\left(y\right) + \int_{[0,\infty)\setminus\mathcal{Y}_\zeta(x^*,\varepsilon)} |\zeta_A(x^*,y) - \beta_A(x^*,y)|\, \mathrm{d}F_{B_i^*}\left(y\right) \\
= &\int_{\mathcal{Y}_\zeta(x^*,\varepsilon)} |1 - \zeta_B(x^*,y) - 1 + \beta_B(x^*,y)|\, \mathrm{d}F_{B_i^*}\left(y\right) + \int_{[0,2X_B]\setminus\mathcal{Y}_\zeta(x^*,\varepsilon)} |1 - \zeta_B(x^*,y) - 1 + \beta_B(x^*,y)|\, \mathrm{d}F_{B_i^*}\left(y\right) \\
= &\int_{\mathcal{Y}_\zeta(x^*,\varepsilon)} |\zeta_B(x^*,y) - \beta_B(x^*,y)|\, \mathrm{d}F_{B_i^*}\left(y\right) + \int_{[0,2X_B]\setminus\mathcal{Y}_\zeta(x^*,\varepsilon)} |\zeta_B(x^*,y) - \beta_B(x^*,y)|\, \mathrm{d}F_{B_i^*}\left(y\right) \\
\leq &\int_{\mathcal{Y}_\zeta(x^*,\varepsilon)} \mathrm{d}F_{B_i^*}\left(y\right) + \int_{[0,2X_B]\setminus\mathcal{Y}_\zeta(x^*,\varepsilon)} \varepsilon \mathrm{d}F_{B_i^*}\left(y\right) \\
\leq &\delta + \varepsilon.
\end{aligned} \tag{C.8}
$$

Here, the second-to-last inequality comes from the fact that $0 \leq \zeta_B(x,y), \beta_B(x,y) \leq 1$ for any x, y and the definition of $\mathcal{Y}_\zeta(x^*,\varepsilon)$ while the last inequality is due to the definition of $\Delta_{\gamma^*}(\zeta, \varepsilon)$.

Now, in order to prove (C.3), we proceed similarly as in (C.8) to show that

$$
\begin{aligned}
&\left|\int\limits_0^\infty \zeta_A\left(x^*, y\right) \mathrm{d}F_{B_i^n}\left(y\right) - \int\limits_0^\infty \beta_A\left(x^*, y\right) \mathrm{d}F_{B_i^n}\left(y\right)\right| \\
\leq &\int_{\mathcal{Y}_\zeta(x^*,\varepsilon)} \mathrm{d}F_{B_i^n}\left(y\right) + \int_{[0,2X_B]\setminus\mathcal{Y}_\zeta(x^*,\varepsilon)} \varepsilon \mathrm{d}F_{B_i^n}\left(y\right) \\
\leq &\int_{\mathcal{Y}_\zeta(x^*,\varepsilon)} \mathrm{d}F_{B_i^n}\left(y\right) + \varepsilon.
\end{aligned} \tag{C.9}
$$

Finally, by Lemma 5.2, for any $n \geq L_0 \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$ and $\delta \in \Delta_{\gamma^*}(\zeta, \varepsilon)$, we have $\int_{\mathcal{Y}_\zeta(x^*,\varepsilon)} \mathrm{d}F_{B_i^n}\left(y\right) \leq \varepsilon + \delta$. Combine this with (C.9), we conclude that (C.3) holds for any $n \geq L_0 \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$ and $\delta \in \Delta_{\gamma^*}(\zeta, \varepsilon)$. Take $L_1 := L_0$, we conclude the proof.

$\square$

Appendix C.4. Proof of Lemma C2

In this proof, we use the notation $\mathbb{E}h(X,y) := \int_0^\infty h(x,y)\mathrm{d}F_X(x)$ and $\mathbb{E}h(x,Y) := \int_0^\infty h(x,y)\mathrm{d}F_Y(y)$ where X, Y are arbitrary non-negative random variables and h is any function.

Proof of (C.4): For any $i \in [n]$ and $x \geq 0$, we have

$$
\begin{aligned}
&\left|\int\limits_0^\infty \zeta_A\left(x, y\right) \mathrm{d}F_{B_i^n}\left(y\right) - \int\limits_0^\infty \zeta_A\left(x, y\right) \mathrm{d}F_{B_i^*}\left(y\right)\right| \\
\leq &\left|\mathbb{E}\zeta_A(x, B_i^n) - \mathbb{E}\beta_A(x, B_i^n)\right| + \left|\mathbb{E}\beta_A(x, B_i^n) - \mathbb{E}\beta_A(x, B_i^*)\right| + \left|\mathbb{E}\beta_A(x, B_i^*) - \mathbb{E}\zeta_A(x, B_i^*)\right|.
\end{aligned} \tag{C.10}
$$

We notice that upper-bounds of the first and third terms in the right-hand-side of (C.10) are given by (C.3) and (C.2) from Lemma C1. We focus on finding an upper-bound of the second term of (C.10); to do this, we rewrite this term as follows.

$$\mathbb{E}\beta_A(x, B_i^n) = \int_{y<x} \mathrm{d}F_{B_i^n}(y) + \alpha\mathbb{P}(B_i^n = x) = F_{B_i^n}(x) - (1-\alpha)\mathbb{P}(B_i^n = x), \tag{C.11}$$

$$\text{and } \mathbb{E}\beta_A(x, B_i^*) = \int_{y<x} \mathrm{d}F_{B_i^*}(y) + \alpha\mathbb{P}(B_i^* = x) = F_{B_i^*}(x) - (1-\alpha)\mathbb{P}(B_i^* = x). \tag{C.12}$$

If $\alpha = 1$, we trivially have $|\mathbb{E}\beta_A(x, B_i^n) - \mathbb{E}\beta_A(x, B_i^*)| = \left|F_{B_i^n}(x) - F_{B_i^*}(x)\right|$. In the following, we assume that $\alpha < 1$ and consider three cases:

Case 1: If $x = 0$. From Lemma B1-(i), we have $\mathbb{P}(B_i^n = 0) = \mathbb{P}(B_i^* = 0)$ and thus

$$|\mathbb{E}\beta_A(0, B_i^n) - \mathbb{E}\beta_A(0, B_i^*)| = \left|\int_{y<0} \mathrm{d}F_{B_i^n}(y) - \int_{y<0} \mathrm{d}F_{B_i^*}(y) + \alpha\mathbb{P}(B_i^n = 0) - \alpha\mathbb{P}(B_i^* = 0)\right| = 0.$$

Case 2: If $x > 0$, $\mathbb{P}(B_i^* = x) = 0$ by definition. On the other hand, from Results (ii) and (iii) of Lemma B1, we have $\mathbb{P}(B_i^n = x) \le D^{n-1}$ where we define $D := \left(1 - \frac{\lambda}{\bar{\lambda}}\frac{w^2}{\bar{w}^2}\right)$. Following (B.14), for any $n \ge C_0 \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$ (here, C_0 is defined as in Appendix B.2), we have $D^{n-1} \le \frac{\varepsilon}{2(1-\alpha)}$. Therefore, for any $n \ge C_0 \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$, we have

$$\begin{aligned}
&|\mathbb{E}\beta_A(x, B_i^n) - \mathbb{E}\beta_A(x, B_i^*)| \\
\le & |F_{B_i^n}(x) - F_{B_i^*}(x)| + (1-\alpha)\,|\mathbb{P}(B_i^n = x)| \qquad \text{(due to (C.11) – (C.12))} \\
\le & \sup_{x\in[0,\infty)} |F_{B_i^n}(x) - F_{B_i^*}(x)| + (1-\alpha)\frac{\varepsilon}{2(1-\alpha)} \\
= & \sup_{x\in[0,\infty)} |F_{B_i^n}(x) - F_{B_i^*}(x)| + \frac{\varepsilon}{2}.
\end{aligned}$$

In conclusion, $|\mathbb{E}\beta_A(x, B_i^n) - \mathbb{E}\beta_A(x, B_i^*)| \le \sup_{x\in[0,\infty)} |F_{B_i^n}(x) - F_{B_i^*}(x)| + \varepsilon/2$ for any $x \ge 0$, $\alpha \in [0,1]$ and $n \ge C_0 \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$. Now, let us define $C_1' = C_1 \cdot 4(\ln(2)+1)$ (where C_1 is indicated in Lemma B4); take $\varepsilon_1 := \varepsilon/2$, we have $C_1'\varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right) \ge C_1\varepsilon_1^{-2} \ln\left(\frac{1}{\min\{\varepsilon_1, 1/e\}}\right)$. Therefore, for any $n \ge C_1'\varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$, we have $n \ge C_1\varepsilon_1^{-2} \ln\left(\frac{1}{\min\{\varepsilon_1, 1/e\}}\right)$ and apply Lemma B4, we have $\sup_{x\in[0,\infty)} |F_{B_i^n}(x) - F_{B_i^*}(x)| \le \varepsilon_1 = \varepsilon/2$.

We deduce that for any $x \ge 0$, for any $n \ge \max\{C_0, C_1'\}\varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$ and $i \in [n]$, we have:

$$|\mathbb{E}\beta_A(x, B_i^n) - \mathbb{E}\beta_A(x, B_i^*)| \le \varepsilon/2 + \varepsilon/2 = \varepsilon. \tag{C.13}$$

Finally, apply Lemma C1 to (C.10) to bounds the first and third term of its right-hand-side, use (C.13) to bound its second-term and take $L_{(C.4)} = \max\{L_1, C_0, C_1'\}$, we deduce that for any $n \ge L_{(C.4)}\varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$ and $\delta \in \Delta_{\gamma^*}(\zeta, \varepsilon)$,

$$\left|\int_0^\infty \zeta_A(x,y)\,\mathrm{d}F_{B_i^n}(y) - \int_0^\infty \zeta_A(x,y)\,\mathrm{d}F_{B_i^*}(y)\right| \le (\delta + 2\varepsilon) + \varepsilon + (\delta + \varepsilon) = 2\delta + 4\varepsilon.$$

Proof of (C.5): To prove this inequality, we note that similar to the proof of (C.2) in Lemma C1 (by replacing $F_{B_i^*}$ by $F_{A_i^*}$ and replacing $\zeta_A(x^*, y), \beta_A(x^*, y)$ by $\zeta_A(x, y^*), \beta_A(x, y^*)$), we can prove that for any n, $i \in [n]$, $\delta \in \Delta_{\gamma^*}(\zeta, \varepsilon)$ and $y^* \in [0, 2X_B]$, the following inequality holds

$$\left|\int_0^\infty \zeta_A(x, y^*)\,\mathrm{d}F_{A_i^*}(x) - \int_0^\infty \beta_A(x, y^*)\,\mathrm{d}F_{A_i^*}(x)\right| \le \delta + \varepsilon. \tag{C.14}$$

Using this, we have

$$\left|\int_0^\infty\int_0^\infty \zeta_A(x,y)\,\mathrm{d}F_{A_i^*}(x)\mathrm{d}F_{B_i^*}(y)-\int_0^\infty\int_0^\infty \zeta_A(x,y)\,\mathrm{d}F_{A_i^*}(x)\mathrm{d}F_{B_i^n}(y)\right|$$

$$\leq\int_0^\infty\left|\int_0^\infty \zeta_A(x,y)\mathrm{d}F_{A_i^*}(x)-\int_0^\infty \beta_A(x,y)\mathrm{d}F_{A_i^*}(x)\right|\mathrm{d}F_{B_i^*}(y)$$

$$+\left|\int_0^\infty\int_0^\infty \beta_A(x,y)\mathrm{d}F_{A_i^*}(x)\mathrm{d}F_{B_i^*}(y)-\int_0^\infty\int_0^\infty \beta_A(x,y)\mathrm{d}F_{A_i^*}(x)\mathrm{d}F_{B_i^n}(y)\right|$$

$$+\int_0^\infty\left|\int_0^\infty \beta_A(x,y)\mathrm{d}F_{A_i^*}(x)-\int_0^\infty \zeta_A(x,y)\mathrm{d}F_{A_i^*}(x)\right|\mathrm{d}F_{B_i^n}(y)$$

$$\leq\int_0^\infty(\delta+\varepsilon)\mathrm{d}F_{B_i^*}(y)+\left|\int_0^\infty \mathbb{E}\beta_A(x,B_i^*)\mathrm{d}F_{A_i^*}(x)-\int_0^\infty \mathbb{E}\beta_A(x,B_i^n)\mathrm{d}F_{A_i^*}(x)\right|+\int_0^\infty(\delta+\varepsilon)\mathrm{d}F_{B_i^n}(y)$$

$$\leq 2\delta+2\varepsilon+\int_0^\infty\left|\mathbb{E}\beta_A(x,B_i^n)-\mathbb{E}\beta_A(x,B_i^*)\right|\mathrm{d}F_{A_i^*}(x).$$

Finally, take $L_{(\mathrm{C}.5)}=\max\{C_0,C_1'\}$ and apply (C.13), we deduce that for any $n\geq L_{(\mathrm{C}.5)}\varepsilon^{-2}\ln\left(\frac{1}{\min\{\varepsilon,1/e\}}\right)$, (C.5) holds.

Proof of (C.6) To prove this inequality, we note that similar to the proof of (C.3) in Lemma C1 (by replacing $F_{B_i^n}$ by $F_{A_i^n}$ and replacing $\zeta_A(x^*,y),\beta_A(x^*,y)$ by $\zeta_A(x,y^*),\beta_A(x,y^*)$), we can prove that for $n\geq L_1\varepsilon^{-2}\ln\left(\frac{1}{\min\{\varepsilon,1/e\}}\right)$, $i\in[n]$ and $\delta\in\Delta_{\gamma^*}(\zeta,\varepsilon)$,

$$\left|\int_0^\infty \zeta_A(x,y^*)\,\mathrm{d}F_{A_i^n}(x)-\int_0^\infty \beta_A(x,y^*)\,\mathrm{d}F_{A_i^n}(x)\right|\leq\delta+2\varepsilon. \tag{C.15}$$

Now, similar to the proof leading to (C.13), we can prove that for any $n\geq\max\{C_0,C_1'\}\varepsilon^{-2}\ln\left(\frac{1}{\min\{\varepsilon,1/e\}}\right)$, $i\in[n]$ and $y\geq 0$, we have

$$\left|\mathbb{E}\beta_A(A_i^*,y)-\mathbb{E}\beta_A(A_i^n,y)\right|\leq\varepsilon. \tag{C.16}$$

Finally, take $L_{(\mathrm{C}.6)}=\max\{L_1,C_0,C_1'\}$, for any $n\geq L_{(\mathrm{C}.6)}\varepsilon^{-2}\ln\left(\frac{1}{\min\{\varepsilon,1/e\}}\right)$, $i\in[n]$ and $\delta\in\Delta_{\gamma^*}(\zeta,\varepsilon)$, we have

$$\left|\int_0^\infty\int_0^\infty \zeta_A(x,y)\,\mathrm{d}F_{B_i^n}(y)\mathrm{d}F_{A_i^*}(x)-\int_0^\infty\int_0^\infty \zeta_A(x,y)\,\mathrm{d}F_{B_i^n}(y)\mathrm{d}F_{A_i^n}(x)\right|$$

$$\leq\int_0^\infty\left|\int_0^\infty \zeta_A(x,y)\mathrm{d}F_{A_i^*}(x)-\int_0^\infty \beta_A(x,y)\mathrm{d}F_{A_i^*}(x)\right|\mathrm{d}F_{B_i^n}(y)$$

$$+\left|\int_0^\infty\int_0^\infty \beta_A(x,y)\mathrm{d}F_{A_i^*}(x)\mathrm{d}F_{B_i^n}(y)-\int_0^\infty\int_0^\infty \beta_A(x,y)\mathrm{d}F_{A_i^n}(x)\mathrm{d}F_{B_i^n}(y)\right|$$

$$+\int_0^\infty\left|\int_0^\infty \beta_A(x,y)\mathrm{d}F_{A_i^n}(x)-\int_0^\infty \zeta_A(x,y)\mathrm{d}F_{A_i^n}(x)\right|\mathrm{d}F_{B_i^n}(y)$$

$$\leq \int_0^\infty (\delta+\varepsilon)\mathrm{d}F_{A_i^*}(x) + \int_0^\infty \left|\mathbb{E}\beta_A(A_i^*,y) - \mathbb{E}\beta_A(A_i^n,y)\right| \mathrm{d}F_{B_i^n}(y) + \int_0^\infty (\delta+2\varepsilon)\mathrm{d}F_{A_i^n}(x) \quad \text{(due to (C.14) and (C.15))}$$
$$\leq 2\delta + 4\varepsilon \qquad \text{(due to (C.16))}.$$

In conclusion, take $L_2 := \max\{L_{(\mathrm{C.4})}, L_{(\mathrm{C.5})}, L_{(\mathrm{C.6})}\}$, we conclude the proof of this lemma. □

Appendix C.5. Remark on the Lottery Blotto games with continuous CSFs

In this section, we present and prove the remark stating that under the additional assumption that the CSFs ζ_A and ζ_B are Lipschitz continuous on $[0, 2X_B] \times [0, 2X_B]$, the statements in Theorem 5.3 also hold with (C.17) and (C.18) (see below) in places of (5.4) and (5.5). For the sake of completeness, we formally state this result as follows.

Remark C3. *For any CSF ζ_A and ζ_B that are Lipschitz continuous on $[0, 2X_B] \times [0, 2X_B]$, the following results hold (here, we denote $\zeta := (\zeta_A, \zeta_B)$):*

(i) In any game $\mathcal{LB}_n(\zeta)$, there exists a positive number $\varepsilon \leq \tilde{\mathcal{O}}(n^{-1/2})$ such that for any $\gamma^* \in \mathcal{S}_n$ and $\delta \in \Delta_{\gamma^*}(\zeta, \varepsilon)$, the following inequalities hold for any pure strategy $\boldsymbol{x}^A$ and $\boldsymbol{x}^B$ of players A and B:

$$\Pi^A_\zeta(\boldsymbol{x}^A, \mathrm{IU}^{\gamma^*}_B) \leq \Pi^A_\zeta(\mathrm{IU}^{\gamma^*}_A, \mathrm{IU}^{\gamma^*}_B) + (2\delta + 5\varepsilon)\, W_A, \tag{C.17}$$
$$\Pi^B_\zeta(\mathrm{IU}^{\gamma^*}_A, \boldsymbol{x}^B) \leq \Pi^B_\zeta(\mathrm{IU}^{\gamma^*}_A, \mathrm{IU}^{\gamma^*}_B) + (2\delta + 5\varepsilon)\, W_B. \tag{C.18}$$

(ii) For any $\varepsilon \in (0,1]$, there exists a constant $L_\zeta > 0$ (that depends on ζ but does not depend on ε) such that in any game $\mathcal{LB}_n(\zeta)$ where $n \geq L_\zeta \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$, (C.17) and (C.18) hold for any $\gamma^* \in \mathcal{S}_n$, $\delta \in \Delta_{\gamma^*}(\zeta, \varepsilon)$ and any pure strategy $\boldsymbol{x}^A, \boldsymbol{x}^B$ of players A and B.

Proof. We define the Lipschitz constant of ζ_A, ζ_B respectively by $\mathcal{L}_{\zeta_A}, \mathcal{L}_{\zeta_B}$ and let $\mathcal{L}_\zeta := \max\{\mathcal{L}_{\zeta_A}, \mathcal{L}_{\zeta_B}\}$. We focus on proving Result (*ii*) of this Remark; Result (*i*) can be deduced from Result (*ii*) and thus is omitted.

Step 1: We prove that for any $x^*, y^* \in [0, 2X_B]$, there exists a constant C_ζ (that does not depend on ε nor x^*, y^*) such that for any $n \geq C_\zeta \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$, the following inequalities hold:

$$\left| \int_0^\infty \zeta_A(x, y^*)\, \mathrm{d}F_{A_i^n}(x) - \int_0^\infty \zeta_A(x, y^*)\, \mathrm{d}F_{A_i^*}(x) \right| \leq \varepsilon, \tag{C.19}$$
$$\left| \int_0^\infty \zeta_A(x^*, y)\, \mathrm{d}F_{B_i^n}(y) - \int_0^\infty \zeta_A(x^*, y)\, \mathrm{d}F_{B_i^*}(y) \right| \leq \varepsilon. \tag{C.20}$$

The proof of this statement is quite similar to the proof of Lemma B5 (see Appendix B.5). We present here the proof of (C.19); the proof of (C.20) can be done similarly.

Fix $y^* \in [0, 2X_B]$; to simplify the notation, we define $f(x) := \zeta_A(x, y^*)$ and $\tilde{\varepsilon}_1 := \frac{\varepsilon}{4 + 4X_B\mathcal{L}_\zeta}$. From Lemma A1, $F_{A_i^n}(x) = F_{A_i^*}(x) = 1, \forall x > 2X_B$; therefore, the left-hand-side of (C.19) can be rewritten as follows.

$$\left| \int_0^\infty \zeta_A(x, y^*)\, \mathrm{d}F_{A_i^n}(x) - \int_0^\infty \zeta_A(x, y^*)\, \mathrm{d}F_{A_i^*}(x) \right| = \left| \int_0^{2X_B} f(x) \mathrm{d}F_{A_i^n}(x) - \int_0^{2X_B} f(x) \mathrm{d}F_{A_i^*}(x) \right|. \tag{C.21}$$

Let us define $K := \lceil \frac{2X_B \mathcal{L}_\zeta}{\tilde{\varepsilon}_1} \rceil$ and $K+1$ points x_j such that $x_0 := 0$ and $x_j := x_{j-1} + \frac{2X_B}{K}, \forall j \in [K]$. In other words, we have the partitions $[0, 2X_B] = \bigcup_{j=1}^K P_j$ where we denote by P_1 the interval $[x_0, x_1]$ and by P_j the interval $(x_{j-1}, x_j]$ for $j = 2, \dots, K$. For any $x, x' \in P_j, \forall j \in [K]$, since f is Lipschitz continuous, we have

$$|f(x) - f(x')| \leq \mathcal{L}_\zeta |x - x'| \leq \mathcal{L}_\zeta \frac{2X_B}{K} \leq \tilde{\varepsilon}_1. \tag{C.22}$$

Now, we define the function $g(x) := \sum_{j=1}^K f(x_j) \mathbf{1}_{P_j}(x)$. Here, $\mathbf{1}_{P_j}$ is the indicator function of the set P_j. From this definition and Inequality (C.22), we have $|f(x) - g(x)| \leq \tilde{\varepsilon}_1$, $\forall x \in [0, 2X_B]$. Therefore,

$$\left| \int_0^{2X_B} f(x) \mathrm{d}F_{A_i^n}(x) - \int_0^{2X_B} g(x) \mathrm{d}F_{A_i^n}(x) \right| \leq \int_0^{2X_B} \tilde{\varepsilon}_1 \mathrm{d}F_{A_i^n}(x) \leq \tilde{\varepsilon}_1, \tag{C.23}$$

$$\left| \int_0^{2X_B} f(x) \mathrm{d}F_{A_i^*}(x) - \int_0^{2X_B} g(x) \mathrm{d}F_{A_i^*}(x) \right| \leq \int_0^{2X_B} \tilde{\varepsilon}_1 \mathrm{d}F_{A_i^*}(x) \leq \tilde{\varepsilon}_1. \tag{C.24}$$

Now, we note that for any $j \in [K]$, $f(x_j) = \sum_{m=0}^{j} [f(x_m) - f(x_{m-1})]$; here, by convention, we denote by x_{-1} an arbitrary negative number and set $f(x_{-1}) = 0$. Using this, we have:

$$\begin{aligned}
&\left| \int_0^{2X_B} g(x) \mathrm{d}F_{A_i^n}(x) - \int_0^{2X_B} g(x) \mathrm{d}F_{A_i^*}(x) \right| \\
=& \left| \sum_{j=1}^K f(x_j) \left[\int_0^{2X_B} \mathbf{1}_{P_j}(x) \mathrm{d}F_{A_i^n}(x) - \int_0^{2X_B} \mathbf{1}_{P_j}(x) \, \mathrm{d}F_{A_i^*}(x) \right] \right| \\
=& \left| \sum_{j=1}^K f(x_j) \left[\mathbb{P}(A_i^n \in P_j) - \mathbb{P}(A_i^* \in P_j) \right] \right| \\
=& \left| \sum_{j=1}^K \left(\sum_{m=0}^{j} [f(x_m) - f(x_{m-1})] \left[\mathbb{P}(A_i^n \in P_j) - \mathbb{P}(A_i^* \in P_j) \right] \right) \right| \\
\leq& \left| [f(x_0) - f(x_{-1})] \sum_{j=1}^K \left[\mathbb{P}(A_i^n \in P_j) - \mathbb{P}(A_i^* \in P_j) \right] \right| \\
&\quad + \left| \sum_{m=1}^K \left([f(x_m) - f(x_{m-1})] \sum_{j=m}^K \left[\mathbb{P}(A_i^n \in P_j) - \mathbb{P}(A_i^* \in P_j) \right] \right) \right|.
\end{aligned} \tag{C.25}$$

Note that $\mathbb{P}(A_i^n \in P_j) - \mathbb{P}(A_i^* \in P_j) = F_{A_i^n}(x_j) - F_{A_i^n}(x_{j-1}) - F_{A_i^*}(x_j) + F_{A_i^*}(x_{j-1})$.[31] Now, we can rewrite the first term in (C.25) as follows.

$$\begin{aligned}
&\left| [f(x_0) - f(x_{-1})] \sum_{j=1}^K \left[\mathbb{P}(A_i^n \in P_j) - \mathbb{P}(A_i^* \in P_j) \right] \right| \\
=& \left| f(0) \cdot \left[\sum_{j=1}^K \left(F_{A_i^n}(x_j) - F_{A_i^n}(x_{j-1}) - F_{A_i^*}(x_j) + F_{A_i^*}(x_{j-1}) \right) \right] \right|
\end{aligned}$$

[31] For any $j \geq 2$, this is trivially since $P_j := (x_{j-1}, x_j]$. For $P_1 = [0, x_1]$, we have that $\mathbb{P}(A_i^n \in P_1) - \mathbb{P}(A_i^* \in P_1) = \mathbb{P}(A_i^n \in (0, x_1]) - \mathbb{P}(A_i^* \in (0, x_1]) + \mathbb{P}(A_i^n = 0) - \mathbb{P}(A_i^* = 0)$; moreover, due to Lemma B1-(i), we also note that $\mathbb{P}(A_i^n = 0) = \mathbb{P}(A_i^* = 0)$.

$$
\begin{aligned}
&= \left| f(0) \cdot \left[F_{A_i^n}(x_K) - F_{A_i^n}(x_0) - F_{A_i^*}(x_K) + F_{A_i^*}(x_0) \right] \right| \\
&\leq 2 \sup_{x \in [0,\infty)} \left| F_{A_i^n}(x) - F_{A_i^*}(x) \right|. \qquad (C.26)
\end{aligned}
$$

Here, the last inequality comes from the fact that $f(x) \leq 1, \forall x \in [0, 2X_B]$ (since it is a CSF).

Now, we recall that for any $m \in [K]$, $f(x_m) - f(x_{m-1}) \leq \frac{2X_B \mathcal{L}_\zeta}{K}$. Therefore, the second term in (C.25) is

$$
\begin{aligned}
&\left| \sum_{m=1}^{K} \left([f(x_m) - f(x_{m-1})] \sum_{j=m}^{K} [\mathbb{P}(A_i^n \in P_j) - \mathbb{P}(A_i^* \in P_j)] \right) \right| \\
=&\left| \sum_{m=1}^{K} \left([f(x_m) - f(x_{m-1})] \sum_{j=m}^{K} \left[F_{A_i^n}(x_j) - F_{A_i^n}(x_{j-1}) - F_{A_i^*}(x_j) + F_{A_i^*}(x_{j-1}) \right] \right) \right| \\
=&\left| \sum_{m=1}^{K} \left([f(x_m) - f(x_{m-1})] \left[F_{A_i^n}(x_K) - F_{A_i^n}(x_{m-1}) - F_{A_i^*}(x_K) + F_{A_i^*}(x_{m-1}) \right] \right) \right| \\
\leq& \sum_{m=1}^{K} \frac{2X_B \mathcal{L}_\zeta}{K} \cdot 2 \sup_{x \in [0,\infty)} \left| F_{A_i^n}(x) - F_{A_i^*}(x) \right| \\
=& 4X_B \mathcal{L}_\zeta \sup_{x \in [0,\infty)} \left| F_{A_i^n}(x) - F_{A_i^*}(x) \right|. \qquad (C.27)
\end{aligned}
$$

Inject (C.26) and (C.27) into (C.25), we obtain that

$$
\left| \int_0^{2X_B} g(x) \mathrm{d}F_{A_i^n}(x) - \int_0^{2X_B} g(x) \mathrm{d}F_{A_i^*}(x) \right| \leq (2 + 4X_B \mathcal{L}_\zeta) \sup_{x \in [0,\infty)} \left| F_{A_i^n}(x) - F_{A_i^*}(x) \right|. \qquad (C.28)
$$

Apply the triangle inequality and combine (C.23), (C.24), (C.28), we have that:

$$
\left| \int_0^{2X_B} f(x) \mathrm{d}F_{A_i^n}(x) - \int_0^{2X_B} f(x) \mathrm{d}F_{A_i^*}(x) \right| \leq 2\tilde{\varepsilon}_1 + (2 + 4X_B \mathcal{L}_\zeta) \sup_{x \in [0,\infty)} \left| F_{A_i^n}(x) - F_{A_i^*}(x) \right|. \qquad (C.29)
$$

Recall the constant C_1 indicated in Lemma B4, we define $C_\zeta := C_1 \cdot (4 + 4X_B \mathcal{L}_\zeta)^2 [\ln(4 + 4X_B \mathcal{L}_\zeta) + 1]$ (note that C_ζ does not depend on n nor ε) and deduce that $C_\zeta \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right) \geq C_1 \tilde{\varepsilon}_1^{-2} \ln\left(\frac{1}{\min\{\tilde{\varepsilon}_1, 1/e\}}\right)$.[32] Take $\varepsilon_1 := \tilde{\varepsilon}_1$, for any $n \geq C_\zeta \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$, we have $n \geq C_1 \varepsilon_1^{-2} \ln\left(\frac{1}{\min\{\varepsilon_1, 1/e\}}\right)$ and by applying Lemma B4, we obtain that $\sup_{x \in [0,\infty)} \left| F_{A_i^n}(x) - F_{A_i^*}(x) \right| \leq \varepsilon_1 = \tilde{\varepsilon}_1$ and thus by (C.21) and (C.29), we have:

$$
\left| \int_0^\infty \zeta_A(x, y^*) \mathrm{d}F_{A_i^n}(x) - \int_0^\infty \zeta_A(x, y^*) \mathrm{d}F_{A_i^*}(x) \right| \leq 2\tilde{\varepsilon}_1 + (2 + 4X_B \mathcal{L}_\zeta) \tilde{\varepsilon}_1 = (4 + 4X_B \mathcal{L}_\zeta) \tilde{\varepsilon}_1 = \varepsilon.
$$

This is exactly (C.19).

<u>*Step 2*</u>: Based on (C.19) and (C.20), we can trivially deduce that the following inequalities hold for any $n \geq C_\zeta \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$ and $i \in [n]$:

$$
\left| \int_0^\infty \zeta_A(x, y) \mathrm{d}F_{B_i^n}(y) - \int_0^\infty \zeta_A(x, y) \mathrm{d}F_{B_i^*}(y) \right| \leq \varepsilon, \forall x \geq 0, \qquad (C.30)
$$

[32] Apply Lemma A2, $C_\zeta \varepsilon^{-2} \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right) = C_1 \left(\frac{4+4X_B \mathcal{L}_\zeta}{\varepsilon}\right)^2 [\ln(4+4X_B \mathcal{L}_\zeta)+1] \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right) \geq C_1 \left(\frac{4+4X_B \mathcal{L}_\zeta}{\varepsilon}\right)^2 \ln\left(\frac{4+4X_B \mathcal{L}_\zeta}{\varepsilon}\right)$. Moreover, since $\frac{\varepsilon}{4+4X_B \mathcal{L}_\zeta} = \min\left\{\frac{\varepsilon}{4+4X_B \mathcal{L}_\zeta}, \frac{1}{e}\right\} = \min\left\{\tilde{\varepsilon}_1, \frac{1}{e}\right\}$ (due to the fact that $\tilde{\varepsilon}_1 = \frac{\varepsilon}{4+4X_B \mathcal{L}_\zeta} < \frac{1}{e}$).

$$\left|\int_0^\infty\int_0^\infty \zeta_A(x,y)\,\mathrm{d}F_{B_i^*}(y)\,\mathrm{d}F_{A_i^*}(x) - \int_0^\infty\int_0^\infty \zeta_A(x,y)\,\mathrm{d}F_{B_i^n}(y)\,\mathrm{d}F_{A_i^*}(x)\right| \le \varepsilon, \tag{C.31}$$

$$\left|\int_0^\infty\int_0^\infty \zeta_A(x,y)\,\mathrm{d}F_{A_i^*}(x)\,\mathrm{d}F_{B_i^n}(y) - \int_0^\infty\int_0^\infty \zeta_A(x,y)\,\mathrm{d}F_{A_i^n}(x)\,\mathrm{d}F_{B_i^n}(y)\right| \le \varepsilon. \tag{C.32}$$

We notice that the left-hand-sides of these inequalities are exactly the terms considered in Lemma C2; moreover, the upper-bounds given in(C.30), (C.31) and (C.32) are smaller than that in (C.4), (C.5) and (C.6) of Lemma C2.

Step 3: To complete the proof of Remark C3, we follow the proof of Theorem 5.3 where we use (C.30), (C.31) and (C.32) instead of (C.4), (C.5) and (C.6). By doing this, we obtain (C.17) and (C.18). □

Appendix D. PROOF OF LEMMA 5.6 AND THEOREM 5.7

Appendix D.1. Proof of Lemma 5.6

(i) We first consider the games $\mathcal{LB}_n(\mu^R)$.

Step 1: We want to prove that there exists $\delta_0 = \mathcal{O}(\varepsilon^{-1/R} - 1)$ such that $\mathcal{X}_{\mu^R}(y^*, \varepsilon) \subset [y^* - \delta_0, y^* + \delta_0]$ for any $y^* \in [0, 2X_B]$. Note that this is trivial if $\mathcal{X}_{\mu^R}(y^*, \varepsilon) = \emptyset$. In the following, we consider the case where $\mathcal{X}_{\mu^R}(y^*, \varepsilon) \neq \emptyset$. We denote by $f : [0, 2X_B] \times [0, 2X_B] \to [0, 1]$ the function:

$$f(x, y^*) := |\mu_A^R(x, y^*) - \beta_A(x, y^*)| = \begin{cases} \frac{\alpha x^R}{\alpha x^R + (1-\alpha)(y^*)^R}, \text{ if } x < y^* \\ 0, \text{ if } x = y^* \\ 1 - \frac{\alpha x^R}{\alpha x^R + (1-\alpha)(y^*)^R}, \text{ if } x > y^* \end{cases}.$$

Trivially, $y^* \notin \mathcal{X}_{\mu^R}(y^*, \varepsilon)$. Take an arbitrary $x \in \mathcal{X}_{\mu^R}(y^*, \varepsilon)$. If $x < y^*$, we have

$$f(x, y^*) \ge \varepsilon \Rightarrow \frac{\alpha x^R}{\alpha x^R + (1-\alpha)(y^*)^R} \ge \varepsilon \Rightarrow \frac{x}{y^*} \ge \left(\frac{\varepsilon}{1-\varepsilon}\frac{1-\alpha}{\alpha}\right)^{1/R}.$$

Therefore, $0 < y^* - x \le y^* \left[1 - \left(\frac{\varepsilon}{1-\varepsilon}\frac{1-\alpha}{\alpha}\right)^{1/R}\right]$. Here, we note that the right-hand side is positive (due to the condition $\varepsilon < \alpha$); moreover, it is upper-bounded by $\mathcal{O}(1 - \varepsilon^{1/R}) \le \mathcal{O}(\varepsilon^{-1/R} - 1)$.

On the other hand, if $x > y^*$, we have:

$$f(x, y^*) \ge \varepsilon \Rightarrow 1 - \frac{\alpha x^R}{\alpha x^R + (1-\alpha)(y^*)^R} \ge \varepsilon \Rightarrow \frac{x}{y^*} \le \left(\frac{1-\varepsilon}{\varepsilon}\frac{1-\alpha}{\alpha}\right)^{1/R}.$$

Therefore we have $0 < x - y^* \le y^* \left[\left(\frac{1-\varepsilon}{\varepsilon}\frac{1-\alpha}{\alpha}\right)^{1/R} - 1\right]$. Here the right-hand side is positive (due to the condition $\alpha + \varepsilon < 1$) and is upper-bounded by $\mathcal{O}(\varepsilon^{-1/R} - 1)$.

In conclusion, for any $\varepsilon < \min\{\alpha, 1-\alpha\}$, there exists $\delta_0 = \mathcal{O}(\varepsilon^{-1/R} - 1)$ such that $\mathcal{X}_{\mu^R}(y^*, \varepsilon) \subset [y^* - \delta_0, y^* + \delta_0]$. Note that a similar proof can be done to prove that there exists $\hat{\delta}_0 = \mathcal{O}(\varepsilon^{-1/R} - 1)$ such that for any $x^* \in [0, 2X_B]$, $\mathcal{Y}_{\mu^R}(x^*, \varepsilon) \subset [x^* - \hat{\delta}_0, x^* + \hat{\delta}_0]$.

Step 2: For any $y^* \in [0, 2X_B]$ and $\delta_0 \ge 0$, let us define the set $I_0(y^*) := [y^* - \delta_0, y^* + \delta_0] \bigcap [0, 2X_B]$; we want to show that $\int_{x \in I_0(y^*)} \mathrm{d}F_{A_i^*}(x) \le \frac{2n\bar{\lambda}\delta_0\bar{w}}{\underline{w}}, \forall i \in [n]$.

Case 1: For $i \in \Omega_A(\gamma^*)$, then $A_i^* A_{\gamma^*,i}^S$, we have that

$$\int_{x \in I_0(y^*)} \mathrm{d}F_{A_i^*}(x) \le F_{A_{\gamma^*,i}^S}(y^* + \delta_0) - F_{A_{\gamma^*,i}^S}(y^* - \delta_0)$$

$$
= \begin{cases} \frac{(y^*+\delta_0)\lambda_B^*}{v_i^B} \leq \frac{2\delta_0\lambda_B^*}{v_i^B}, \text{ if } 0 \leq y^* \leq \delta_0 \\ \frac{(y^*+\delta_0)\lambda_B^*}{v_i^B} - \frac{(y^*-\delta_0)\lambda_B^*}{v_i^B} = \frac{2\delta_0\lambda_B^*}{v_i^B}, \text{ if } \delta_0 \leq y^* < \frac{v_i^B}{\lambda_B^*} - \delta_0 \\ 1 - \frac{(y^*-\delta_0)\lambda_B^*}{v_i^B} = \frac{v_i^B - y^*\lambda_B^* + \delta_0\lambda_B^*}{v_i^B} \leq \frac{2\delta_0\lambda_B^*}{v_i^B}, \text{ if } \frac{v_i^B}{\lambda_B^*} - \delta_0 \leq y^* \leq \frac{v_i^B}{\lambda_B^*} + \delta_0 \\ 1 - 1 = 0, \text{ otherwise} \end{cases}
$$

$$
\leq \frac{2n\bar{\lambda}\delta_0\bar{w}}{\underline{w}}.
$$

Case 2: For $i \notin \Omega_A(\gamma^*)$, then $A_i^* = A_{\gamma^*,i}^W$. We have

$$
\int_{x \in I_0(y^*)} \mathrm{d}F_{A_i^*}(x) \leq F_{A_{\gamma^*,i}^W}(y^*+\delta_0) - F_{A_{\gamma^*,i}^W}(y^*-\delta_0)
$$

$$
= \begin{cases} \frac{(y^*+\delta_0)\lambda_B^*}{v_i^B} \leq \frac{2n\bar{\lambda}\delta_0\bar{w}}{\underline{w}}, \text{ if } 0 \leq y^* \leq \delta_0 \\ \frac{(y^*+\delta_0)\lambda_B^*}{v_i^B} - \frac{(y^*-\delta_0)\lambda_B^*}{v_i^B} = \frac{2\delta_0\lambda_B^*}{v_i^B}, \text{ if } \delta_0 < y^* < \frac{v_i^A}{\lambda_A^*} - \delta_0 \\ 1 - \frac{\frac{v_i^B}{\lambda_B^*} - \frac{v_i^A}{\lambda_A^*}}{\frac{v_i^B}{\lambda_B^*}} - \frac{(y^*-\delta_0)\lambda_B^*}{v_i^B} = \frac{v_i^A\frac{\lambda_B^*}{\lambda_A^*} - y^*\lambda_B^* + \delta_0\lambda_B^*}{v_i^B} \leq \frac{2\delta_0\lambda_B^*}{v_i^B}, \text{ if } \frac{v_i^A}{\lambda_A^*} - \delta_0 \leq y^* \leq \frac{v_i^A}{\lambda_A^*} + \delta_0 \\ 1 - 1 = 0, \text{ otherwise} \end{cases}
$$

$$
\leq \frac{2n\bar{\lambda}\delta_0\bar{w}}{\underline{w}}.
$$

Note that we also can similarly prove that for any $x^* \in [0, 2X_B]$ and $\delta_0 \geq 0$, for any $i \in [n]$, we also have $\int_{y \in I_0(x^*)} \mathrm{d}F_{B_i^*}(y) \leq \frac{2n\bar{\lambda}\delta_0\bar{w}}{\underline{w}}$.

Step 3: Conclusion. We note that all random variable $A_i^*, B_i^*, i \in [n]$ are bounded in $[0, 2X_B]$; therefore, for any $x^*, y^* \in [0, 2X_B]$ and $\delta_0 \geq 0$, we have:

$$
\int_{x \in [y^*-\delta, y^*+\delta_0]} \mathrm{d}F_{A_i^*}(x) = \int_{x \in I_0(y^*)} \mathrm{d}F_{A_i^*}(x) \text{ and } \quad \int_{y \in [x^*-\delta, x^*+\delta_0]} \mathrm{d}F_{B_i^*}(y) = \int_{y \in I_0(x^*)} \mathrm{d}F_{B_i^*}(x).
$$

Let us define $\delta_\mu := min\{1, \frac{2n\bar{\lambda}\delta_0\bar{w}}{\underline{w}}\} = \mathcal{O}\left(n(\varepsilon^{-1/R} - 1)\right)$ and we conclude that:

$$
\max\left\{\max_{y^* \in [0,2X_B]} \int_{\mathcal{X}_{\mu^R}(y^*,\varepsilon)} \mathrm{d}F_{A_i^*}(x), \max_{x^* \in [0,2X_B]} \int_{\mathcal{Y}_{\mu^R}(x^*,\varepsilon)} \mathrm{d}F_{B_i^*}(y)\right\} \leq \delta_\mu.
$$

This implies that $\delta_\mu \in \Delta_{\gamma^*}(\mu^R, \varepsilon)$.

(ii) We now turn our focus on the games $\mathcal{LB}_n(\nu^R)$. We first prove the existence of $\delta_1 > 0$ such that $\mathcal{X}_{\nu^R}(y^*, \varepsilon) \subset [y^* - \delta_1, y^* + \delta_1]$ for any $y^* \in [0, 2X_B]$. Similar to step 1 in the above analysis for the game $\mathcal{LB}_n(\mu^R)$, we denote by $g : [0, 2X_B] \times [0, 2X_B] \to [0, 1]$ the function:

$$
g(x, y^*) := |\nu_A^R(x, y^*) - \beta_A(x, y^*)| = \begin{cases} \frac{\alpha e^{xR}}{\alpha e^{xR} + (1-\alpha)e^{y^*R}} \text{ , if } x < y^*, \\ 0 \text{ , if } x = y^*, \\ 1 - \frac{\alpha e^{xR}}{\alpha e^{xR} + (1-\alpha)e^{y^*R}} \quad \text{ , if } x > y^*. \end{cases}
$$

Trivially, $y^* \notin \mathcal{X}_{\nu^R}(y^*, \varepsilon)$. Take an arbitrary $x \in \mathcal{X}_{\mu^R}(y^*, \varepsilon)$. If $x < y^*$, we have

$$
g(x, y^*) \geq \varepsilon \Rightarrow \frac{\alpha e^{xR}}{\alpha e^{xR} + (1-\alpha)e^{y^*R}} \geq \varepsilon.
$$

Therefore, $0 < y^* - x \leq \frac{1}{R} \ln\left(\frac{1-\varepsilon}{\varepsilon} \frac{\alpha}{1-\alpha}\right)$. Here, we note that the right-hand side is positive (due to the condition $\varepsilon < \alpha$).

On the other hand, if $x > y^*$, we have:

$$g(x, y^*) \geq \varepsilon \Rightarrow 1 - \frac{\alpha e^{xR}}{\alpha e^{xR} + (1-\alpha)e^{y^* R}} \geq \varepsilon.$$

Therefore, $0 < x - y^* \leq \frac{1}{R} \ln\left(\frac{1-\varepsilon}{\varepsilon} \frac{1-\alpha}{\alpha}\right)$. Here, the right-hand side is positive (due to the condition $\alpha + \varepsilon < 1$).

In conclusion, let us denote $\delta_1 = \mathcal{O}(R^{-1} \ln(\varepsilon^{-1}))$, we have proved that $\mathcal{X}_{\nu^R}(y^*, \varepsilon) \subset [y^* - \delta_1, y^* + \delta_1]$ for any $y^* \in [0, 2X_B]$. Now, we define $I_1(y^*) := [y^* - \delta_1, y^* + \delta_1] \bigcap [0, 2X_B]$. Similar to step 2 of the above analysis regarding the game $\mathcal{LB}_n(\mu^R)$, we can prove that $\int_{I_1(y^*)} \mathrm{d}F_{A_i^*}(x) \leq 2n\bar{\lambda}\delta_1 \bar{w}/\underline{w}$ for any $y^* \in [0, 2X_B]$. Therefore,

$$\max\left\{ \max_{y^* \in [0, 2X_B]} \int_{\mathcal{X}_{\nu^R}(y^*, \varepsilon)} \mathrm{d}F_{A_i^*}(x), \max_{x^* \in [0, 2X_B]} \int_{\mathcal{Y}_{\nu^R}(x^*, \varepsilon)} \mathrm{d}F_{B_i^*}(y) \right\} \leq \delta_\nu,$$

where $\delta_\nu := \min\{1, \frac{2n\bar{\lambda}\delta_1\bar{w}}{\underline{w}}\} = \mathcal{O}\left(nR^{-1}\ln(\varepsilon^{-1})\right)$ and $\delta_\nu \in \Delta_{\gamma^*}(\nu^R, \varepsilon)$.

Appendix D.2. Proof of Theorem 5.7

Theorem 5.7. ***(Approximate equilibria of the ratio-form Lottery Blotto games)*** *For any $\bar{\varepsilon} > 0$ and $\alpha \in (0,1)$ such that $\bar{\varepsilon} < \min\{\alpha, 1-\alpha\}$, there exists $\tilde{L} > 0$ such that for any $n \geq \tilde{L}\bar{\varepsilon}^{-2} \ln\left(\frac{1}{\min\{\bar{\varepsilon}, 1/e\}}\right)$, $R \geq \mathcal{O}\left(\frac{n}{\bar{\varepsilon}} \ln\left(\frac{1}{\bar{\varepsilon}}\right)\right)$ and $\gamma^* \in \mathcal{S}_n$, the IU^{γ^*} strategy is an $\bar{\varepsilon}W$-equilibrium of any game $\mathcal{LB}_n(\mu^R)$ and $\mathcal{LB}_n(\nu^R)$ having α as the tie-breaking-rule parameter.*

Proof. Take $\varepsilon = \bar{\varepsilon}/21$ and $\tilde{L} = L^* 21^2(\ln(21) + 1)$ (where L^* is indicated in Theorem 5.3). We note that $\tilde{L}\bar{\varepsilon}^{-2} \ln\left(\frac{1}{\min\{\bar{\varepsilon}, 1/e\}}\right) \geq L^* \varepsilon \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$;[33] therefore, for any $n \geq \tilde{L}\bar{\varepsilon}^{-2} \ln\left(\frac{1}{\min\{\bar{\varepsilon}, 1/e\}}\right)$, we have $n \geq L^* \varepsilon \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$ and thus, apply Theorem 5.3-(ii), for any $R > 0$, the IU^{γ^*} strategy is an $(8\delta_\mu + 13\varepsilon)W$-equilibrium of the game $\mathcal{LB}_n(\mu^R)$ (recall that $W := \max\{W_A, W_B\}$). Similarly, the IU^{γ^*} strategy is an $(8\delta_\nu + 13\varepsilon)W$-equilibrium of the game $\mathcal{LB}_n(\nu^R)$).

We first consider the game $\mathcal{LB}_n(\mu^R)$. Apply Lemma 5.6, for any $R \geq \mathcal{O}\left(\ln\left(\frac{1}{\varepsilon}\right) \ln\left(\frac{\varepsilon}{n} + 1\right)\right) = \mathcal{O}\left(\ln\left(\frac{1}{\varepsilon}\right) \frac{n}{\varepsilon}\right)$ and $\gamma^* \in \mathcal{S}_n$, we have $\delta_\mu \leq \varepsilon$. Therefore, for any $n \geq \tilde{L}\bar{\varepsilon}^{-2} \ln\left(\frac{1}{\min\{\bar{\varepsilon}, 1/e\}}\right)$, $R \geq \mathcal{O}\left(\frac{n}{\bar{\varepsilon}} \ln\left(\frac{1}{\bar{\varepsilon}}\right)\right)$, the IU^{γ^*} strategy is an $21\varepsilon W$-equilibrium (i.e., $\bar{\varepsilon}W$-equilibrium) of the game $LB(\mu^R)$.

Similarly, apply Lemma 5.6, for any $\gamma^* \in \mathcal{S}_n$ and $R \geq \mathcal{O}\left(\frac{n}{\bar{\varepsilon}} \ln\left(\frac{1}{\bar{\varepsilon}}\right)\right)$, we have $\delta_\nu \leq \varepsilon$. Therefore, for any $n \geq \tilde{L}\bar{\varepsilon}^{-2} \ln\left(\frac{1}{\min\{\bar{\varepsilon}, 1/e\}}\right)$, $R \geq \mathcal{O}\left(\frac{n}{\bar{\varepsilon}} \ln\left(\frac{1}{\bar{\varepsilon}}\right)\right)$, the IU^{γ^*} strategy is an $21\varepsilon W$-equilibrium (i.e., $\bar{\varepsilon}W$-equilibrium) of the game $LB(\nu^R)$.

[33] Note that $\varepsilon = \bar{\varepsilon}/21$ and apply Lemma A2 to have that $\tilde{L}\bar{\varepsilon}^{-2} \ln\left(\frac{1}{\min\{\bar{\varepsilon}, 1/e\}}\right) \geq L^* \left(\frac{21}{\bar{\varepsilon}}\right)^2 \ln\left(\frac{21}{\bar{\varepsilon}}\right)$; moreover, we recall that $\frac{\bar{\varepsilon}}{21} < \frac{1}{e}$; therefore, $\ln\left(\frac{21}{\bar{\varepsilon}}\right) = \ln\left(\frac{1}{\min\{\bar{\varepsilon}/21, 1/e\}}\right) = \ln\left(\frac{1}{\min\{\varepsilon, 1/e\}}\right)$.