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# Asymptotic analysis of a thin linearly elastic plate equipped with a periodic distribution of stiffeners



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## ABSTRACT

We derive several models of thin plates equipped with a periodic distribution of stiffeners. Depending on the orders of magnitude of the different parameters involved, diverse situations arise, from classical Kirchhoff–Love behaviour with additional energy term to full rigidification.

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## 1. Introduction

To mathematically derive the Reissner–Mindlin model of thin linearly elastic plates (see [1]), we considered a periodic distribution of plates abutted together through thin and soft adhesive layers. So, it is natural now to examine the case when the adhesive layers are stiff. One of the main motivations is the rigidification of plates through a distribution of parallel stiffeners. But it is also a first step in the study of the optimization of plates. In that respect, a more distant goal lies in the relation between the design of the plate and the improvement of some selected aspects of its mechanical performances, without altering the total quantity of material employed (see [2] for example). We will present here six models indexed by  $p = (p_1, p_2)$  in  $\{1, 2, 3\} \times \{1, 2\}$ , where  $p_1$  is a geometric parameter linked to the stiffeners layout, while  $p_2$  accounts for the order of magnitude of the rigidity of the stiffeners.

More precisely, as usual we make no difference between the Euclidean physical space and  $\mathbb{R}^3$  with orthonormal basis  $\{e_1, e_2, e_3\}$  and, for all  $\xi = (\xi_1, \xi_2, \xi_3)$  in  $\mathbb{R}^3$ , we define  $\hat{\xi} := (\xi_1, \xi_2)$  and denote the standard Euclidean distance by  $dist$ . Let  $(\tau^k, \nu^k) := (e_{3-k}, e_k)$ , for  $k = 1, 2$  and  $(\tau^3, \nu^3) := \frac{1}{\sqrt{2}}(e_1 + e_2, e_1 - e_2)$ ,  $(h, \varepsilon, \eta)$  three small positive real numbers and  $\omega$  a domain of  $\mathbb{R}^2$  with a Lipschitz-continuous boundary  $\partial\omega$ . For all  $k$  in  $\{1, 2, 3\}$ , we define:

$$\Sigma_i^k := \{i\varepsilon\nu^k + \mathbb{R}\tau^k\} \cap \omega, \quad I^k := \{i \in \mathbb{Z}; \Sigma_i^k \neq \emptyset\}, \quad \Sigma^{k,h} := \left( \bigcup_{i \in I^k} \Sigma_i^k \right) \times (-h, h), \quad \Sigma_{p_1}^h := \bigcup_{k \leq p_1} \Sigma^{k,h}$$

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The following two subsets of  $\Omega^h := \omega \times (-h, h)$  denoted by

$$B_{\eta,\varepsilon,p_1}^h := \left\{ x \in \Omega^h; \text{dist}(x, \Sigma_{p_1}^h) < \eta\varepsilon \right\}, \quad P_{\eta,\varepsilon,p_1}^h := \Omega^h \setminus B_{\eta,\varepsilon,p_1}^h$$

are occupied by linearly elastic materials with strain energy densities  $\mu W_I$  and  $W$  respectively, where  $\mu$  is a large stiffness parameter. Introducing  $\mathbb{S}^N$  as the space of all  $N \times N$  symmetric matrices equipped with the usual inner product and norm denoted as for  $\mathbb{R}^N$  by  $\cdot$  and  $|\cdot|$ ,  $W$  and  $W_I$  are two positive quadratic forms on  $\mathbb{S}^3$ .

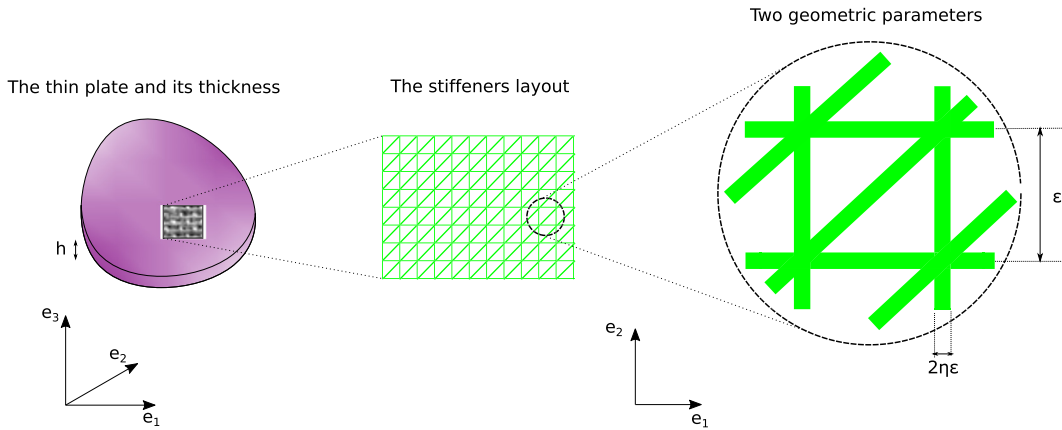


Fig. 1. The thin plate, the triplet of geometric parameters  $(\eta, \varepsilon, h)$  and the stiffeners' layout in the case  $p_1 = 3$ .

The structure made of these two parts perfectly bonded together is clamped on  $\Gamma_D^h := \partial\omega \times (-h, h)$  and subjected to body forces and surface forces on  $\Gamma_{\pm}^h := \omega \times \{\pm h\}$  of densities  $f^h$  and  $g^h$ . Hence, the equilibrium of the structure involves a quadruplet  $s := (\mu, \eta, \varepsilon, h)$  of data and leads to:

$$\left( \mathcal{P}_p^s \right) \quad \text{Min} \left\{ J_p^s(v); v \in H_{\Gamma_D^h}^1(\Omega^h; \mathbb{R}^3) \right\}$$

where, classically, for all domain  $G$  in  $\mathbb{R}^N$  and all smooth part  $\gamma$  of its boundary  $\partial G$ ,  $H_{\gamma}^1(G; \mathbb{R}^N)$  denotes the subspace of the Sobolev space  $H^1(G; \mathbb{R}^N)$  made of the elements with vanishing trace on  $\gamma$ ,

$$J_p^s(v) := \int_{P_{\eta,\varepsilon,p_1}^h} W(e(v)) \, dx + \mu \int_{B_{\eta,\varepsilon,p_1}^h} W_I(e(v)) \, dx - L^h(v)$$

$$L^h(v) := \int_{\Omega^h} f^h \cdot v \, dx + \int_{\Gamma_{\pm}^h \cup \Gamma_D^h} g^h \cdot v \, d\hat{x}$$

$e(v)$  being the strain tensor associated with the displacement field  $v$ .

Clearly, if  $(f^h, g^h)$  belongs to  $L^2(\Omega^h \times (\Gamma_+^h \cup \Gamma_-^h); \mathbb{R}^3)$ ,  $(\mathcal{P}_p^s)$  has a unique solution  $u_p^s$  and, considering the data  $s$  as a parameter, we are interested in the asymptotic behaviour of  $u_p^s$  when  $s$  takes values in a countable set of  $(0, +\infty)^4$  with  $\bar{s} := (+\infty, 0, 0, 0)$  as a unique limit point. As in the mathematical derivation of Kirchhoff–Love theory of plates (cf. [3,4]), it is convenient to introduce the linear mappings  $\Pi^h$  and  $S_h$ :

$$\xi = (\hat{\xi}, \xi_3) \in \mathbb{R}^3 \mapsto \Pi^h \xi = (\hat{\xi}, h\xi_3) \in \mathbb{R}^3$$

$$v \in L^2(\Omega^h; \mathbb{R}^3) \mapsto S_h v \in L^2(\Omega; \mathbb{R}^3) \text{ s.t. } (S_h v)(x) = \frac{1}{h} \Pi^h(v(\Pi^h x)), \quad \forall x \in \Omega := \omega \times (-1, 1)$$

We make the following assumption on the loading:

$$(H1) \left\{ \begin{array}{l} \exists (f, g) \in L^2(\Omega \times (\Gamma_+ \cup \Gamma_-); \mathbb{R}^3) \text{ s.t.} \\ f^h(\Pi^h x) = h \Pi^h f(x) \text{ a.e. } x \in \Omega, \quad g^h(\Pi^h x) = h^2 \Pi^h g(x) \text{ a.e. } x \in \Gamma_{\pm} \end{array} \right.$$

therefore,  $u_{s,p} := S_h u_p^s$  is the unique solution to

$$\left( \mathcal{P}_{s,p} \right) \quad \text{Min} \left\{ J_{s,p}(v); v \in H_{\Gamma_D}^1(\Omega; \mathbb{R}^3) \right\}$$

where

$$J_{s,p}(v) := \int_{P_{\eta,\varepsilon,p_1}} W(e(h, v)) \, dx + \mu \int_{B_{\eta,\varepsilon,p_1}} W_l(e(h, v)) \, dx - L(v)$$

$$L(v) := \int_{\Omega} f \cdot v \, dx + \int_{\Gamma_+ \cup \Gamma_-} g \cdot v \, d\hat{x}$$

$$e_{\alpha\beta}(h, v) = e_{\alpha\beta}(v), \quad e_{\alpha 3}(h, v) = \frac{1}{h} e_{\alpha 3}(v), \quad 1 \leq \alpha, \beta \leq 2, \quad e_{33}(h, v) = \frac{1}{h^2} e_{33}(v)$$

with  $\Gamma_D$  the reciprocal image by  $\Pi^h$  of  $\Gamma_D^h$  and, similarly, index  $h$  is dropped for the image by  $(\Pi^h)^{-1}$  of  $\Gamma_{\pm}^h, \Omega^h, B_{\eta,\varepsilon,p_1}^h, P_{\eta,\varepsilon,p_1}^h, \Sigma^{k,h}$  and  $\Sigma_{p_1}^h$ .

**2. A convergence result**

We assume that

$$(H2) \quad \begin{cases} \exists \bar{\mu} \in (0, +\infty] \text{ s.t. } \bar{\mu} := \lim_{s \rightarrow \bar{s}} (2\mu\eta), & \bar{\mu} \in (0, +\infty) \text{ if } p_2 = 1, \quad \bar{\mu} = +\infty \text{ if } p_2 = 2 \\ \lim_{s \rightarrow \bar{s}} \frac{\eta\varepsilon}{h} = 0, & \lim_{s \rightarrow \bar{s}} \frac{h^2}{\eta\varepsilon^2} = 0 \end{cases}$$

and introduce the space  $V_{KL}(\Omega)$  of Kirchhoff–Love displacements vanishing on  $\Gamma_D$ :

$$V_{KL}(\Omega) := \left\{ v \in H^1_{\Gamma_D}(\Omega; \mathbb{R}^3) \text{ s.t. } e_{i3}(v) = 0 \text{ in } \Omega, \quad 1 \leq i \leq 3 \right\}$$

and the positive definite quadratic form on  $\mathbb{S}^2$  defined by

$$W_{KL}(q) := \text{Min}\{W(e); e \in \mathbb{S}^3 \text{ s.t. } \hat{e} = q\}$$

where  $\hat{e}_{\alpha\beta} = e_{\alpha\beta}, 1 \leq \alpha, \beta \leq 2$ , for all  $e$  in  $\mathbb{S}^3$ .

Let  $(\tau, \nu)$  in  $\{(\tau^k, \nu^k), k = 1, 2, 3\}$ . We perform the change of coordinates

$$x = (x_1, x_2, x_3) \in \mathbb{R}^3 \xrightarrow{\theta} \mathbb{R}^3 \ni y = (y_\tau, y_\nu, y_3) := (x \cdot \tau, x \cdot \nu, x_3)$$

and, for all  $v$  in  $H^1(\Omega; \mathbb{R}^3)$ , define  $v_\tau$  by  $v_\tau(y) = v(\theta^{-1}(y)) \cdot \tau$ . To shorten notations, we write  $\partial_\tau v_\tau$  for the derivative in the sense of distributions  $\partial_{y_\tau} v_\tau$ . Note that  $\partial_{\tau 1} v_{\tau 1} = e_{22}(v), \partial_{\tau 2} v_{\tau 2} = e_{11}(v), \partial_{\tau 3} v_{\tau 3} = \frac{1}{2}(e_{11}(v) + 2e_{12}(v) + e_{22}(v))$ . For all  $k$  in  $\{1, 2, 3\}$ , we define the real convex quadratic function  $W_l^k$  by:

$$W_l^k(t) := \text{Inf} \left\{ W_l \left( (Q^k)^\top e Q^k \right); e \in \mathbb{S}^3, e_{11} = t \right\}$$

where  $Q^k = \tau^k \otimes e_1 + \nu^k \otimes e_2 + e_3 \otimes e_3$  and  $(Q^k)^\top$  denotes the transpose of  $Q^k$ .

Let

$$V_p := V_{KL}(\Omega) \text{ if } p_2 = 1, \quad V_{(p_1,2)} := V_{KL}(\Omega) \cap \left( \bigcap_{k \leq p_1} \{ \partial_{\tau^k} v_{\tau^k} = 0 \} \right)$$

$$\bar{J}_p(v) := \int_{\Omega} \left[ W_{KL}(\hat{e}(v)) + (2 - p_2) \bar{\mu} \sum_{k \leq p_1} W_l^k(\partial_{\tau^k} v_{\tau^k}) \right] dx - L(v)$$

Then we have the following result.

**Theorem 2.1.** Under assumptions (H1) and (H2), as  $s$  goes to  $\bar{s}$ ,  $u_{s,p}$  converges strongly in  $H^1(\Omega; \mathbb{R}^3)$  toward the unique solution  $u_p$  to

$$\left( \bar{\mathcal{P}}_p \right) \quad \text{Min} \left\{ \bar{J}_p(v), v \in V_p \right\}$$

and

$$\bar{J}_p(u_p) = \lim_{s \rightarrow \bar{s}} J_{s,p}(u_{s,p}) \tag{1}$$

$$\int_{\Omega} W_{KL}(\hat{e}(u_p)) \, dx = \lim_{s \rightarrow \bar{s}} \int_{P_{\eta,\varepsilon,p_1}} W(e(u_{s,p})) \, dx \tag{2}$$

$$\bar{\mu} \int_{\Omega} \sum_{k \leq p_1} W_l^k(\partial_{\tau^k}(u_p)_{\tau^k}) \, dx = \lim_{s \rightarrow \bar{s}} \mu \int_{B_{\eta,\varepsilon,p_1}} W_l(e(u_{s,p})) \, dx \text{ when } p_2 = 1 \tag{3}$$

The elementary proof is achieved in two steps through a standard method of variational convergence.

Step 1 (asymptotic behaviour of  $u_{s,p}$ )

**Proposition 2.1.** *When  $s$  goes to  $\bar{s}$ ,  $u_{s,p}$  (up to a not relabelled subsequence) weakly converges in  $H^1(\Omega; \mathbb{R}^3)$  toward some  $u_p$  in  $V_p$  such that*

$$\bar{J}_p(u_p) \leq \varliminf_{s \rightarrow \bar{s}} J_{s,p}(u_{s,p})$$

**Proof.** As, clearly,  $u_{s,p}$  is bounded in  $H^1(\Omega; \mathbb{R}^3)$ , we deduce that there exists some  $u_p$  in  $H^1_{\Gamma_D}(\Omega; \mathbb{R}^3)$  such that, up to a not relabelled subsequence,  $u_{s,p}$  weakly converges in  $H^1(\Omega; \mathbb{R}^3)$  toward  $u_p$ , which does belong to  $V_{KL}(\Omega)$ . Moreover, the very definitions of  $W_{KL}$ ,  $W_l^k$ ,  $1 \leq k \leq 3$ , and Jensen's inequality imply:

$$\begin{aligned} \int_{P_{\eta,\varepsilon,p_1}} W(e(u_{s,p})) \, dx &\geq \int_{P_{\eta,\varepsilon,p_1}} W_{KL}(\hat{e}(u_{s,p})) \, dx \\ \mu \int_{B_{\eta,\varepsilon,p_1}} W_l(e(u_{s,p})) \, dx &\geq 2\mu\eta \int_{\mathbb{R}^2 \times (-1,1)} \sum_{k \leq p_1} W_l^k(\partial_{\tau^k}(\langle u \rangle_{s,p}^k)_{\tau^k}) \, dx \end{aligned}$$

where

- $\langle u \rangle_{s,p}^k := \sum_{i \in I^k} \frac{1}{2\eta\varepsilon} \int_{-\eta\varepsilon}^{\eta\varepsilon} \widetilde{u}_{s,p}((x \cdot \tau^k)\tau^k + (i\varepsilon + t)v^k) \, dt \chi_{\varepsilon,i}^k$
- $\chi_{\varepsilon,i}^k$  is the characteristic function of  $\{(i+t)\varepsilon v^k, 0 < t < 1\} \times \mathbb{R} \times (-1, 1)$
- $\widetilde{v}$  is the extension by 0 to  $H^1(\mathbb{R}^2 \times (-1, 1); \mathbb{R}^3)$  of all  $v$  in  $H^1_{\Gamma_D}(\Omega; \mathbb{R}^3)$

As  $\langle u \rangle_{s,p}^k$  has the same strong limit  $\widetilde{u}_p$  in  $L^2(\Omega; \mathbb{R}^3)$  as  $\widetilde{u}_{s,p}$ , a standard lower semi-continuity argument yields the result.  $\square$

Step 2 (identification of  $u_p$ )

**Proposition 2.2.** *For all  $v$  in  $V_p$ , there exists a sequence  $v_s$  in  $H^1_{\Gamma_D}(\Omega; \mathbb{R}^3)$  such that*

$$\varliminf_{s \rightarrow \bar{s}} J_{s,p}(v_s) \leq \bar{J}_p(v)$$

**Proof.** It is straightforward by using test functions like

$$w_s(x) = h\Pi^h \rho(x) + \varepsilon \sum_{k \leq p_1} \left( \delta_{\eta}(x \cdot v^k/\varepsilon)\Pi^h \rho(x) + \varphi_{\eta}(x \cdot v^k/\varepsilon)(\psi^k, 0) \right)$$

with

$(\rho, \psi^k)$  in  $C^\infty(\bar{\Omega}; \mathbb{R}^3 \times \mathbb{R}^2)$  vanishing on  $\Gamma_D$ ,  
 $\delta_{\eta}, \varphi_{\eta}$  the 1-periodic functions such that

$$\delta_{\eta}(t) = \begin{cases} 0 & \text{if } t < \eta \\ (t - \eta)/\eta & \text{if } \eta \leq t \leq 2\eta \\ 1 & \text{if } 2\eta \leq t \leq 1/2 \\ \delta_{\eta}(1 - t) & \text{if } 1/2 \leq t \leq 1 \end{cases}, \quad \varphi_{\eta}(t) = \begin{cases} t & \text{if } |t| < \eta \\ \frac{\eta(1 - 2t)}{1 - 2\eta} & \text{if } \eta < t < 1 - \eta \end{cases}$$

to check that  $v_s$  defined by

$$\begin{aligned} v_s \in H^1_{\Gamma_D}(\Omega; \mathbb{R}^3); \quad & \int_{P_{\eta,\varepsilon,p_1}} DW(e(h, v_s)) \cdot e(h, w) \, dx + \mu \int_{B_{\eta,\varepsilon,p_1}} DW_l(e(h, v_s)) \cdot e(h, w) \, dx = \\ & = \int_{\Omega} DW_{KL}(\hat{e}(v_s)) \cdot \hat{e}(w) \, dx + (2 - p_2) \bar{\mu} \int_{\Omega} \sum_{k \leq p_1} DW_l^k(\partial_{\tau^k}(v_s)_{\tau^k}) \cdot \partial_{\tau^k} w_{\tau^k} \, dx, \quad \forall w \in H^1_{\Gamma_D}(\Omega; \mathbb{R}^3) \end{aligned}$$

satisfies the assertion.

Thus  $u_p$  is the unique minimizer in  $V_p$  of  $\bar{J}_p$  and satisfies (1) and, consequently, (2) and (3). Hence, the whole sequence  $u_{s,p}$  weakly converges in  $H^1(\Omega; \mathbb{R}^3)$ , but also strongly because

$$\overline{\lim}_{s \rightarrow \bar{s}} \int_{P_{\eta,\varepsilon,p_1}} W_{KL}(\hat{e}(u_{s,p})) \, dx \leq \overline{\lim}_{s \rightarrow \bar{s}} \int_{P_{\eta,\varepsilon,p_1}} W(e(u_{s,p})) \, dx = \int_{\Omega} W_{KL}(\hat{e}(u_p)) \, dx \leq \underline{\lim}_{s \rightarrow \bar{s}} \int_{P_{\eta,\varepsilon,p_1}} W_{KL}(\hat{e}(u_{s,p})) \, dx$$

and

$$\int_{B_{\eta,\varepsilon,p_1}} |\hat{e}(u_{s,p})|^2 \, dx \leq \frac{C}{\mu}, \quad \lim_{s \rightarrow \bar{s}} \int_{\Omega} |e_{i3}(u_{s,p})|^2 \, dx = 0, \quad 1 \leq i \leq 3 \quad \square$$

As quoted in [5,6], to make more precise the asymptotic behaviour of  $u_p^s$ , we develop a variant of Theorem 2.1. As no ambiguity ensues, we use the same symbol  $\hat{e}$  for an element  $e$  of  $\mathbb{S}^3$  such that its non vanishing entries are  $e_{\alpha\beta} = \hat{e}_{\alpha\beta}$ ,  $1 \leq \alpha, \beta \leq 2$ , and let  $e^\perp := e - \hat{e}$ . Then we have the following theorem.

**Theorem 2.2.** *There exists a unique  $z_p$  in  $L^2(\Omega; \mathbb{R}^3)$  such that  $e(h, u_p^s)$  converges strongly in  $L^2(\Omega; \mathbb{S}^3)$  towards  $\hat{e}(u_p) + z_p \otimes_s e_3$ . Moreover,*

- $(DW(\hat{e}(u_p) + z_p \otimes_s e_3))^\perp = 0$
- $\int_{\Omega} W(\hat{e}(u_p) + z_p \otimes_s e_3) \, dx = \int_{\Omega} W_{KL}(\hat{e}(u_p)) \, dx$
- $(u_p, z_p)$  is solution to

$$(\mathcal{Q}_p) \quad \text{Min} \left\{ \int_{\Omega} \left[ W(\hat{e}(v) + z \otimes_s e_3) \, dx + (2 - p_2) \bar{\mu} \sum_{k \leq p_1} W_l^k(\partial_{\tau^k} v_{\tau^k}) \right] dx - L(v); (v, z) \in V_p \times L^2(\Omega; \mathbb{R}^3) \right\}$$

**Proof.** As  $e(h, u_{s,p})$  is bounded in  $L^2(\Omega; \mathbb{S}^3)$ , it converges weakly toward some  $\hat{e}(u_p) + z_p \otimes_s e_3$  up to a not relabelled subsequence. Moreover  $(u_p, z_p)$  appears as the unique solution to  $(\mathcal{Q}_p)$  because for all element  $z$  of  $C^\infty(\bar{\Omega}; \mathbb{R}^3)$  vanishing

on  $\Gamma_D$ ,  $e(h, Z_h)$ , with  $Z_h = h\Pi^h Z$ ,  $Z(\hat{x}, x_3) := \int_0^{x_3} z(\hat{x}, t) \, dt$ , converges strongly in  $L^2(\Omega; \mathbb{S}^3)$  toward  $z \otimes_s e_3$ . Hence, the whole sequence  $e(h, u_{s,p})$  converges weakly in  $L^2(\Omega; \mathbb{S}^3)$ , but also strongly because

$$\begin{aligned} \int_{B_{\eta,\varepsilon,p_1}} |e(h, u_{s,p})|^2 \, dx & \leq \frac{C}{\mu}, \\ \overline{\lim}_{s \rightarrow \bar{s}} \int_{P_{\eta,\varepsilon,p_1}} W(e(h, u_{s,p})) \, dx & = \int_{\Omega} W_{KL}(\hat{e}(u_p)) \, dx \leq \int_{\Omega} W(\hat{e}(u_p) + z_p \otimes_s e_3) \, dx \leq \underline{\lim}_{s \rightarrow \bar{s}} \int_{P_{\eta,\varepsilon,p_1}} W(e(h, u_{s,p})) \, dx \quad \square \end{aligned}$$

Now these mathematical results can immediately be rephrased in terms related to the genuine physical problem  $(\mathcal{P}_p^s)$ , which will supply our asymptotic model. Let

$$\begin{aligned} V_{KL}(\Omega^h) & := \left\{ v \in H^1_{\Gamma_D^h}(\Omega^h; \mathbb{R}^3); e_{i3}(v) = 0, 1 \leq i \leq 3 \right\} \\ V_p^h & := V_{KL}(\Omega^h) \text{ if } p_2 = 1, \quad V_{(p_1,2)}^h := V_{KL}(\Omega^h) \cap \left( \bigcap_{k \leq p_1} \{ \partial_{\tau^k} v_{\tau^k} = 0 \} \right) \end{aligned}$$

$$\begin{aligned} \bar{u}_p^h &:= S_h^{-1} u_p \\ \bar{z}_p^h(x) &:= z_p \left( (\Pi^h)^{-1} x \right) \text{ a.e. } x \in \Omega^h \end{aligned}$$

then we have

**Theorem 2.3.** *The fields  $\bar{u}_p^h$  and  $\bar{z}_p^h$  are solutions to*

$$\begin{aligned} (\bar{P}_p^h) \quad & \text{Min} \left\{ \int_{\Omega^h} \left[ W_{KL}(\hat{e}(v)) + (2 - p_2) \bar{\mu} \sum_{k \leq p_1} W_l^k(\partial_{\tau^k} v_{\tau^k}) \right] dx - L^h(v); v \in V_p^h \right\} \\ (\bar{Q}_p^h) \quad & \text{Min} \left\{ \int_{\Omega^h} \left[ W(\hat{e}(v) + z \otimes_s e_3) + (2 - p_2) \bar{\mu} \sum_{k \leq p_1} W_l^k(\partial_{\tau^k} v_{\tau^k}) \right] dx - L^h(v); (v, z) \in V_p^h \times L^2(\Omega^h; \mathbb{R}^3) \right\} \end{aligned}$$

and

$$\lim_{s \rightarrow \bar{s}} \frac{1}{h^3} \int_{\Omega^h} |\hat{e}(u_p^s) - \hat{e}(\bar{u}_p^h)|^2 dx = 0, \quad \int_{\Omega^h} |e^\perp(u_p^s)|^2 dx \leq Ch^3 \tag{4}$$

$$\int_{\Omega^h} |e^\perp(u_p^s) - h\bar{z}_p^h \otimes_s e_3|^2 dx = o(h^3) \tag{5}$$

### 3. Physical interpretation

The second line of hypothesis (H2) refers to the design of the stiffeners but also to their layout. The condition  $\frac{\eta \varepsilon}{h} \rightarrow 0$  is clear: the stiffeners have to be slender. As to  $\frac{h^2}{\eta \varepsilon^2} \rightarrow 0$ , it encloses various information. On the one hand, because  $\frac{h^2}{\eta \varepsilon^2} = \frac{h}{\varepsilon} \frac{\eta \varepsilon}{h}$ , it says that the slenderness of the microscopic plates constituting the genuine plate  $\Omega^h$  (see Fig. 1) is lesser than the one of the stiffeners. On the other hand, the thickness  $h$  of the plate being given, it yields that the distance between two nearest parallel stiffeners has to be large enough (more precisely, the condition is  $\varepsilon \gg \frac{h}{\eta \varepsilon}$ ).

Theorem 2.3 tells us that, when the order of magnitude of the rigidity of the stiffeners is  $\frac{1}{2\eta}$ , the asymptotic behaviour of the structure is the one of Kirchhoff–Love type. The stiffeners supply an additional term  $\bar{\mu} \sum_{k \leq p_1} W_l^k(\partial_{\tau^k} v_{\tau^k})$  to the classical term  $W_{KL}(\hat{e}(v))$  stemming from the sole  $W$ . When the rigidity is of an order of magnitude larger than  $\frac{1}{2\eta}$ , the periodic distribution of stiffeners of direction  $\tau^k$  implies a vanishing stretch in this direction. Hence, to get a full rigidity, it suffices to use three families of stiffeners, in our case a fourth direction like  $\frac{e_1 - e_2}{2}$  is not necessary.

To go to the essential we assumed that the stiffness of each family of layers was the same, it is easy if not tedious to consider  $p_k^h \in \{1, 2\}$ ,  $k = 1, 2, 3! \dots$

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