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Optimal prevention of large risks with two types of claims

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Abstract

In this paper, we propose and study a risk model with two types of claims in which the insurer may invest into a prevention plan which decreases the large claims intensity without impacting the small claims. In this setting, we prove that prevention is advantageous when claim severities for small and large claims are ordered in the sense of the Harmonic-Mean-Residual-

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Lifetime (HMRL) order. In addition, we show that the optimal prevention amount is the lowest when there is no initial surplus. Finally, we characterize the asymptotic optimal prevention strategy when the initial surplus tends to infinity in the two main cases where both claim types are light-tailed and where one of them is light-tailed and the other one is heavy-tailed.

Key words: Ruin theory, Prevention, Optimal prevention strategy, Insurance.

1. Introduction

Prevention has become very popular in insurance. It is more and more used by insurers for marketing purposes, in relation with their corporate social responsibility. However, it is not yet regarded as a powerful risk management tool by insurance managers, in particular because they feel that prevention in insurance has more impact on extreme risks than on more frequent claims. As some of the extreme health risks (like long term affections) are taken care of by social security in several countries, and as the policyholder might switch insurer in the meantime, insurers fear that prevention might benefit to the state or to their competitors. In the near future, it might become possible for insurers of some countries to claim reimbursement of their prevention expenses by the state if they prove that their efforts have an impact on the claims paid by social security. In property and casualty insurance, prevention is often efficient to manage extreme claims arising from consequences of dramatic fire or flooding episodes. Prevention has however less impact on less severe events. It is therefore interesting to study optimal prevention strategies in presence of two types of claims: large claims for which prevention has

an impact and small claims for which the effect of prevention is limited or does not exist.

The impact of prevention (sometimes combined with self-insurance, that can be assimilated to reinsurance in our framework) has been investigated in economics by Ehrlich and Becker (Ehrlich and Becker, 1972), Dionne and Eeckhoudt (Dionne and Eeckhoudt, 1985) and Courbage (Courbage, 2001), among others. In ruin theory, a first risk model with prevention has been recently proposed by Gauchon et al. (Gauchon et al., 2019), with a unique set of claims for which prevention has some effect. The authors consider an insurance company with initial surplus $U(0) = u$. This company receives premiums at a rate c per unit of time and invests a fixed amount p in prevention per unit of time. The aggregate claim amount up to time t is given by a compound Poisson process $S(t) = \sum_{i=1}^{N(t)} X_i$, where N is a Poisson process of arrival intensity $\lambda(p)$ and the $(X_i)_{i \in \mathbb{N}^*}$ are i.i.d. random variables, independent from N , and with cumulative distribution function F_X , such that $\mathbb{E}(X) = \mu < \infty$. (Gauchon et al., 2019) assume that $\lambda(\cdot)$ is a decreasing, strictly convex, positive, and \mathcal{C}^2 function defined on $[0, c]$. They determine the optimal prevention investment for different risk indicators, and in particular for the ruin probability. This optimal prevention strategy does not depend on the initial surplus level.

A common point of the approaches of Ehrlich and Becker (Ehrlich and Becker, 1972), Gauchon et al. (Gauchon et al., 2019) and of all other papers (except Courbage and Rey who consider a second risk with lotteries on a two periods model (Courbage and Rey, 2012)) addressing prevention (to the best of our

knowledge) is that they consider that prevention has the same effect on all claims. In the present paper, we introduce two types of claims, and assume that prevention only works for the large claims. We show that in this case, the optimal prevention investment does depend on the initial surplus level. One could think that it would be optimal to do less prevention if one has a high initial surplus level. In the present paper, we show that it is the opposite: when the initial surplus is very large, under very reasonable conditions, optimising the ruin probability leads to implement more prevention than without an initial surplus. In the conclusion, we propose an intuitive explanation for this surprising result.

Our main contribution is threefold. First, we propose and analyse a first risk model with two types of claims, where prevention has some impact on the large claims only. Second, we show that prevention is advantageous when claim severities of small claims and large claims are ordered in the sense of the so-called Harmonic Mean Residual Lifetime (HMRL) order, and that the optimal prevention effort is higher with a positive initial surplus than when the initial surplus is zero. Third, we characterize the asymptotic optimal prevention strategy when the initial surplus tends to infinity in the two main cases where both claim types are light-tailed and where one of them is light-tailed and the other one is heavy-tailed. In addition, on the occasion of a natural example of the second case, we provide a necessary and sufficient condition to order a translated Exponential random variable and a Pareto one in the HMRL order.

The paper is organized as follows. In Section 2, we describe our model with

two types of claims. In Section 3, we recall the definition of the HMRL order and we notably provide a necessary and sufficient condition to order a translated Exponential random variable and a Pareto one in the HMRL order. In Section 4, we show that, under some HMRL ordering condition, the optimal prevention level is positive and higher than when the initial surplus is zero. In Section 5, we study the asymptotic optimal prevention strategy when both claim types are light tailed. Section 6 is devoted to the case where the second type of claims is heavy tailed.

2. The model

Let us consider an insurer facing two types of claims, namely small claims and large claims. In such a case, the number of claims $N(t)$ up to time t can be written as $N_1(t) + N_2(t)$, where N_1 (resp. N_2) is a Poisson process of arrival intensity $\lambda_1(p)$ (resp. $\lambda_2(p)$) that represents the number of small claims (resp. large claims), such that $\lambda(p) = \lambda_1(p) + \lambda_2(p)$. The Poisson processes N_1 and N_2 are assumed to be independent. Hence, the aggregate claim amount $S(t) = \sum_{k=1}^{N(t)} X_k$ up to time t can be decomposed as

$$S(t) = \sum_{k=1}^{N(t)} X_k = \sum_{i=1}^{N_1(t)} X_i^{(1)} + \sum_{j=1}^{N_2(t)} X_j^{(2)},$$

where the $X_i^{(1)}$ s (resp. $X_j^{(2)}$ s) are the claim severities for small claims (resp. large claims) and are assumed to be independent and identically distributed as $X^{(1)}$ (resp. $X^{(2)}$) with mean μ_1 (resp. μ_2) such that $\mu_1 \leq \mu_2$. Furthermore, the claim severities $X_i^{(1)}$ and $X_j^{(2)}$ are supposed to be independent and independent of the number of claims $N_1(t)$ and $N_2(t)$. Such a model amounts

to consider the claim severities X_k with cumulative distribution function

$$F_X(x) = \frac{\lambda_1(p)F_{X^{(1)}}(x) + \lambda_2(p)F_{X^{(2)}}(x)}{\lambda(p)}, \quad x \geq 0, \quad (1)$$

where $F_{X^{(1)}}(\cdot)$ and $F_{X^{(2)}}(\cdot)$ are the cumulative distribution functions of $X^{(1)}$ and $X^{(2)}$, respectively.

Considering that the prevention amount p only influences $\lambda(p)$ and not $F_X(\cdot)$, as supposed in Gauchon et al. (Gauchon et al., 2019), means that $\lambda_1(p)$ and $\lambda_2(p)$ must be impacted in the same way by p in order to keep constant both ratios $\frac{\lambda_1(p)}{\lambda(p)}$ and $\frac{\lambda_2(p)}{\lambda(p)}$ in (1). In other words, a decrease of $x\%$ for $\lambda(\cdot)$ because of prevention also means a decrease of $x\%$ for both $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$.

In practice, we can have some situations where prevention only impacts the number of large claims. Let us think about DNA test leading to preventive mastectomy, only preventing breast cancer and not smaller risks, like influenza. That is why in this paper the function $\lambda_1(\cdot)$ is considered to be constant, meaning that prevention only impacts the large claims. So, in the following, we consider an insurer with a surplus process given by

$$\begin{aligned} U(t, p) &= u + (c - p)t - \sum_{k=1}^{N(t)} X_k \\ &= u + (c - p)t - \sum_{i=1}^{N_1(t)} X_i^{(1)} - \sum_{j=1}^{N_2(t)} X_j^{(2)} \end{aligned} \quad (2)$$

where $\lambda_1(p) = \lambda_1 > 0$ and $\lambda_2(p)$ is a decreasing, strictly convex, positive, and \mathcal{C}^2 function defined on $[0, c]$. These constraints on $\lambda_2(\cdot)$ are similar to the ones imposed on $\lambda(\cdot)$ in (Gauchon et al., 2019) and we refer the reader to this latter paper for a discussion about these assumptions. In order to avoid

ruin with certainty, we define $p_{lim} \in [0, c]$ as the solution of

$$\frac{\lambda_1 \mu_1 + \lambda_2(p_{lim}) \mu_2}{c - p_{lim}} = 1 \quad (3)$$

and we require $p < p_{lim}$. Since $\lambda_2(\cdot)$ is continuous and decreasing, the intermediate value theorem ensures the existence of p_{lim} when $\frac{\lambda_1 \mu_1 + \lambda_2(0) \mu_2}{c} < 1$, which is assumed in the rest of the paper.

In this model, we notice that the prevention does not only act on the number of claims but also on the distribution F_X of the claim severities X_k . In the following, we denote by $X(p)$ the random variable X to make explicit the link with the prevention amount p .

In this paper, we are interested in the prevention amount $p^*(u)$ that minimizes the ruin probability

$$\psi(u, p) = \mathbb{P}(\exists t > 0 \text{ such that } U(t, p) < 0).$$

It is well-known that $\psi(u, p)$ coincides with the tail function of a compound geometric distribution, namely

$$\psi(u, p) = \mathbb{P}\left(\sum_{j=1}^M D_j > u\right), \quad (4)$$

where M follows the geometric distribution with success probability $\varphi(0, p) = 1 - \psi(0, p)$ and where the random variables D_1, D_2, \dots are called ladder heights and are independent and identically distributed as the random variable D , also denoted $D(p)$ in the following to make explicit the dependence with respect to p . In our compound Poisson model, the cumulative distribution function of $D(p)$ is given by the integrated tail distribution

$$F_{D(p)}(u) = \int_0^u \frac{1 - F_X(x)}{\mathbb{E}(X)} dx, \quad u > 0. \quad (5)$$

From (4), we get the well-known Pollaczec Khinchin formula

$$\varphi(u, p) = \varphi(0, p) \sum_{n=0}^{\infty} (1 - \varphi(0, p))^n F_{D(p)}^{*n}(u), \quad (6)$$

where $F_{D(p)}^{*n}$ is the n -fold convolution of $F_{D(p)}$. Let us remark that this formula guarantees the existence of the derivative $\frac{\partial \varphi(u, p)}{\partial p}$ when $\frac{\partial F_{D(p)}(u)}{\partial p}$ exists.

Denoting by $M_X(\cdot)$ the moment generating function of X , we can define for our model the well-known adjustment coefficient $\kappa(p)$, which verifies

$$\lambda(p) + (c - p)\kappa(p) = \lambda(p)M_X(\kappa(p)). \quad (7)$$

It always exists when

$$M_X(s) < \infty \quad \text{for all } s \in \mathbb{R} \quad (8)$$

or when there exists $s^* > 0$ such that

$$M_X(s) < \infty \quad \text{for all } s < s^* \quad \text{and} \quad M_X(s) = \infty \quad \text{for all } s \geq s^*. \quad (9)$$

The reader could refer to the book of Asmussen and Albrecher (Asmussen and Albrecher, 2010) for more details.

Notice that models with multiple risks have been introduced by Cramér in 1955 (Cramér, 1955) and have been used for example by Dickson and Gray (Dickson and Gray, 1984), Bowers et al. (Bowers et al., 1984) or Gerber et al. (Gerber et al., 1987).

3. Preliminaries

Before studying the optimal prevention amount $p^*(u)$, we recall in this section the notion of Harmonic Mean Residual Life (HMRL) order (more details can

be found in Shaked and Shanthikumar (Shaked and Shanthikumar, 2007)). Also, we prove two results that will be useful in the subsequent analysis

Definition 1. Given two non-negative random variables $X^{(1)}$ and $X^{(2)}$ with respective cumulative distribution functions $F_{X^{(1)}}$ and $F_{X^{(2)}}$, recall that $X^{(1)}$ is said to be smaller than $X^{(2)}$ in the harmonic mean residual life order (denoted as $X^{(1)} \preceq_{\text{hmrl}} X^{(2)}$) when

$$\frac{\int_t^\infty \bar{F}_{X^{(1)}}(u) du}{\mathbb{E}(X^{(1)})} \leq \frac{\int_t^\infty \bar{F}_{X^{(2)}}(u) du}{\mathbb{E}(X^{(2)})} \quad \text{for all } t \geq 0, \quad (10)$$

where $\bar{F}_{X^{(1)}} = 1 - F_{X^{(1)}}$ and $\bar{F}_{X^{(2)}} = 1 - F_{X^{(2)}}$. The inequality in (10) can be equivalently written as

$$\frac{\mathbb{E}((X^{(1)} - t)_+)}{\mathbb{E}(X^{(1)})} \leq \frac{\mathbb{E}((X^{(2)} - t)_+)}{\mathbb{E}(X^{(2)})} \quad \text{for all } t \geq 0. \quad (11)$$

Below we give some examples taken from Heilmann and Schröter (Heilmann and Schröter, 1991) where $X^{(1)} \preceq_{\text{hmrl}} X^{(2)}$ holds true:

- If $X^{(1)}$ is Uniformly distributed over the interval $[a, b]$ and $X^{(2)}$ is Uniformly distributed over the interval (a', b') , then $X^{(1)} \preceq_{\text{hmrl}} X^{(2)}$ if and only if $a + b \leq a' + b'$ and $b \leq b'$.
- If $X^{(1)}$ is Exponentially distributed with mean $1/a$ and $X^{(2)}$ is Exponentially distributed with mean $1/a'$ then $X^{(1)} \preceq_{\text{hmrl}} X^{(2)}$ if and only if $a \geq a'$.
- If $X^{(1)}$ is Pareto distributed with parameters a and b , that is, $X^{(1)}$ has distribution function $1 - (\frac{a}{x})^b$, $x > a$, and if $X^{(2)}$ is Pareto distributed with parameters a' and b' with $\min(b, b') > 1$, then $X^{(1)} \preceq_{\text{hmrl}} X^{(2)}$ if and only if $\frac{b-1}{b'-1} \geq \max(\frac{a}{a'}, 1)$

Next to these examples, we can also add the next result that will be of interest later on.

Proposition 2. *If $X^{(1)}$ follows a translated Exponential distribution over the interval $[a, \infty]$ with mean $a + \frac{1}{\lambda}$ and $X^{(2)}$ is Pareto distributed with parameters a and b , then $X^{(1)} \preceq_{\text{hmrl}} X^{(2)}$ if and only if $b \leq \lambda a + 1$.*

Proof. Since

$$F_{X^{(1)}}(x) = 1 - e^{-\lambda(x-a)}, \quad x \geq a \quad (12)$$

and

$$F_{X^{(2)}}(x) = 1 - \left(\frac{a}{x}\right)^b \quad x \geq a, \quad (13)$$

inequality (10) tells us that $X^{(1)} \preceq_{\text{hmrl}} X^{(2)}$ if and only if

$$\frac{e^{\lambda(a-t)}}{\lambda a + 1} \leq \left(\frac{a}{t}\right)^{b-1} \frac{1}{b}, \quad \text{for all } t \geq a. \quad (14)$$

In particular, inequality (14) must hold for $t = a$. A necessary condition for $X^{(1)} \preceq_{\text{hmrl}} X^{(2)}$ is thus given by

$$\frac{1}{\lambda a + 1} \leq \frac{1}{b}. \quad (15)$$

Let us now show that (15) is actually a sufficient condition for $X^{(1)} \preceq_{\text{hmrl}} X^{(2)}$. Inequality 14 can be rewritten as

$$\log\left(\frac{b}{\lambda a + 1}\right) + \lambda(a-t) - (b-1)[\log(a) - \log(t)] \leq 0 \quad \text{for all } t \geq a. \quad (16)$$

Considering $t > a$ and introducing the function $g(t) = \log\left(\frac{b}{\lambda a + 1}\right) + \lambda(a-t) - (b-1)[\log(a) - \log(t)]$, we easily see that $g'(t) < 0$ if and only if $\frac{b-1}{\lambda} < t$, which is always true since (15) yields $\frac{b-1}{\lambda} \leq a < t$. So, it comes $g'(t) < 0$ for

all $t > a$. Now, as (15) implies that $g(a) < 0$, we finally get $g(t) < 0$ for all $t \geq a$. ■

Notice that if $b > 1$, $\mathbb{E}(X^{(1)}) \leq \mathbb{E}(X^{(2)})$ if and only if $1 + \lambda a \leq b \frac{a\lambda}{b-1}$. Indeed, multiplying both sides of the last inequality by $b - 1$ shows that $X^{(1)} \preceq_{\text{hmrl}} X^{(2)}$ is equivalent to $\mathbb{E}(X^{(1)}) \leq \mathbb{E}(X^{(2)})$.

The next proposition will be useful to prove Proposition 5. Recall that one says that $X^{(1)}$ is smaller than $X^{(2)}$ in the usual stochastic order, denoted $X^{(1)} \preceq_{\text{st}} X^{(2)}$, when $F_{X^{(1)}}(t) \geq F_{X^{(2)}}(t)$ for all t .

Proposition 3. *Let $p_1 \leq p_2$. We have $X^{(1)} \preceq_{\text{hmrl}} X^{(2)}$ if and only if $D(p_2) \preceq_{\text{st}} D(p_1)$.*

Proof. We know from Theorem 2.B.2. in Shaked and Shanthikumar (Shaked and Shanthikumar, 2007) that

$$X(p_2) \preceq_{\text{hmrl}} X(p_1) \Leftrightarrow D(p_2) \preceq_{\text{st}} D(p_1). \quad (17)$$

So, it suffices to show that $X^{(1)} \preceq_{\text{hmrl}} X^{(2)} \Leftrightarrow X(p_2) \preceq_{\text{hmrl}} X(p_1)$. Let us denote $l(p) = \frac{\lambda_1 \mu_1}{\lambda_1 \mu_1 + \lambda_2(p) \mu_2}$ for $p \in (0, 1)$. We notice that $l(p_1) \leq l(p_2)$ since $\lambda_2(p_1) \geq \lambda_2(p_2)$. Considering the “ \Rightarrow ” part, as (10) holds, we clearly have

$$\begin{aligned} \frac{\int_t^\infty \bar{F}_{X(p_2)}(u) du}{\mathbb{E}(X(p_2))} &= \frac{\lambda_1 \int_t^\infty \bar{F}_{X^{(1)}}(u) du + \lambda_2(p_2) \int_t^\infty \bar{F}_{X^{(2)}}(u) du}{\lambda_1 \mu_1 + \lambda_2(p_2) \mu_2} \\ &= l(p_2) \frac{\int_t^\infty \bar{F}_{X^{(1)}}(u) du}{\mu_1} + (1 - l(p_2)) \frac{\int_t^\infty \bar{F}_{X^{(2)}}(u) du}{\mu_2} \\ &\leq l(p_1) \frac{\int_t^\infty \bar{F}_{X^{(1)}}(u) du}{\mu_1} + (1 - l(p_1)) \frac{\int_t^\infty \bar{F}_{X^{(2)}}(u) du}{\mu_2} \\ &= \frac{\int_t^\infty \bar{F}_{X(p_1)}(u) du}{\mathbb{E}(X(p_1))}. \end{aligned}$$

Turning to the “ \Leftarrow ” part, it suffices to notice that $\frac{\int_t^\infty \overline{F}_{X(p_2)}(u) du}{\mathbb{E}(X(p_2))} \leq \frac{\int_t^\infty \overline{F}_{X(p_1)}(u) du}{\mathbb{E}(X(p_1))}$ yields

$$(l(p_2) - l(p_1)) \left(\frac{\int_t^\infty \overline{F}_{X(2)}(u) du}{\mu_2} - \frac{\int_t^\infty \overline{F}_{X(1)}(u) du}{\mu_1} \right) \geq 0, \quad (18)$$

which ends the proof. ■

4. Optimal prevention amount $p^*(u)$

Let us first consider an insurer with no initial surplus. We then have the following result.

Proposition 4. *The optimal prevention amount $p^*(0)$ is positive if and only if*

$$-\lambda'_2(0) > \frac{\lambda_1\mu_1 + \lambda_2(0)\mu_2}{\mu_2c}. \quad (19)$$

In this case, $p^*(0)$ satisfies

$$-\lambda'_2(p^*(0)) = \frac{\lambda_1\mu_1 + \lambda_2(p^*(0))\mu_2}{\mu_2(c - p^*(0))}. \quad (20)$$

Proof. Starting from

$$\varphi(0, p) = 1 - \frac{\mu_1\lambda_1 + \mu_2\lambda_2(p)}{c - p}, \quad (21)$$

we get

$$\frac{\partial\varphi(0, p)}{\partial p} = -\frac{\lambda_1\mu_1}{(c - p)^2} - \frac{\lambda_2(p)\mu_2}{(c - p)^2} - \frac{\lambda'_2(p)\mu_2}{c - p} \quad (22)$$

and

$$\frac{\partial^2\varphi(u, p)}{\partial p^2} = \frac{2}{c - p} \frac{\partial\varphi(0, p)}{\partial p} - \frac{\lambda''_2(p)\mu_2}{c - p}. \quad (23)$$

Now, similarly to the proof of Proposition 1 in (Gauchon et al., 2019), it suffices to see that Equations (22) and (23) enable us to prove that if $\frac{\partial\varphi(0,0)}{\partial p} \leq 0$, then $\frac{\partial\varphi(0,p)}{\partial p} < 0$ and $\frac{\partial^2\varphi(0,p)}{\partial p^2} < 0$ for all $p > 0$. ■

Hence, if Condition (19) is fulfilled, an insurer with no initial surplus can decrease its risk by investing a part of its premiums in prevention.

Notice that Condition (19) is slightly easier to fulfill than Condition (10) in (Gauchon et al., 2019), suggesting that in some cases, it is better to concentrate the effort for preventing large claims rather than trying to invest money for preventing both small and large claims.

Next to Condition (19), the following result gives an additional sufficient condition ensuring that an insurer with a positive initial surplus can also decrease its risk through prevention.

Proposition 5. *If Condition (19) is verified and if $X^{(1)} \preceq_{hmrl} X^{(2)}$, then $p^*(u) > 0$ for all $u \geq 0$.*

Proof. By definition, $\varphi(u, p_{lim}) = 0$ and $p < p_{lim}$ ensures that $\varphi(u, 0) > 0$. As Condition (19) holds true, we know from the boundedness theorem that $\varphi(u, \cdot)$ reaches a maximum for one $p \in [0, p_{lim}]$.

Now, it suffices to prove that $\frac{\partial\varphi(0,p)}{\partial p} > 0$ implies $\frac{\partial\varphi(u,p)}{\partial p} > 0$ and the proof will be completed. From the Pollaczeck Khinchin formula (6), we get

$$\begin{aligned} \frac{\partial\varphi(u,p)}{\partial p} &= \frac{\partial\varphi(0,p)}{\partial p} \sum_{n=0}^{\infty} (1 - \varphi(0,p))^n F_{D(p)}^{*n}(u) \left(1 - \frac{\varphi(0,p)}{1 - \varphi(0,p)} n \right) \\ &\quad + \varphi(0,p) \sum_{n=0}^{\infty} (1 - \varphi(0,p))^n \frac{\partial F_{D(p)}^{*n}(u)}{\partial p}. \end{aligned} \quad (24)$$

In a first time, let us show that

$$\sum_{n=0}^{\infty} (1 - \varphi(0, p))^n F_{D(p)}^{*n}(u) \left(1 - \frac{\varphi(0, p)}{1 - \varphi(0, p)} n\right) \geq 0. \quad (25)$$

We have

$$\begin{aligned} \sum_{n=0}^{\infty} (1 - \varphi(0, p))^n \left(1 - \frac{\varphi(0, p)}{1 - \varphi(0, p)} n\right) &= \sum_{n=0}^{\infty} (1 - \varphi(0, p))^n \left(1 + n - \frac{n}{1 - \varphi(0, p)}\right) \\ &= \sum_{n=0}^{\infty} (1 - \varphi(0, p))^n (1 + n) - \sum_{n=0}^{\infty} (1 - \varphi(0, p))^{n-1} n \\ &= 1 + \sum_{n=1}^{\infty} (1 - \varphi(0, p))^n (1 + n) - \sum_{n=1}^{\infty} (1 - \varphi(0, p))^{n-1} n \\ &= 1 - 1 + \sum_{n=1}^{\infty} (1 - \varphi(0, p))^{n-1} n - \sum_{n=1}^{\infty} (1 - \varphi(0, p))^{n-1} n \\ &= 0 \\ &= 1 + \sum_{n=1}^{\infty} (1 - \varphi(0, p))^n \left(1 - \frac{\varphi(0, p)}{1 - \varphi(0, p)} n\right). \end{aligned}$$

Moreover, we notice that $F_{D(p)}^{*n+1}(u) \leq F_{D(p)}^{*n}(u) \leq 1$ for all $n \in \mathbb{N}^+$. Hence,

$$\sum_{n=0}^{\infty} (1 - \varphi(0, p))^n F_{D(p)}^{*n}(u) \left(1 - \frac{\varphi(0, p)}{1 - \varphi(0, p)} n\right) = 1 + \sum_{n=1}^{\infty} (1 - \varphi(0, p))^n F_{D(p)}^{*n}(u) \left(1 - \frac{\varphi(0, p)}{1 - \varphi(0, p)} n\right) \geq 0 \quad (26)$$

so that

$$\sum_{n=0}^{\infty} (1 - \varphi(0, p))^n F_{D(p)}^{*n}(u) \left(1 - \frac{\varphi(0, p)}{1 - \varphi(0, p)} n\right) \geq 0. \quad (27)$$

In a second time, it is easy to see that

$$\varphi(0, p) \sum_{n=0}^{\infty} (1 - \varphi(0, p))^n \frac{\partial F_{D(p)}^{*n}(u)}{\partial p} \geq 0. \quad (28)$$

Indeed, as the usual stochastic order is closed under convolution, Proposition 3 directly leads to $\frac{\partial F_{D(p)}^{*n}(u)}{\partial p} \geq 0$, which completes the proof. ■

Thus, in case (19) and $X^{(1)} \preceq_{\text{hml}} X^{(2)}$ hold true, we know that $p^*(u) > 0$ for all u . We learn from the next proposition that the minimum of $p^*(u)$

is reached in $u = 0$. Notice that contrary to (Gauchon et al., 2019), one observes that $p^*(u)$ is not constant anymore in the present context.

Proposition 6. *If (19) holds true and if we have $X^{(1)} \preceq_{hmrl} X^{(2)}$, then $p^*(0) < p^*(u)$ for all $u > 0$.*

Proof. From Equation (24) and inequalities (27) and (28), one sees that $\frac{\partial \varphi(0, p^*(u))}{\partial p} < 0$ is a necessary condition for $\frac{\partial \varphi(u, p^*(u))}{\partial p} = 0$. Now, Equations (22) and (23) imply that $\varphi(0, \cdot)$ is increasing on $[0, p^*(0)]$ and decreasing on $[p^*(0), p_{lim}]$. Thus, $\frac{\partial \varphi(0, p^*(u))}{\partial p} < 0$ is only possible when $p^*(u) > p^*(0)$. ■

In particular, as we will see later on, the asymptotic values derived for $p^*(\cdot)$ in Propositions 7 and 8 are indeed larger than $p^*(0)$, which enables to think that $p^*(\cdot)$ is increasing with u . However, we are not in position to prove this assertion.

5. $p^*(u)$ when $\kappa(p)$ exists for all $p \in [0, c]$

In this section, we consider claim severities $X^{(1)}$ and $X^{(2)}$ such that the adjustment coefficient $\kappa(p)$ exists for all $p \in [0, c]$. This requirement is for example met when both random variables $X^{(1)}$ and $X^{(2)}$ are Exponentially distributed, which is a particular case of models considered in Gerber et al. (Gerber et al., 1987) and Dufresne and Gerber (Dufresne and Gerber, 1988).

When the initial surplus u goes to infinity, we show in the next proposition that the optimal prevention amount $p^*(u)$ converges to the one that maximizes the adjustment coefficient $\kappa(p)$, denoted p_κ^* .

Proposition 7. *The prevention amount p_κ^* maximizing the adjustment coefficient $\kappa(p)$ is solution of the equation*

$$-\lambda_2'(p)[(c-p) + \lambda_1 \mathbb{E}(e^{\kappa(p)X(p)}(X^{(1)} - X^{(2)}))] = \lambda_1 + \lambda_2(p). \quad (29)$$

Moreover, we have

$$\lim_{u \rightarrow \infty} p^*(u) = p_\kappa^*. \quad (30)$$

Proof. The adjustment coefficient $\kappa(p)$ verifies

$$\lambda_1 + \lambda_2(p) + (c-p)\kappa(p) = (\lambda_1 + \lambda_2(p))M_{X(p)}(\kappa(p)). \quad (31)$$

Deriving (31) with respect to p yields

$$\begin{aligned} \lambda_2'(p)(1 - M_X(\kappa(p))) - \kappa(p) &= (\lambda_1 + \lambda_2(p)) \left(\frac{\lambda_1 \kappa(p)}{\lambda_1 + \lambda_2(p)} \right)' M_{X^{(1)}} \left(\frac{\lambda_1 \kappa(p)}{\lambda_1 + \lambda_2(p)} \right) M_{X^{(2)}} \left(\frac{\lambda_2(p)\kappa(p)}{\lambda_1 + \lambda_2(p)} \right) \\ &\quad + (\lambda_1 + \lambda_2(p)) \left(\frac{\lambda_2(p)\kappa(p)}{\lambda_1 + \lambda_2(p)} \right)' M_{X^{(1)}} \left(\frac{\lambda_1 \kappa(p)}{\lambda_1 + \lambda_2(p)} \right) M_{X^{(2)}} \left(\frac{\lambda_2(p)\kappa(p)}{\lambda_1 + \lambda_2(p)} \right) \end{aligned}$$

Now, Equation (31) can be rewritten as

$$(M_X(\kappa(p)) - 1) = \frac{(c-p)\kappa(p)}{\lambda_1 + \lambda_2(p)}. \quad (32)$$

So, combining (32) and (33) and taking $p = p_\kappa^*$, we get

$$-\lambda_2'(p_\kappa^*)[(c-p_\kappa^*) + \lambda_1 \mathbb{E}(e^{\kappa(p_\kappa^*)X(p_\kappa^*)}(X^{(1)} - X^{(2)}))] = \lambda_1 + \lambda_2(p_\kappa^*) \quad (34)$$

since $\kappa'(p_\kappa^*) = 0$. Notice that Equation (34) admits a unique solution when Condition (19) holds true.

Let us now prove that $\lim_{u \rightarrow \infty} p^*(u) = p_\kappa^*$. It is well-known that $\varphi(u, p)$ can be written as

$$\varphi(u, p) = 1 - e^{-\kappa(p)u} \mathbb{E}_{\kappa(p)}(e^{-\kappa(p)\xi(u)}), \quad (35)$$

where $\xi(u)$ is the random variable representing the overshoot in case of ruin and $\mathbb{E}_{\kappa(p)}$ is the expected value computed under a change of probability measure using the exponential families (for more details, we refer the reader to Section 4.4 and Equations (5.4) to (5.6) in Section 4.5 of Albrecher and Asmussen (Asmussen and Albrecher, 2010)). Thus, for all $u > 0$, we have

$$\begin{aligned} \frac{\partial \varphi(u, p^*(u))}{\partial p} &= -\kappa'(p^*(u))u e^{-\kappa(p^*(u))u} \mathbb{E}_{\kappa(p)}(e^{-\kappa(p^*(u))\xi(u)}) \\ &\quad + e^{-\kappa(p^*(u))u} \frac{\partial \mathbb{E}_{\kappa(p)}(e^{-\kappa(p^*(u))\xi(u)})}{\partial p} \\ &= 0, \end{aligned} \quad (36)$$

from which we deduce

$$\frac{\frac{\partial \mathbb{E}_{\kappa(p)}(e^{-\kappa(p^*(u))\xi(u)})}{\partial p}}{u \mathbb{E}_{\kappa(p)}(e^{-\kappa(p^*(u))\xi(u)})} = \kappa'(p^*(u)). \quad (37)$$

Moreover, the Cramer-Lundberg approximation shows that

$$\lim_{u \rightarrow \infty} \mathbb{E}_{\kappa(p)}(e^{-\kappa(p)\xi(u)}) = C(p) = (c - p) \frac{\varphi(0, p)}{(\lambda_1 + \lambda_2(p))M'_X(\kappa(p)) - c + p} > 0. \quad (38)$$

Now, since p is bounded, $C(p)$ is bounded as well. Furthermore, we get

$$\lim_{u \rightarrow \infty} \frac{\partial \mathbb{E}_{\kappa(p)}(e^{-\kappa(p^*(u))\xi(u)})}{\partial p} = C'(p_\kappa^*) \in \mathbb{R}. \quad (39)$$

Then, combining (38) and (39) leads to

$$\lim_{u \rightarrow \infty} \frac{\frac{\partial \mathbb{E}_{\kappa(p)}(e^{-\kappa(p^*(u))\xi(u)})}{\partial p}}{u \mathbb{E}_{\kappa(p)}(e^{-\kappa(p^*(u))\xi(u)})} = \lim_{u \rightarrow \infty} \frac{C'(p^*(u))}{u C(p^*(u))} = 0. \quad (40)$$

As Equation (37) holds true for all $u > 0$, (40) shows that

$$\lim_{u \rightarrow \infty} \kappa'(p^*(u)) = 0.$$

Because p_κ^* is the unique positive solution of $\kappa'(p) = 0$, we then get $\lim_{u \rightarrow \infty} p^*(u) = p_\kappa^*$. ■

6. $p^*(u)$ when $\kappa(p)$ does not exist

Let us now consider the case where $X^{(2)}$ follows a sub-exponential distribution and $X^{(1)}$ a light-tailed distribution. This is for example the case when $X^{(1)}$ follows a translated Exponential distribution over the interval $[a, \infty]$ with mean $a + \frac{1}{\lambda}$ and $X^{(2)}$ a Pareto distribution of parameters a and b . Then, the stochastic inequality $X^{(1)} \preceq_{\text{hmrl}} X^{(2)}$ holds true when $b \leq \lambda a + 1$, as proved in Proposition 2.

In such a situation, the moment generating function $M_X(\cdot)$ does not exist anymore. However, we can still determine the optimal prevention amount when u goes to infinity, as shown in the next proposition.

Proposition 8. *If Condition (19) holds true, if $X^{(1)} \preceq_{\text{hmrl}} X^{(2)}$ and if $D(p)$ follows a sub-exponential distribution for all p , then $\lim_{u \rightarrow \infty} p^*(u) = p_\infty > 0$, where p_∞ is solution of the equation*

$$-\lambda_2'(p) \left[\frac{\mu_2}{\varphi(0, p)} + \frac{\lambda_1 \mu_1}{\lambda_2(p)} \right] = \frac{\lambda_2(p) \mu_2 + \lambda_1 \mu_1}{\varphi(0, p)(c - p)}. \quad (41)$$

Proof. Proposition 5 guarantees the existence of an optimal prevention

amount for all u . As seen in (24), we have

$$\begin{aligned} \frac{\partial \psi(u, p)}{\partial p} &= \frac{\partial \varphi(0, p)}{\partial p} \sum_{n=0}^{\infty} (1 - \varphi(0, p))^n \bar{F}_{D(p)}^{*n}(u) \left(1 - \frac{\varphi(0, p)}{1 - \varphi(0, p)} n\right) \\ &\quad + \varphi(0, p) \sum_{n=0}^{\infty} (1 - \varphi(0, p))^n \frac{\partial \bar{F}_{D(p)}^{*n}(u)}{\partial p}. \end{aligned} \quad (42)$$

Moreover,

$$F_{D(p)}(u) = \frac{1}{\lambda_1 \mu_1 + \lambda_2(p) \mu_2} \int_0^u \lambda_1 \bar{F}_{X^{(1)}}(t) + \lambda_2(p) \bar{F}_{X^{(2)}}(t) dt, \quad (43)$$

which yields

$$\frac{\partial F_{D(p)}(u)}{\partial p} = \frac{\lambda_2'(p) \lambda_1 \mu_1}{\lambda_2(p) (\lambda_1 \mu_1 + \lambda_2(p) \mu_2)} \left(F_{D(p)}(u) - F_{D_{X^{(1)}}}(u) \right) \quad (44)$$

with $F_{D_{X^{(1)}}}(u) = \frac{1}{\mu_1} \int_0^u \bar{F}_{X^{(1)}}(t) dt$.

Also, we have

$$\frac{\partial \frac{F_{D(p)}^{*n}(u)}{F_{D(p)}(u)}}{\partial p} = \frac{\frac{\partial F_{D(p)}^{*n}(u)}{\partial p} F_{D(p)}(u) - \frac{\partial F_{D(p)}(u)}{\partial p} F_{D(p)}^{*n}(u)}{F_{D(p)}^2(u)}, \quad (45)$$

which leads to

$$\frac{\frac{\partial F_{D(p)}^{*n}(u)}{\partial p}}{F_{D(p)}(u)} = \frac{\frac{\partial \frac{F_{D(p)}^{*n}(u)}{F_{D(p)}(u)}}{\partial p} F_{D(p)}^2(u) + \frac{\partial F_{D(p)}(u)}{\partial p} F_{D(p)}^{*n}(u)}{F_{D(p)}^2(u)}. \quad (46)$$

Hence, combining (42), (44) and (46) and letting u goes to infinity, we finally obtain

$$\begin{aligned} \lim_{u \rightarrow \infty} \frac{\frac{\partial \psi(u, p)}{\partial p}}{\bar{F}_{D(p)}(u)} &= \frac{\partial \varphi(0, p)}{\partial p} \sum_{n=0}^{\infty} (1 - \varphi(0, p))^n n - \frac{\partial \varphi(0, p)}{\partial p} \sum_{n=0}^{\infty} (1 - \varphi(0, p))^{n-1} n^2 \varphi(0, p) \\ &\quad + \varphi(0, p) \frac{\lambda_2'(p) \lambda_1 \mu_1}{\lambda_2(p) (\lambda_1 \mu_1 + \lambda_2(p) \mu_2)} \sum_{n=1}^{\infty} (1 - \varphi(0, p))^n n \\ &= -\frac{\frac{\partial \varphi(0, p)}{\partial p}}{\varphi(0, p)^2} + \frac{\lambda_2'(p) \lambda_1 \mu_1}{\lambda_2(p) (\lambda_1 \mu_1 + \lambda_2(p) \mu_2)} \frac{(1 - \varphi(0, p))}{\varphi(0, p)}. \end{aligned} \quad (47)$$

Now, taking $\frac{\partial \psi(u, p)}{\partial p} = 0$ gives the announced result. ■

7. Conclusion

In this paper, we have proposed a risk model with two types of risks where prevention reduces the claim intensity of the most severe risk. In this context, we have studied the ruin probability as a function of the prevention effort. A sufficient condition for prevention to be efficient has been provided, and both cases of light and heavy-tailed claims have been considered.

Moreover, we have shown that an insurer should invest more in prevention when its initial reserves are large enough. It is likely due to the fact that, when the reserves are huge, the small claims are less important since ruin is most likely to occur due to an extreme claim.

Finally, let us notice that it would be interesting to consider a model where the impact of prevention is not constant over time and where the uncertainty on prevention efficiency is taken into account. This is left for further research.

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