HAL open science

On Siegel's problem for E-functions

S Fischler, T. Rivoal

▶ To cite this version:

S Fischler, T. Rivoal. On Siegel's problem for E-functions. 2019. hal-02314834v1

HAL Id: hal-02314834

https://hal.science/hal-02314834v1

Preprint submitted on 14 Oct 2019 (v1), last revised 29 May 2020 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On Siegel's problem for E-functions

S. Fischler and T. Rivoal

October 13, 2019

Abstract

Siegel defined in 1929 two classes of power series, the E-functions and G-functions, which generalize the Diophantine properties of the exponential and logarithmic functions respectively. In 1949, he asked whether any E-function can be represented as a polynomial with algebraic coefficients in a finite number of confluent hypergeometric series with rational parameters. The case of E-functions of differential order less than 2 was settled in the affirmative by Gorelov in 2004, but Siegel's question is open for higher order. We prove here that if Siegel's question has a positive answer, then the ring $\mathbf{G}$ of values taken by analytic continuations of G-functions at algebraic points must be a subring of the relatively "small" ring $\mathbf{H}$ generated by algebraic numbers, $1/\pi$ and the values of the derivatives of the Gamma function at rational points. Because that inclusion seems unlikely (and contradicts standard conjectures), this points towards a negative answer to Siegel's question in general. As intermediate steps, we first prove that any element of $\mathbf{G}$ is a coefficient of the asymptotic expansion of a suitable E-function, which completes previous results of ours. We then prove that the coefficients of the asymptotic expansion of a confluent hypergeometric series with rational parameters are in $\mathbf{H}$.

1 Introduction

Siegel [20, p. 223] introduced in 1929 the notion of E-function as a generalization of the exponential and Bessel functions. We fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$.

Definition 1. *A power series $F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \in \overline{\mathbb{Q}}[[z]]$ is an E-function if*

(i) $F(z)$ is solution of a non-zero linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$.

(ii) There exists $C > 0$ such that for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and any $n \geq 0$, $|\sigma(a_n)| \leq C^{n+1}$.

(iii) There exists $D > 0$ and a sequence of integers d_n, with $1 \leq d_n \leq D^{n+1}$, such that $d_n a_m$ are algebraic integers for all $m \leq n$.

Siegel's original definition was in fact slightly more general than above and we shall make some remarks about this in §2.1. Note that (i) implies that the a_n's all lie in a certain number field $\mathbb{K}$, so that in (ii) there are only finitely many Galois conjugates $\sigma(a_n)$ of a_n to consider, with $\sigma \in \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ (assuming for simplicity that $\mathbb{K}$ is a Galois extension of $\mathbb{Q}$). E-functions are entire, and they form a ring stable under $\frac{d}{dz}$ and $\int_0^z$. A power series $\sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[[z]]$ is said to be a G-function if $\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ is an E-function. G-functions also form a ring stable under $\frac{d}{dz}$ and $\int_0^z$; they are not entire in general but they can be analytically continued in suitably slit planes.

The generalized hypergeometric series is defined as

$$ {}_pF_q\left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z\right] := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(1)_n (b_1)_n \cdots (b_q)_n} z^n \tag{1.1} $$

where $p, q \geq 0$ and $(a)_0 := 1$, $(a)_n := a(a+1)\cdots(a+n-1)$ if $n \geq 1$. The parameters a_j and b_j are in $\mathbb{C}$, with the restriction that $b_j \notin \mathbb{Z}_{\leq 0}$ so that $(b_j)_n \neq 0$ for all $n \geq 0$. We shall also denote it by ${}_pF_q[a_1, \dots, a_p; b_1, \dots, b_q; z]$. Siegel proved that, for any integer $p \geq 1$, the *confluent* hypergeometric series

$$ {}_pF_p\left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_p \end{matrix}; z\right] \tag{1.2} $$

is an E-function (in the sense of this paper) when $a_j \in \mathbb{Q}$ et $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ for all j. The simplest example is ${}_1F_1[1; 1; z] = \exp(z)$. If $a_j \in \mathbb{Z}_{\leq 0}$ for some j, then the series reduces to a polynomial. Any polynomial with coefficients in $\overline{\mathbb{Q}}$ of hypergeometric functions of the form ${}_pF_p[a_1, \dots, a_p; b_1, \dots, b_p; \lambda z]$, with parameters $a_j, b_j \in \mathbb{Q}$ and $\lambda \in \overline{\mathbb{Q}}$, is an E-function.

The E-functions

$$ L(z) := \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{n} \Big) \frac{z^n}{n!}, \; H(z) := \sum_{n=0}^{\infty} \Big(\sum_{k=1}^{n} \frac{1}{k} \Big) \frac{z^n}{n!}, \; J_0(z) := \sum_{n=0}^{\infty} \frac{(iz/2)^{2n}}{n!^2} $$

are not of the hypergeometric type (1.2), even with z changed to λz for some $\lambda \in \overline{\mathbb{Q}}$, but we have

$$ \begin{aligned} L(z) &= e^{(3-2\sqrt{2})z} \cdot {}_1F_1\big[1/2; 1; 4\sqrt{2}z\big], \\ H(z) &= z e^{z} \cdot {}_2F_2\big[1, 1; 2, 2; -z\big], \\ J_0(z) &= e^{-iz} \cdot {}_1F_1[1/2; 1; 2iz]. \end{aligned} $$

(See [1, p. 509, 13.6.1] and [18].) These puzzling identities, amongst others, naturally suggest to study further the role played by hypergeometric series in the theory of E-functions. In fact, Siegel had already stated in [21] a problem that we reformulate as the following question.

Question (Siegel)**.** *Is it possible to write any E-function as a polynomial with coefficients in $\overline{\mathbb{Q}}$ of hypergeometric functions of the form ${}_pF_p[a_1, \dots, a_p; b_1, \dots, b_p; \lambda z]$, with parameters $a_j, b_j \in \mathbb{Q}$ and $\lambda \in \overline{\mathbb{Q}}$?*

It must be understood that λ and p can take various values in the polynomial. Siegel's original statement is given in §2.1 along with some comments. Gorelov [10, p. 514, Theorem 1] proved that the answer to Siegel's question is positive if the E-function (in the above sense, not Siegel's original one) satisfies a linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$ of order ≤ 2. He used the pioneering results of André [2] on E-operators. Gorelov's theorem was reproved in [18] with a method also based on André's results, but somewhat different in the details. It seems difficult to generalize any one of these two approaches when the order is ≥ 3, though Gorelov [11] also obtained partial results in the case of E-functions solution of a linear inhomogeneous differential equation of order 2 with coefficients in $\overline{\mathbb{Q}}(z)$, like $H(z)$ above.

In this paper, we adopt another point of view on Siegel's question. Let us first define two subrings of $\mathbb{C}$; the former was introduced and studied in [6].

Definition 2. $\mathbf{G}$ *denotes the ring of G-values, i.e. the values taken at algebraic points by the analytic continuations of all G-functions.*

$\mathbf{H}$ *denotes the ring generated by $\overline{\mathbb{Q}}$, $1/\pi$ and the values $\Gamma^{(n)}(r)$, $n \geq 0$, $r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$.*

Here, $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}dt$ is the usual Gamma function that can be analytically continued to $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. We can now state our main result.

Theorem 1. *At least one of the following statements is true:*

(*i*) $\mathbf{G} \subset \mathbf{H}$*;*

(*ii*) *Siegel's question has a negative answer.*

We provide in §2.2 another description of the ring $\mathbf{H}$, and explain there why the inclusion $\mathbf{G} \subset \mathbf{H}$ (and therefore a positive answer to Siegel's question) seems very unlikely; as Y. André, F. Brown and J. Fresán pointed out to us, this inclusion contradicts standard conjectures.

The paper is organized as follows. In §2, we comment on Siegel's original formulation of his problem and make some remarks on the ring $\mathbf{H}$. In §3, we prove that any element of $\mathbf{G}$ is a coefficient of the asymptotic expansion of a suitable E-function (Theorem 3). In §4, we prove that the coefficients of the asymptotic expansion of hypergeometric series ${}_pF_p$ with rational parameters are in $\mathbf{H}$ (Theorem 4). We complete the proof of Theorem 1 in §5 by comparing the results of the previous sections.

Acknowledgements. We warmly thank Yves André, Francis Brown and Javier Fresán for their comments on a previous version of this paper, and in particular for explaining to us why the inclusion $\mathbf{G} \subset \mathbf{H}$ cannot hold under the standard conjectures on (exponential) periods.

2 Comments on Theorem 1

2.1 Siegel's formulation of his problem

In [21, Chapter II, §9], Siegel proved that the hypergeometric series of the type (1.2) with rational parameters are E-functions, and named them "hypergeometric E-functions". He then wrote on page 58: *Performing the substitution $x \mapsto \lambda x$ for arbitrary algebraic λ and taking any polynomial in x and finitely many hypergeometric E-functions, with algebraic coefficients, we get again an E-function satisfying a homogeneous linear differential equation whose coefficients are rational function of x. It would be interesting to find out whether all such E-functions can be constructed in the preceding manner.*

Siegel obviously considered E-functions in his sense, which we recall here: in Definition 1, (i) is unchanged but (ii) and (iii) have to be replaced by

(ii') For any $\varepsilon > 0$ and for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, there exists $N(\varepsilon, \sigma) \in \mathbb{N}$ such that for any $n \geq N(\varepsilon, \sigma)$, $|\sigma(a_n)| \leq n!^{\varepsilon}$.

(iii') There exists a sequence of integers $d_n \neq 0$ such that $d_n a_m$ are algebraic integers for all $m \leq n$ and such that for any $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that for any $n \geq N(\varepsilon)$, $|d_n| \leq n!^{\varepsilon}$.

Again, by (i), there are only finitely many σ to consider for a given E-function. We have chosen to formulate his problem for E-functions in the restricted sense of Definition 1 because the proof of Theorem 1 is based on results which are currently proven only in this sense. However, *a fortiori*, Theorem 1 obviously holds *verbatim* if one considers E-functions in Siegel's sense. Note also that the function $1 - z$ is equal to the hypergeometric series ${}_1F_1[-1; 1; z]$ so that Siegel could have formulated his problem in terms of hypergeometric series only, as we did. Despite the apparences, the E-function $\sinh(z) = \frac{1}{2z}(e^z - e^{-z})$ is not a counter-example to Siegel's problem because $\frac{1}{2z}(e^z - 1) = {}_1F_1[1; 2; z]$; there is no unicity of the representation of E-functions by polynomials in hypergeometric ones.

Moreover, the series in (1.2) may be an E-function even if some of its parameters are not rational numbers. For instance, for every $\alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Z}_{\leq 0}$,

$$ {}_1F_1\left[\begin{matrix}\alpha+1\\ \alpha\end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(\alpha+1)_n}{(1)_n(\alpha)_n} z^n = \sum_{n=0}^{\infty} \frac{\alpha+n}{\alpha} \cdot \frac{z^n}{n!} = \big(1 + \frac{z}{\alpha}\big)e^z $$

is an E-function. Thus, even though Siegel did not consider such examples, the notion of "hypergeometric E-functions" could be interpreted in a broader way than he did in his problem. Galochkin [9] proved the following non-trivial characterization, where E-functions are understood in Siegel's sense. (See [17] for a different proof for E-functions in the sense of the present paper).

Theorem (Galochkin)**.** *Let $p \geq 1$, $a_1, \ldots, a_p, b_1, \ldots, b_p \in (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})^{2p}$ be such that $a_i \neq b_j$ for all i, j. Then, the hypergeometric series ${}_pF_p[a_1, \ldots, a_p; b_1, \ldots, b_p; z]$ is an E-function if and only if the following two conditions hold:*

(i) *The a_j's and b_j's are all in $\overline{\mathbb{Q}}$;*

(ii) *The a_j's and b_j's which are not rational (if any) can be grouped in $k \leq p$ pairs $(a_{j_1}, b_{j_1}), \ldots, (a_{j_k}, b_{j_k})$ such that $a_{j_\ell} - b_{j_\ell} \in \mathbb{N}$.*

It follows that hypergeometric E-functions with arbitrary parameters are in fact $\overline{\mathbb{Q}}$-linear combinations of hypergeometric E-functions with rational parameters. Hence, there is no loss of generality in considering the latter instead of the former in Siegel's problem.

Another generalization of Siegel's problem is the following. When $q \geq p \geq 1$, $r := q - p + 1 \geq 1$, $a_j \in \mathbb{Q}$ and $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, the function

$$ {}_pF_q \left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} ; z^{q-p+1} \right] := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n (rn)!}{(1)_n (b_1)_n \cdots (b_q)_n} \frac{z^{rn}}{(rn)!} \tag{2.1} $$

is an E-function. The special case $q = p$ is that of confluent hypergeometric series, but this family includes also Bessel's function

$$ J_0(z) := \sum_{n=0}^{\infty} (-1)^n \frac{(z/2)^{2n}}{n!^2} = {}_1F_2 \left[\begin{matrix} 1, 1 \\ 1 \end{matrix} ; (iz/2)^2 \right]. $$

We recall that $J_0(z) = e^{-iz} \cdot {}_1F_1[1/2; 1; 2iz]$ so that $J_0(z)$ is an example for Siegel's problem but this is not known for other parameters in the function (2.1) in general. It is natural to ask the following question: is it possible to write any E-function as a polynomial with coefficients in $\overline{\mathbb{Q}}$ of functions of the form (2.1) with z replaced with λz, $\lambda \in \overline{\mathbb{Q}}$? It must be understood that λ, p, q and $q - p$ can take various values in the polynomial.

2.2 The ring H

For $x \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, we define the Digamma function

$$ \Psi(x) := \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \sum_{n=0}^{\infty} \Big(\frac{1}{n+1} - \frac{1}{n+x} \Big), $$

where γ is Euler's constant $\lim_{n \to +\infty} (\sum_{k=1}^{n} 1/k - \log(n))$, and the Hurwitz zeta function

$$ \zeta(s, x) := \frac{(-1)^s}{(s-1)!} \Psi^{(s-1)}(x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \quad s \in \mathbb{N},\ s \geq 2. $$

The polylogarithms are defined by

$$ \mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s}, \quad s \in \mathbb{N}^* = \mathbb{N} \setminus \{0\}, $$

where the series converges for $|z| \leq 1$ (except at $z = 1$ if $s = 1$). The Beta function is defined as

$$\mathrm{B}(x, y) := \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

for $x, y \in \mathbb{C}$ which are not singularities of Beta coming from the poles of Γ at non-positive integers.

In this section, we shall prove the following result.

Proposition 1. *The ring* $\mathbf{H}$ *is generated by* $\overline{\mathbb{Q}}$, γ, $1/\pi$, $\mathrm{Li}_s(e^{2i\pi r})$ $(s \in \mathbb{N}^*,\ r \in \mathbb{Q},\ (s, e^{2i\pi r}) \neq (1,1))$, $\log(q)$ $(q \in \mathbb{N}^*)$ *and* $\Gamma(r)$ $(r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0})$.

For any $r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, $\Gamma(r)$ *is a unit of* $\mathbf{H}$.

Proof. We first prove that for any $r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, $\Gamma(r)$ is a unit of $\mathbf{H}$. Indeed, if $r \in \mathbb{N}^*$, then $\Gamma(r) \in \mathbb{N}^*$ and $1/\Gamma(r) \in \mathbb{Q} \subset \mathbf{H}$. If $r \in \mathbb{Q} \setminus \mathbb{Z}$, then by the reflection formula [4, p. 9, Theorem 1.2.1], we have

$$\frac{1}{\Gamma(r)} = \frac{1}{\pi}\sin(\pi r)\Gamma(1-r) \in \mathbf{H}$$

because $1/\pi \in \mathbf{H}$, $\sin(\pi r) \in \overline{\mathbb{Q}} \subset \mathbf{H}$ and $\Gamma(1-r) \in \mathbf{H}$.

From the identity $\Gamma'(x) = \Gamma(x)\Psi(x)$ we obtain that, for any integer $s \geq 1$ and any $r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$,

$$\Psi^{(s)}(r) = \frac{\Gamma^{(s+1)}(r)}{\Gamma(r)} - \sum_{k=0}^{s-1} \binom{s}{k} \frac{\Gamma^{(s-k)}(r)}{\Gamma(r)} \Psi^{(k)}(r).$$

Since $\Gamma(r)$ is a unit of $\mathbf{H}$, we have $\psi(r) \in \mathbf{H}$ and it follows immediately by induction on s that $\zeta(s, r) = \frac{(-1)^s}{(s-1)!}\Psi^{(s-1)}(r) \in \mathbf{H}$ for any $s \geq 2$ and any $r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. In particular $\gamma = -\Psi(1)$ and the values of the Riemann zeta function $\zeta(s) = \zeta(s, 1)$ $(s \geq 2)$ are all in $\mathbf{H}$. Note that γ is not expected to be in $\mathbf{G}$ but that $\zeta(s) \in \mathbf{G}$ for all $s \geq 2$.

We have for any $x \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ and any $n \in \mathbb{N}$,

$$\Psi(x+n) = \Psi(x) + \sum_{k=0}^{n-1} \frac{1}{k+x}, \tag{2.2}$$

and the identity $\Gamma'(x) = \Gamma(x)\Psi(x)$ also implies by induction that, for any $x \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, we have

$$\Gamma^{(s)}(x) = \Gamma(x) P_s\big(\Psi(x), \zeta(2, x), \ldots, \zeta(s, x)\big) \tag{2.3}$$

for some $P_s \in \mathbb{Q}[X_1, \ldots, X_s]$. Furthermore, set $p, q \in \mathbb{N}$, $0 < p \leq q$, and $\mu := \exp(2i\pi/q)$.

Then,

$$\Psi\Big(\frac{p}{q}\Big) = -\gamma - \log(q) - \sum_{n=1}^{q-1} \mu^{-np}\,\mathrm{Li}_1(\mu^n), \tag{2.4}$$

$$\mathrm{Li}_1(\mu^p) = -\frac{1}{q}\sum_{n=1}^{q} \mu^{np}\,\Psi\Big(\frac{n}{q}\Big), \quad p \neq q \tag{2.5}$$

$$\zeta\Big(s, \frac{p}{q}\Big) = q^{s-1}\sum_{n=1}^{q} \mu^{-np}\,\mathrm{Li}_s(\mu^n), \quad s \geq 2 \tag{2.6}$$

$$\mathrm{Li}_s(\mu^p) = \frac{1}{q^s}\sum_{n=1}^{q} \mu^{np}\,\zeta\Big(s, \frac{n}{q}\Big), \quad s \geq 2. \tag{2.7}$$

We refer to [4, p. 14] for details. From (2.5) and (2.7), we deduce that $\mathrm{Li}_s(\mu^p) \in \mathbf{H}$ for any $s \geq 1$ (with $(s, \mu^p) \neq (1,1)$); then (2.4) implies in turn that $\log(q) \in \mathbf{H}$. The numbers $\log(q)$ and $\mathrm{Li}_s(\mu^p)$ are also in $\mathbf{G}$.

The set of Identities (2.2)–(2.7) shows that $\mathbf{H}$ coincides with the ring generated by $\overline{\mathbb{Q}}$, $\gamma = -\Psi(1)$, $1/\pi$, $\mathrm{Li}_s(e^{2i\pi r})$ ($s \in \mathbb{N}^*$, $r \in \mathbb{Q}$, $(s, e^{2i\pi r}) \neq (1,1)$), $\log(q)$ ($q \in \mathbb{N}^*$) and $\Gamma(r)$ ($r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$). □

Other units of $\mathbf{H}$ can be easily identified, which are also units of $\mathbf{G}$ (see [6, §2.2]): the values of the Beta function $\mathrm{B}(x,y)$ at rational numbers x, y at which it is defined and non-zero. It follows that $\pi = \Gamma(1/2)^2 = \mathrm{B}(1/2, 1/2)$ and more generally $\Gamma(a/b)^b = (a-1)!\prod_{j=1}^{b-1}\mathrm{B}(a/b, ja/b)$, $a, b \in \mathbb{N}^*$, are units of $\mathbf{H}$. By the Chowla-Selberg formula [15, p. 230, Corollary 2], periods of CM elliptic curves defined over $\mathbb{Q}$ are also units of $\mathbf{H}$.

If Siegel's problem has a positive answer, Theorem 1 yields $\mathbf{G} \subset \mathbf{H}$: any element of $\mathbf{G}$ can be written as a polynomial, with algebraic coefficients, in the numbers γ, $1/\pi$, $\mathrm{Li}_s(e^{2i\pi r})$, $\log(q)$ and $\Gamma(r)$ of Proposition 1. This seems extremely doubtful: we recall that $\mathbf{G}$ contains all the multiple zeta values

$$\zeta(s_1, s_2, \ldots, s_n) := \sum_{k_1 > k_2 > \cdots > k_n \geq 1} \frac{1}{k_1^{s_1} k_2^{s_2} \cdots k_n^{s_n}},$$

where the integers s_j are such that $s_1 \geq 2, s_2 \geq 1, \ldots, s_n \geq 1$, all values at algebraic points of (multiple) polylogarithms, all elliptic and abelian integrals, etc. For now, we have proved that $\mathbf{G} \cap \mathbf{H}$ contains the ring generated by $\overline{\mathbb{Q}}$, $1/\pi$ and all the values $\mathrm{Li}_s(e^{2i\pi r})$, $\log(q)$ and $\mathrm{B}(x,y)$, and it is in fact possible that both rings are equal.

It is interesting to know what can be deduced from the standard conjectures in the domain, such as the Bombieri-Dwork conjecture "G-functions come from geometry", Grothendieck's periods conjecture, its extension to exponential periods by Fresán-Jossen, and the Rohrlich-Lang conjecture on the values of the Gamma function; see [3, Partie III] and [8, p. 201, Conjecture 8.2.5]. In a private communication to the authors, Y. André wrote the following argument, which he has autorised us to reproduce here. It shows that $\mathbf{G} \subset \mathbf{H}$ cannot hold under these standard conjectures:

Because of the presence of γ, the inclusion $\mathbf{G} \subset \mathbf{H}$ does not contradict Grothendieck's period conjecture but it certainly contradicts its extension to exponential motives. More precisely, in the description of $\mathbf{H}$ given in Proposition 1, we find γ (a period of an exponential motive E_γ, which is a non-classical extension of the Tate motive [8, §12.8]), $1/\pi$ (a period of the Tate motive), $\mathrm{Li}_s(e^{2i\pi r})$ (periods of a mixed Tate motive over $\mathbb{Z}[1/r]$), $\log(q)$ (a period of a 1-motive over $\mathbb{Q}$), and $\Gamma(r)$ whose suitable powers are periods of Abelian varieties with complex multiplication by $\mathbb{Q}(e^{2i\pi r})$. On the one hand, let M be the Tannakian category of classical mixed motives over $\overline{\mathbb{Q}}$ generated by all these motives. On the other hand, consider a non CM elliptic curve over $\overline{\mathbb{Q}}$ and E its motive. The periods of E are in $\mathbf{G}$: indeed, it is enough to consider the Gauss hypergeometric solutions centered at $1/2$, and to observe that the periods of the fiber at $1/2$ of the Legendre family can be expressed using values of the Beta function at rational points by the Chowla-Selberg formula, and in particular are algebraic in π and $\Gamma(1/4)$. If $\mathbf{G} \subset \mathbf{H}$, the periods of E are in $\mathbf{H}$. By the exponential periods conjecture, E would be in M, which is impossible since the motivic Galois group of M is pro-resoluble, while that of E is GL_2.

We conclude this section with a question of J. Fresán: at which differential order can we expect to find a counter-example to Siegel's problem? Based on the above remarks, it seems unlikely, for any fixed integer $s \geq 1$, that all the values $\mathrm{Li}_s(\alpha)$ are in $\mathbf{H}$, where $\alpha \in \overline{\mathbb{Q}}$, $|\alpha| < 1$. From the proof of Theorem 3 below, we deduce that if $\mathrm{Li}_s(\alpha) \notin \mathbf{H}$, then the E-function

$$\sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} \frac{\alpha^k}{k^s} \right) \frac{z^n}{n!}$$

is such a counter-example. It is of differential order at most $s+2$ because it is in the kernel of the differential operator $P(\theta-2) + zQ(\theta-1) + z^2R(\theta) \in \overline{\mathbb{Q}}[z, \frac{d}{dz}]$, where $\theta := z\frac{d}{dz}$ and

$$P(x) := (x+2)(x+1)^{s+1}, \quad Q(x) := (x+1)(\alpha x^s - (x+1)^s), \quad R(x) := \alpha x^s.$$

It is thus possible that a counter-example to Siegel's problem already exists at the order 3. However, the function $H(z) := \sum_{n=0}^{\infty} (\sum_{k=1}^{n} \frac{1}{k}) \frac{z^n}{n!}$ is an example of order 3 to the problem (see the Introduction) and this shows that one must be careful and not draw hasty conclusions here.

3 Elements of G as coefficients of asymptotic expansions of E-functions

3.1 Definition of asymptotic expansions

As in [7], the asymptotic expansions used throughout this paper are defined as follows.

Definition 3. *Let $\theta \in \mathbb{R}$, and $\Sigma \subset \mathbb{C}$, $S \subset \mathbb{Q}$, $T \subset \mathbb{N}$ be finite subsets. Given complex numbers $c_{\rho,\alpha,i,n}$, we write*

$$f(x) \approx \sum_{\rho\in\Sigma} e^{\rho x} \sum_{\alpha\in S} \sum_{i\in T} \sum_{n=0}^{\infty} c_{\rho,\alpha,i,n} x^{-n-\alpha} (\log(1/x))^i \tag{3.1}$$

and say that the right hand side is the asymptotic expansion of $f(x)$ in a large sector bisected by the direction θ, if there exist $\varepsilon, R, B, C > 0$ and, for any $\rho \in \Sigma$, a function $f_\rho(x)$ holomorphic on

$$U = \Big\{ x \in \mathbb{C},\ |x| \geq R,\ \theta - \frac{\pi}{2} - \varepsilon \leq \arg(x) \leq \theta + \frac{\pi}{2} + \varepsilon \Big\},$$

such that

$$f(x) = \sum_{\rho\in\Sigma} e^{\rho x} f_\rho(x)$$

and

$$\Big| f_\rho(x) - \sum_{\alpha\in S} \sum_{i\in T} \sum_{n=0}^{N-1} c_{\rho,\alpha,i,n} x^{-n-\alpha} (\log(1/x))^i \Big| \leq C^N N! |x|^{B-N}$$

for any $x \in U$ and any $N \geq 1$.

This means (see [16, §§2.1 and 2.3]) that for any $\rho \in \Sigma$,

$$\sum_{\alpha\in S} \sum_{i\in T} \sum_{n=0}^{N-1} c_{\rho,\alpha,i,n} x^{-n-\alpha} (\log(1/x))^i \tag{3.2}$$

is 1-summable in the direction θ and its sum is $f_\rho(x)$. Using a result of Watson (see [16, §2.3]), the sum $f_\rho(x)$ is then determined by its asymptotic expansion (3.2). Therefore the expansion on the right hand side of (3.1) determines $f(x)$, up to analytic continuation. The converse holds too: [7, Lemma 1] asserts that a given function $f(x)$ can have at most one asymptotic expansion in the sense of Definition 3. Of course we assume implicitly (throughout this paper) that Σ, S and T in (3.1) cannot trivially be made smaller, and that for any α there exist ρ and i with $c_{\rho,\alpha,i,0} \neq 0$.

3.2 Computing asymptotic expansions of E-functions

In this section, we state [7, Theorem 5] which enables one to determine the asymptotic expansion of an E-function. We refer to [7] for more details.

Let $E(x) = \sum_{n=1}^{\infty} a_n x^n$ be an E-function such that $E(0) = 0$; consider $g(z) = \sum_{n=1}^{\infty} \frac{a_n}{z^{n+1}}$. Denoting by $\overline{\mathcal{F}} : \mathbb{C}[z, \frac{\mathrm{d}}{\mathrm{d}z}] \to \mathbb{C}[x, \frac{\mathrm{d}}{\mathrm{d}x}]$ the Fourier transform of differential operators, i.e. the morphism of $\mathbb{C}$-algebras defined by $\overline{\mathcal{F}}(z) = \frac{\mathrm{d}}{\mathrm{d}x}$ and $\overline{\mathcal{F}}(\frac{\mathrm{d}}{\mathrm{d}z}) = -x$, there exists a G-operator $\mathcal{D}$ such that $\overline{\mathcal{F}}\mathcal{D}E = 0$, and we have $(\frac{\mathrm{d}}{\mathrm{d}z})^\delta \mathcal{D} g = 0$ where δ is the degree of $\mathcal{D}$. We denote by Σ the set of all finite singularities of $\mathcal{D}$ and let

$$\mathcal{S} = \mathbb{R} \setminus \{\arg(\rho - \rho'), \rho, \rho' \in \Sigma, \rho \neq \rho'\} \tag{3.3}$$

where all the values modulo 2π of the argument of $\rho-\rho'$ are considered, so that $\mathcal{S}+\pi=\mathcal{S}$.

We fix $\theta\in\mathbb{R}$ with $-\theta\in\mathcal{S}$ (so that the direction θ is not anti-Stokes, i.e. not singular, see for instance [13, p. 79]). For any $\rho\in\Sigma$ we denote by $\Delta_\rho=\rho-e^{-i\theta}\mathbb{R}_+$ the closed half-line of angle $-\theta+\pi$ mod 2π starting at ρ. Since $-\theta\in\mathcal{S}$, no singularity $\rho'\neq\rho$ of $\mathcal{D}$ lies on Δ_ρ: these half-lines are pairwise disjoint. We shall work in the simply connected cut plane obtained from $\mathbb{C}$ by removing the union of these half-lines. We agree that for $\rho\in\Sigma$ and z in the cut plane, $\arg(z-\rho)$ will be chosen in the open interval $(-\theta-\pi,-\theta+\pi)$. This enables one to define $\log(z-\rho)$ and $(z-\rho)^\alpha$ for any $\alpha\in\mathbb{Q}$.

Now let us fix $\rho\in\Sigma$. Combining theorems of André, Chudnovski and Katz (see [2, p. 719]), there exist (non necessarily distinct) rational numbers $t_1^\rho,\dots,t_{J(\rho)}^\rho$, with $J(\rho)\geq 1$, and G-functions $g_{j,k}^\rho$, for $1\leq j\leq J(\rho)$ and $0\leq k\leq K(\rho,j)$, such that a basis of local solutions of $(\frac{\mathrm{d}}{\mathrm{d}z})^\delta\mathcal{D}$ around ρ (in the above-mentioned cut plane) is given by the functions

$$f_{j,k}^\rho(z-\rho)=(z-\rho)^{t_j^\rho}\sum_{k'=0}^{k}g_{j,k-k'}^\rho(z-\rho)\frac{(\log(z-\rho))^{k'}}{k'!}\tag{3.4}$$

for $1\leq j\leq J(\rho)$ and $0\leq k\leq K(\rho,j)$. Since $(\frac{\mathrm{d}}{\mathrm{d}z})^\delta\mathcal{D}g=0$ we can expand g in this basis:

$$g(z)=\sum_{j=1}^{J(\rho)}\sum_{k=0}^{K(\rho,j)}\varpi_{j,k}^\rho f_{j,k}^\rho(z-\rho)$$

with connection constants $\varpi_{j,k}^\rho$; Theorem 2 of [6] yields $\varpi_{j,k}^\rho\in\mathbf{G}$.

We denote by $\{u\}\in[0,1)$ the fractional part of a real number u, and agree that all derivatives of this or related functions taken at integers will be right-derivatives. We let

$$y_{\alpha,i}(z)=\sum_{n=0}^{\infty}\frac{1}{i!}\frac{\mathrm{d}^i}{\mathrm{d}y^i}\Big(\frac{\Gamma(1-\{y\})}{\Gamma(-y-n)}\Big)_{|y=\alpha}z^n\in\mathbb{Q}[[z]]\tag{3.5}$$

for $\alpha\in\mathbb{Q}$ and $i\in\mathbb{N}$. We also denote by $\star$ the Hadamard (coefficientwise) product of formal power series in z, and we consider

$$\eta_{j,k}^\rho(1/x)=\sum_{m=0}^{k}(y_{t_j^\rho,m}\star g_{j,k-m}^\rho)(1/x)\in\overline{\mathbb{Q}}[[1/x]]\tag{3.6}$$

for any $1\leq j\leq J(\rho)$ and $0\leq k\leq K(j,\rho)$. Then [7, Theorem 5] is the following result, where $\widehat{\Gamma}=1/\Gamma$.

Theorem 2. *In a large sector bisected by the direction θ we have the following asymptotic expansion:*

$$E(x)\approx$$
$$\sum_{\rho\in\Sigma}e^{\rho x}\sum_{j=1}^{J(\rho)}\sum_{k=0}^{K(j,\rho)}\varpi_{j,k}^\rho x^{-t_j^\rho-1}\sum_{i=0}^{k}\Big(\sum_{\ell=0}^{k-i}\frac{(-1)^\ell}{\ell!}\widehat{\Gamma}^{(\ell)}(1-\{t_j^\rho\})\eta_{j,k-\ell-i}^\rho(1/x)\Big)\frac{(\log(1/x))^i}{i!}.\tag{3.7}$$

3.3 G-values as coefficients of asymptotic expansions

We can now state and prove the main result of this section.

Theorem 3. *For any $\xi \in \mathbf{G}$, there exists an E-function $E(z)$ such that for any $\theta \in [-\pi, \pi)$ outside a finite set, ξ is a coefficient of the asymptotic expansion of $E(x)$ in a large sector bisected by θ.*

Proof. Let $\xi \in \mathbf{G}$; we may assume $\xi \neq 0$. Using [6, Theorem 1], there exists a G-function $h(z)$ holomorphic at $z = 1$ such that $h(1) = \xi$. Let $g(z) = \frac{h(1/z)}{z(z-1)}$. This function has a Taylor expansion at ∞ of the form $\sum_{n=1}^{\infty} \frac{a_n}{z^{n+1}}$, and $E(x) = \sum_{n=1}^{\infty} \frac{a_n}{n!} x^n$ is an E-function. Using the results of [7] recalled in §3.2 we shall compute (partially) its asymptotic expansion at infinity in a large sector bisected by the direction θ, for any $\theta \in [-\pi, \pi)$ outside a finite set; we shall prove that the coefficient of e^x in this expansion is equal to ξ. With this aim in mind, we keep the notation of §3.2, including $\mathcal{D}$ and θ.

We let $\rho = 1$ (eventhough we still write ρ for better readability), and consider a basis of local solutions of $(\frac{\mathrm{d}}{\mathrm{d}z})^{\delta}\mathcal{D}$ around ρ with functions $f_{j,k}^{\rho}$ and $g_{j,k}^{\rho}$ as in §3.2. By Frobenius' method, upon shifting t_j^{ρ} by an integer we may assume that $g_{j,0}^{\rho}(0) \neq 0$. Moreover, upon performing $\overline{\mathbb{Q}}$-linear combinations of the basis elements and a permutation of the indices, we may assume that $t_1^{\rho} < \ldots < t_{J(\rho)}^{\rho}$ so that the solutions $f_{j,k}^{\rho}$ have pairwise distinct asymptotic behaviours at 0, namely $f_{j,k}^{\rho}(s) \sim \frac{g_{j,0}^{\rho}(0)}{k!} s^{t_j^{\rho}} (\log s)^k$. At last, dividing each $f_{j,k}^{\rho}$ with $g_{j,0}^{\rho}(0)$ we may assume that $g_{j,0}^{\rho}(0) = 1$ for any j.

Now consider the expansion

$$g(z) = \sum_{j=1}^{J(\rho)} \sum_{k=0}^{K(\rho,j)} \varpi_{j,k}^{\rho} f_{j,k}^{\rho}(z - \rho). \tag{3.8}$$

Let $T = \{(j,k), \varpi_{j,k}^{\rho} \neq 0\}$. Since g is not identically zero, T is not empty. Let $j_0 \in \{1, \ldots, J(\rho)\}$ be the minimal value such that $(j_0, k) \in T$ for some k, and let k_0 be the maximal value such that $(j_0, k_0) \in T$. Then on the right hand side of Eq. (3.8), the leading term as $z \to \rho$ is given by $(j,k) = (j_0, k_0)$, so that

$$g(z) \sim \frac{\varpi_{j_0,k_0}^{\rho}}{k_0!} (z - \rho)^{t_{j_0}^{\rho}} (\log(z - \rho))^{k_0} \tag{3.9}$$

since $g_{j_0,0}^{\rho}(0) = 1$. Now recall that $g(z) = \frac{h(1/z)}{z(z-1)}$ with $h(1) = \xi \neq 0$ and $\rho = 1$; therefore $g(z) \sim \frac{\xi}{z-1}$. Comparing this with Eq. (3.9) yields $t_{j_0}^{\rho} = -1$, $k_0 = 0$, and $\varpi_{j_0,0}^{\rho} = \xi$.

Let us consider the asymptotic expansion given by Theorem 2, and especially the coefficient of e^x that we denote by α. This coefficient comes from the multiple sum in Eq. (3.7). In this sum, we have $\varpi_{j,k}^{\rho} = 0$ for any $j < j_0$ and any k (by definition of j_0), so that these terms do not contribute to the value of α. For any $j > j_0$ we have $t_j^{\rho} > t_{j_0}^{\rho} = -1$ so that $-t_j^{\rho} - 1 < 0$ and the corresponding terms do not contribute either. Therefore the value of

α is given only by the terms corresponding to $j = j_0$ (with $t^\rho_{j_0} = -1$):

$$\alpha = \sum_{k=0}^{K(\rho,j_0)} \varpi^\rho_{j_0,k} \sum_{\ell=0}^{k} \frac{(-1)^\ell}{\ell!} \widehat{\Gamma}^{(\ell)}(1) \eta^\rho_{j_0,k-\ell}(0).$$

Now recall that by definition, $k_0 = 0$ is the maximal value of k such that $\varpi^\rho_{j_0,k} \neq 0$. Therefore the previous sum has (at most) one non-zero term: the one corresponding to $k = 0$. Since $\widehat{\Gamma}(1) = 1$ and $\varpi^\rho_{j_0,0} = \xi$ we have $\alpha = \xi \eta^\rho_{j_0,0}(0) = \xi y_{-1,0}(0) g^\rho_{j_0,0}(0) = \xi$ using Eqns. (3.5) and (3.6). This concludes the proof that the coefficient of e^x in the asymptotic expansion of $E(x)$ is equal to ξ.

4 Asymptotic expansion of the generalized hypergeometric series

In this section, we prove the following result (recall that asymptotic expansions have been defined in §3.1).

Theorem 4. *Let $\theta \in (-\pi, \pi) \setminus \{0\}$, and $f(z) = {}_pF_p[a_1, \ldots, a_p; b_1, \ldots, b_p; z]$ be a hypergeometric series with parameters $a_j \in \mathbb{Q}$ and $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. Then $f(z)$ has an asymptotic expansion*

$$f(x) \approx \sum_{\rho \in \Sigma} e^{\rho x} \sum_{\alpha \in S} \sum_{i \in T} \sum_{n=0}^{\infty} c_{\rho,\alpha,i,n} x^{-n-\alpha} (\log(1/x))^i$$

in a large sector bisected by θ, with $\Sigma \subset \{0, 1\}$, $S \subset \mathbb{Q}$ and $T \subset \mathbb{N}$ both finite, and coefficients $c_{\rho,\alpha,i,n}$ in $\mathbf{H}$.

Proof. If one of the a_j's is in $\mathbb{Z}_{\leq 0}$, the hypergeometric series is in $\mathbb{C}[z]$ and the conclusion clearly holds with $c_{\rho,\alpha,i,n}$ in $\overline{\mathbb{Q}}$. From now on, as for the b_j's, we assume that none of the a_j's is in $\mathbb{Z}_{\leq 0}$.

Let

$$R(s) = R(\underline{a}, \underline{b}; s) := \frac{\prod_{j=1}^p \Gamma(a_j + s)}{\prod_{j=1}^p \Gamma(b_j + s)} \Gamma(-s).$$

The poles of $R(s)$ are located at $-a_j - k$, $k \in \mathbb{Z}_{\geq 0}$, $j = 1, \ldots, p$, and at $\mathbb{Z}_{\geq 0}$. We define the series

$$L_p(\underline{a}, \underline{b}; z) = \sum_{j=1}^{p} \sum_{k=0}^{\infty} \mathrm{Residue}\big(R(s) z^{-s}, s = -a_j - k\big).$$

Set $\nu := \sum_{j=1}^p a_j - \sum_{j=1}^p b_j$, $b_{p+1} := 1$ and

$$e_{k,m} := e_{k,m}(\underline{a}, \underline{b}) := \sum_{j=1}^{p+1} (1 - \nu + b_j + m)_{k-m} \frac{\prod_{i=1}^p (a_i - b_j)}{\prod_{i=1, i \neq j}^{p+1} (b_i - b_j)}.$$

We define a sequence $C_k := C_k(\underline{a}, \underline{b})$ by induction:

$$C_0 := 1, \quad C_k := \frac{1}{k} \sum_{m=0}^{k-1} e_{k,m} C_m,$$

and the formal series

$$K_p(\underline{a}, \underline{b}; z) := e^z \sum_{k=0}^{\infty} C_k(\underline{a}, \underline{b}) z^{\nu-k}.$$

By [12, p. 283, Theorem], reproved in [19, p. 113, Theorem 4.1, Eq. (4.4)], we have in fact

$$C_k(\underline{a}, \underline{b}) = \sum_{k_1 \geq 0, \ldots, k_p \geq 0, K_p = k} \frac{(1-a_p)_{k_p} \prod_{j=1}^{p-1} (a_{j+1} + b_{j+1} - a_j)_{k_j} \prod_{j=1}^{p} (B_j + K_{j-1})_{k_j}}{\prod_{j=1}^{p} k_j!},$$

where for any j, $B_j = \sum_{m=1}^{j} b_m$ and $K_j = \sum_{m=1}^{j} k_m$.

Note that $K_p(\underline{a}, \underline{b}; z) \in e^z z^\nu \mathbb{Q}[\underline{a}, \underline{b}][[1/z]]$.

In [14, p. 212], it is shown that as $z \to \infty$ in the sector $-\frac{3\pi}{2} < \arg(z) < \frac{\pi}{2}$, we have the asymptotic expansion

$${}_pF_p \left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_p \end{matrix}; z \right] \approx \frac{\prod_{j=1}^{p} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)} \Big(L_p(\underline{a}, \underline{b}; e^{i\pi} z) + K_p(\underline{a}, \underline{b}; z) \Big),$$

while if $z \to \infty$ in the sector $-\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}$, we have

$${}_pF_p \left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_p \end{matrix}; z \right] \approx \frac{\prod_{j=1}^{p} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)} \Big(L_p(\underline{a}, \underline{b}; e^{-i\pi} z) + K_p(\underline{a}, \underline{b}; z) \Big).$$

These two expansions satisfy Definition 3 above: they hold in a *large sector* bisected by any $\theta \in (-\pi, 0)$, respectively any $\theta \in (0, \pi)$, and $L_p(\underline{a}, \underline{b}; e^{\pm i\pi} z)$ and $e^{-z} K_p(\underline{a}, \underline{b}; z)$ are 1-summable in the direction θ. Indeed, it is well-known that any hypergeometric series ${}_pF_p[\underline{a}, \underline{b}; z]$ admits an asymptotic expansion ([1]) in the sense of Definition 3, while Lemma 1 of [7] ensures that a function admits at most one expansion of this type in any given large sector bisected by a given direction.

These asymptotic expansions are refined versions of Barnes and Wright's fundamental works [5, 22] and are consequences of the general expansion of Meijer G-function [14, Chapter V]. Note that Meijer G-function is not related to Siegel's G-functions, though by specialization of its parameter the former provides examples of the latter. In the next two subsections, we provide more explicit expressions for the function $L_p(\underline{a}, \underline{b}; z)$ under the assumption that the a_j's and b_j's are in $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, in order to prove that all coefficients of the asymptotic expansion belong to $\mathbf{H}$.

[1] When the a_j's and b_j's are in $\mathbb{Q}$ as in our application, ${}_pF_p[\underline{a}, \underline{b}; z]$ is an E-function and Theorem 2 above proves the existence of this expansion. But the proof of this theorem does not need this assumption for the hypergeometric series, so that the general case follows as well.

4.1 R has simple poles

If the a_j's are pairwise distinct modulo $\mathbb{Z}$, then the poles of $R(s)$ are simple, and we have

$$L_p(\underline{a},\underline{b};z)=\sum_{j=1}^{p}\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(a_j+k)\prod_{i=1,i\neq j}^{p}\Gamma(a_i-a_j-k)}{k!\prod_{i=1}^{p}\Gamma(b_i-a_j-k)}z^{-a_j-k}.$$

When the a_j's and b_j's are in $\mathbb{Q}\setminus\mathbb{Z}_{\leq 0}$, $\frac{\prod_{j=1}^{p}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}L_p(\underline{a},\underline{b};z)$ is thus equal to a finite sum

$$\sum_j z^{-a_j}f_j(z)$$

with $f_j(z)\in\mathbf{H}[[1/z]]$. Note that the element $1/\pi\in\mathbf{H}$ appears through the use of the reflection formula $\frac{1}{\Gamma(s)}=\frac{1}{\pi}\sin(\pi s)\Gamma(1-s)$ for rational values of s.

4.2 R has multiple poles

We assume that the a_j's and b_j's are in $\mathbb{Q}\setminus\mathbb{Z}_{\leq 0}$. Up to reordering the a_j's, we can group them in ℓ groups as follows: for $m=0,\dots,\ell-1$, we have

$$a_{j_m+1},a_{j_m+2},\dots,a_{j_{m+1}}\text{ equal mod }\mathbb{Z},\quad a_{j_m+1}\text{ the smallest one in the group,}$$

where the a_{j_m} are pairwise distinct mod $\mathbb{Z}$ for $m=1,\dots,\ell$, and $0=j_0<j_1<j_2<\cdots<j_\ell=p$.

Then, for every $j\in\{j_m+1,\dots,j_{m+1}\}$, we have

$$\Gamma(a_j+s)=(a_{j_m+1}+s)_{a_j-a_{j_m+1}}\Gamma(a_{j_m+1}+s).$$

Set $d_m:=j_m-j_{m-1}\geq 1$, $c_m:=a_{j_{m-1}+1}$ and

$$P(s):=\prod_{m=0}^{\ell-1}\left(\prod_{j=j_m+1}^{j_{m+1}}(a_{j_m+1}+s)_{a_j-a_{j_m+1}}\right)\in\mathbb{Q}[s].$$

Hence,

$$R(s)=P(s)\Gamma(-s)\frac{\prod_{m=1}^{\ell}\Gamma(c_m+s)^{d_m}}{\prod_{m=1}^{p}\Gamma(b_m+s)}.$$

To compute the residue of $R(s)z^{-s}$ at $s=-c_n-k$ for given $n\in\{1,\dots,\ell\}$ and $k\in\mathbb{Z}_{\geq 0}$, we write

$$\Gamma(c_n+s)=\frac{\Gamma(c_n+s+k+1)}{(c_n+s)_k(c_n+s+k)}$$

and define

$$\Phi_{c_n,k}(s):=z^{-s}P(s)\Gamma(-s)\frac{\prod_{m=1,m\neq n}^{\ell}\Gamma(c_m+s)^{d_m}}{\prod_{m=1}^{p}\Gamma(b_m+s)}\cdot\frac{\Gamma(c_n+s+k+1)^{d_n}}{(c_n+s)_k^{d_n}}$$

which is holomorphic at $s = -c_n - k$. We thus deduce from

$$R(s)z^{-s} = \frac{\Phi_{c_n,k}(s)}{(c_n + s + k)^{d_n}}$$

that

$$\mathrm{Residue}\big(R(s)z^{-s}, s = -c_n - k\big) = \frac{1}{(d_n - 1)!}\Phi_{c_n,k}^{(d_n-1)}(-c_n - k).$$

It follows that $\frac{\prod_{j=1}^{p}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}L_p(\underline{a}, \underline{b}; z)$ is equal to a finite sum

$$\sum_{j,\ell} z^{-a_j}\log(z)^{\ell} f_{j,\ell}(z)$$

with $f_{j,\ell}(z) \in \mathbf{H}[[1/z]]$. This concludes the proof of Theorem 4.

5 Application to Siegel's problem

We now complete the proof of Theorem 1. Assume that Siegel's question has an affirmative answer, and let $\xi \in \mathbf{G}$. Theorem 3 provides an E-function $E(z)$ and a finite set $S \subset (-\pi, \pi)$ such that for any $\theta \in (-\pi, \pi) \setminus S$, ξ is a coefficient of the asymptotic expansion of $E(z)$ in a large sector bisected by θ. Now an affirmative answer to Siegel's question yields n ${}_pF_p$ hypergeometric series $f_1, \ldots, f_n$ with rational parameters, n algebraic numbers $\lambda_1, \ldots, \lambda_n$, and a polynomial $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_n]$, such that $E(z) = P(f_1(\lambda_1 z), \ldots, f_n(\lambda_n z))$. Choose $\theta \in (-\pi, \pi) \setminus S$ such that $\theta + \arg(\lambda_i) \notin \pi\mathbb{Z}$ for any $i \in \{1, \ldots, n\}$. Then Theorem 4 implies that for any i, the asymptotic expansion of $f_i(\lambda_i z)$ in a large sector bisected by θ has coefficients in $\mathbf{H}$. The same holds for $E(z) = P(f_1(\lambda_1 z), \ldots, f_n(\lambda_n z))$ because $\mathbf{H}$ is a $\overline{\mathbb{Q}}$-algebra. Since such an asymptotic expansion is unique (see §3.1), the coefficient ξ belongs to $\mathbf{H}$. This concludes the proof.

References

[1] M. Abramowitz, I. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover Publications, 9th edition, 1970.

[2] Y. André, *Séries Gevrey de type arithmétique I. Théorèmes de pureté et de dualité*, Annals of Math. **151** (2000), 705–740.

[3] Y. André, *Une introduction aux motifs (motifs purs, motifs mixtes, périodes)*, Panoramas et Synthèses **17**, Soc. Math. de France, Paris, ix + 261 pages, 2004.

[4] G. E. Andrews, R. Askey, R. Roy, *Special Functions*, Encyclopedia of mathematics and its applications **71**, Cambridge Univ. Press, xvi + 664 pages, 1999.

[5] E. W. Barnes, *The asymptotic expansion of integral functions defined by generalized hypergeometric series*, Proc. London Math. Soc. **5** (1907), 59–116.

[6] S. Fischler, T. Rivoal, *On the values of G-functions*, Commentarii Math. Helv. **29**.2 (2014), 313–341.

[7] S. Fischler, T. Rivoal, *Arithmetic theory of E-operators*, Journal de l'École polytechnique – Mathématiques **3** (2016), 31–65.

[8] J. Fresán, P. Jossen, *Exponential motives*, 268 pages, 2019. Preliminary version available at `http://javier.fresan.perso.math.cnrs.fr/expmot.pdf`

[9] A. I. Galochkin, *Hypergeometric Siegel functions and E-functions*, Math. Notes **29** (1981), 3–8; english translation of Mat. Zametki **29**.1 (1981), 3–14 (in russian).

[10] V. A. Gorelov, *On the Siegel conjecture for second-order homogeneous linear differential equations*, Math. Notes **75**.4 (2004), 513–529.

[11] V. A. Gorelov, *On the structure of the set of E-functions satisfying linear differential equations of second order*, Math. Notes **78**.3 (2005), 304–319.

[12] C. M. Joshi, J. B. McDonald, *Some finite summation theorems and an asymptotic expansion for the generalized hypergeometric series*, J. Math. Anal. Appl. **40** (1972), 278–285.

[13] M. Loday-Richaud, *Séries formelles provenant de systèmes différentiels linéaires méromorphes*, in: Séries divergentes et procédés de resommation, Journées X-UPS, 1991, 69–100.

[14] Y. L. Luke, *Mathematical functions and their approximations*, Academic Press, 584 pages, 1975.

[15] C. J. Moreno, *The Chowla-Selberg Formula*, J. Number Theory **17** (1983), 226–245.

[16] J. P. Ramis, *Séries Divergentes et Théories Asymptotiques*, Panoramas et Synthèses **21** (1993), Soc. Math. France, Paris.

[17] T. Rivoal, *A new proof of Galochkin's characterization of hypergeometric G-functions*, preprint (2018), 12 pages.

[18] T. Rivoal, J. Roques, *Siegel's problem for E-functions of order 2*, preprint (2016), 14 pages.

[19] H. Volkmer, J. J. Wood, *A note on the asymptotic expansion of generalized hypergeometric functions*, Analysis and Applications **12**.1 (2014), 107–115.

[20] C. L. Siegel, *Über einige Anwendungen diophantischer Approximationen*, vol. 1 S. Abhandlungen Akad., Berlin, 1929, 209–266.

[21] C. L. Siegel, *Transcendental Numbers*, Annals of Mathematical Studies, Princeton 1949.

[22] E. M. Wright, *The asymptotic expansion of the generalized hypergeometric function*, Proc. London Math. Soc (2) **46**.1 (1940), 389–408.

S. Fischler, Laboratoire de Mathématiques d'Orsay, Univ. Paris-Sud, CNRS, Université Paris-Saclay, 91405 Orsay, France.

T. Rivoal, Institut Fourier, CNRS et Université Grenoble Alpes, CS 40700, 38058 Grenoble cedex 9, France.

Keywords: Hypergeometric series, Asymptotic expansions, Siegel's E- and G-functions.

MSC 2010: 33C15 (Primary); 41A60, 11J91 (Secondary).