



**HAL**  
open science

# MODULAR TRANSFORMATIONS OF APPELL-LERCH SERIES BY MEANS OF MORDELL INTEGRALS

Changgui Zhang

► **To cite this version:**

Changgui Zhang. MODULAR TRANSFORMATIONS OF APPELL-LERCH SERIES BY MEANS OF MORDELL INTEGRALS. 2019. hal-02314599

**HAL Id: hal-02314599**

**<https://hal.science/hal-02314599v1>**

Preprint submitted on 12 Oct 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# MODULAR TRANSFORMATIONS OF APPELL-LERCH SERIES BY MEANS OF MORDELL INTEGRALS

CHANGGUI ZHANG

ABSTRACT. The goal of this paper is to establish a general modular transformation formula for the first order Appell-Lerch series in terms of one finite number of Mordell integrals. Generalizing the classic transformation laws known for the theta functions of Jacobi, this formula contains Lerch's and Mordell's results on the same subject.

## CONTENTS

1.	Introduction	1
2.	Main result and commentaries	3
3.	$q$ -analogs of Borel-Laplace summation	7
4.	Proof of Theorem 3.1	9
5.	From Theorem 3.1 to Theorem 2.1	10
6.	About the main transformation formula	14
7.	Applications and remarks	19
	References	21

## 1. INTRODUCTION

In his paper [11], Mordell gave a series of results previously obtained in several works on the analytic theory of numbers. Here are two of those results:

$$(1.1) \quad \int_{-\infty}^{\infty} \frac{e^{\pi i \tau t^2 - 2\pi t z}}{e^{2\pi t} - 1} dt = \frac{f\left(\frac{z}{\tau} \mid -\frac{1}{\tau}\right) + i\tau f(z \mid \tau)}{\tau \vartheta_1(z \mid \tau)},$$

$$(1.2) \quad \int_{-\infty}^{\infty} \frac{te^{\pi i \tau t^2}}{e^{2\pi t} - 1} dt = -2\Omega(\tau) + \frac{2}{\tau^2} \sqrt{-i\tau} \Omega(-1/\tau) + \frac{1}{4} \vartheta_0^3(0 \mid \tau);$$

see [11, (3.1) & (2.18)]. In the above,  $\Im(\tau) > 0$ ; the path of integration may be taken as the real axis of  $t$  indented by the lower half of a small circle described about the origin as centre;  $f$  is the integral function defined by the series

$$(1.3) \quad if(z \mid \tau) = \sum_{m \text{ odd}}^{\pm\infty} \frac{(-1)^{\frac{1}{2}(m-1)} e^{\frac{1}{4}m^2\pi i\tau + m\pi iz}}{1 + e^{m\pi i\tau}};$$

and  $\Omega(\tau)$  denotes the generating function of the sequence  $\{F(n)\}_{n \geq 0}$  with respect to the variable  $q = e^{\pi i\tau}$ , where  $F(n)$  equals to the number of uneven classes of

positive, definite binary quadratic forms of given determinant  $-n$ . In addition,  $\vartheta_0 = \vartheta_{00}$  and  $\vartheta_1 = \vartheta_{11}$  are the theta functions defined by the following relations:

$$(1.4) \quad \vartheta_0(z|\tau) = \sum_{n=-\infty}^{\infty} e^{n^2\pi i\tau + 2n\pi iz}, \quad \vartheta_1(z|\tau) = -i \sum_{m \text{ odd}}^{\pm\infty} (-1)^{\frac{1}{2}(m-1)} e^{\frac{1}{4}m^2\pi i\tau + m\pi iz}.$$

By following [4, (43.1) p. 52], the integral in (1.1) is called *Mordell integral*, and it will be denoted as follows:

$$(1.5) \quad \phi(z|\tau) = \int_{-\infty}^{\infty} \frac{e^{\pi i\tau t^2 + 2\pi tz}}{e^{2\pi t} - 1} dt.$$

This is intimately connected with theta functions and mock theta functions on one hand and with the Riemann zeta function on the other. See also [6, 18–20] and [5, p. 170].

As said in [11] by himself, Mordell's formula stated in (1.1) can be deduced from the following result obtained by Lerch [9, 10]:

$$(1.6) \quad \sqrt{\frac{i}{\tau}} \vartheta_0(z|\tau) \Psi(v|\tau) = R(z, z+v|\tau) - \frac{1}{\tau} e^{\frac{\pi i}{\tau}(v^2 - z^2)} R\left(\frac{z}{\tau}, \frac{z+v}{\tau} \mid -\frac{1}{\tau}\right);$$

see [11, (7.8)]. Recall on the one hand that Mordell there defined the above function  $\Psi(v|\tau)$  by the integral along the real axis

$$(1.7) \quad \Psi(v|\tau) = \int_{-\infty}^{\infty} \frac{e^{-\frac{\pi i}{\tau}(t + \frac{i}{2} - iv)^2}}{e^{2\pi t} + 1} dt.$$

A simple calculation shows that  $\Psi$  may be expressed by means of the above integral  $\phi$  as follows:

$$(1.8) \quad \Psi(v|\tau) = -e^{\frac{\pi i}{\tau}(v-1)^2} \phi\left(-\frac{v}{\tau} + \frac{1}{\tau} \mid -\frac{1}{\tau}\right) = e^{\frac{\pi i}{\tau}v^2} \phi\left(\frac{v}{\tau} + 1 \mid -\frac{1}{\tau}\right).$$

On the other hand, the function  $R(z, v|\tau) = R_1(z, v|\tau)$  is given by the below series for  $k=1$ :

$$(1.9) \quad R_k(z, v|\tau) = \sum_{n=-\infty}^{\infty} \frac{e^{kn^2\pi i\tau + 2kn\pi iz}}{1 - e^{2n\pi i\tau + 2\pi iv}},$$

what is currently called *Appell-Lerch series* of order  $k$ .

Several particular cases of the Mordell integrals may be found in the works [13–15] of Ramanujan. In [1], Andrews considered functions explicitly appeared in the ‘‘Lost’’ Notebook, and gave for them basic modular type transformations, one of which may be expressed as follows :

$$(1.10) \quad c(\tau) \Psi\left(\frac{2-\tau}{4} \mid \frac{\tau}{2}\right) = \frac{\tau}{i} R_2\left(\frac{\tau}{4}, \frac{1}{2} \mid \tau\right) - 2e^{-\frac{\pi i}{2\tau}} R_2\left(-\frac{1}{2\tau} + \frac{1}{4}, -\frac{1}{2\tau} \mid -\frac{1}{\tau}\right),$$

where  $c(\tau) = \vartheta_0(\frac{1}{2} \mid -\frac{2}{\tau}) e^{-\frac{\pi i\tau}{8}}$ . It should be noted that (1.10) is for a couple of second order Appell-Lerch series while both (1.1) and (1.6) concern the first order cases.

Mordell ended his paper [11] by some problems, the first one of which consists of finding the general transformation formula for  $f(z|\tau)$ , *i.e.* a simple result for  $f\left(\frac{z}{\gamma\tau + \delta} \mid \frac{\alpha\tau + \beta}{\gamma\tau + \delta}\right)$ , where  $\alpha, \beta, \gamma, \delta$  are any integers satisfying  $\alpha\delta - \beta\gamma = 1$ . The goal of our paper is to give a general transformation formula on any Appell-Lerch

series of the first order. In this way, we will obtain a complete answer to the above-mentioned problem of Mordell for  $f(z|\tau)$ .

The rest of the paper will contain six paragraphs that will be organized as follows.

In §2, we will start by stating the main result of this paper, Theorem 2.1, which gives one generalization of (1.6) to the case of an arbitrary modular matrix  $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}(2; \mathbb{Z})$  with  $\gamma > 0$ . After that, it will be made clear in what manner one can deduce from Theorem 2.1 the well-known linear transformation formula of the theta function  $\vartheta_1$  on the one hand and why and how one can remove the restrictive condition  $\gamma > 0$  from Theorem 2.1 on the other. See (2.17) for a more general statement of Theorem 2.1.

A proof of Theorem 2.1 will be completed in §5, after having established results about one first order non-homogenous  $q$ -difference equation in §3 and §4. Namely, our approaches will be based on the analytic theory of singular  $q$ -difference equations. Since Appell et Hermite [2, 3, 7], the so-called Appell-Lerch series have been considered in some way as one part of the general elliptic functions. The study of such almost doubly periodic functions may be done inside the analytic theory of singular irregular  $q$ -difference equations developed in [17, 21, 22]; see also [16] for more details.

In §3, we will consider the first order  $q$ -difference equation (3.2) from which two analytic functions will be constructed by following different summation procedures about its unique divergent power series solution  $\hat{E}(x; q)$ ; see (3.3). A Stokes analysis about these sum-functions will lead us to obtain Theorem 3.1, which will be proved in §4.

In §5, we will interpret the above-mentioned functions in terms of Appell-Lerch series on the one side and of Mordell integral on the other side. This allows one to deduce Theorem 2.1 from Theorem 3.1.

Some functional relations will be established in §6 for the automorphic factor  $\tilde{A}_M(z, v|\tau)$  and the remaining term  $\tilde{\Psi}_M(v|\tau)$ , both coming from the transformation formula stated in Theorem 2.1; see (2.17) and (6.2). And a more general relation than (1.1) will be found in §7 for the above function  $f(z|\tau)$ , which is defined in (1.3). In a forthcoming paper, we hope to be able to apply Theorem 2.1 to other functions coming from the number theory, like as the function  $\Omega(\tau)$  considered in (1.2) by Mordell.

This work has been done in direct line with our previous work [23], all being aimed at understanding the mock theta functions of Ramanujan.

## 2. MAIN RESULT AND COMMENTARIES

Throught the whole paper, we will write  $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}(2; \mathbb{Z})$ , and let  $\tau' = \tau'_M$  to be the *modular* variable associated with  $M$  by the relation  $\tau' = M\tau = \frac{\alpha\tau + \beta}{\gamma\tau + \delta}$ . As usual, it will always be assumed that  $\Im(\tau) > 0$ ; so, it follows that also  $\Im(\tau') > 0$ . Define

$$(2.1) \quad z' = Mz = \frac{z}{\gamma\tau + \delta}, \quad v' = Mv = \frac{v}{\gamma\tau + \delta}$$

for any complex numbers  $z$  and  $v$ , each of which may be given by an expression of the form  $a\tau + b$  where  $(a, b) \in \mathbb{C}^2$ .

For simplify, we will suppose that  $\gamma > 0$ . The cases for which  $\gamma \leq 0$  will be discussed at the end of this paragraph; see (2.17).

**Theorem 2.1.** *The following transformation formula holds for any complex numbers  $z$  and  $v$  such that  $v \notin \mathbb{Z} \oplus \tau\mathbb{Z}$ :*

$$(2.2) \quad \begin{aligned} R(z + \frac{\tau}{2} + \frac{1}{2}, v | \tau) - \frac{1}{\gamma\tau + \delta} e^{(\frac{\gamma v(v-2z)}{\gamma\tau + \delta} - v + v')\pi i} R(z' + \frac{\tau'}{2} + \frac{1}{2}, v' | \tau') \\ = \vartheta_0(z + \frac{\tau}{2} + \frac{1}{2} | \tau) \sqrt{\frac{\gamma i}{\gamma\tau + \delta}} \sum_{k=0}^{\gamma-1} (-1)^k e^{-k(k+1)\pi i\tau} \Psi_{M;k}(v - z | \tau), \end{aligned}$$

where the functions  $\Psi_{M;k}$ , depending upon the entries  $\gamma, \delta$  of the modular matrix  $M$ , are given by the following integral along the real axis

$$(2.3) \quad \Psi_{M;k}(v | \tau) = \int_{-\infty}^{\infty} e^{-\frac{\gamma}{\gamma\tau + \delta}(s - vi + (k + \frac{1}{2})(\tau + \frac{\delta}{\gamma})i)^2 \pi i} \frac{ds}{1 + e^{2\gamma\pi s}}.$$

Putting  $s = \frac{t}{\gamma}$  into the integral of (2.3) and considering the function  $\Psi(v | \tau)$  defined in (1.7) yields the following expression of  $\Psi_{M;k}$ :

$$(2.4) \quad \Psi_{M;k}(v | \tau) = \frac{1}{\gamma} \Psi(\gamma v - (k + \frac{1}{2})(\gamma\tau + \delta) + \frac{1}{2} | \gamma(\gamma\tau + \delta)).$$

Our first commentary concerns the fact that the transformation formula given in (2.2) contains Lerch's formula (1.6). Indeed, if  $\gamma = 1$  and  $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ , one has  $\tau' = -\frac{1}{\tau}$ . Since  $\delta = 0$ , it follows from (2.4) that

$$\Psi_{M;0}(v | \tau) = \Psi(v - \frac{\tau}{2} + \frac{1}{2} | \tau).$$

Thus, replacing  $(z, z')$  with  $(z - \frac{\tau}{2} + \frac{1}{2}, z' - \frac{1}{2} - \frac{\tau'}{2})$  in (2.2) yields that

$$R(z, v | \tau) - \frac{1}{\tau} e^{\frac{1}{2}v(v-2z)\pi i} R(z', v' | \tau') = \vartheta_0(z | \tau) \sqrt{\frac{i}{\tau}} \Psi(v - z | \tau),$$

what is clearly the same as the basic transformation formula stated in (1.6).

Our second commentary focus on whether the above transformation formula (2.2) is related with the general modular relations on the Jacobi's theta functions. For this, we consider the theta function  $\vartheta_1$  defined in (1.4); see also [12, (76.1)]. By making use of [12, (78.32)], in which  $q^2$  will be replaced with  $q = e^{2\pi i\tau}$ , it follows that

$$\vartheta_1(v | \tau) = -i q^{\frac{1}{8}} e^{\pi i v} (q, qe^{2\pi i v}, e^{-2\pi v}; q)_{\infty}.$$

Thus, one gets the following decomposition relation:

$$(2.5) \quad \frac{1}{\vartheta_1(v | \tau)} = \frac{i q^{-\frac{1}{8}} e^{-\pi i v}}{(q; q)_{\infty}^3} \sum_{n=-\infty}^{\infty} \frac{(-1)^{n-1} q^{\frac{1}{2}n(n-1)}}{1 - q^n e^{2\pi i v}}.$$

In the above, the notation  $(\dots; q)_{\infty}$  is defined as follows: for any given  $n$  complex numbers  $x_1, \dots, x_n$ ,

$$(x_1, \dots, x_n; q)_{\infty} = \prod_{k=0}^{\infty} (1 - x_1 q^k) \dots (1 - x_n q^k).$$

As usual, let  $\eta(\tau)$  to denote the Dedekind etafunction. By using the definition of  $R_k(z, v | \tau)$  in (1.9) with  $k = 1$ , the above decomposition formula in (2.5) can be read as follows:

$$\frac{1}{\vartheta_1(v | \tau)} = -i \frac{e^{-\pi iv}}{\eta(\tau)^3} R\left(-\frac{\tau}{2} + \frac{1}{2}, v | \tau\right)$$

or, equivalently:

$$(2.6) \quad R\left(-\frac{\tau}{2} + \frac{1}{2}, v | \tau\right) = i e^{\pi iv} \frac{\eta(\tau)^3}{\vartheta_1(v | \tau)}.$$

Thanks to [12, (74.92)], one knows that

$$(2.7) \quad \eta(\tau') = \epsilon(\alpha, \beta, \gamma, \delta) \sqrt{\frac{\gamma\tau + \delta}{i}} \eta(\tau).$$

In addition, if one defines

$$(2.8) \quad \epsilon_{1,M} = \epsilon_1(\alpha, \beta, \gamma, \delta) = -i \epsilon(\alpha, \beta, \gamma, \delta)^3,$$

it is well-known that the function  $\vartheta_1$  satisfies the following fractional transformation formula [12, (80.8)]:

$$(2.9) \quad \vartheta_1(v' | \tau') = \epsilon_{1,M} \sqrt{\frac{\gamma\tau + \delta}{i}} e^{\frac{\gamma v^2}{\gamma\tau + \delta} \pi i} \vartheta_1(v | \tau).$$

Now, put together (2.7) and (2.9) into (2.6), and consider the relation between  $\epsilon$  and  $\epsilon_{1,M}$  given in (2.8). This implies the following transformation formula for the first order Appell-Lerch series  $R(z, v | \tau)$  at the half-period  $z = -\frac{\tau}{2} + \frac{1}{2}$ :

$$(2.10) \quad e^{-\pi iv'} R\left(-\frac{\tau'}{2} + \frac{1}{2}, v' | \tau'\right) = (\gamma\tau + \delta) e^{-\frac{\gamma v^2}{\gamma\tau + \delta} \pi i - \pi iv} R\left(-\frac{\tau}{2} + \frac{1}{2}, v | \tau\right).$$

Furthermore, a direct computation gives that the Appell-Lerch series  $R(z, v | \tau)$  satisfies the functional relation:

$$(2.11) \quad R(z, v | \tau) - e^{2\pi iv} R(z + \tau, v | \tau) = \vartheta_0(z | \tau).$$

As  $\vartheta_0(-\frac{\tau'}{2} + \frac{1}{2} | \tau') = \vartheta_0(-\frac{\tau}{2} + \frac{1}{2} | \tau) = 0$ , the relation in (2.10) can be written into the following form:

$$(2.12) \quad e^{\pi iv'} R\left(\frac{\tau'}{2} + \frac{1}{2}, v' | \tau'\right) = (\gamma\tau + \delta) e^{\frac{\gamma v^2}{\gamma\tau + \delta} \pi i + \pi iv} R\left(\frac{\tau}{2} + \frac{1}{2}, v | \tau\right).$$

It is obvious to see that this last modular relation in (2.12), which is equivalent to the classical transformation formula (2.9) for  $\vartheta_1$ , can also be deduced from (2.2) by putting there  $z = 0$ , for again  $\vartheta_0(\frac{\tau}{2} + \frac{1}{2}) = 0$ .

Our last commentary will be about the remaining cases in which the entry  $\gamma$  is zero or negative in the matrix  $M$ . When  $\gamma = 0$ , one has  $\alpha = \delta = 1$  or  $-1$ , which, according to (2.1), implies that  $(\tau', z', v') = (\tau + \beta, z, v)$  or  $(\tau - \beta, -z, -v)$  respectively. By using the definition of  $R(z, v | \tau)$  given in (1.9) for  $k = 1$ , it follows that

$$(2.13) \quad R(z, -v | \tau) = -e^{2\pi iv} R(\tau - z, v | \tau).$$

As  $R(z, v | \tau + \beta) = R(z, v | \tau)$  for any integer  $\beta$ , one deduces immediately that, for both cases in which  $(\gamma, \delta) = (0, \pm 1)$ :

$$(2.14) \quad R\left(z + \frac{\tau}{2} + \frac{1}{2}, v | \tau\right) = \frac{1}{\delta} e^{(-v+v')\pi i} R\left(z' + \frac{\tau'}{2} + \frac{1}{2}, v' | \tau'\right).$$

When  $\gamma$  is negative, one considers the opposite matrix  $\tilde{M} = -M$ , for which Theorem 2.1 works. Let  $(\tilde{z}, \tilde{v}) = (\tilde{M}z, \tilde{M}v)$ , and note that  $(z', v') = (-\tilde{z}, -\tilde{v})$ . By applying (2.2) to the matrix  $\tilde{M}$ , one finds that

$$(2.15) \quad \begin{aligned} & R\left(z + \frac{\tau}{2} + \frac{1}{2}, v \mid \tau\right) + \frac{1}{\gamma\tau + \delta} e^{(\frac{\gamma v(v-2z)}{\gamma\tau + \delta} - v - v')\pi i} R\left(-z' + \frac{\tau'}{2} + \frac{1}{2}, -v' \mid \tau'\right) \\ &= \vartheta_0\left(z + \frac{\tau}{2} + \frac{1}{2} \mid \tau\right) \sqrt{\frac{\gamma i}{\gamma\tau + \delta}} \sum_{k=0}^{|\gamma|-1} (-1)^k e^{-k(k+1)\pi i\tau} \Psi_{\tilde{M};k}(v - z \mid \tau). \end{aligned}$$

By applying the functional relation in (2.13) to the second term containing  $R(-z' + \frac{\tau'}{2} + \frac{1}{2}, -v' \mid \tau')$  in (2.15) in the above, one can notice that this term equals to

$$-\frac{1}{\gamma\tau + \delta} e^{(\frac{\gamma v(v-2z)}{\gamma\tau + \delta} - v + v')\pi i} R\left(z' + \frac{\tau'}{2} + \frac{1}{2}, v' \mid \tau'\right).$$

Beside, by replacing  $M$  with  $\tilde{M}$  in (2.3) and noticing the identity  $\frac{1}{1 + e^{2\gamma\pi s}} + \frac{1}{1 + e^{-2\gamma\pi s}} = 1$ , it follows that

$$\Psi_{\tilde{M};k}(v \mid \tau) + \Psi_{M;k}(v \mid \tau) = \int_{-\infty}^{\infty} e^{-\frac{\gamma}{\gamma\tau + \delta}(s - vi + (k + \frac{1}{2})(\tau + \frac{\delta}{\gamma})i)^2 \pi i} ds.$$

Thus, one reduces from the above that, for any integer  $k$ :

$$(2.16) \quad \Psi_{\tilde{M};k}(v \mid \tau) = -\Psi_{M;k}(v \mid \tau) + \sqrt{\frac{\gamma\tau + \delta}{\gamma i}}.$$

In summary, by considering both (2.14) and (2.15) together with (2.16), one can extend the transformation formula (2.2) into the following form for any given

modular matrix  $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}(2; \mathbb{Z})$ :

$$(2.17) \quad R\left(z + \frac{\tau}{2} + \frac{1}{2}, v \mid \tau\right) = A_M(z, v \mid \tau) R\left(z' + \frac{\tau'}{2} + \frac{1}{2}, v' \mid \tau'\right) + \vartheta_0\left(z + \frac{\tau}{2} + \frac{1}{2} \mid \tau\right) \Psi_M(v - z \mid \tau),$$

where both  $z'$  and  $v'$  are given in (2.1) and where the factors  $A_M$  and  $\Psi_M$  are defined as follows:

$$(2.18) \quad A_M(z, v \mid \tau) = \frac{1}{\gamma\tau + \delta} e^{(\frac{\gamma v(v-2z)}{\gamma\tau + \delta} - v + v')\pi i}$$

and

$$(2.19) \quad \begin{aligned} \Psi_M(v \mid \tau) &= \text{Sgn}(\gamma) \sqrt{\frac{\gamma i}{\gamma\tau + \delta}} \sum_{k=0}^{|\gamma|-1} (-1)^k e^{-k(k+1)\pi i\tau} \Psi_{M;k}(v \mid \tau) \\ &+ \mathbf{1}_{\mathbb{Z}_{<0}}(\gamma) \sum_{k=0}^{|\gamma|-1} (-1)^k e^{-k(k+1)\pi i\tau}. \end{aligned}$$

In the above,  $\text{Sgn}$  is the sign function,  $\Psi_{M;k}$  is the Mordell integral defined in (2.3), and  $\mathbf{1}_{\mathbb{Z}_{<0}}$  denotes the indicator function of the subset  $\mathbb{Z}_{<0}$  of  $\mathbb{Z}$ .

3.  $q$ -ANALOGS OF BOREL-LAPLACE SUMMATION

As in the previous paragraph, let  $\tau' = M\tau = \frac{\alpha\tau + \beta}{\gamma\tau + \delta}$  to be the modular variable associated with the matrix  $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{SL}(2; \mathbb{Z})$ , where  $\Im(\tau) > 0$  and  $\Im(\tau') > 0$ . Let  $q = e^{2\pi i\tau}$ , and write  $q' = e^{2\pi i\tau'}$ . For simplify, we will suppose that  $\gamma \geq 1$ . Furthermore, we define

$$(3.1) \quad \zeta = e^{-\frac{2\delta}{\gamma}\pi i}, \quad \rho = e^{\frac{2(\gamma\tau + \delta)}{\gamma}\pi i}, \quad \zeta' = e^{\frac{2\alpha}{\gamma}\pi i}, \quad \rho' = e^{-\frac{2}{\gamma(\gamma\tau + \delta)}\pi i}.$$

It is obvious to see that  $q = \rho\zeta$  and  $0 < |q| < 1$ . Moreover, by noticing that  $\tau' = -\frac{1}{\gamma(\gamma\tau + \delta)} + \frac{\alpha}{\gamma}$ , it follows that  $q' = \rho'\zeta'$ .

In line with [24], we shall consider the following first order  $q$ -difference equation in the complex plane of  $x$ :

$$(3.2) \quad x y\left(\frac{x}{q}\right) + y(x) = 1.$$

Let  $\hat{E}(x) = \hat{E}(x; q)$  be the formal power series of  $x$  given by the relation

$$(3.3) \quad \hat{E}(x) = \sum_{n=0}^{\infty} (-1)^n q^{-\frac{1}{2}n(n-1)} x^n.$$

One can find that  $\hat{E}(x)$  satisfies term-by-term the functional equation stated in (3.2). As  $|q| < 1$ , the power series  $\hat{E}(x)$  diverges for any  $x \neq 0$ . So, one finds that (3.2) admits no analytic solution at  $x = 0$ , and this can be viewed as one consequence of the fact that  $x = 0$  represents an irregular singular point for (3.2). See [16, 21] for the definition of an irregular singular point of any given linear  $q$ -difference equation.

As one will see in the below, one can, however, construct for equation (3.2) or its derivatives some analytic solutions whose asymptotic expansion as  $x \rightarrow 0$  is exactly the divergent power series  $\hat{E}(x)$ , and a suitable Stokes analysis will yields relations linking all these solutions.

First, by applying the  $q$ -Borel-Laplace transform introduced in [22] and [17], one finds that equation (3.2) is satisfied by the function  $L(x, u; q)$  defined in the following manner:

$$(3.4) \quad L(x, u; q) = \frac{1}{\theta\left(-\frac{u}{x}; q\right)} \sum_{n=-\infty}^{\infty} \frac{q^{\frac{1}{2}n(n-1)}}{1 - uq^n} \left(-\frac{u}{x}\right)^n.$$

In the above,  $\theta(x; q)$  is the theta function defined by the series

$$(3.5) \quad \theta(x; q) = \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n(n-1)} x^n,$$

and  $u$  denotes any given non-zero parameter such that  $u \neq q^n$  for all integer  $n$ . The function  $L(x, u; q)$  given in (3.4) is well-defined for any  $x \in \mathbb{C}^* \setminus \{uq^n : n \in \mathbb{Z}\}$ .

Secondly, one knows that  $q = \rho\zeta$ , where  $\zeta$  is a  $\gamma$ th root of unity. By applying the  $\rho$ -Borel transform to  $\hat{E}(x)$ , one gets the function  $h(\xi; \zeta)$  given by the power series

$$h(\xi; \zeta) = \sum_{n=0}^{\infty} (-1)^n \zeta^{-\frac{1}{2}n(n-1)} \xi^n,$$



that is convergent in the unit disk  $|\xi| < 1$ . Since  $\zeta^{-\frac{1}{2}(n+\gamma)(n+\gamma-1)}(-\xi)^{n+\gamma} = \zeta^{-\frac{1}{2}n(n-1)}(-\xi)^n(-\xi^\gamma)$  for any integer  $n$ , it follows that

$$(3.6) \quad h(\xi; \zeta) = \frac{1}{1 + \xi^\gamma} \sum_{k=0}^{\gamma-1} \zeta^{-\frac{1}{2}k(k-1)} (-\xi)^k.$$

By using the root  $\zeta'$  introduced in (3.1), one can find that, for any integer  $k$ ,

$$(3.7) \quad \text{Res}\left(\frac{1}{\xi} h(\xi; \zeta) : \xi = e^{-\frac{1}{\gamma}(2k+1)\pi i}\right) = (-e^{-\frac{1}{\gamma}\pi i})^k \zeta'^{-\frac{1}{2}k(k+1)} S(\zeta),$$

where  $S(\zeta)$  is the Gauss sum satisfying the following relations:

$$(3.8) \quad S(\zeta) = \text{Res}\left(\frac{1}{\xi} h(\xi; \zeta) : \xi = e^{-\frac{1}{\gamma}\pi i}\right) = \frac{1}{\gamma} \sum_{k=0}^{\gamma-1} (-e^{-\frac{1}{\gamma}\pi i})^k \zeta^{-\frac{1}{2}k(k-1)}.$$

To see (3.7), one can notice the relation  $e^{-\frac{1}{\gamma}(2k+1)\pi i} = e^{-\frac{1}{\gamma}\pi i} \zeta^{k\alpha}$ , which is a consequence of  $\alpha\delta - \beta\gamma = 1$ . Thus, one can obtain (3.7) from (3.8) by iterating several times the following functional equation at the special value  $\xi = e^{-\frac{1}{\gamma}\pi i}$ :

$$\sum_{k=0}^{\gamma-1} \zeta^{-\frac{1}{2}k(k-1)} \left(-\frac{\xi}{\zeta}\right)^k = -\frac{1}{\xi} \sum_{k=1}^{\gamma} \zeta^{-\frac{1}{2}k(k-1)} (-\xi)^k.$$

In what follows, it will be assumed that the following condition is fulfilled:

(H): both  $\rho$  and  $\rho'$  belong to the interval  $(0, 1)$ .

This is equivalent to suppose that  $\tau$  satisfies the condition  $\frac{\gamma\tau + \delta}{i} \in (0, +\infty) \subset \mathbb{R}$ . Notice that, by taking into account the argument of the analytic continuation with respect to  $q$  inside the unit disc  $0 < |q| < 1$ , it is enough to only prove (2.2) under (H).

Let  $\log$  to denote the principal branch of the logarithm in the universal covering of  $\mathbb{C}^*$ , and define

$$(3.9) \quad \omega(t; \rho) = \frac{1}{\sqrt{2\pi \ln(1/\rho)}} e^{\frac{1}{2\ln \rho} \log^2(\frac{t}{\sqrt{\rho}})},$$

$$(3.10) \quad G(x; \rho, \zeta) = \int_0^\infty h(\xi; \zeta) \omega\left(\frac{\xi}{x}; \rho\right) \frac{d\xi}{\xi},$$

where the path of integration may be the half line starting from the origin whose angle with the positive real axis is less than  $\frac{\pi}{\gamma}$ .

**Theorem 3.1.** *Given any  $(z, v) \in \mathbb{C}^2$  such that  $v \notin \mathbb{Z} \oplus \tau\mathbb{Z}$  and  $z - v \notin \mathbb{Z} \oplus \tau\mathbb{Z}$ , if*

$$(3.11) \quad x = e^{2\pi iz} \quad u = e^{2\pi iv}, \quad x' = e^{2\pi iz'}, \quad u' = e^{2\pi iv'},$$

where  $z' = Mz$ ,  $v' = Mv$  are defined in (2.1), then:

$$(3.12) \quad G(x; \rho, \zeta) = L(x, u; q) - 2\pi i S(\zeta) \omega\left(\frac{e^{-\frac{1}{\gamma}\pi i}}{x}; \rho\right) L\left(\frac{x'}{q'}, u'; q'\right).$$

In view of the definition of  $\rho$  given in (3.1), one can notice that  $\ln \rho = \frac{2}{\gamma}(\gamma\tau + \delta)\pi i$ . Thus, one finds from (2.1) and (3.11) that

$$(3.13) \quad x' = e^{\frac{2 \log x}{\gamma \ln q} \pi i}, \quad u' = e^{\frac{2 \log u}{\gamma \ln q} \pi i}.$$

## 4. PROOF OF THEOREM 3.1

The functional equation in (3.2) can be put into the following form:

$$y(x) = (-x)y\left(\frac{x}{q}\right) + 1.$$

By iterating this last equation  $(\gamma - 1)$  times, one finds that the power series  $\hat{E}(x)$  satisfies the functional equation

$$y(x) = q^{-\frac{1}{2}\gamma(\gamma-1)} (-x)^\gamma y\left(\frac{x}{q^\gamma}\right) + \sum_{k=0}^{\gamma-1} q^{-\frac{1}{2}k(k-1)} (-x)^k.$$

As  $q = \rho\zeta$  and  $\zeta = e^{-\frac{2\delta}{\gamma}\pi i}$ , one can notice that  $q^\gamma = \rho^\gamma$  and  $q^{-\frac{1}{2}\gamma(\gamma-1)} = (-1)^\gamma \rho^{-\frac{1}{2}\gamma(\gamma-1)}$ . Thus, one deduces from the above that the divergent power series  $\hat{E}(x)$ , as well as its sum-functions  $L(x, u; q)$  and  $G(x; \rho, \zeta)$ , are solution to the non-homogenous  $\rho$ -difference equation

$$(4.1) \quad \rho^{-\frac{1}{2}\gamma(\gamma-1)} x^\gamma y\left(\frac{x}{\rho^\gamma}\right) + y(x) = \sum_{k=0}^{\gamma-1} q^{-\frac{1}{2}k(k-1)} (-x)^k.$$

Thus, if  $g(x) = G(x; \rho, \zeta) - L(x, u; q)$ , then:

$$(4.2) \quad g(x) = -x^\gamma \rho^{-\frac{1}{2}\gamma(\gamma-1)} g\left(\frac{x}{\rho^\gamma}\right).$$

Further, as  $L(x, u; q)$  is left invariant for the monodromy operator  $x \mapsto xe^{2\pi i}$ , one has

$$(4.3) \quad g(xe^{2\pi i}) - g(x) = G(xe^{2\pi i}; \rho, \zeta) - G(x; \rho, \zeta).$$

In view of the definition of  $G(x; \rho, \zeta)$  in (3.10), one can observe that

$$G(xe^{2\pi i}; \rho, \zeta) = \int_0^{\infty e^{-2\pi i}} h(\xi; \zeta) \omega\left(\frac{\xi}{x}; \rho\right) \frac{d\xi}{\xi},$$

where the integration path  $(0, \infty e^{-2\pi i})$  may be obtained by turning through a full rotation the positive real-axis in the clockwise direction around the origin. By

applying the Cauchy's theorem to the contour-integral  $(\int_0^{\infty e^{-2\pi i}} - \int_0^\infty)$ , it follows

from the residues formula (3.7) that

$$(4.4) \quad G(xe^{2\pi i}; \rho, \zeta) - G(x; \rho, \zeta) = 2\pi i S(\zeta) \sum_{k=0}^{\gamma-1} (-e^{-\frac{1}{\gamma}\pi i})^k \zeta'^{-\frac{1}{2}k(k+1)} \omega\left(\frac{e^{-\frac{1}{\gamma}(2k+1)\pi i}}{x}; \rho\right).$$

From the definition of  $\omega(t; \rho)$  in (3.9), one obtains that

$$\omega\left(\frac{e^{-\frac{1}{\gamma}(2k+1)\pi i}}{x}; \rho\right) = e^{\frac{ab}{2\ln\rho}} \omega\left(\frac{e^{-\frac{1}{\gamma}\pi i}}{x}; \rho\right),$$

where  $a = \log(e^{\frac{2k}{\gamma}\pi i}) = \frac{2k}{\gamma}\pi i$  and  $b = \log(e^{\frac{2k+2}{\gamma}\pi i}\rho x^2) = \frac{2k+2}{\gamma}\pi i + \ln\rho + 2\log x$ . Observe that

$$ab = -\frac{4}{\gamma^2}k(k+1)\pi^2 + \left(\frac{2\ln\rho}{\gamma} + \frac{4\log x}{\gamma}\right)k\pi i.$$

From the definition of  $\rho$  and  $\rho'$  given in (3.1), it follows that  $\rho' = e^{\frac{4\pi^2}{\gamma^2 \ln \rho}}$ . By taking into account the relation in (3.13) for  $x'$ , one finds from the above that

$$(4.5) \quad \omega\left(\frac{e^{-\frac{1}{\gamma}(2k+1)\pi i}}{x}; \rho\right) = (x' e^{\frac{1}{\gamma}\pi i})^k \rho'^{-\frac{1}{2}k(k+1)} \omega\left(\frac{e^{-\frac{1}{\gamma}\pi i}}{x}; \rho\right).$$

As  $q' = \rho' \zeta'$ , it results from combining (4.4) and (4.5) that equation (4.3) takes the following form:

$$(4.6) \quad g(xe^{2\pi i}) - g(x) = 2\pi i S(\zeta) \omega\left(\frac{e^{-\frac{1}{\gamma}\pi i}}{x}; \rho\right) \sum_{k=0}^{\gamma-1} \left(-\frac{x'}{q'}\right)^k q'^{-\frac{1}{2}k(k-1)}.$$

Note that  $g(x)$  is defined and analytic on the universal covering of  $\mathbb{C}^*$  excepted at  $x = e^{2\pi i(v+n+m\tau)}$  for  $(n, m) \in \mathbb{Z}^2$ . Moreover, the map  $x \mapsto x'$  is one-to-one between both lattices  $e^{2\pi i(v+\mathbb{Z}\oplus\tau\mathbb{Z})}$  and  $e^{2\pi i(v'+\mathbb{Z}\oplus\tau'\mathbb{Z})}$ . Thus, if  $\Delta$  is the function defined by the relation

$$(4.7) \quad g(x) = -2\pi i S(\zeta) \omega\left(\frac{e^{-\frac{1}{\gamma}\pi i}}{x}; \rho\right) \Delta(x'),$$

$\Delta$  is well-defined and analytic on the universal covering of  $\mathbb{C}^*$  excepted in the multiplicative lattice  $e^{2\pi i(v'+\mathbb{Z}\oplus\tau'\mathbb{Z})}$ .

By (3.11), it follows that  $(x\rho^{-\gamma})' = x'e^{-2\pi i}$  and  $(xe^{2\pi i})' = \rho'^{-\gamma}x'$ . On the one hand, since

$$\omega\left(\frac{e^{-\frac{1}{\gamma}\pi i}}{x\rho^{-\gamma}}; \rho\right) = e^{-\pi i} \rho^{\frac{1}{2}\gamma(\gamma-1)} x^{-\gamma} \omega\left(\frac{e^{-\frac{1}{\gamma}\pi i}}{x}; \rho\right),$$

equation (4.2) becomes:

$$(4.8) \quad \Delta(x') = \Delta(x'e^{-2\pi i}).$$

On the other hand, letting  $k = \gamma$  into (4.5) yields that

$$\omega\left(\frac{e^{-\frac{1}{\gamma}\pi i}}{xe^{2\pi i}}; \rho\right) = -x'^{\gamma} \rho'^{-\frac{1}{2}\gamma(\gamma+1)} \omega\left(\frac{e^{-\frac{1}{\gamma}\pi i}}{x}; \rho\right).$$

In this way, (4.6) is transformed as follows:

$$(4.9) \quad \rho'^{-\frac{1}{2}\gamma(\gamma-1)} \left(\frac{x'}{q'}\right)^{\gamma} \Delta\left(\frac{x'}{\rho'^{\gamma}}\right) + \Delta(x') = \sum_{k=0}^{\gamma-1} \left(-\frac{x'}{q'}\right)^k q'^{-\frac{1}{2}k(k-1)}.$$

The above functional equation in (4.8) implies that  $\Delta$  represents an analytic function on the complex plane excepted on  $\{0\} \cup q'^{\mathbb{Z}}$ , while equation (4.9) is really a dual form of (4.1). By the uniqueness of solutions with simple poles on an only lattice like as in the theory of elliptic functions, one obtains that  $\Delta(x') = L\left(\frac{x'}{q'}, u'; q'\right)$ , which together with (4.7) gives (3.12) and then finishes the proof of Theorem 3.1.

## 5. FROM THEOREM 3.1 TO THEOREM 2.1

We will interpret formula (3.12) into a relation about Appell-Lerch series and Mordell integrals. We shall start by recalling a classic link existing between on a side the Gauss sum  $S(\zeta)$  used in (3.12), which is defined in (3.8), and on the other side the modular relations for theta functions.

Thanks to [8, (28)], the transformation formula (2.9) for  $\vartheta_1$  can also be written into the following form:

$$(5.1) \quad \vartheta_1(v' | \tau') = S_M \sqrt{\frac{\gamma\tau + \delta}{i}} e^{\frac{\gamma v^2}{\gamma\tau + \delta} \pi i} \vartheta_1(v | \tau),$$

where  $S_M$  is the following Gauss sum related with the matrix  $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ :

$$(5.2) \quad S_M = \frac{1}{i\sqrt{\gamma}} \sum_{k=0}^{\gamma-1} e^{\frac{1}{\gamma}(\alpha k^2 + (\alpha - \gamma + 1)k + \frac{\alpha + \delta}{4} - \frac{\gamma - 1}{2})\pi i}.$$

It should be useful to mention that, in the above and also in the below for all similar situations, the summation on the index  $k$  can be made over any full system of integers with respect to the congruence relation for  $\gamma$ .

**Lemma 5.1.** *Consider the Gauss sum  $S(\zeta)$  defined in (3.8). One has:*

$$(5.3) \quad S_M = i\sqrt{\gamma} e^{\frac{1}{\gamma}(\frac{\alpha + \delta}{4} - \frac{\gamma - 1}{2})\pi i} S(\zeta).$$

*Proof.* Let  $\zeta' = e^{\frac{2\alpha}{\gamma}\pi i}$  be as in (3.1), and define  $S(\zeta'^{\pm})$  in the same way as what done in (3.8) for  $S(\zeta)$ , that is to say:

$$(5.4) \quad S(\zeta') = \frac{1}{\gamma} \sum_{k=0}^{\gamma-1} (-e^{-\frac{1}{\gamma}\pi i})^k \zeta'^{-\frac{1}{2}k(k-1)}, \quad S(\zeta'^{-1}) = \frac{1}{\gamma} \sum_{k=0}^{\gamma-1} (-e^{-\frac{1}{\gamma}\pi i})^k \zeta'^{\frac{1}{2}k(k-1)}.$$

As  $\alpha$  and  $\gamma$  are coprime, one has  $(\alpha + 1)(\gamma + 1) \equiv 0 \pmod{2}$ . Thus, replacing  $k$  with  $(\gamma - k)$  in (5.2) yields that

$$S_M = \frac{e^{(\frac{\alpha + \delta}{4} - \frac{\gamma - 1}{2})\pi i}}{i\sqrt{\gamma}} \sum_{k=1}^{\gamma} e^{\frac{1}{\gamma}(\alpha k^2 - (\alpha - \gamma + 1)k)\pi i}.$$

This is to say that

$$(5.5) \quad S_M = -i\sqrt{\gamma} e^{\frac{1}{\gamma}(\frac{\alpha + \delta}{4} - \frac{\gamma - 1}{2})\pi i} S(\zeta'^{-1}).$$

We will express  $S_M$  by means of  $S(\zeta)$ . On the one hand, one can observe that

$$\begin{aligned} S(\zeta') S(\zeta'^{-1}) &= \frac{1}{\gamma^2} \sum_{k, \ell=0}^{\gamma-1} (-e^{-\frac{1}{\gamma}\pi i})^{k+\ell} \zeta'^{\frac{1}{2}(k-\ell)(k+\ell-1)} \\ &= \frac{1}{\gamma^2} \sum_{N=0}^{\gamma-1} \sum_{k=0}^{\gamma-1} (-e^{-\frac{1}{\gamma}\pi i})^N \zeta'^{\frac{1}{2}(2k-N)(N-1)}. \end{aligned}$$

For any given integer  $N$  belonging to  $[0, \gamma)$ , the last summation for  $k$  from 0 to  $(\gamma - 1)$  equals zero if  $N \neq 1$ , and equals to  $\gamma$  otherwise. So, it follows that

$$(5.6) \quad S(\zeta') S(\zeta'^{-1}) = -\frac{1}{\gamma} e^{-\frac{1}{\gamma}\pi i}.$$

On the other hand, by considering all residus of the rational function  $\frac{1}{\xi} h(\xi; \zeta)$  over the whole complex plane, one obtains that

$$\sum_{k=0}^{\gamma-1} \operatorname{Res}\left(\frac{1}{\xi} h(\xi; \zeta) : \xi = e^{-\frac{1}{\gamma}(2k+1)\pi i}\right) = -\operatorname{Res}\left(\frac{1}{\xi} h(\xi; \zeta) : \xi = 0\right) = -1.$$

This together with (3.7) implies that

$$\sum_{k=0}^{\gamma-1} (-e^{-\frac{1}{\gamma}\pi i})^k \zeta'^{-\frac{1}{2}k(k+1)} S(\zeta) = -1.$$

Therefore, one deduces from the definition of  $S(\zeta')$  given in (5.4) that  $S(\zeta') S(\zeta) = \frac{1}{\gamma} e^{-\frac{1}{\gamma}\pi i}$ , which, by (5.6), implies that  $S(\zeta) = -S(\zeta'^{-1})$ . By taking into account (5.5), one finds easily (5.3).  $\square$

Now, we will write Theorem 2.1 by means of Appell-Lerch series. Note that the definition of  $L(x, u; q)$  in (3.4) can be read as follows:

$$L(x, u; q) = \frac{1}{\theta(-\frac{u}{x}; q)} R(v - z - \frac{\tau}{2} + \frac{1}{2}, v | \tau),$$

where  $x = e^{2\pi iz}$  and  $u = e^{2\pi iv}$ . By noticing that  $L(\frac{x'}{q'}, u'; q') = \frac{1}{x'} (1 - L(x', u'; q'))$ , one deduces from (3.12) that

$$(5.7) \quad \theta(-\frac{u}{x}; q) G(x; \rho, \zeta) = R(v - z - \frac{\tau}{2} + \frac{1}{2}, v | \tau) - \Theta_M(z, v | \tau) \times \left( R(v' - z' - \frac{\tau'}{2} + \frac{1}{2}, v' | \tau') - \theta(-\frac{u'}{x'}; q') \right),$$

where

$$(5.8) \quad \Theta_M(z, v | \tau) = -2\pi i S(\zeta) \omega\left(\frac{e^{-\frac{1}{\gamma}\pi i}}{x}; \rho\right) \frac{\theta(-\frac{u}{x}; q)}{\theta(-\frac{u'}{x'}; q')} \frac{1}{x'}.$$

By (3.1), one knows that  $\ln \rho = \frac{2}{\gamma}(\gamma\tau + \delta)\pi i$ . If  $t = e^{2\pi s}$ , the definition of  $\omega(t; \rho)$  given in (3.9) implies that

$$(5.9) \quad \omega(e^{2\pi s}; \rho) = \frac{1}{2\pi} \sqrt{\frac{\gamma i}{\gamma\tau + \delta}} e^{-\frac{\gamma}{\gamma\tau + \delta}(s - \frac{\gamma\tau + \delta}{2\gamma}i)^2 \pi i}.$$

Letting in the above  $s = -i(\frac{1}{2\gamma} + z)$  yields that

$$(5.10) \quad \omega\left(\frac{e^{-\frac{1}{\gamma}\pi i}}{x}; \rho\right) = \frac{1}{2\pi} \sqrt{\frac{\gamma i}{\gamma\tau + \delta}} e^{\frac{\gamma}{\gamma\tau + \delta}(z + \frac{1}{2\gamma} + \frac{\gamma\tau + \delta}{2\gamma}i)^2 \pi i}.$$

Further, comparing (3.5) with (1.4) gives that

$$\theta(-e^{2\pi iv}; e^{2\pi i\tau}) = -i e^{\pi i(v - \frac{\tau}{4})} \vartheta_1(v | \tau).$$

Thus, in view of (5.10), the factor  $\Theta_M(z, v | \tau)$  defined in (5.8) can be written as follows:

$$(5.11) \quad \Theta_M(z, v | \tau) = -i S(\zeta) \sqrt{\frac{\gamma i}{\gamma\tau + \delta}} e^{\lambda\pi i} \frac{\vartheta_1(v - z | \tau)}{\vartheta_1(v' - z' | \tau')},$$

where

$$(5.12) \quad \lambda = \frac{\gamma}{\gamma\tau + \delta} \left( z + \frac{1}{2\gamma} + \frac{\gamma\tau + \delta}{2\gamma}i \right)^2 + v - z - \frac{\tau}{4} - v' - z' + \frac{\tau'}{4}.$$

By applying the transformation formula (5.1), one deduces from (5.11) that

$$\Theta_M(z, v | \tau) = \frac{S(\zeta)}{S_M} \frac{\sqrt{\gamma}}{\gamma\tau + \delta} e^{(\lambda - \frac{\gamma(v-z)^2}{\gamma\tau + \delta})\pi i}.$$

Using (5.3) yields that

$$(5.13) \quad \Theta_M(z, v | \tau) = -\frac{i}{\gamma\tau + \delta} e^{(\lambda - \frac{\gamma(v-z)^2}{\gamma\tau + \delta} - \frac{1}{\gamma}(\frac{\alpha + \delta}{4} - \frac{\gamma - 1}{2}))\pi i}.$$

It remains to simplify the exponent employed in the right-hand side of (5.13). For this, one notices the following expansion:

$$\frac{\gamma}{\gamma\tau + \delta} \left( z + \frac{1}{2\gamma} + \frac{\gamma\tau + \delta}{2\gamma} \right)^2 = \frac{\gamma z^2}{\gamma\tau + \delta} + z' + z + \frac{1}{4\gamma(\gamma\tau + \delta)} + \frac{\gamma\tau + \delta}{4\gamma} + \frac{1}{2\gamma}.$$

Furthermore, since  $\frac{1}{\gamma\tau + \delta} = -\gamma\tau' + \alpha$ , one deduces from (5.12) that

$$\lambda = \frac{\gamma z^2}{\gamma\tau + \delta} + v - v' + \frac{\alpha}{4\gamma} + \frac{\delta}{4\gamma} + \frac{1}{2\gamma}.$$

So, the expression of  $\Theta_M$  given in (5.13) can be simplified as follows:

$$(5.14) \quad \Theta_M(z, v | \tau) = \frac{1}{\gamma\tau + \delta} e^{(\frac{\gamma}{\gamma\tau + \delta} v(2z - v) + v - v')\pi i}.$$

While replacing  $z$  with  $(v - z)$  in (5.7), one makes use of the functions  $\theta(-x; q)$  and  $\theta(-x'; q)$  instead of  $\theta(-\frac{u}{x}; q)$  and  $\theta(-\frac{u'}{x'}; q)$ . In view of the expression of  $\Theta_M$  given in (5.14), one obtains from (5.7) that

$$(5.15) \quad \theta(-x; q) G\left(\frac{u}{x}; \rho, \zeta\right) = R\left(z - \frac{\tau}{2} + \frac{1}{2}, v | \tau\right) - \frac{1}{\gamma\tau + \delta} e^{(\frac{\gamma v(v-2z)}{\gamma\tau + \delta} + v - v')\pi i} \times \left( R\left(z' - \frac{\tau'}{2} + \frac{1}{2}, v' | \tau'\right) - \theta(-x'; q') \right).$$

By using the definition of the Appell-Lerch series  $R_k$  in (1.9) with  $k = 1$ , a direct calculation shows that

$$R\left(z - \frac{\tau}{2} + \frac{1}{2}, v | \tau\right) - \theta(-x; q) = e^{2\pi i v} R\left(z + \frac{\tau}{2} + \frac{1}{2}, v | \tau\right).$$

Therefore, formula (5.15) can also be expressed as follows:

$$(5.16) \quad \theta(-x; q) \left( G\left(\frac{u}{x}; \rho, \zeta\right) - 1 \right) = e^{2\pi i v} R\left(z + \frac{\tau}{2} + \frac{1}{2}, v | \tau\right) - \frac{1}{\gamma\tau + \delta} e^{(\frac{\gamma v(v-2z)}{\gamma\tau + \delta} + v + v')\pi i} R\left(z' + \frac{\tau'}{2} + \frac{1}{2}, v' | \tau'\right).$$

In addition, be considering the expression obtained in (5.9) for the heat-kernel  $\omega(e^{2\pi s}; \rho)$ , it follows from (3.10) that

$$(5.17) \quad G\left(\frac{u}{x}; \rho, \zeta\right) - 1 = \sqrt{\frac{\gamma i}{\gamma\tau + \delta}} \int_{-\infty}^{\infty} (h(e^{2\pi s}; \zeta) - 1) e^{-\frac{\gamma}{\gamma\tau + \delta}(s - vi + zi - \frac{\tau}{2}i - \frac{\delta}{2\gamma}i)^2 \pi i} ds,$$

where the path of integration may be a line parallel with the real axis with a distance less than  $\frac{1}{2\gamma}$  from the real axis. By putting  $\xi = e^{2\pi s}$  into the right-hand

side of (3.6), it follows that

$$(5.18) \quad h(e^{2\pi s}; \zeta) - 1 = \frac{1}{1 + e^{2\gamma\pi s}} \sum_{k=1}^{\gamma} (-1)^k \zeta^{-\frac{1}{2}k(k-1)} e^{2k\pi s}.$$

At this time, let  $a = -vi + zi - \frac{\tau}{2}i - \frac{\delta}{2\gamma}i$ , and note that

$$e^{-\frac{\gamma}{\gamma\tau+\delta}(s+a)^2\pi i+2k\pi s} = e^{\left(-\frac{\gamma}{\gamma\tau+\delta}(s+a+\frac{k}{\gamma}(\gamma\tau+\delta))i\right)^2+2kai-\frac{k^2}{\gamma}(\gamma\tau+\delta)}\pi i.$$

Since

$$a + \frac{k}{\gamma}(\gamma\tau + \delta)i = (z - u)i + \left(k - \frac{1}{2}\right)\left(\tau + \frac{\delta}{\gamma}\right)i$$

and

$$2kai - \frac{k^2}{\gamma}(\gamma\tau + \delta) = 2k(v - z) - k(k - 1)\left(\tau + \frac{\delta}{\gamma}\right),$$

one finds that

$$e^{-\frac{\gamma}{\gamma\tau+\delta}(s+a)^2\pi i+2k\pi s} = e^{-\frac{\gamma}{\gamma\tau+\delta}(s+(z-v)i+(k-\frac{1}{2})(\tau+\frac{\delta}{\gamma})i)^2\pi i} \frac{u}{x} q^{-\frac{1}{2}k(k-1)} \zeta^{\frac{1}{2}k(k-1)}.$$

It follows from (5.18) that

$$\begin{aligned} & (h(e^{2\pi s}; \zeta) - 1) e^{-\frac{\gamma}{\gamma\tau+\delta}(s-vi+zi-\frac{\tau}{2}i-\frac{\delta}{2\gamma}i)^2\pi i+2k\pi s} \\ &= \frac{u}{x(1 + e^{2\gamma\pi s})} \sum_{k=1}^{\gamma} (-1)^k e^{-\frac{\gamma}{\gamma\tau+\delta}(s+(z-v)i+(k-\frac{1}{2})(\tau+\frac{\delta}{\gamma})i)^2\pi i-k(k-1)\pi i\tau}. \end{aligned}$$

By considering (5.17) and the relation  $\theta(-x; q) = -x\theta(-qx; q)$ , one deduces that

$$(5.19) \quad \theta(-x; q) \left(G\left(\frac{u}{x}; \rho, \zeta\right) - 1\right) = -u\theta(-qx; q) \sqrt{\frac{\gamma i}{\gamma\tau + \delta}} \times \sum_{k=1}^{\gamma} (-1)^k q^{-\frac{1}{2}k(k-1)} \int_{-\infty}^{\infty} e^{-\frac{\gamma}{\gamma\tau+\delta}(s+(z-v)i+(k-\frac{1}{2})(\tau+\frac{\delta}{\gamma})i)^2\pi i} \frac{ds}{1 + e^{2\gamma\pi s}}.$$

In view of the definition of  $\vartheta_0$  given in (1.4), one sees that

$$\theta(-qx; q) = \vartheta_0\left(z + \frac{\tau}{2} + \frac{1}{2} \mid \tau\right).$$

In this way, one obtains (2.2) from (5.16), and achieves the proof of Theorem 2.1.

## 6. ABOUT THE MAIN TRANSFORMATION FORMULA

Let us consider the transformation formula (2.17), and notice that

$$(6.1) \quad i\vartheta_1(z \mid \tau) = e^{(z+\frac{\tau}{4})\pi i} \vartheta_0\left(z + \frac{\tau}{2} + \frac{1}{2} \mid \tau\right);$$

see (1.4). If one substitutes  $z + v$  to  $v$  in (2.17) and multiplies both sides by the factor  $e^{(z+v)\pi i}$ , one can write this formula into the following form:

$$(6.2) \quad \tilde{R}(z, v \mid \tau) = \tilde{A}_M(z, v \mid \tau) \tilde{R}(z', v' \mid \tau') + \vartheta_1(z \mid \tau) \tilde{\Psi}_M(v \mid \tau).$$

In the above,  $\tilde{R}$ ,  $\tilde{A}_M$  and  $\tilde{\Psi}_M$  are related respectively with  $R$ ,  $A_M$  and  $\Psi_M$  in the following manner:

$$(6.3) \quad \tilde{R}(z, v \mid \tau) = e^{(z+v)\pi i} R\left(z + \frac{\tau}{2} + \frac{1}{2}, z + v \mid \tau\right),$$

$$(6.4) \quad \tilde{A}_M(z, v) = e^{(z+v-z'-v')\pi i} A_M(z, z + v \mid \tau)$$

and

$$(6.5) \quad \tilde{\Psi}_M(v | \tau) = i e^{(v-\frac{\tau}{4})\pi i} \Psi_M(v | \tau).$$

First, by replacing  $v$  with  $z + v$  in (2.18) and considering (6.4), one can notice that

$$(6.6) \quad \tilde{A}_M(z, v | \tau) = \frac{1}{\gamma\tau + \delta} e^{\frac{\gamma}{\gamma\tau + \delta}(v^2 - z^2)\pi i}.$$

In view of the relation  $\alpha\delta - \beta\gamma = 1$ , one can also notice the following identities deduced directly from the definition of  $\tau'$ :

$$(6.7) \quad \frac{\tau}{\gamma\tau + \delta} = \delta\tau' - \beta, \quad \frac{1}{\gamma\tau + \delta} = -\gamma\tau' + \alpha.$$

Thus, one deduces from (6.6) that

$$(6.8) \quad \tilde{A}_M(z, v + 1 | \tau) = (-1)^{\alpha\gamma} e^{(2\gamma v' - \gamma^2 \tau')\pi i} \tilde{A}_M(z, v | \tau).$$

Beside, by writing  $\gamma\tau = (\gamma\tau + \delta) - \delta$ , one gets that

$$(6.9) \quad \frac{\gamma\tau v}{\gamma\tau + \delta} = v - \delta v', \quad \frac{\gamma\tau^2}{\gamma\tau + \beta} = \tau - \frac{\delta\tau}{\gamma\tau + \delta} = \tau - \delta^2\tau' + \beta\delta.$$

So, one can find that

$$(6.10) \quad \tilde{A}_M(z, v - \tau | \tau) = (-1)^{\beta\delta} e^{(-2v + 2\delta v' + \tau - \delta^2\tau')\pi i} \tilde{A}_M(z, v | \tau).$$

In order to express the functional relations on  $\Psi_M$  or  $\tilde{\Psi}_M$ , it will be convenient to introduce the family of functions  $\{P_k\}_{k \in \mathbb{Z}}$  in the following fashion:  $P_0 = 0$ ; for  $k \geq 1$  or  $k \leq -1$ ,

$$(6.11) \quad P_k(v | \tau) = -i e^{(v-\frac{\tau}{4})\pi i} \sum_{\ell=0}^{k-1} (-1)^\ell e^{(2\ell v - \ell(\ell+1)\tau)\pi i}$$

or

$$(6.12) \quad P_k(v | \tau) = i e^{(v-\frac{\tau}{4})\pi i} \sum_{\ell=k}^{-1} (-1)^\ell e^{(2\ell v - \ell(\ell+1)\tau)\pi i}$$

respectively. By comparing both relations in (6.11) and (6.12), one can observe that the following symmetries hold for any integer  $k$ :

$$(6.13) \quad P_{-k}(v | \tau) = -P_k(-v | \tau) = (-1)^{k+1} e^{(-2kv - k^2\tau)\pi i} P_k(v + k\tau | \tau).$$

**Lemma 6.1.** *The following identity holds for any integer  $k$ :*

$$(6.14) \quad \tilde{R}(z, v - k\tau | \tau) = (-1)^k e^{(-2kv + k^2\tau)\pi i} (\tilde{R}(z, v | \tau) + P_k(v | \tau) \vartheta_1(z | \tau)).$$

*Proof.* By considering the definition of  $R(z, v | \tau)$  in (1.9), one finds that, for any integer  $k$ ,

$$(6.15) \quad R(z, v - k\tau | \tau) = e^{(2kz + k^2\tau)\pi i} R(z + k\tau, v | \tau).$$

If one writes (2.11) into the form  $R(z + \tau, v | \tau) = e^{-2v\pi i} (R(z, v | \tau) - \vartheta_0(z | \tau))$ , iterating this gives that, for any positive integer  $k$ :

$$R(z + k\tau, v | \tau) = e^{-2kv\pi i} R(z, v | \tau) - \sum_{\ell=0}^{k-1} e^{-2(k-\ell)v\pi i} \vartheta_0(z + \ell\tau | \tau).$$



By taking into account the definition of  $\vartheta_0(z|\tau)$  in (1.4), one can immediately obtain that  $\vartheta_0(z + \ell\tau|\tau) = e^{-(2\ell z + \ell^2\tau)\pi i} \vartheta_0(z|\tau)$ . So, one deduces from the above that, for positive integer  $k$ :

$$R(z + k\tau, v|\tau) = e^{-2k\pi i v} \left( R(z, v|\tau) - e^{(z-v-\frac{\tau}{4})\pi i} P_k(v - z + \frac{\tau}{2} + \frac{1}{2}|\tau) \vartheta_0(z|\tau) \right),$$

where  $P_k$  is defined in (6.11). This together with (6.15) implies that, for  $k \geq 0$ :

$$\begin{aligned} R(z, v - k\tau|\tau) &= e^{(2k(z-v) + k^2\tau)\pi i} \left( R(z, v|\tau) \right. \\ &\quad \left. - e^{(z-v-\frac{\tau}{4})\pi i} P_k(v - z + \frac{\tau}{2} + \frac{1}{2}|\tau) \vartheta_0(z|\tau) \right). \end{aligned}$$

If one replaces  $(z, v)$  with  $(z + \frac{\tau}{2} + \frac{1}{2}, z + v)$  in the above and considers the relation stated in (6.1) for  $\vartheta_0$  and  $\vartheta_1$ , one deduces immediately from the definition of  $\tilde{R}$  in (6.3) the wanted formula (6.14) for any positive integer  $k$ .

The case for which  $k \leq -1$  can be obtained by substituting  $(v + k\tau)$  to  $v$  into (6.14) and then expanding there  $P_k(v + k\tau|\tau)$  by means of  $P_{-k}(v|\tau)$  with the help of the last relation of (6.13).  $\square$

In what follows, we will make use of the linear transformation formula (2.9) for  $\vartheta_1$ , in which  $\epsilon_{1,M}$  denotes some root of unity associated with the given modular matrix  $M$ . Namely, one deduces from (6.6) that

$$(6.16) \quad \tilde{A}_M(z, v|\tau) \frac{\vartheta_1(z'|\tau')}{\vartheta_1(z|\tau)} = i \epsilon_{1,M} \sqrt{\frac{i}{\gamma\tau + \delta}} e^{\frac{\gamma v^2}{\gamma\tau + \delta} \pi i}.$$

**Theorem 6.1.** *The following functional relations hold for the function  $\tilde{\Psi}_M(v|\tau)$ :*

$$(6.17) \quad \tilde{\Psi}_M(v + 1|\tau) + \tilde{\Psi}_M(v|\tau) = i \epsilon_{1,M} \sqrt{\frac{i}{\gamma\tau + \delta}} e^{\frac{\gamma v^2}{\gamma\tau + \delta} \pi i} P_\gamma(v'|\tau')$$

and

$$(6.18) \quad \tilde{\Psi}_M(v|\tau) + e^{(2v-\tau)\pi i} \tilde{\Psi}_M(v-\tau|\tau) = i e^{(v-\frac{\tau}{4})\pi i} + i \epsilon_{1,M} \sqrt{\frac{i}{\gamma\tau + \delta}} e^{\frac{\gamma v^2}{\gamma\tau + \delta} \pi i} P_\delta(v'|\tau').$$

*Proof.* In view of the second identity given in (6.7), one knows that  $(v + 1)' = v' - \gamma\tau' + \alpha$ . Since  $\tilde{R}(z, v + k|\tau) = (-1)^k \tilde{R}(z, v|\tau)$  for any integer  $k$ , one obtains that  $\tilde{R}(z', (v + 1)'|\tau') = (-1)^\alpha \tilde{R}(z', v' - \gamma\tau'|\tau')$ . Thus, substituting  $(v + 1)$  to  $v$  in the transformation formula (6.2) yields that, if

$$R = (-1)^{\alpha+1} \tilde{A}_M(z, v + 1|\tau) \tilde{R}(z, v' - \gamma\tau'|\tau'),$$

then:

$$(6.19) \quad \tilde{A}_M(z, v|\tau) \tilde{R}(z', v'|\tau') + \vartheta_1(z|\tau) \tilde{\Psi}_M(v|\tau) = R - \vartheta_1(z|\tau) \tilde{\Psi}_M(v + 1|\tau).$$

Furthermore, by applying (6.14), it follows that

$$(6.20) \quad R = (-1)^{\alpha+\gamma+1} \tilde{A}_M(z, v + 1|\tau) e^{(-2\gamma v' + \gamma\tau')\pi i} \left( \tilde{R}(z', v'|\tau') + P_\gamma(v'|\tau') \vartheta_1(z'|\tau') \right).$$

Since  $(\gamma + 1)(\alpha + 1) \equiv 0 \pmod{2}$ , putting (6.8) into (6.20) implies that

$$R = \tilde{A}_M(z, v|\tau) \left( \tilde{R}(z', v'|\tau') + P_\gamma(v'|\tau') \right) \vartheta_1(z'|\tau').$$

Therefore, (6.19) takes the following form:

$$\vartheta(z|\tau) (\tilde{\Psi}_M(v+1|\tau) + \tilde{\Psi}_M(v|\tau)) = \tilde{A}_M(z, v|\tau) \vartheta_1(z'|\tau') P_\gamma(v'|\tau').$$

which is clearly equivalent with (6.17), in view of (6.16).

To obtain (6.18), one can replace  $v$  with  $v - \tau$  in (6.2). Indeed, by the first identity given in (6.7), one knows that  $(v - \tau)' = v' - \delta\tau' + \beta$ . Thus, one deduces from (6.2) that

$$\tilde{R}(z, v - \tau|\tau) - \vartheta_1(z|\tau) \tilde{\Psi}(v - \tau|\tau) = (-1)^\beta \tilde{A}_M(z, v - \tau|\tau) \tilde{R}(z', v' - \delta\tau'|\tau').$$

By taking into account (6.10) and also the congruence relation  $(\beta+1)(\delta+1) \equiv 0 \pmod{2}$ , one sees that

$$(-1)^\beta \tilde{A}_M(z, v - \tau|\tau) = (-1)^{\delta+1} e^{(-2v+\tau)\pi i} \tilde{A}_M(z, v|\tau) e^{(2\delta v' - \delta^2 \tau')\pi i}.$$

Thus, by applying (6.14) to both  $\tilde{R}(z, v - \tau|\tau)$  and  $\tilde{R}(z', v' - \delta\tau'|\tau')$ , one obtains from the above that

$$\begin{aligned} & e^{(-2v+\tau)\pi i} (\tilde{R}(z, v|\tau) + P_1(v|\tau) \vartheta_1(z|\tau)) + \vartheta_1(z|\tau) \tilde{\Psi}(v - \tau|\tau) \\ &= e^{(-2v+\tau)\pi i} \tilde{A}_M(z, v|\tau) (\tilde{R}(z', v'|\tau') + P_\delta(v'|\tau') \vartheta_1(z'|\tau')). \end{aligned}$$

If one considers again (6.2), one may find that

$$\tilde{\Psi}(v|\tau) + P_1(v|\tau) + e^{(2v-\tau)\pi i} \tilde{\Psi}(v - \tau|\tau) = \tilde{A}_M(z, v|\tau) \frac{\vartheta_1(z'|\tau')}{\vartheta_1(z|\tau)} P_\delta(v'|\tau').$$

By letting  $k = 1$  into (6.11), one obtains that  $P_1(v|\tau) = -i e^{(v-\frac{\tau}{4})\pi i}$ . In this way, one arrives at the formula expected in (6.18), with the help of (6.16).  $\square$

Putting  $x = e^{2\pi i z}$  and  $u = e^{2\pi i v}$  into (5.19) implies the following identity:

$$G(e^{2\pi i v}; \rho, \zeta) = e^{2\pi i v} \Psi_M(v|\tau) + 1 = -i e^{(v+\frac{\tau}{4})\pi i} \tilde{\Psi}_M(v|\tau) + 1.$$

One can notice that (6.17) is equivalent with the monodromy relation (4.6) on  $g(x)$ . At the same times, the relation in (6.18) means that  $G(x; \rho, \zeta)$  satisfies one first order  $q$ -difference equation.

In the following, we will deal with the composite structure of the transformation formula stated in (6.2). Let  $M'$  be another modular matrix with  $M' = \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \in \text{SL}(2; \mathbb{Z})$ , write  $M'M = \begin{pmatrix} \alpha'' & \beta'' \\ \gamma'' & \delta'' \end{pmatrix}$ , and set  $\tau'' = M'\tau' = \frac{\alpha'\tau' + \beta'}{\gamma'\tau' + \delta'}$ . Beside, in accordance with (2.1), we define

$$(6.21) \quad z'' = M'z' = \frac{z'}{\gamma'\tau' + \delta'}, \quad v'' = M'v' = \frac{v'}{\gamma'\tau' + \delta'}.$$

It is well-known from the fractional transformation theory that

$$(6.22) \quad \tau'' = (M'M)\tau, \quad z'' = (M'M)z, \quad v'' = (M'M)v.$$

**Theorem 6.2.** *The following relations hold for all  $(z, v) \in \mathbb{C}^2$  and  $M' \in \text{SL}(2; \mathbb{Z})$ :*

$$(6.23) \quad \tilde{A}_{M'M}(z, v|\tau) = \tilde{A}_M(z, v|\tau) \tilde{A}_{M'}(z', v'|\tau')$$

and

$$(6.24) \quad \tilde{\Psi}_{M'M}(v|\tau) = \tilde{\Psi}_M(v|\tau) + i \epsilon_{1, M} \sqrt{\frac{i}{\gamma\tau + \delta}} e^{\frac{\gamma v^2}{\gamma\tau + \delta} \pi i} \tilde{\Psi}_{M'}(v'|\tau').$$

*Proof.* The relation stated in (6.23) can be directly deduced from the expression of  $\tilde{A}_M$  given in (6.6). Indeed, by considering (6.21), one obtains that

$$(6.25) \quad \tilde{A}_M(z, v | \tau) \tilde{A}_{M'}(z', v' | \tau') = a e^{b\pi i},$$

where  $a = \frac{1}{(\gamma\tau + \delta)(\gamma'\tau' + \delta')}$  and where  $b = \gamma'v''v' + \gamma v'v$ . On the one hand, for  $z = 1$ , since  $z' = \frac{1}{\gamma\tau + \delta}$ , it follows from the relation in (6.22) for  $z''$  that  $a = \frac{z'}{\gamma'\tau' + \delta'} = z'' = \frac{1}{\gamma''\tau + \delta''}$ . On the other hand, by (6.7), one knows that  $\frac{1}{\gamma\tau + \delta} = -\gamma\tau' + \alpha$ . Thus, one finds that, for any  $v \in \mathbb{C}$ ,

$$\gamma'v''v' + \gamma v'v = (\gamma'(-\gamma\tau' + \alpha) + \gamma(\gamma'\tau' + \delta'))v''v = \gamma''v''v.$$

In this way, one gets (6.23) from (6.25).

To prove (6.24), we shall apply transformation formula (6.2) to  $\tilde{R}(z', v' | \tau')$  with the modular matrix  $M'$ , and this implies that

$$(6.26) \quad \tilde{R}(z', v' | \tau') = \tilde{A}_{M'}(z', v' | \tau') \tilde{R}(z'', v'' | \tau'') + \vartheta_1(z' | \tau') \tilde{\Psi}_{M'}(v' | \tau').$$

If one combines (6.2) with (6.26) in the above, one finds that

$$(6.27) \quad \tilde{R}(z, v | \tau) = \tilde{A}_M(z, v | \tau) \tilde{A}_{M'}(z', v' | \tau') \tilde{R}(z'', v'' | \tau'') + \Psi,$$

where

$$\Psi = \vartheta_1(z | \tau) \tilde{\Psi}_M(v | \tau) + \tilde{A}_M(z, v | \tau) \vartheta_1(z' | \tau') \tilde{\Psi}_{M'}(v' | \tau').$$

Thus, in view of (6.23), one can deduce from (6.27) that

$$\vartheta_1(z | \tau) \tilde{\Psi}_{M'M}(v | \tau) = \vartheta_1(z | \tau) \tilde{\Psi}_M(v | \tau) + \tilde{A}_M(z, v | \tau) \vartheta_1(z' | \tau') \tilde{\Psi}_{M'}(v' | \tau'),$$

which is equivalent with (6.24); see (6.16).  $\square$

In particular, if  $M'$  is the inverse matrix  $M^{-1}$  of  $M$ , one has  $M'M = I_2$ ,  $\tau'' = \tau$ ,  $z'' = z$  and  $v'' = v$ . As  $\tilde{A}_{I_2}(z, v | \tau) = 1$  and  $\tilde{\Psi}_{I_2}(v | \tau) = 0$ , the relations in (6.23) and (6.24) imply respectively that

$$(6.28) \quad \tilde{A}_{M^{-1}}(z', v' | \tau') = \frac{1}{\tilde{A}_M(z, v | \tau)}$$

and

$$(6.29) \quad \tilde{\Psi}_{M^{-1}}(v' | \tau') = \frac{i}{\epsilon_{1,M}} \sqrt{\frac{\gamma\tau + \delta}{i}} e^{-\frac{\gamma v^2}{\gamma\tau + \delta} \pi i} \tilde{\Psi}_M(v | \tau).$$

Beside, it is easy to see the following symmetry relation for  $\tilde{A}_M(z, v | \tau)$ :

$$(6.30) \quad \tilde{A}_M(-z, v | \tau) = \tilde{A}_M(z, -v | \tau) = \tilde{A}_M(z, v | \tau).$$

At the same time, the function  $\tilde{\Psi}_M(v | \tau)$  satisfies also the following relation:

$$(6.31) \quad \tilde{\Psi}_M(-v | \tau) = \tilde{\Psi}_M(v | \tau).$$

Indeed, by considering the symmetry relation in (2.13) for  $R(z, v | \tau)$ , one obtains that  $R(-z + \frac{\tau}{2} + \frac{1}{2}, -v | \tau) = -e^{2\pi i v} R(z + \frac{\tau}{2} + \frac{1}{2}, v | \tau)$ . This is equivalent to say that  $\tilde{R}(-z, -v | \tau) = -\tilde{R}(z, v | \tau)$ . As  $\vartheta_1(-z | \tau) = -\vartheta_1(z | \tau)$ , one deduces (6.31) by applying (6.2) and (6.30).

Moreover, when one takes the derivative with respect to  $v$  at  $v = 0$  for both sides of (6.31), one finds that

$$(6.32) \quad \frac{d}{dv} \tilde{\Psi}_M(0|\tau) = 0, \quad \frac{d}{dv} \Psi_M(0|\tau) = -\pi i \Psi_M(0|\tau).$$

## 7. APPLICATIONS AND REMARKS

We shall consider the action of any modular matrix on the function  $f(z|\tau)$  given in (1.3), whose basic transformation formula was stated in [11]; see (1.1). As before, let  $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}(2; \mathbb{Z})$ . It is obvious to see that  $M$  contains one or two even integers. By considering the definition of the function  $f(z|\tau)$  in (1.3), one finds that  $f(z|\tau)$  can be expressed as follows:

$$(7.1) \quad f(z|\tau) = e^{(z-\frac{\tau}{4})\pi i} \tilde{R}(z, \frac{\tau}{2} - \frac{1}{2} - z|\tau).$$

For  $w = \frac{\tau}{2} - \frac{1}{2}$ , it follows from (6.7) that

$$(7.2) \quad w' = \frac{\gamma + \delta}{2} \tau' - \frac{\alpha + \beta}{2}.$$

**Theorem 7.1.** *If two of the entries of the matrix  $M \in \mathrm{SL}(2; \mathbb{Z})$  are even, the following transformation formula holds for the function  $f(z|\tau)$  defined in (1.3):*

$$(7.3) \quad f(z|\tau) = \frac{1}{\gamma\tau + \delta} e^{\frac{\pi i}{4}((\alpha+\beta)(\gamma+\delta)-1)} f(z'|\tau') + \Delta(z|\tau),$$

where

$$(7.4) \quad \begin{aligned} \Delta(z|\tau) = & \frac{1}{\gamma\tau + \delta} e^{(z'-\frac{\tau'}{4}+\frac{1}{4}(\alpha+\beta)(\gamma+\delta)-\frac{1}{4})\pi i} \vartheta_1(z'|\tau') P_{\frac{1-\gamma-\delta}{2}}(-z'|\tau') \\ & + e^{(z-\frac{\tau}{4})\pi i} \vartheta_1(z|\tau) \tilde{\Psi}_M(\frac{\tau}{2} - \frac{1}{2} - z|\tau). \end{aligned}$$

*Proof.* As  $\alpha\delta - \beta\gamma = 1$ , the integers  $\alpha$  and  $\beta$ , as well as the couple  $(\delta, \gamma)$ , are coprime. So, when  $M$  contains two even integers among its entries, one can find  $(k, n) \in \mathbb{Z}^2$  such that (7.2) takes the following form:

$$(7.5) \quad w' = w'_0 - k\tau', \quad w'_0 = \frac{\tau'}{2} - \frac{1}{2} + n.$$

Thus, by applying the transformation formula (6.2) to  $\tilde{R}(z, \frac{\tau}{2} - \frac{1}{2} - z|\tau)$ , one obtains from (7.1) that

$$(7.6) \quad f(z|\tau) = a \tilde{R}(z', w'_0 - z' - k\tau'|\tau') + \Psi,$$

where  $a = e^{(z-\frac{\tau}{4})\pi i} \tilde{A}_M(z, \frac{\tau}{2} - \frac{1}{2} - z|\tau)$  and  $\Psi = e^{(z-\frac{\tau}{4})\pi i} \vartheta_1(z|\tau) \tilde{\Psi}_M(\frac{\tau}{2} - \frac{1}{2} - z|\tau)$ .

Furthermore, by considering (6.14), one observes that, if  $b = e^{(2kz'+k(k-1)\tau')\pi i}$ , then:

$$\tilde{R}(z', w'_0 - z' - k\tau'|\tau') = b (\tilde{R}(z', w'_0 - z|\tau') + P_k(w'_0 - z'|\tau') \vartheta_1(z'|\tau')).$$

So, the relation in (7.6) implies that

$$f(z|\tau) = a b \tilde{R}(z', w'_0 - z|\tau') + a b P_k(w'_0 - z'|\tau') \vartheta_1(z'|\tau') + \Psi.$$

In view of (7.3), this is to say that

$$(7.7) \quad f(z|\tau) = a b e^{(-z'+\frac{\tau'}{4})\pi i} f(z'|\tau') + a b P_k(w'_0 - z'|\tau') \vartheta_1(z'|\tau') + \Psi.$$

It remains to express the product  $ab$  in (7.7). By considering (6.6), it follows that

$$\tilde{A}_M(z, \frac{\tau}{2} - \frac{1}{2} - z | \tau) = \frac{1}{\gamma\tau + \delta} e^{\frac{\gamma}{\gamma\tau + \delta}(\frac{\tau^2}{4} - \frac{\tau}{2} + \frac{1}{4} - \tau z + z)\pi i}.$$

In a similar manner as what done in (6.9), one can find that

$$\frac{\gamma}{\gamma\tau + \delta}(\frac{\tau^2}{4} - \tau z + z) = -z + (\gamma + \delta)z' + \frac{\tau}{4} - \frac{\delta^2}{4}\tau' + \frac{\beta\delta}{4}.$$

Thus, by making use of (6.7), one gets that

$$\frac{\gamma}{\gamma\tau + \delta}(\frac{\tau^2}{4} - \frac{\tau}{2} + \frac{1}{4} - \tau z + z) = -z + (\gamma + \delta)z' + \frac{\tau}{4} - \frac{(\gamma + \delta)^2}{4}\tau' + \frac{\beta\delta}{4} + \frac{\beta\gamma}{2} + \frac{\alpha\gamma}{4}.$$

By observing that  $\alpha\gamma + 2\beta\gamma + \beta\delta = (\alpha + \beta)(\gamma + \delta) - 1$ , the factor  $a$  used in (7.6) can be expressed as follows:

$$a = \frac{1}{\gamma\tau + \delta} e^{((\gamma + \delta)z' - \frac{(\gamma + \delta)^2}{4}\tau' + \frac{1}{4}(\alpha + \beta)(\gamma + \delta) - \frac{1}{4})\pi i}.$$

By comparing (7.2) with (7.5), it follows that  $2k = 1 - \gamma - \delta$ . So, one gets from the above that

$$ab = \frac{1}{\gamma\tau + \delta} e^{(z' - \frac{\tau'}{4} + \frac{1}{4}(\alpha + \beta)(\gamma + \delta) - \frac{1}{4})\pi i}.$$

In this way, one deduces immediately (7.3) and (7.4) from (7.7).  $\square$

For  $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ , one has  $(\alpha + \beta)(\gamma + \delta) = -1$ . By noticing that  $P_0 = 0$  and  $\tilde{\Psi}_M(\frac{\tau}{2} - \frac{1}{2} - z | \tau) = e^{(-z + \frac{\tau}{4})\pi i} \Psi_M(\frac{\tau}{2} - \frac{1}{2} - z | \tau)$ , the above relation (7.3) takes the following form:

$$f(z | \tau) = -\frac{i}{\tau} f(\frac{z}{\tau} | -\frac{1}{\tau}) + \vartheta_1(z | \tau) \Psi_M(\frac{\tau}{2} - \frac{1}{2} - z | \tau).$$

While replacing  $(z, \tau)$  with  $(\frac{z}{\tau}, -\frac{1}{\tau})$ , one can easily check this last relation is reduced into (1.1).

If only one entry of  $M$  is even, one can deduce from Theorem 2.1 modular-like relations between  $f(z | \tau)$  and some modified variants of  $f(z' | \tau')$  but not with this function itself. Indeed, the condition in (7.5) may be replaced with the following ones:

$$w' = (m + \frac{1}{2})\tau' + n \quad \text{or} \quad w' = m\tau' + n + \frac{1}{2}$$

for some suitable integers  $m$  and  $n$ ; see also (7.2). Note that this is a situation in some way similar to what happening for the modular relations for other theta functions than  $\vartheta_1$ ; see also (1.10).

In a forthcoming paper, we will make use of the transformation formula (2.17) for establishing a general modular relation including (1.2). For this matter, the relation in (6.32) would be helpful.

## REFERENCES

- [1] G. E. Andrews, Mordell integrals and Ramanujan's "lost" notebook, in *Analytic Number Theory*, M. I. Knopp, ed., Springer-Verlag, LNM **899** (1981), 10-48.
- [2] P. Appell, Sur les fonctions doublement périodiques de troisième espèce, *Ann. Sci. École Norm. Sup. Sér. 3* **1** (1884), 135-164.
- [3] P. Appell, Sur les fonctions doublement périodiques de troisième espèce, *Ann. Sci. École Norm. Sup. Sér. 3* **3** (1884), 9-42.
- [4] R. Bellman, A Brief Introduction to Theta Functions, Holt, Rinehart and Winston, Inc. 1961.
- [5] H. M. Edwards, Riemann's Zeta Function, Dover Publications, INC., New York, 2001.
- [6] B. Gordon and R. J. McIntosh, A survey of classical mock theta functions, in *Partitions, q-series, and modular forms*, 95-144, Dev. Math., 23, Springer, New York, 2012.
- [7] C. Hermite, Remarques sur la décomposition en éléments simples des fonctions doublement périodiques, *Ann. Fac. Sci. Toulouse, Sér. 1*, **2** (1888), 1-12
- [8] G. Landsberg, Zur Theorie der Gauss'schen Summen und der linearen Transformation der Thetafunktionen, *Journal für die reine und angewandte Mathematik* **111** (1893), 234 - 253.
- [9] M. Lerch, Bemerkungen zur Theorie der elliptischen Funktionen, *Jahrbuch über die Fortschritte der Mathematik* 24: 442-445.
- [10] M. Lerch, Essais sur le calcul du nombre des classes de formes quadratiques binaires aux coefficients entiers, *Acta Math.* **29** (1905), no. 1, 333 - 424.
- [11] L. J. Mordell, The Definite Integral  $\int_{-\infty}^{\infty} \frac{e^{ax^2+bx}}{e^{cx+d}} dx$  and the Analytic Theory of Numbers, *Acta Math.* **61** (1933), 323-360.
- [12] H. Rademacher, *Topics in Analytic Number Theory*, Springer-Verlag, 1973.
- [13] S. Ramanujan, Some definite integrals connected with Gauss's sums, *Messenger of Mathematics* **44** (1915), 75-85; reprinted in *Collected Papers of Srinivasa Ramanujan*, Chelsea (1962), 59-67.
- [14] S. Ramanujan, On certain infinite series, *Messenger of Mathematics* **45** (1916), 11-15; reprinted in *Collected Papers of Srinivasa Ramanujan*, Chelsea (1962), 129-132.
- [15] S. Ramanujan, On Some definite integrals, *J. Indian Math. Soc.* **11** (1919), 81-87; reprinted in *Collected Papers of Srinivasa Ramanujan*, Chelsea (1962), 202-207.
- [16] J.P. Ramis, J. Sauloy and C. Zhang, Local analytic classification of q-difference equations, *Astérisque* **355** (2013).
- [17] J.P. Ramis and C. Zhang, Développements asymptotiques q-Gevrey et fonction thêta de Jacobi, *C. R. Acad. Sci. Paris, Ser. I* **335** (2002), 277-280.
- [18] G.N. Watson, Generating functions of class-numbers. *Compositio Mathematica* **1** (1935), 39-68.
- [19] G.N. Watson, The final problem: An account of the mock theta functions. *J. London Math. Soc.* **11**, 55-80 (1936).
- [20] D. Zagier, Ramanujan's mock theta functions and their applications (after Zwegers and Ono-Bringmann), Séminaire Bourbaki. Vol. 2007/2008, *Astérisque* **326** (2009), 143-164.
- [21] C. Zhang, Développements asymptotiques q-Gevrey et séries Gq-sommables, *Ann. Inst. Fourier* **49** (1999), 227-261.
- [22] C. Zhang, Une sommation discrète pour des équations aux q-différences linéaires et à coefficients analytiques: théorie générale et exemples, in *Differential Equations and the Stokes Phenomenon*, Braaksma, B. L. J. and Al. Ed., World Scientific (2002), 309-329.
- [23] C. Zhang, On the mock-theta behavior of Appell-Lerch series, *C. R. Math. Acad. Sci. Paris* **353** (2015), 1067-1073.
- [24] S. Zhou, Z. Luo & C. Zhang, On summability of formal solutions to a Cauchy problem and generalization of Mordell's theorem, *C. R. Math. Acad. Sci. Paris* **348** (2010), 753-758.

LABORATOIRE P. PAINLEVÉ (UMR – CNRS 8524) & CEMPI, DÉP. MATH., FST, UNIVERSITÉ DE LILLE, CITÉ SCIENTIFIQUE, 59655 VILLENEUVE D'ASCQ CEDEX, FRANCE  
 Email address: changgui.zhang@univ-lille.fr