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Small C^1 actions of semidirect products on compact manifolds

CHRISTIAN BONATTI, SANG-HYUN KIM, THOMAS KOBERDA, AND MICHELE TRIESTINO

ABSTRACT. Let T be a compact fibered 3-manifold, presented as a mapping torus of a compact, orientable surface S with monodromy ψ , and let M be a compact Riemannian manifold. Our main result is that if the induced action ψ^* on $H^1(S, \mathbb{R})$ has no eigenvalues on the unit circle, then there exists a neighborhood \mathcal{U} of the trivial action in the space of C^1 actions of $\pi_1(T)$ on M such that any action in \mathcal{U} is abelian. We will prove that the same result holds in the generality of an infinite cyclic extension of an arbitrary finitely generated group H , provided that the conjugation action of the cyclic group on $H^1(H, \mathbb{R}) \neq 0$ has no eigenvalues of modulus one. We thus generalize a result of A. McCarthy, which addressed the case of abelian-by-cyclic groups acting on compact manifolds.

1. INTRODUCTION

In this paper, we consider smooth actions of finitely generated-by-cyclic groups on compact manifolds, motivated by the study of fibered hyperbolic 3-manifold groups. We let S be a compact, orientable surface of negative Euler characteristic, possibly with boundary. Thus, $\pi_1(S)$ is either a finitely generated free group, or the fundamental group of a closed surface of genus g for some $g \geq 2$. If $\phi \in \text{Homeo}(S)$ is a (possibly orientation reversing) homeomorphism, then we may form $T = T_\phi$, the mapping torus of ϕ . We have that $\pi_1(T)$ fits into a short exact sequence of the form

$$1 \rightarrow \pi_1(S) \rightarrow \pi_1(T) \rightarrow \mathbb{Z} \rightarrow 1,$$

where the conjugation action of \mathbb{Z} on $\pi_1(S)$ is by the induced action of ϕ . It is well-known that, up to an inner automorphism of $\pi_1(S)$, this action depends only on the homotopy class of ϕ , and is therefore an invariant of the (extended) mapping class of ϕ . It follows that the isomorphism type of $\pi_1(T)$ depends only on the mapping class of ϕ .

Fibered 3-manifold groups fall into a much larger class of groups which we will be able to investigate with our methods. Here and throughout, we will let M be a compact Riemannian manifold. Recall a short exact sequence of finitely generated groups

$$1 \rightarrow H \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1$$

naturally determines $\psi \in \text{Out}(H)$, and hence, induces a unique automorphism ψ^* of $H^1(H, \mathbb{R})$. Abstractly as groups, we have that

$$G \cong H \rtimes_\psi \mathbb{Z},$$

and the outer automorphism ψ is given by the conjugation action of $\mathbb{Z} \cong G/H$ on H . The map ψ^* is said to be *hyperbolic* if every eigenvalue of ψ^* has modulus different from one. We will study $\text{Hom}(G, \text{Diff}^1(M))$, the space of C^1 actions on M .

1.1. Main result. We will use the symbol 1 to mean the identity map, the trivial group, the identity group element or the real number 1 depending on the context, as it will not cause confusion. The principal result of this paper is the following:

Theorem 1.1. *Suppose we have a short exact sequence of finitely generated groups*

$$1 \rightarrow H \rightarrow G \rightarrow \mathbb{Z} \rightarrow 1,$$

which induces a nontrivial hyperbolic automorphism ψ^ of $H^1(H, \mathbb{R})$. Then there exists a neighborhood $\mathcal{U} \subseteq \text{Hom}(G, \text{Diff}^1(M))$ of the trivial representation such that $\rho(H) = 1$ for all $\rho \in \mathcal{U}$.*

Thus, sufficiently small actions of G on compact manifolds necessarily factor through cyclic groups, provided the monodromy maps are hyperbolic on cohomology. The reader is directed to Subsection 2.1 for a discussion of the topology on $\text{Hom}(G, \text{Diff}^1(M))$. Theorem 1.1 may be viewed as an analogue of a result of A. McCarthy [36], who proved a statement with the same conclusion for abelian-by-cyclic groups.

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For certain manifolds and with certain natural hypotheses, abelian-by-cyclic group actions by diffeomorphisms enjoy rather strong rigidity properties [27].

Note that if H is a left-orderable group then it is not difficult to find a faithful action of G by homeomorphisms of $[0, 1]$ which is arbitrarily close to the identity, so the C^1 regularity assumption in Theorem 1.1 is essential.

By applying Theorem 1.1 to the above short exact sequence for a fibered 3-manifold group, we obtain the following result.

Corollary 1.2. *Let S, ϕ , and T be as above. If ϕ induces a hyperbolic automorphism of $H^1(S, \mathbb{R})$, then there exists a neighborhood \mathcal{U} of the trivial representation in $\text{Hom}(\pi_1(T), \text{Diff}^1(M))$ such that $\rho(H) = 1$ for all $\rho \in \mathcal{U}$.*

The hypotheses in Theorem 1.1 may be contrasted with the following result of Bonatti–Rezaei [7], which generalizes some work of Farb–Franks [19] and Jorquera [28].

Theorem 1.3 (Bonatti–Rezaei). *Every finitely generated, residually torsion-free nilpotent group G admits a faithful representation $\rho: G \rightarrow \text{Diff}^1([0, 1])$ that is C^1 -close to the identity.*

Here, a group is *residually torsion-free nilpotent* if every nontrivial element $g \in G$ survives in a torsion-free nilpotent quotient of G . An element $\rho \in \text{Hom}(G, \text{Diff}^1(M))$ is said to be *C^1 -close to the identity* if for every $\epsilon > 0$, there is an element $h = h_\epsilon \in \text{Diff}^1(M)$ such that h conjugates ρ into an ϵ -neighborhood of the trivial representation of G , in the C^1 -topology on $\text{Hom}(G, \text{Diff}^1(M))$.

It is not difficult to check that if G satisfies the hypotheses of Theorem 1.1 then the only torsion-free nilpotent quotient admitted by G is $\mathbb{Z} = G/H$. Thus, the hyperbolicity of the map ψ^* plays a crucial role in the dynamics of the group G .

1.2. Unipotent monodromy maps and virtually special groups. An essential feature of Theorem 1.1 is its “unstable” nature, in the sense that it does not remain true after passing to finite index subgroups of G . Indeed, we have the following fairly easy fact:

Proposition 1.4. *Let N be a hyperbolic 3-manifold with finite volume. Then a finite index subgroup of $\pi_1(N)$ admits a faithful representation ρ into $\text{Diff}^1(M)$ such that ρ is C^1 -close to the identity.*

Proof. By the work of Agol and Wise [1, 44], there is a finite index subgroup $G_0 < G$ such that G_0 is *special*. In particular, G_0 embeds in a right-angled Artin group. These groups are always residually torsion-free nilpotent [17]. Thus, the proposition follows from Theorem 1.3. \square

Thus if G is a group satisfying the hypotheses of Theorem 1.1, then passing to a finite index subgroup G_0 , one often obtains a group satisfying the hypotheses of Theorem 1.3. In such a case, one can build a C^1 action of G on n copies of $[0, 1]$, where here $n = [G : G_0]$, by an analogue of the induced representation of a finite index subgroup. Such an action will permute the components of this manifold transitively. This does not contradict Theorem 1.1, since any such action will be outside of a fixed neighborhood of the trivial representation of G .

One can produce many fibered 3-manifold groups, even hyperbolic ones, which are residually torsion-free nilpotent, without using the deep results of Agol and Wise. Indeed, it suffices to use monodromy maps ϕ such that ϕ^* is unipotent (i.e. has all eigenvalues equal to one). In this case, the resulting G will always be residually torsion-free nilpotent [32]. In fact, a semidirect product of \mathbb{Z} with a finitely generated, residually torsion-free nilpotent group H will again be residually torsion-free nilpotent if the \mathbb{Z} -action on $H^1(H, \mathbb{R})$ is unipotent.

It is not true that if G is residually torsion-free nilpotent then ψ^* is unipotent. Indeed, considering a fibered hyperbolic 3-manifold group $\pi_1(T)$ satisfying the hypotheses of Theorem 1.1 and passing to a finite index subgroup which is special, we can obtain a new mapping torus structure on a finite cover T_0 of T with monodromy ϕ_0 , and with a fiber S_0 which covers S . The action of ϕ_0^* on $H^1(S_0, \mathbb{R})$ will not be unipotent, since $H^1(S, \mathbb{R})$ will naturally sit inside $H^1(S_0, \mathbb{R})$ via pullback and will be invariant under ϕ_0^* .

The regularity assumption in Theorem 1.3 is subtle. Nonabelian nilpotent groups cannot admit faithful C^2 actions on any compact one-manifold [40]. Right-angled Artin groups and specialness do not provide any help in producing higher regularity actions, since in dimension one they almost never admit faithful C^2 actions on compact manifolds [3, 31]. The compactness of the manifold acted upon here is also essential; see [2].

1.3. General group actions on compact manifolds. A robust trend in the theory of group actions on manifolds is that “large” groups should not act on “small” manifolds. Among the striking results in this area are the facts that irreducible lattices in higher rank semisimple Lie groups do not admit

infinite image C^1 actions (and often even C^0 actions) on compact 1-manifolds [12, 24, 45]. For higher dimensional manifolds, the work of Brown–Fisher–Hurtado shows that for $n \geq 3$, groups commensurable with $\mathrm{SL}_n(\mathbb{Z})$ do not admit faithful C^1 actions on m -dimensional compact manifolds for $m < n - 2$, and for $m < n - 1$ if the actions preserve a volume form [11]. They obtain similar results for cocompact lattices in $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{Sp}_{2n}(\mathbb{R})$, $\mathrm{SO}(n, n)$, and $\mathrm{SO}(n, n + 1)$ [10].

Lattices in rank one Lie groups often do admit faithful smooth actions on compact one manifolds. By [4], many arithmetic lattices in $\mathrm{SO}(n, 1)$ are virtually special, which by virtue of Proposition 1.4 furnishes many faithful C^1 actions by such lattices.

McCarthy’s result [36] furnishes a class of solvable groups which admit no small C^1 actions on compact manifolds whatsoever. Topologically, her groups arise as fundamental groups of torus bundles over the circle, with no restrictions on the dimension. Our main result identifies a larger class of such groups, including ones within the much more dimensionally restricted and algebraically different class of compact 3-manifolds groups. For fibered 3-manifolds groups acting without at least some smallness assumptions, we can only make much weaker statements:

Proposition 1.5. *If T is a hyperbolic fibered 3-manifold, then the universal circle action of $\pi_1(T)$ on S^1 is not topologically conjugate to a C^3 action.*

Proposition 1.5 follows immediately from the work of Miyoshi [37]. We will deduce Proposition 1.5 from a stronger fact (Proposition 4.3) in Section 4 for the convenience of the reader.

There is no hope of establishing a result as sweeping as the Brown–Fisher–Hurtado resolution of many cases of the Zimmer Conjecture for 3-manifold groups acting on the circle, even with maximal regularity assumptions:

Proposition 1.6 (e.g. [13]). *There exist finite volume hyperbolic 3-manifold subgroups of $\mathrm{PSL}_2(\mathbb{R})$.*

Any such groups act by projective (and hence analytic) diffeomorphisms on S^1 . We remark that Proposition 1.6 seems well-known to experts.

1.4. Uniqueness of the presentation of G . We remark briefly that if $G = \pi_1(T)$ satisfies the hypotheses of Corollary 1.2 then there is an essentially unique homomorphism $G \rightarrow \mathbb{Z}$ whose kernel is isomorphic to a finitely generated group, and in particular the fibered 3-manifold structure on T is unique (see [43, 41]). Thus, the induced map ψ^* is canonically defined, and one may therefore speak of *the* monodromy action. For fibered 3-manifold groups with $b_1 > 1$ this is no longer the case.

2. PRELIMINARIES

In this section, we gather the tools we will need to establish the principal result of this paper.

2.1. The space of C^1 actions of G . Recall that in our notation, M denotes a fixed compact Riemannian manifold. We denote by $\mathrm{Diff}^1(M)$ the group of C^1 -diffeomorphisms of M . For $f \in \mathrm{Diff}^1(M)$, we will write

$$D_x f: T_x M \rightarrow T_{f(x)} M$$

for the Jacobian of f .

It will be convenient for us to assume that M is C^1 -isometrically embedded in a Euclidean space \mathbb{R}^N for some $N \gg 0$. This is always possible, by the Nash Embedding Theorem [38]. For brevity, we let $\|X\|$ denote the ℓ^∞ norm when X is a function, a vector, a matrix or a tensor. We may replace distances in M by distances in \mathbb{R}^N , and the Jacobian acquires the ℓ^∞ -norm $\|D_x f\|$ as a multi-linear map. Note that if $V \cong \mathbb{R}^N$ is a vector space equipped with the ℓ^∞ norm and $T \in \mathrm{End}(V)$, then we have the estimate

$$\|Tv\| \leq N\|T\|\|v\|$$

for all $v \in V$. We will make use of this estimate in the sequel.

We define the C^1 -metric on $\mathrm{Diff}^1(M)$ by

$$d(f, g) = \|f - g\| + \sup_{x \in M} \|D_x f - D_x g\|,$$

where all these distances and norms are now interpreted in the ambient Euclidean space.

If G is generated by a finite set S , then we may define a metric d_S on $\mathrm{Hom}(G, \mathrm{Diff}^1(M))$ via

$$d_S(\rho, \rho') = \max_{s \in S} d(\rho(s), \rho'(s)).$$

This metric d_S determines the C^1 -topology of $\mathrm{Hom}(G, \mathrm{Diff}^1(M))$, and this topology is independent of the choice of the generating set S .

For an arbitrary group G , we will write $\rho_0 \in \text{Hom}(G, \text{Diff}^1(M))$ for the trivial representation of G . We see that in order to prove Theorem 1.1, it suffices to find some $\epsilon > 0$ such that every $\rho \in \text{Hom}(G, \text{Diff}^1(M))$ satisfying $d_S(\rho, \rho_0) < \epsilon$ maps H to the identity 1.

2.2. Hyperbolic monodromies. Here, we recall some basic facts from linear algebra of hyperbolic automorphisms of a real vector space. Let V be a d -dimensional vector space over \mathbb{R} , and let $\|\cdot\|_d$ be a fixed norm on V . If $A \in \text{GL}(V)$, we say that A is *hyperbolic* if every eigenvalue of A has modulus different from one.

Lemma 2.1. *Let $A \in \text{GL}(V)$ be hyperbolic. Then there is an A -invariant splitting $V = E^- \oplus E^+$ and a positive integer p_0 such that the following conclusions hold for all $p \geq p_0$:*

(1) if $v \in E^-$ then

$$\|A^p v\|_d \leq \frac{1}{2} \|v\|_d;$$

(2) if $v \in E^+$ then

$$\|A^p v\|_d \geq 2 \|v\|_d.$$

We omit the proof of the lemma, which is well-known; see [29, Chapter 1] for instance. As is standard from dynamics, E^- and E^+ are the *stable* and *unstable* subspaces of V associated to A . In the sequel, we will use the notation π_+ and π_- to denote projections $V \rightarrow E^+$ and $V \rightarrow E^-$ with kernels E^- and E^+ respectively. Observe that invariance of the splitting implies that A commutes with each projection π_+ and π_- .

2.3. Approximate linearization. A fundamental tool for proving Theorem 1.1 is the following result of Bonatti [5, 6], which arose as an interpretation of Thurston Stability [42], and which we refer to as *approximate linearization*.

Lemma 2.2. *Let M be a compact manifold, let $\eta > 0$, and let $k \in \mathbb{N}$. Then there exists a neighborhood of the identity $\mathcal{U} \subseteq \text{Diff}^1(M)$ such that for all $x \in M$, for all $f_1, \dots, f_k \in \mathcal{U}$ and for all $\epsilon_1, \dots, \epsilon_k \in \{-1, 1\}$, we have the following:*

$$\left\| f_k^{\epsilon_k} \circ \dots \circ f_1^{\epsilon_1}(x) - x - \sum_{i=1}^k \epsilon_i (f_i(x) - x) \right\| \leq \eta \max_{i=1, \dots, k} \|f_i(x) - x\|.$$

Throughout the rest of this paper, we will often suppress the notation $\rho \in \text{Hom}(G, \text{Diff}^1(M))$ and just write $g_x = g(x) = \rho(g)(x)$ for $g \in G$ and $x \in M$. We define a *displacement vector* for g at x as

$$\Delta_x^\rho(g) := \rho(g)(x) - x,$$

regarded as an N -dimensional row vector. More generally, if $B = \{b_1, \dots, b_n\} \subseteq G$ is a finite set then we define an $n \times N$ matrix

$$\Delta_x^\rho(B) := (\Delta_x^\rho(b_i))_{1 \leq i \leq n}.$$

We often write Δ_x for Δ_x^ρ when the meaning is clear. Then the above lemma asserts that

$$\left\| \Delta_x(g_k^{\epsilon_k} \circ \dots \circ g_1^{\epsilon_1}) - \sum_{i=1}^k \epsilon_i \Delta_x(g_i) \right\| \leq \eta \|\Delta_x(\{g_1, \dots, g_k\})\|,$$

in the case when $g_i \in G$ and $\rho(g_i) \in \mathcal{U}$.

2.4. First homology and cohomology groups. We briefly recall for the reader unfamiliar with group homology that the first homology group of a group H is given by the abelianization

$$H_1(H, \mathbb{Z}) = H/[H, H].$$

When $R \in \{\mathbb{Z}, \mathbb{R}\}$, the first cohomology group $H^1(H, R)$ coincides with the abelian group of homomorphisms from H to R . In particular, $H^1(H, \mathbb{Z})$ is a free abelian group of the same rank as $H_1(H, \mathbb{Z})$.

3. PROOF OF THEOREM 1.1

We are now ready to give a proof of Theorem 1.1. For this, we will fix $\psi \in \text{Aut}(H)$ such that G can be written as

$$G = \langle H, t \mid tht^{-1} = \psi(h) \text{ for all } h \in H \rangle.$$

3.1. Reducing to homologically independent generators. We first establish Lemma 3.1 below, which will say that we may more or less assume that H is finitely generated and free abelian.

Let $d \geq 1$ be the rank of $H^1(H, \mathbb{Z})$. We can find a finite generating set

$$S = S_0 \sqcup S_1$$

of H such that all of the following hold.

- The image of S_0 in $H_1(H, \mathbb{Z}) = H/[H, H]$ is a basis for the free part.
- The image of each element in S_1 is torsion or trivial in $H_1(H, \mathbb{Z})$.

We pick $K \geq 2$ so that $\tau^K = 0$ for all

$$\tau \in \ker\{H_1(H, \mathbb{Z}) \rightarrow H_1(H, \mathbb{R}) = H_1(H, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}\},$$

where here the map between the homology groups is the tensoring map. We enumerate $S_0 = \{s_1, s_2, \dots, s_d\}$, and regard S_0 as an ordered set. Let $A := (\alpha_{ij})$ be the matrix of the hyperbolic linear map

$$\psi^*: H^1(H, \mathbb{Z}) \rightarrow H^1(H, \mathbb{Z})$$

with respect to the basis which is dual to S_0 , viewed as real homology classes. The action ψ_* on $H_1(H, \mathbb{Z})$ is then given by the transpose (α_{ji}) . In this case, we can write each $\psi(s_j)$ as

$$(3.1) \quad \psi(s_j) = ts_j t^{-1} = \prod_{i=1}^d s_i^{\alpha_{ji}} \tau_j$$

for some $\tau_j \in H$ such that $\tau_j^K \in [H, H]$. It will be convenient for us to define a set

$$S' := \{u^K : u \in S_1 \cup \{\tau_1, \dots, \tau_d\}\} \subseteq [H, H].$$

Observe that each $h \in [H, H]$ can be expressed as a product of commutators in S . It follows that h can be expressed as a *balanced* word in S , which is to say that all generators in S occur with exponent sum zero. Since $S' \subseteq [H, H]$, we can find an integer $k_0 \geq K$ such that every element in S' is a balanced word of length at most k_0 in S . Recall our convention $\|A\| := \max_{i,j} |\alpha_{ij}|$. We set

$$(3.2) \quad k := k_0 + d\|A\|.$$

Lemma 3.1. *Let $0 < \eta < 1$. Then there exists a neighborhood $\mathcal{U} \subseteq \text{Hom}(G, \text{Diff}^1(M))$ of ρ_0 such that each of the following relations hold for all $\rho \in \mathcal{U}$ and $x \in M$.*

- (1) $\|\Delta_x^\rho(S')\| \leq \eta \|\Delta_x^\rho(S)\|$;
- (2) $\|\Delta_x^\rho(S_1 \cup \{\tau_1, \dots, \tau_d\})\| \leq \eta \|\Delta_x^\rho(S)\|$;
- (3) $\|\Delta_x^\rho(S)\| = \|\Delta_x^\rho(S_0)\|$;
- (4) $\|\Delta_x^\rho(\psi(S_0)) - A\Delta_x^\rho(S_0)\| \leq 2\eta \|\Delta_x^\rho(S_0)\|$.

Proof. Let k be defined as in (3.2). We have an identity neighborhood $\mathcal{V} \subseteq \text{Diff}^1(M)$ furnished by Lemma 2.2 for η and k . We define \mathcal{U} by

$$\mathcal{U} = \{\rho \in \text{Hom}(G, \text{Diff}^1(M)) : \rho(S \cup \{\tau_1, \dots, \tau_d\}) \subseteq \mathcal{V}\}.$$

We now fix $\rho \in \mathcal{U}$, and we suppress ρ from the notation by writing $g(x) := \rho(g)(x)$. Similarly, we write $\Delta_x(g) := \Delta_x^\rho(g)$, and we propagate this notation functorially in g and in x .

(1) Let $u \in S'$, so that u can be expressed as a balanced word in S with length at most $k_0 < k$. We see from Lemma 2.2 that

$$\|\Delta_x(u)\| \leq \eta \|\Delta_x(S)\|.$$

This proves part (1).

(2) Let $u \in S_1 \cup \{\tau_1, \dots, \tau_d\}$. Since $u \in \mathcal{V}$ by assumption, we again use Lemma 2.2 to see that

$$\|\Delta_x(u^K) - K\Delta_x(u)\| \leq \eta \|\Delta_x(u)\|.$$

Using the triangle inequality and part (1), we see that

$$K\|\Delta_x(u)\| \leq \|\Delta_x(u^K)\| + \eta \|\Delta_x(u)\| \leq \eta \|\Delta_x(S)\| + \eta \|\Delta_x(u)\|.$$

Since $K \geq 2$, we obtain the desired conclusion as

$$\|\Delta_x(u)\| \leq \frac{\eta}{K - \eta} \|\Delta_x(S)\| \leq \eta \|\Delta_x(S)\|.$$

Part (3) is obvious from the previous parts. For part (4), let us pick an arbitrary $s_j \in S_0$. From the expression (3.1) for $\psi(s_j) = ts_j t^{-1}$ and from Lemma 2.2, we can deduce that

$$\left\| \Delta_x(\psi(s_j)) - \sum_{i=1}^d \alpha_{ji} \Delta_x(s_i) - \Delta_x(\tau_j) \right\| \leq \eta \|\Delta_x(S \cup \{\tau_j\})\| = \eta \|\Delta_x(S_0)\|.$$

The triangle inequality and the second and third parts of the lemma imply the conclusion of part (4). \square

3.2. McCarthy's Lemma. Retaining previous notation, we have a group G presented as $H \rtimes_{\psi} \langle t \rangle$. Another ingredient for the proof of the main theorem is the following lemma, which was proved by McCarthy [36, Lemmas 4.1 and 4.2] in the case when H is abelian:

Lemma 3.2 (cf. Lemmas 4.1 and 4.2 of [36]). *For all $\eta \in (0, 1/3)$, there exists a neighborhood $\mathcal{U} \subseteq \text{Hom}(G, \text{Diff}^1(M))$ of the trivial representation ρ_0 such that whenever $\rho \in \mathcal{U}$ and $x \in M$ we have*

$$\left\| \Delta_{\rho(t^{-1}(x))}^{\rho}(S_0) - A\Delta_x^{\rho}(S_0) \right\| \leq \eta \|\Delta_x^{\rho}(S_0)\|.$$

Roughly speaking, under the above hypothesis if we denote the displacement matrix of S_0 at x as v , then Av will be near from the displacement matrix of S_0 at $t^{-1}x$. Thus, one can apply hyperbolic dynamics to estimate the change of displacement matrices as points are moved under iterations of t :

$$x \mapsto t^{-1}x \mapsto t^{-2}x \mapsto \dots \mapsto t^{-n}x \mapsto \dots$$

Since McCarthy's arguments concerned the case where H abelian and hence do not apply in this situation, let us reproduce proofs here which work for general groups.

Proof of Lemma 3.2. Fix $\eta' \in (0, \eta)$, which will be nailed down later. We pick a sufficiently small neighborhood $\mathcal{U} \subseteq \text{Hom}(G, \text{Diff}^1(M))$ of ρ_0 , which is at least as small as the set \mathcal{U} in Lemma 3.1 for this choice of η' . We let $\rho \in \mathcal{U}$, and again suppress the notation ρ in expressions. We also fix $x \in M$, and set $y := t^{-1}x$.

Suppose we have $s \in S_0$. From the definition of the derivative, we have that

$$\Delta_x(\psi(s)) = \Delta_{ty}(tst^{-1}) = ts(y) - t(y) = D_y t(\Delta_y(s)) + o(\|\Delta_y(s)\|).$$

Replacing \mathcal{U} by a smaller neighborhood if necessary, we may assume that (with a slight abuse of notation)

$$o(\|\Delta_y(s)\|) < \eta' \|\Delta_y(s)\|$$

in norm, and that $N\|D_x t - 1\| \leq \eta'$, where here 1 denotes the identity map, and N is the dimension of the Euclidean space where M is embedded. It then follows that

$$(3.3) \quad \|\Delta_x(\psi(s)) - \Delta_y(s)\| \leq N\|D_x t - 1\| \cdot \|\Delta_y(s)\| + o(\|\Delta_y(s)\|) \leq 2\eta' \|\Delta_y(s)\|.$$

Here, we are using the ℓ^∞ norm estimate

$$\|Tv\| \leq N\|T\|\|v\|$$

for arbitrary vectors v and linear maps $T: \mathbb{R}^N \rightarrow \mathbb{R}^N$.

Applying the triangle inequality, Lemma 3.1 (4) and (3.3), we deduce that

$$(3.4) \quad \|\Delta_y(S_0) - A\Delta_x(S_0)\| \leq 2\eta' \|\Delta_y(S_0)\| + \|\Delta_x(\psi(S_0)) - A\Delta_x(S_0)\| \leq 2\eta' (\|\Delta_y(S_0)\| + \|\Delta_x(S_0)\|).$$

From the inequality (3.4), we note that

$$(3.5) \quad (1 - 2\eta') \|\Delta_y(S_0)\| \leq \|A\Delta_x(S_0)\| + 2\eta' \|\Delta_x(S_0)\| \leq (d\|A\| + 2\eta') \|\Delta_x(S_0)\|.$$

We will now choose $\eta' \in (0, \eta)$ sufficiently small so that

$$\left(\frac{d\|A\| + 2\eta'}{1 - 2\eta'} + 1 \right) \cdot 2\eta' \leq \left(\frac{d\|A\| + 2/3}{1/3} + 1 \right) \cdot 2\eta' \leq \eta.$$

Combining inequalities (3.4) and (3.5) we obtain the desired conclusion as

$$\|\Delta_y(S_0) - A\Delta_x(S_0)\| \leq 2\eta' \left(\frac{d\|A\| + 2\eta'}{1 - 2\eta'} + 1 \right) \|\Delta_x(S_0)\| \leq \eta \|\Delta_x(S_0)\|. \quad \square$$

3.3. Finishing the proof. We can now complete the proof of the main result.

Proof of Theorem 1.1. Let ρ be sufficiently near from ρ_0 . By Lemma 3.1 (3), it suffices for us to prove that the $d \times N$ matrix $\Delta_x(S_0)$ is equal to 0 for all $x \in M$.

The hyperbolic automorphism ψ^* on $H^1(H, \mathbb{Z})$ induces an invariant splitting

$$\mathbb{R}^d = \bigoplus_{i=1}^d \mathbb{R}s_i = E^+ \oplus E^-,$$

as in Lemma 2.1. We may assume $p_0 = 1$ in that lemma after replacing ψ^* by a sufficiently large power; this is the same as passing to the kernel of the natural map $G \rightarrow \mathbb{Z}/p\mathbb{Z}$ given by reducing G/H modulo p .

Let us pick $x \in M$ such that the quantity

$$\max(\|\pi_+ \Delta_z(S_0)\|, \|\pi_- \Delta_z(S_0)\|)$$

is itself maximized at $z = x$. Here, π_{\pm} is regarded as a map from $\bigoplus_{i=1}^N \mathbb{R}^d$ to $\bigoplus_{i=1}^N E^{\pm}$.

For a proof by contradiction, we will suppose that this maximum is nonzero. We may further assume the maximum occurs for the unstable direction. Since the stable and unstable subspaces of a hyperbolic matrix are symmetric under inversion, the case where the maximum is in the stable direction is analogous.

Let us choose $\eta \in (0, 1/3)$ and an identity neighborhood $\mathcal{U} \subseteq \text{Hom}(G, \text{Diff}^1(M))$ so that the conclusion of Lemma 3.2 holds for $\rho \in \mathcal{U}$. With this choice, using also the contraction property of π_+ , we estimate

$$\begin{aligned} \|\pi_+ \Delta_{t^{-1}x}(S_0) - \pi_+ A \Delta_x(S_0)\| &\leq \|\Delta_{t^{-1}x}(S_0) - A \Delta_x(S_0)\| \leq \eta \|\Delta_x(S_0)\| \\ &\leq \eta (\|\pi_+ \Delta_x(S_0)\| + \|\pi_- \Delta_x(S_0)\|) \leq 2\eta \|\pi_+ \Delta_x(S_0)\|. \end{aligned}$$

On the other hand, applying the triangle inequality and Lemma 2.1 (2) we have

$$\|\pi_+ \Delta_{t^{-1}x}(S_0) - A \pi_+ \Delta_x(S_0)\| \geq \|A \pi_+ \Delta_x(S_0)\| - \|\pi_+ \Delta_{t^{-1}x}(S_0)\| \geq 2\|\pi_+ \Delta_x(S_0)\| - \|\pi_+ \Delta_{t^{-1}x}(S_0)\|.$$

Combining the above chains of inequalities, and using that $A \pi_+ = \pi_+ A$, we obtain

$$\|\pi_+ \Delta_{t^{-1}x}(S_0)\| \geq 2(1 - \eta) \|\pi_+ \Delta_x(S_0)\| > \|\pi_+ \Delta_x(S_0)\|.$$

This contradicts the maximality of our choices. \square

4. GENERAL GROUP ACTIONS AND QUESTIONS

As remarked in the introduction, there is no hope of ruling out highly regular faithful actions of 3-manifold groups on low dimensional manifolds. Thus, Theorem 1.1 can be viewed as a local rigidity phenomenon of $\text{Hom}(G, \text{Diff}^1(M))$ near ρ_0 rather than as a global statement about this space of actions. In this section we discuss actions of 3-manifold groups on the circle which are not small, and thus are much less constrained.

4.1. Universal circle actions. First, we show that for certain types of faithful actions of 3-manifold groups, some regularity constraints persist. Let T be a fibered 3-manifold with closed, orientable fiber S and monodromy $\psi \in \text{Mod}(S, p)$. We assume that $\chi(S) < 0$. Here, we have equipped S with a basepoint p , and we assume that elements of $\text{Mod}(S, p)$ preserve p , as do isotopies between them.

We have that $\pi_1(S)$ naturally sits in $\text{Mod}(S, p)$ as the kernel of the homomorphism $\text{Mod}(S, p) \rightarrow \text{Mod}(S)$ which forgets the basepoint p [20]. The short exact sequence

$$1 \rightarrow \pi_1(S) \rightarrow \text{Mod}(S, p) \rightarrow \text{Mod}(S) \rightarrow 1$$

is known as the Birman Exact Sequence. The mapping class group $\text{Mod}(S, p)$ has a natural faithful action on S^1 by homeomorphisms, known as Nielsen's action (see [15]). This action of $\text{Mod}(S, p)$ is not conjugate to a C^1 action, and even after passing to finite index subgroups it is known not to be conjugate to a C^2 action [18, 3, 31, 39, 34]. Moreover, this action is not absolutely continuous, as can be easily seen from Proposition 4.1 below. However, one can topologically conjugate Nielsen's action to a bi-Lipschitz one; this is a general fact for countable groups acting on the circle [16]. We remark that Nielsen's action, as it is constructed by extensions of quasi-isometries of \mathbb{H}^2 to S^1 , enjoys a regularity property known as quasi-symmetry. See [15, 25, 21].

If $\psi \in \text{Mod}(S, p)$ then the conjugation action of ψ on the group

$$\pi_1(S) = \ker \{\text{Mod}(S, p) \rightarrow \text{Mod}(S)\}$$

makes the group $\langle \psi, \pi_1(S) \rangle$ isomorphic to $\pi_1(T)$. We thus obtain an action is called the *universal circle action* of $\pi_1(T)$ (see [14]). While it follows that $\pi_1(T)$ admits a natural faithful action on S^1 by absolutely continuous homeomorphisms, the higher regularity properties of this action are somewhat mysterious.

We now give a proof of Proposition 1.5, which asserts that this action is not topologically conjugate to a C^3 action. As stated in the introduction, this result is known from the work of Miyoshi. The proof of Proposition 1.5 given in [37] follows similar lines to the argument given here. Proposition 1.5 is easily implied by the following two results:

Proposition 4.1. *Let S be a closed surface and $\rho : \pi_1(S) \rightarrow \text{PSL}_2(\mathbb{R})$ be a faithful discrete representation. Then the normalizer of $\rho(\pi_1(S))$ in $\text{Homeo}^{\text{ac}}(S^1)$ is a discrete subgroup of $\text{PSL}_2(\mathbb{R})$ which contains $\rho(\pi_1(S))$ as a finite-index subgroup.*

Proof. Let g be an absolutely continuous homeomorphism of the circle which normalizes $\rho(\pi_1(S))$. Then by an argument of Sullivan (see [22, Prop. III.4.1]), we see that g is actually contained in $\text{PSL}_2(\mathbb{R})$. Now, it follows from standard facts about Zariski dense subgroups of simple Lie groups that the normalizer of a Fuchsian group in $\text{PSL}_2(\mathbb{R})$ is necessarily Fuchsian [30], and contains the original Fuchsian group with finite index. \square

Proposition 4.1 in fact implies that every pseudo-Anosov homeomorphism of $\text{Mod}(S, p)$, other than those arising from the copy of $\pi_1(S)$ in the Birman Exact Sequence, fails to act by an absolutely continuous homeomorphism on S^1 under Nielsen's action.

The following result is known as Ghys' differentiable rigidity of Fuchsian actions [23].

Theorem 4.2. *Let S be a closed surface and let $\rho : \pi_1(S) \rightarrow \text{Diff}^r(S^1)$ for $r \geq 3$ be a representation which is topologically conjugate to a Fuchsian subgroup of $\text{PSL}_2(\mathbb{R})$. Then ρ is conjugate to a Fuchsian subgroup of $\text{PSL}_2(\mathbb{R})$ by a C^r diffeomorphism.*

Proposition 1.5 is an immediate consequence of the following, which in turn is an obvious corollary of Proposition 4.1 and Theorem 4.2.

Proposition 4.3. *Let T be a hyperbolic fibered 3-manifold with a fiber S . If an action*

$$\rho : \pi_1(T) = \pi_1(S) \rtimes \langle t \rangle \rightarrow \text{Homeo}^+(S^1)$$

satisfies that $\rho(\pi_1(S))$ is topologically conjugate to a Fuchsian action, then either $\rho(\pi_1(S)) \not\leq \text{Diff}_+^3(S^1)$ or $\rho(t)$ is not absolutely continuous.

We remark that universal circle actions enjoy a strong C^0 rigidity property, namely that actions in the same connected component of the representation variety of $\pi_1(T) \rightarrow \text{Homeo}^+(S^1)$ are semi-conjugate to the standard action [8]. In that paper, the precise notion of equivalence is “weak conjugacy”, and not semi-conjugacy.

4.2. Analytic actions. Finally, we discuss faithful analytic actions of fibered 3-manifold groups on S^1 . By Agol's resolution of the virtual fibering conjecture [1], we have that every hyperbolic 3-manifold virtually fibers over the circle. Thus, if $\Gamma < \text{PSL}_2(\mathbb{C})$ is discrete (i.e. a Kleinian group) with finite covolume, then Γ has a finite index subgroup which is $\pi_1(T)$ for some fibered 3-manifold T . Now, if the matrix entries of Γ are contained in a number field $K \supset \mathbb{Q}$ such that K has a real place (i.e. a Galois embedding $\sigma : K \rightarrow \mathbb{C}$ such that $\sigma(K) \subseteq \mathbb{R}$), then Γ can be identified with a subgroup of $\text{PSL}_2(\mathbb{R})$.

Therefore, in order to establish Proposition 1.6, it suffices to produce such a Kleinian group. If Γ has matrix entries in a field K of odd degree over \mathbb{Q} then K has at least one real place, since the number of complex places is even. Many such arithmetic Kleinian groups of finite covolume exist; see Section 13.7 of [33], for example.

4.3. Questions. There are several natural questions which arise from the discussion in this paper.

Question 4.4 (J. Souto). *Let T be a fibered 3-manifold and let $G = \pi_1(T)$. Is there a finite index subgroup $G_0 < G$ such that $G_0 < \text{Diff}^2(I)$? What about $G_0 < \text{Diff}^\infty(I)$?*

In [35], Marquis and Souto constructed a faithful C^∞ action of closed orientable surface groups, for genus $g \geq 2$, on the unit interval.

Question 4.5. *Is the universal circle action of a fibered 3-manifold group topologically conjugate to a C^1 action?*

In other words, Question 4.5 asks if we can replace the C^3 conclusion in Proposition 1.5 with a C^1 conclusion. We note that for arbitrary $\alpha < 1$, there are $C^{1+\alpha}$ actions of $\pi_1(S)$ that are C^0 conjugate to a Fuchsian action, but that are not conjugate to a Fuchsian action by an absolutely continuous homeomorphism; see [26]. Other instances of this phenomenon arise from the theory of Hitchin representations [9].

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