Symmetric binary Steinhaus triangles and parity-regular Steinhaus graphs

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Abstract

A binary Steinhaus triangle is a triangle of zeroes and ones that points down and with the same local rule than the Pascal triangle modulo 2. A binary Steinhaus triangle is said to be rotationally symmetric, horizontally symmetric or dihedrally symmetric if it is invariant under the 120 degrees rotation, the horizontal reflection or both, respectively. The first part of this paper is devoted to the study of linear subspaces of rotationally symmetric, horizontally symmetric and dihedrally symmetric binary Steinhaus triangles. We obtain simple explicit bases for each of them by using elementary properties of the binomial coefficients. A Steinhaus graph is a simple graph with an adjacency matrix whose upper-triangular part is a binary Steinhaus triangle. A Steinhaus graph is said to be even or odd if all its vertex degrees are even or odd, respectively. One of the main results of this paper is the existence of an isomorphism between the linear subspace of even Steinhaus graphs and a certain linear subspace of dihedrally symmetric binary Steinhaus triangles. This permits us to give, in the second part of this paper, an explicit basis for even Steinhaus graphs and for the vector space of parity-regular Steinhaus graphs, that is the linear subspace of Steinhaus graphs that are even or odd. Finally, in the last part of this paper, we consider the generalized Pascal triangles, that are triangles of zeroes and ones, that point up now, and always with the same local rule than the Pascal triangle modulo 2. New simple bases for each linear subspace of symmetric generalized Pascal triangles are deduced from the results of the first part.

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1 Introduction

A *binary Steinhaus triangle (or Steinhaus triangle for short) of size n* is a triangle $(a_{i,j})_{1\leqslant i\leqslant j\leqslant n}$ of 0's and 1's verifying the same local rule than the Pascal triangle modulo 2, that is,

$$a_{i,j} \equiv a_{i-1,j-1} + a_{i-1,j} \pmod 2, \tag{LR}$$

for all integers i, j such that $2 \leqslant i \leqslant j \leqslant n$. Note that (0) and (1) are the Steinhaus triangles of size 1 and $\emptyset$ is the Steinhaus triangle of size 0. An example of Steinhaus triangle of size 7 is depicted in Figure 1.

It is clear that a Steinhaus triangle $(a_{i,j})_{1\leqslant i\leqslant j\leqslant n}$ is completely determined by its first row $(a_{1,j})_{1\leqslant j\leqslant n}$. Indeed, by induction on i and using (LR), we obtain that

$$a_{i,j} \equiv \sum_{k=0}^{i-1} \binom{i-1}{k} a_{1,j-k} \pmod 2, \tag{1}$$

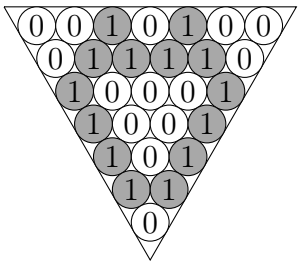


Figure 1: A Steinhaus triangle of size 7

for all integers i, j such that $1 \leqslant i \leqslant j \leqslant n$, where the binomial coefficient $\binom{a}{b}$ is the coefficient of the monomial X^b in the expansion of $(1+X)^a$, for all non-negative integers a and b such that $b \leqslant a$. In the sequel, the Steinhaus triangle whose first row is the sequence S is denoted by ∇S. The Steinhaus triangle in Figure 1 is then $\nabla(0010100)$.

Since the set $\mathcal{ST}(n)$ of binary Steinhaus triangles of size n is closed under addition modulo 2, it follows that $\mathcal{ST}(n)$ is a vector space over $\mathbb{Z}/2\mathbb{Z}$. Moreover, since a Steinhaus triangle is uniquely determined by its first row, the dimension of $\mathcal{ST}(n)$ is n, for all non-negative integers n.

This kind of binary triangles has been introduced by Hugo Steinhaus in his problem book [26, 27], where he posed, as an unsolved problem, the following

Question. *Does there exist, for every non-negative integer n such that $n \equiv 0$ or $3 \bmod 4$, a Steinhaus triangle of size n containing as many 0's as 1's?*

The Steinhaus triangle $\nabla(0010100)$ depicted in Figure 1 solves this problem for $n = 7$, since it contains 14 zeroes and 14 ones. Note that, since a triangle of size n contains $\binom{n+1}{2}$ elements, the condition $n \equiv 0$ or $3 \bmod 4$ is a necessary and sufficient condition for having a triangle of size n with an even number of terms. The Steinhaus problem was solved for the first time by Heiko Harborth in 1972 [23]. Since then, many solutions of this problem have appeared in the litterature [21, 22, 20, 13]. Generalizations of this problem in $\mathbb{Z}/m\mathbb{Z}$, for all $m \geqslant 2$, can be found in [24, 9, 11, 14] and in higher dimensions in [12].

The local rule (LR) can also be written as

$$a_{i-1,j-1} \equiv a_{i-1,j} + a_{i,j} \pmod 2 \quad \text{or} \quad a_{i,j} \equiv a_{i-1,j} + a_{i-1,j-1} \pmod 2,$$

for all integers i, j such that $2 \leqslant i \leqslant j \leqslant n$. This is the reason why the 120 degrees rotation and the horizontal reflection of a Steinhaus triangle are also Steinhaus triangles, of the same size. We denote by r and h the corresponding automorphisms of $\mathcal{ST}(n)$, that are,

$$\begin{array}{rccc} r: & \mathcal{ST}(n) & \longrightarrow & \mathcal{ST}(n) \\ & (a_{i,j})_{1\leqslant i\leqslant j\leqslant n} & \longmapsto & (a_{j-i+1,n-i+1})_{1\leqslant i\leqslant j\leqslant n} \end{array}$$

and

$$\begin{array}{rccc} h: & \mathcal{ST}(n) & \longrightarrow & \mathcal{ST}(n) \\ & (a_{i,j})_{1\leqslant i\leqslant j\leqslant n} & \longmapsto & (a_{i,n-j+i})_{1\leqslant i\leqslant j\leqslant n} \end{array}$$

for all non-negative integers n. These automorphisms verify the following identities

$$r^3 = h^2 = hrhr = id_{\mathcal{ST}(n)},$$

where $id_{\mathcal{ST}(n)}$ is the identity map on $\mathcal{ST}(n)$. Therefore, the subgroup $\langle r, h\rangle$ generated by r and h, of the automorphism group of $\mathcal{ST}(n)$, is isomorphic to the dihedral group D_3.

This induces a faithful representation of D_3 on $\mathcal{ST}(n)$, for all non-negative integers n. In the sequel, the automorphism subgroup $\langle r, h\rangle$ is simply denoted by D_3. For instance, for $S = (11001)$ and for all $g \in D_3$, the Steinhaus triangles $g(\nabla S)$ are depicted in Figure 2.

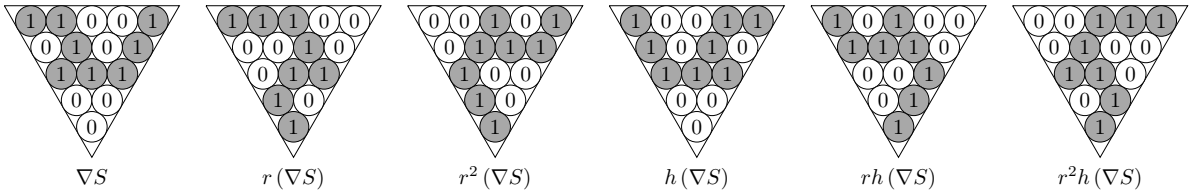


Figure 2: Action of D_3 on $\nabla(11001)$

For any subgroup G of D_3 and any non-negative integer n, we consider the linear subspace of invariant triangles of $\mathcal{ST}(n)$ under G, that is,

$$\mathcal{ST}(n)^G = \{\nabla \in \mathcal{ST}(n) \mid \forall g \in G, g(\nabla) = \nabla\}.$$

It is well known that there are exactly 6 subgroups of D_3, that are $\left\{id_{\mathcal{ST}(n)}\right\}$, $\langle h\rangle$, $\langle rh\rangle$, $\langle r^2h\rangle$, $\langle r\rangle$ and D_3. Obviously, we have $\mathcal{ST}(n)^G = \mathcal{ST}(n)$ for the trivial subgroup $G = \left\{id_{\mathcal{ST}(n)}\right\}$. Moreover, by the linear maps

$$\begin{array}{ccc} \mathcal{ST}(n)^{\langle h\rangle} & \longrightarrow & \mathcal{ST}(n)^{\langle rh\rangle} \\ \nabla & \longmapsto & r^2(\nabla) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{ST}(n)^{\langle h\rangle} & \longrightarrow & \mathcal{ST}(n)^{\left\langle r^2h\right\rangle} \\ \nabla & \longmapsto & r(\nabla) \end{array}$$

it is clear that the three linear subspaces $\mathcal{ST}(n)^{\langle h\rangle}$, $\mathcal{ST}(n)^{\langle rh\rangle}$ and $\mathcal{ST}(n)^{\left\langle r^2h\right\rangle}$ are isomorphic to each other. Therefore, for all non-negative integers n, we only consider the linear subspaces $\mathcal{ST}(n)^{\langle h\rangle}$, $\mathcal{ST}(n)^{\langle r\rangle}$ and $\mathcal{ST}(n)^{D_3}$, that will be denoted by $\mathcal{HST}(n)$, $\mathcal{RST}(n)$ and $\mathcal{DST}(n)$, respectively, in the sequel of this paper. Obviously, these vector spaces simply correspond to $\ker\left(h - id_{\mathcal{ST}(n)}\right)$, $\ker\left(r - id_{\mathcal{ST}(n)}\right)$ and $\ker\left(r - id_{\mathcal{ST}(n)}\right) \cap \ker\left(h - id_{\mathcal{ST}(n)}\right)$, respectively.

A Steinhaus triangle ∇ of $\mathcal{HST}(n)$, $\mathcal{RST}(n)$ or $\mathcal{DST}(n)$ is said to be *horizontally symmetric*, *rotationally symmetric* or *dihedrally symmetric*, respectively, and verifies $h(\nabla) = \nabla$, $r(\nabla) = \nabla$ or $r(\nabla) = h(\nabla) = \nabla$, respectively. Examples of such symmetric Steinhaus triangles appear in Figure 3.

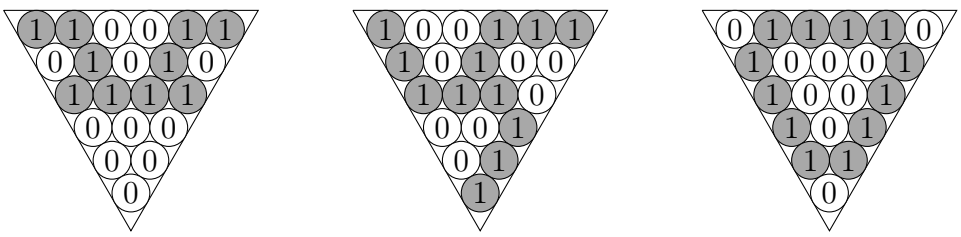


Figure 3: Triangles of $\mathcal{HST}(6)$, $\mathcal{RST}(6)$ and $\mathcal{DST}(6)$.

In [3], it was proved that

- $\dim \mathcal{HST}(n) = \left\lceil \frac{n}{2} \right\rceil$,

- $\dim \mathcal{RST}(n) = \left\lfloor \frac{n}{3} \right\rfloor + \delta_{1,(n \bmod 3)}$,
- $\dim \mathcal{DST}(n) = \left\lfloor \frac{n+3}{6} \right\rfloor + \delta_{1,(n \bmod 6)}$,

for all non-negative integers n, where $\delta_{i,(n \bmod j)}$ is equal to 1, if $n \equiv i \bmod j$, and 0 otherwise. Bases of $\mathcal{HST}(n)$, $\mathcal{RST}(n)$ and $\mathcal{DST}(n)$, for all non-negative integers n, are obtained in [7]. In this paper, we give new bases, for each of these three linear subspaces, which are simpler than those mentioned. They are obtained by considering elementary properties of generalized binomial coefficients.

A *Steinhaus graph* of order $n \geqslant 1$ is a simple graph whose adjacency matrix has an upper-triangular part which is a binary Steinhaus triangle of size $n-1$. For any sequence $S = (a_1, a_2, \ldots, a_{n-1})$ of 0's and 1's of length $n-1$, its associated Steinhaus graph $\mathrm{G}(S)$ is the simple graph of order n whose adjacency matrix $\mathrm{M}(S) = (a_{i,j})_{1 \leqslant i,j \leqslant n}$ verifies

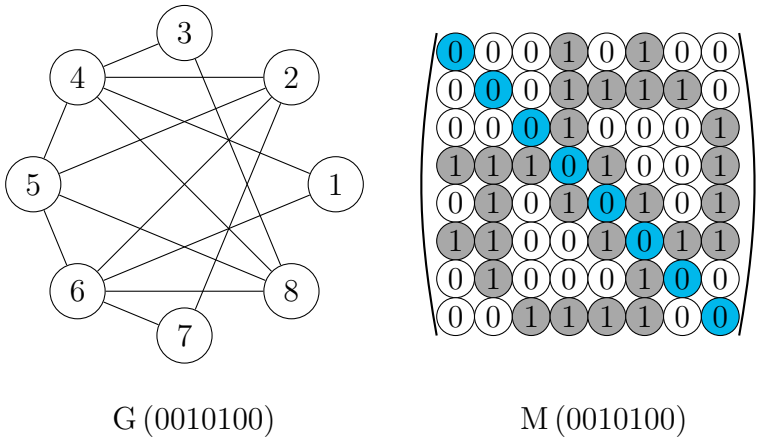
i) $a_{i,j} = a_{j,i}$, for all $i, j \in \{1, \ldots, n\}$, (symmetry)

ii) $a_{i,i} = 0$, for all $i \in \{1, \ldots, n\}$, (diagonal of zeroes)

iii) $a_{1,j} = a_{j-1}$, for all $j \in \{2, \ldots, n-1\}$, (sequence S)

iv) $a_{i,j} = a_{i-1,j-1} + a_{i-1,j}$, for all integers i, j such that $2 \leqslant i < j \leqslant n$, (local rule of ∇S)

where $\{x, \ldots, y\}$ denotes the set of integers $\{i \in \mathbb{Z} \mid x \leqslant i \leqslant y\}$, for any integers x and y. For example, for $S = (0010100)$, the Steinhaus graph $\mathrm{G}(S)$ and its adjacency matrix $\mathrm{M}(S)$ are depicted in Figure 4.



G (0010100) M (0010100)

Figure 4: The Steinhaus graph G (0010100) and its adjacency matrix M (0010100)

The set of Steinhaus graphs of order n is denoted by $\mathcal{SG}(n)$, for all positive integers n. It is clear that there is a natural correspondence between $\mathcal{SG}(n)$ and $\mathcal{ST}(n-1)$. Therefore, for all positive integers n, the set $\mathcal{SG}(n)$ is a vector space over $\mathbb{Z}/2\mathbb{Z}$ of dimension $n-1$.

The family of Steinhaus graphs has been introduced in [24]. In [15], it was proved that any simple graph of order n is isomorphic to an induced subgraph of a Steinhaus graph of order $\binom{n}{2}+1$. A general problem on Steinhaus graphs is to characterize those, or their associated binary sequences, having a given graph property such as connectedness, planarity, bipartition, regularity, etc. It is easy to see that a Steinhaus graph is either connected or totally disconnected (the edgeless graph). The bipartite Steinhaus graphs are characterized in [17, 18, 8] and the planar ones in [19]. In [16, 2], it was conjectured that there is only one regular Steinhaus graph of odd degree, the complete graph $K_2 =$

$\mathrm{G}(1)$, and that the regular Steinhaus graphs of even degrees are the edgeless graphs $\overline{K_n} = \mathrm{G}(00\cdots0)$ of orders n, for all positive integers n, and the non-trivial graphs $\mathrm{G}(110110\cdots110)$ of orders $n = 3m+1$, for all positive integers m. This conjecture was verified up to 117 vertices in [1] and up to 1500 vertices in [10] for the odd case.

A Steinhaus graph is said to be *even* (resp. *odd*) if every vertex has even degree (resp. odd degree). Examples of even and odd Steinhaus graphs are given in Figure 5. For all positive integers n, the sets of even Steinhaus graphs and of odd Steinhaus graphs of order n are denoted by $\mathcal{ESG}(n)$ and $\mathcal{OSG}(n)$, respectively. In [16], it was proved that $\mathcal{ESG}(n)$ is a linear subspace of $\mathcal{SG}(n)$ of dimension $\left\lfloor \frac{n-1}{3} \right\rfloor$, for all positive integers n, and $\mathcal{OSG}(n)$ is an affine subspace of direction $\mathcal{ESG}(n)$, for all even numbers n. Obviously, since the number of vertices of odd degrees is always even, $\mathcal{OSG}(n) = \emptyset$ when n is odd. According to the terminology used in [1], a *parity-regular* Steinhaus graph is a Steinhaus graph that is even or odd. For all positive integers n, the set of parity-regular Steinhaus graphs of order n is denoted by $\mathcal{PRSG}(n)$, that is, $\mathcal{PRSG}(n) = \mathcal{ESG}(n) \cup \mathcal{OSG}(n)$. As shown in [1], the set $\mathcal{PRSG}(n)$ is a linear subspace of $\mathcal{SG}(n)$ of dimension $\left\lceil \frac{n}{3} \right\rceil - \delta_{1,(n \bmod 2)}$, for all positive integers n. Bases of $\mathcal{PRSG}(n)$ have been computed, for $n \leqslant 30$, in [1]. In this paper, we determine bases of $\mathcal{ESG}(n)$ and $\mathcal{PRSG}(n)$, for all positive integers n. This is achieved by showing that the vector space $\mathcal{ESG}(n)$ is isomorphic to a particular linear subspace of $\mathcal{DST}(2n-1)$, for all positive integers n.

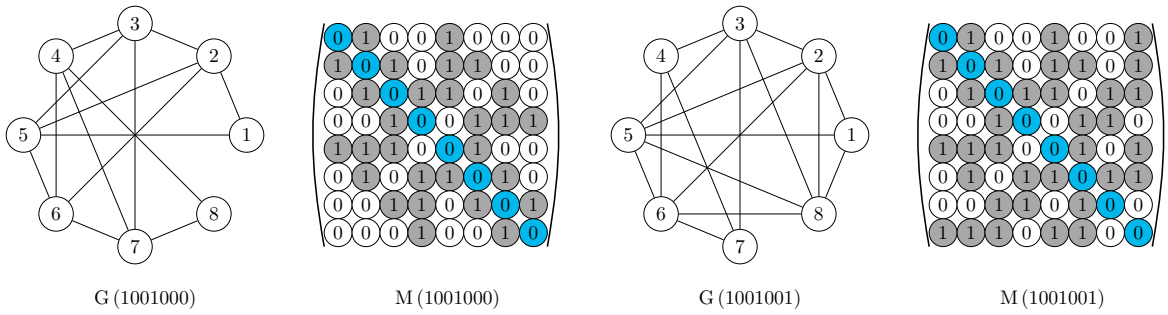


Figure 5: Even and odd Steinhaus graphs

This paper is organized as follows. In Section 2, some basic properties on generating index sets of Steinhaus triangles and on derived and antiderived binary sequences are introduced. After that, for all non-negative integers n, the linear subspaces $\mathcal{RST}(n)$, $\mathcal{HST}(n)$ and $\mathcal{DST}(n)$ are studied in detail, with the determination of generating index sets and bases for each of them: for $\mathcal{RST}(n)$ in Section 3, for $\mathcal{HST}(n)$ in Section 4 and for $\mathcal{DST}(n)$ in Section 5. For any positive integer n, a certain correspondence between the vector spaces $\mathcal{ESG}(n)$ and $\mathcal{DST}(2n-1)$ is established and bases of $\mathcal{ESG}(n)$ and $\mathcal{PRSG}(n)$ are given in Section 6. In Section 7, we deal with symmetric generalized Pascal triangles, that are binary triangles which point up and always with the same local rule. Finally, open problems on generalizations of symmetric binary triangles are proposed in Section 8.

2 Preliminary results

We introduce, in this section, the notions of generating index sets of Steinhaus triangles and of derived and antiderived binary sequences.

2.1 Generating index sets

Let n be a positive integer. We denote by $\nabla(n)$ the index set of Steinhaus triangles of size n, that is,

$$\nabla(n) = \left\{(i,j) \in \mathbb{N}^2 \;\middle|\; 1 \leqslant i \leqslant j \leqslant n\right\}.$$

A subset G of $\nabla(n)$ is said to be a *generating index set of* $\mathcal{ST}(n)$ if the knowledge of the values $a_{i,j}$, for all $(i,j) \in G$, uniquely determines the whole Steinhaus triangle $(a_{i,j})_{1\leqslant i\leqslant j\leqslant n}$, i.e., if the linear map

$$\begin{array}{rccc} \pi_G : & \mathcal{ST}(n) & \longrightarrow & \{0,1\}^G \\ & (a_{i,j})_{1\leqslant i\leqslant j\leqslant n} & \longmapsto & (a_{i,j})_{(i,j)\in G} \end{array}$$

is an isomorphism. Since $\dim \mathcal{ST}(n) = n$, we deduce that the cardinality of a generating index set of $\mathcal{ST}(n)$ is always n. From (1), it is clear that the set of top row indices of a Steinhaus triangle of size n, that is,

$$G_1 = \{(1,1),(1,2),\ldots,(1,n)\},$$

is a generating index set of $\mathcal{ST}(n)$. Note that ${\pi_{G_1}}^{-1}(S) = \nabla S$, for all $S \in \{0,1\}^{G_1}$. It follows that the set G is a generating index set of $\mathcal{ST}(n)$ if and only if the linear map $\pi_G \circ {\pi_{G_1}}^{-1} : \{0,1\}^{G_1} \to \{0,1\}^G$ is an isomorphism. For instance, the 16 generating index sets of $\mathcal{ST}(3)$ (4 up to the action of the dihedral group D_3) are depicted in Figure 6, where a disk is either black if its position is in the generating index set or white otherwise.

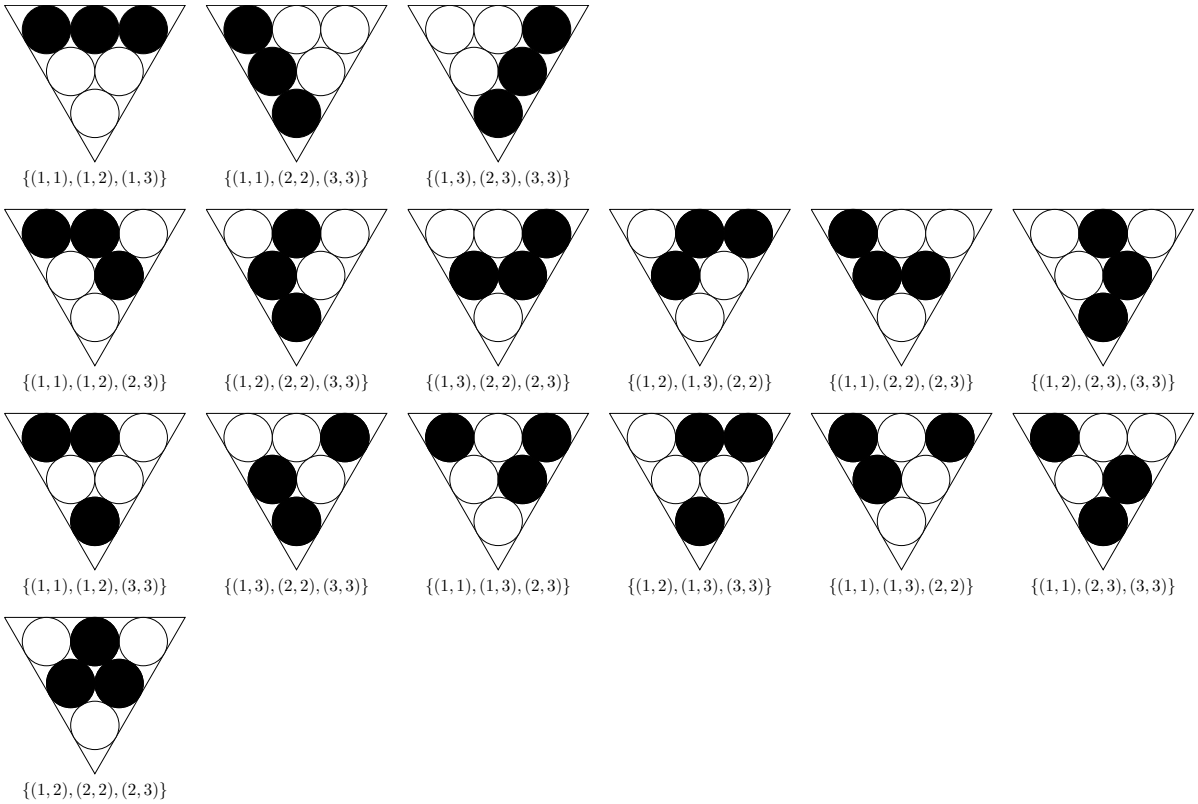


Figure 6: Generating index sets of $\mathcal{ST}(3)$

Since the sets of right side indices,

$$G_r = \{(1,n),(2,n),\ldots,(n,n)\},$$

and left side indices,

$$G_l = \{(1,1),(2,2),\ldots,(n,n)\},$$

of a Steinhaus triangle ∇ of size n can be seen as the sets of top row indices of the Steinhaus triangles $r(\nabla)$ and $r^2(\nabla)$, respectively, it follows that G_r and G_l are generating index sets of $\mathcal{ST}(n)$ too. Therefore, each element of a Steinhaus triangle can be expressed in function of the terms of its first row, of its right side or of its left side.

For any non-negative integers a and b such that $b \leqslant a$, the binomial coefficient $\binom{a}{b}$ is the coefficient of the monomial X^b in the polynomial expansion of the binomial power $(1+X)^a$. It corresponds to the number of ways to choose b elements in a set of a elements. Here, we extend this notation by supposing that $\binom{a}{b} = 0$, for all integers b such that $b < 0$ or $b > a$. For this generalization, the Pascal identity

$$\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b}$$

holds, for all positive integers a and all integers b.

Lemma 2.1. *Let $(a_{i,j})_{1\leqslant i\leqslant j\leqslant n}$ be a binary Steinhaus triangle of size n. Then, we have*

$$a_{i,j} \equiv \sum_{k=1}^{n} \binom{i-1}{j-k} a_{1,k} \equiv \sum_{k=1}^{n} \binom{n-j}{k-i} a_{k,n} \equiv \sum_{k=1}^{n} \binom{j-i}{k-i} a_{k,k} \pmod 2,$$

for all integers i, j such that $1 \leqslant i \leqslant j \leqslant n$.

Proof. As for (1), by induction on i and using the local rule (LR). □

Proposition 2.2. *Let $G = \{(i_1,j_1),(i_2,j_2),\ldots,(i_n,j_n)\}$ be a subset of $\nabla(n)$ whose cardinality is $|G| = n$. Then, the set G is a generating index set of $\mathcal{ST}(n)$ if and only if* $\det(M_G) \equiv 1 \bmod 2$*, where*

$$M_G = \left(\binom{i_k - 1}{j_k - l}\right)_{1\leqslant k,l\leqslant n}.$$

Proof. From Lemma 2.1, we know that

$$a_{i_k,j_k} \equiv \sum_{l=1}^{n} \binom{i_k - 1}{j_k - l} a_{1,l} \pmod 2,$$

for all $k \in \{1,\ldots,n\}$. It follows that

$$\pi_G \circ {\pi_{G_1}}^{-1}(S) \equiv S.{M_G}^t \pmod 2,$$

for all $S \in \{0,1\}^{G_1}$. Finally, the linear map $\pi_G \circ {\pi_{G_1}}^{-1}$ is an isomorphism if and only if $\det(M_G) \equiv 1 \bmod 2$. □

The notion of generating index sets and the result of Proposition 2.2 appear in a more general context in [5, 6], where it is also proved that the set of generating index sets of $\mathcal{ST}(n)$ define a matroid called the Pascal matroid modulo 2. Note that a generating index set is simply called a generating set in [5, 6].

The definition of generating index sets can be extended to any linear subspace $\mathcal{V}$ of $\mathcal{ST}(n)$. A subset G of $\nabla(n)$ is said to be a *generating index set of* $\mathcal{V}$ if the linear map

$$\begin{array}{rccc} \pi_G : & \mathcal{V} & \longrightarrow & \{0,1\}^G \\ & (a_{i,j})_{1\leqslant i\leqslant j\leqslant n} & \longmapsto & (a_{i,j})_{(i,j)\in G} \end{array}$$

is an isomorphism. Note that $|G| = \dim \mathcal{V}$, for any generating index set G of $\mathcal{V}$. In this paper, we consider generating index sets of the linear subspaces $\mathcal{RST}(n)$, $\mathcal{HST}(n)$ and $\mathcal{DST}(n)$, for all non-negative integers n.

2.2 Derived and antiderived sequences

Let $S = (a_j)_{1 \leqslant j \leqslant n}$ be a sequence of 0's and 1's of length n.

The *derived sequence* ∂S of S is the sequence

$$\partial S = (a_j + a_{j+1} \bmod 2)_{1 \leqslant j \leqslant n-1} \tag{2}$$

of length $n-1$, when $n \geqslant 2$, and the empty sequence, when $n \leqslant 1$. The *iterated derived sequences* $\partial^i S$ of S are recursively defined by $\partial^i S = \partial(\partial^{i-1} S)$, for all $i \geqslant 1$, with $\partial^0 S = S$. The Steinhaus triangle ∇S can then be seen as the collection $(\partial^i S)_{0 \leqslant i \leqslant n-1}$, where, for every $i \in \{1, \ldots, n\}$, the ith row of ∇S corresponds to the derived sequence $\partial^{i-1} S$.

The set of binary sequences of length n can be seen as a vector space over $\mathbb{Z}/2\mathbb{Z}$ of dimension n. Indeed, for two binary sequences $S_1 = (a_j)_{1 \leqslant j \leqslant n}$ and $S_2 = (b_j)_{1 \leqslant j \leqslant n}$ of the same length $n \geqslant 1$, their sum is the sequence $S_1 + S_2 = (a_j + b_j)_{1 \leqslant j \leqslant n}$ of length n. Therefore, it is clear that the derivation map ∂ is linear, i.e., $\partial(S_1 + S_2) = \partial S_1 + \partial S_2$ for all binary sequences S_1 and S_2 of same length.

For any $i \in \{1, \ldots, n+1\}$ and any $x \in \{0, 1\}$, the *antiderived sequence of* $S = (a_j)_{1 \leqslant j \leqslant n}$ *whose ith term is* x is the sequence $\int_{i,x} S = (b_j)_{1 \leqslant j \leqslant n+1}$ of length $n+1$ defined by

$$b_j = \begin{cases} x + \displaystyle\sum_{k=j}^{i-1} a_k \pmod 2 & \text{for } j \in \{1, \ldots, i-1\}, \\ x & \text{for } j = i, \\ x + \displaystyle\sum_{k=i}^{j-1} a_k \pmod 2 & \text{for } j \in \{i+1, \ldots, n+1\}. \end{cases}$$

In a more concise way, we have

$$b_j = x + \sum_{k=1}^{i-1} a_k + \sum_{k=1}^{j-1} a_k \pmod 2, \tag{3}$$

for all $j \in \{1, \ldots, n+1\}$. Further, it is straightforward to obtain a fundamental theorem of calculus.

For any non-negative integer n, the constant sequence of length n equal to x is denoted by $(x)_n$. For $n = 1$, the sequence $(x)_1$ is simply denoted (x).

Proposition 2.3. *Let* $S = (a_j)_{1 \leqslant j \leqslant n}$ *be a binary sequence of length* n. *For any* $i \in \{1, \ldots, n+1\}$ *and any* $x \in \{0, 1\}$, *we have that*

i) $\partial\left(\int_{i,x} S\right) = S$,

ii) $\int_{i,x} (\partial S) = S + (a_i + x \bmod 2)_n$.

Proof. For i), let $\int_{i,x} = (b_j)_{1 \leqslant j \leqslant n+1}$ and $\partial\left(\int_{i,x} S\right) = (c_j)_{1 \leqslant j \leqslant n}$. By definition, from (2) and (3), we obtain that

$$c_j \equiv b_j + b_{j+1} \equiv \left(x + \sum_{k=1}^{i-1} a_k + \sum_{k=1}^{j-1} a_k\right) + \left(x + \sum_{k=1}^{i-1} a_k + \sum_{k=1}^{j} a_k\right) \equiv a_j \pmod 2,$$

for all $j \in \{1, \ldots, n\}$. Therefore $\partial\left(\int_{i,x} S\right) = S$.

Now, for ii), let $\partial S = (b_j)_{1\leqslant j\leqslant n-1}$ and $\int_{i,x}(\partial S) = (c_j)_{1\leqslant j\leqslant n}$. By definition, from (2) and (3), we obtain that

$$c_j \equiv x + \sum_{k=1}^{i-1} b_k + \sum_{k=1}^{j-1} b_k \equiv x + \sum_{k=1}^{i-1}(a_k + a_{k+1}) + \sum_{k=1}^{j-1}(a_k + a_{k+1})$$

$$\equiv x + (a_1 + a_i) + (a_1 + a_j) \equiv a_j + (a_i + x) \pmod 2,$$

for all $j \in \{1, \ldots, n\}$. Therefore $\int_{i,x}(\partial S) = S + (a_i + x \bmod 2)_n$. □

A similar result has been obtained for infinite binary sequences in [25]. It follows that every binary sequence S of length n admits only two different antiderived sequences, that are, the sequences $\int_{i,0} S$ and $\int_{i,1} S$ for some $i \in \{1, \ldots, n+1\}$. Moreover, it is easy to see that $\int_{i,0} S + \int_{i,1} S = (1)_n$. For example, the sequence $S = (0100)$ admits the two antiderived sequences (00111) and (11000).

3 Rotationally symmetric Steinhaus triangles

In this section, after characterizing rotationally symmetric Steinhaus triangles, we determine, for all non-negative integers n, generating index sets and bases of $\mathcal{RST}(n)$.

3.1 Characterizations of $\mathcal{RST}(n)$

First, by definition of the automorphism r, we have

$$r\left((a_{i,j})_{1\leqslant i\leqslant j\leqslant n}\right) = (a_{j-i+1,n-i+1})_{1\leqslant i\leqslant j\leqslant n} = \nabla(a_{j,n})_{1\leqslant j\leqslant n},$$

for any Steinhaus triangle $(a_{i,j})_{1\leqslant i\leqslant j\leqslant n} = \nabla(a_{1,j})_{1\leqslant j\leqslant n}$. Therefore, a Steinhaus triangle $(a_{i,j})_{1\leqslant i\leqslant j\leqslant n}$ is rotationally symmetric if and only if its first row $(a_{1,j})_{1\leqslant j\leqslant n}$ and its right side $(a_{j,n})_{1\leqslant j\leqslant n}$ correspond.

Proposition 3.1. *The Steinhaus triangle $(a_{i,j})_{1\leqslant i\leqslant j\leqslant n}$ is rotationally symmetric if and only if $(a_{1,j})_{1\leqslant j\leqslant n} = (a_{j,n})_{1\leqslant j\leqslant n}$.*

For two binary sequences $S_1 = (a_1, a_2, \ldots, a_{n_1})$ and $S_2 = (b_1, b_2, \ldots, b_{n_2})$ of length n_1 and n_2, respectively, we denote by $S_1 \cdot S_2$ *the concatenated sequence* of length $n_1 + n_2$ defined by $S_1 \cdot S_2 = (a_1, a_2, \ldots, a_{n_1}, b_1, b_2, \ldots, b_{n_2})$.

Let H be the linear map that assigns, to each Steinhaus triangle of order $n \geqslant 3$, its subtriangle of order $n-3$ obtained by removing its first row and its left and right sides, that is,

$$\begin{array}{rccc} \mathrm{H}: & \mathcal{ST}(n) & \longrightarrow & \mathcal{ST}(n-3) \\ & (a_{i,j})_{1\leqslant i\leqslant j\leqslant n} & \longmapsto & (a_{1+i,2+j})_{1\leqslant i\leqslant j\leqslant n-3} \end{array}$$

Note that the linear map H is surjective. Indeed, for any $\nabla S' \in \mathcal{ST}(n-3)$, it is easy to verify that $\nabla S' = \mathrm{H}(\nabla S)$ if and only if S is one of the eight sequences of the form $S = (x_1) \cdot \int_{i,x} S' \cdot (x_2)$, where $x_1, x_2 \in \{0, 1\}$ and $\int_{i,x} S'$ is one of the two antiderived sequences of S'. Examples of a Steinhaus triangle ∇S and its subtriangle $\mathrm{H}(\nabla S)$ are depicted in Figure 7.

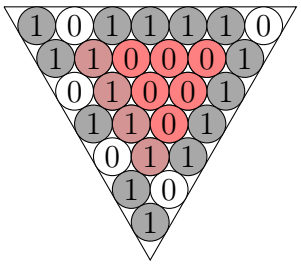


Figure 7: $\mathrm{H}\left(\nabla(1011110)\right) = \nabla(1000)$

For any binary sequence $S = (a_j)_{1 \leqslant j \leqslant n}$, we denote by $\sigma(S)$ its sum $\sigma(S) = \sum_{j=1}^{n} a_j$, i.e., the number of ones in S, and by $\sigma_2(S)$ its sum modulo 2.

For any positive integer $n \geqslant 3$, by definition of $\mathcal{RST}(n)$ and H, it is clear that $\mathrm{H}\left(\mathcal{RST}(n)\right) \subset \mathcal{RST}(n-3)$. The precise relationship between a rotationally symmetric Steinhaus triangle ∇S and its subtriangle $\mathrm{H}\left(\nabla S\right)$ is given in the following

Proposition 3.2. *Let S be a finite binary sequence of length $n \geqslant 3$. The Steinhaus triangle ∇S is rotationally symmetric if and only if $\mathrm{H}\left(\nabla S\right) = \nabla S'$ is rotationally symmetric and $S = \left(\sigma_2\left(S'\right)\right) \cdot \int_{i,x} S' \cdot \left(\sigma_2\left(S'\right)\right)$, for some $i \in \{1, \ldots, n-2\}$ and some $x \in \{0, 1\}$.*

Proposition 3.2 appears in [4] in a more general context. For the convenience of the reader, a proof is given here.

Proof. Let $\nabla S = (a_{i,j})_{1 \leqslant i \leqslant j \leqslant n} \in \mathcal{ST}(n)$ such that $\nabla S' = \mathrm{H}(\nabla S) = (a_{1+i,2+j})_{1 \leqslant i \leqslant j \leqslant n-3} \in \mathcal{RST}(n-3)$. Since $\nabla S'$ is rotationally symmetric, its top row $(a_{2,j})_{3 \leqslant j \leqslant n-1}$, its right side $(a_{i,n-1})_{2 \leqslant i \leqslant n-2}$ and the reverse of its left side $(a_{n-i,n-i+1})_{2 \leqslant i \leqslant n-2}$ correspond by Proposition 3.1. Moreover, since $(a_{1,j})_{2 \leqslant j \leqslant n-1}$, $(a_{i,n})_{2 \leqslant i \leqslant n-1}$ and $(a_{n-i+1,n-i+1})_{2 \leqslant i \leqslant n-1}$ are antiderived sequences of the sequences $(a_{2,j})_{3 \leqslant j \leqslant n-1}$, $(a_{i,n-1})_{2 \leqslant i \leqslant n-2}$ and $(a_{n-i,n-i+1})_{2 \leqslant i \leqslant n-2}$, respectively, we deduce that they correspond if and only if there exist $i_1, i_2 \in \{2, \ldots, n-1\}$ such that $a_{1,i_1} = a_{i_1,n}$ and $a_{1,i_2} = a_{n-i_2+1,n-i_2+1}$. Since $a_{1,1} \equiv a_{1,2} + a_{2,2} \bmod 2$ and $a_{1,n} \equiv a_{1,n-1} + a_{2,n} \bmod 2$, it follows from Proposition 3.1 again that the Steinhaus triangle ∇S is rotationally symmetric if and only if $a_{1,1} = a_{1,n} \equiv a_{1,2} + a_{1,n-1} \bmod 2$. Finally, using the local rule (LR), we have that

$$\sigma\left(S'\right) = \sum_{j=3}^{n-1} a_{2,j} \equiv \sum_{j=3}^{n-1} a_{1,j-1} + a_{1,j} \equiv a_{1,2} + a_{1,n-1} \pmod 2.$$

This completes the proof. □

3.2 Generating index sets of $\mathcal{RST}(n)$

We are now ready to determine generating index sets of the linear subspace of rotationally symmetric Steinhaus triangles, for every non-negative integer n.

Theorem 3.3. *Let n be a non-negative integer. The set*

$$G_R = \left\{ (i, j_i) \;\middle|\; i \in \left\{ 1, \ldots, \left\lfloor \frac{n}{3} \right\rfloor + \delta_{1,(n \bmod 3)} \right\} \right\},$$

where $j_i \in \{2i, \ldots, n-i\}$ for all $i \in \left\{1, \ldots, \left\lfloor \frac{n}{3} \right\rfloor\right\}$ and $j_{\frac{n+2}{3}} = \frac{2n+1}{3}$ when $n \equiv 1 \bmod 3$, is a generating index set of $\mathcal{RST}(n)$.

Proof. By induction on n.

For $n = 0$ and $n = 2$, it is clear that $\emptyset$ and $\nabla(00)$ are the only rotationally symmetric Steinhaus triangles of sizes 0 and 2. Therefore, the empty set $\emptyset$ is a generating index set of $\mathcal{RST}(0)$ and $\mathcal{RST}(2)$. For $n = 1$, the Steinhaus triangles $\nabla(0)$ and $\nabla(1)$ are both rotationally symmetric and thus the set $\{(1, 1)\}$ is a generating index set of $\mathcal{RST}(1)$.

Suppose now that the result is true for the sets of rotationally symmetric Steinhaus triangles of size strictly lesser than $n \geqslant 3$. Let $m = \left\lfloor \frac{n}{3} \right\rfloor + \delta_{1,(n \bmod 3)}$. We consider the subset $\mathrm{H}(G_R) \subset \nabla(n-3)$ defined by

$$\mathrm{H}(G_R) = \{(i-1, j_i - 2) \mid i \in \{2, \ldots, m\}\}$$

and the linear maps f_1 and f_2 defined by

$$\begin{array}{rccc} f_1 : & \mathcal{RST}(n) & \longrightarrow & \{0,1\} \times \mathcal{RST}(n-3) \\ & \nabla S = (a_{i,j})_{1 \leqslant i \leqslant j \leqslant n} & \longmapsto & (a_{1,j_1}, \mathrm{H}(\nabla S)) \end{array}$$

and

$$\begin{array}{rccc} f_2 : & \{0,1\} \times \mathcal{RST}(n-3) & \longrightarrow & \{0,1\}^m \\ & (x, \nabla S') & \longmapsto & (x) \cdot \pi_{\mathrm{H}(G_R)}(\nabla S') \end{array}$$

Then, for any $(a_{i,j})_{1 \leqslant i \leqslant j \leqslant n} \in \mathcal{RST}(n)$, we have

$$\begin{aligned} f_2 f_1 ((a_{i,j})_{1 \leqslant i \leqslant j \leqslant n}) &= f_2 (a_{1,j_1}, (a_{1+i,2+j})_{1 \leqslant i \leqslant j \leqslant n-3}) \\ &= (a_{1,j_1}) \cdot \pi_{\mathrm{H}(G_R)} ((a_{1+i,2+j})_{1 \leqslant i \leqslant j \leqslant n-3}) \\ &= (a_{1,j_1}) \cdot (a_{i,j_i})_{2 \leqslant i \leqslant m} \\ &= (a_{i,j_i})_{1 \leqslant i \leqslant m} = \pi_{G_R} ((a_{i,j})_{1 \leqslant i \leqslant j \leqslant n}). \end{aligned}$$

Therefore $f_2 f_1 = \pi_{G_R}$. From Proposition 3.2, we know that f_1 is an isomorphism whose inverse is defined by ${f_1}^{-1}(x, \nabla S') = \nabla\Big((\sigma_2(S')) \cdot \int_{j_1 - 1, x} S' \cdot (\sigma_2(S'))\Big)$, for all $(x, \nabla S') \in \{0,1\} \times \mathcal{RST}(n-3)$. Moreover, since we have

$$1 \leqslant i - 1 \leqslant \frac{n-3}{3} \quad \text{and} \quad 2(i-1) \leqslant j_i - 2 \leqslant (n-3) - (i-1),$$

for all $i \in \left\{2, \ldots, \left\lfloor \frac{n}{3} \right\rfloor\right\}$, and

$$m - 1 = \frac{(n-3)+2}{3} \quad \text{and} \quad j_m - 2 = \frac{2(n-3)+1}{3},$$

when $n \equiv 1 \bmod 3$, the set $\mathrm{H}(G_R)$ is a generating index set of $\mathcal{RST}(n-3)$ by induction hypothesis. Therefore $\pi_{\mathrm{H}(G_R)}$ and thus f_2 are isomorphisms. Finally, since the linear map $\pi_{G_R} = f_2 f_1$ is an isomorphism, the set G_R is a generating index set of $\mathcal{RST}(n)$. □

Corollary 3.4. *Let n be a non-negative integer. The set*

$$G_R := \left\{ \left(i, n - \left\lfloor \frac{n}{3} \right\rfloor \right) \;\middle|\; i \in \left\{ 1, \ldots, \left\lfloor \frac{n}{3} \right\rfloor + \delta_{1,(n \bmod 3)} \right\} \right\}$$

is a generating index set of $\mathcal{RST}(n)$.

Proof. From Theorem 3.3, since $2i \leqslant n - \left\lfloor \frac{n}{3} \right\rfloor \leqslant n - i$ for all $i \in \left\{1, \ldots, \left\lfloor \frac{n}{3} \right\rfloor\right\}$ and $n - \left\lfloor \frac{n}{3} \right\rfloor = \frac{2n+1}{3}$ when $n \equiv 1 \bmod 3$. □

Since the dimension of $\mathcal{RST}(n)$ corresponds to the cardinality of a generating index set G_R, it is straightforward to obtain the following

Corollary 3.5. $\dim \mathcal{RST}(n) = \left\lfloor \frac{n}{3} \right\rfloor + \delta_{1,(n \bmod 3)}$, *for all non-negative integers n.*

3.3 Bases of $\mathcal{RST}(n)$

In the end of this section, using the generating index sets G_R introduced before, we determine bases of the linear subspace of rotationally symmetric Steinhaus triangles.

First, we consider the linear map $\rho : \mathcal{ST}(n) \longrightarrow \mathcal{RST}(n)$ defined by $\rho = r^2+r+id_{\mathcal{ST}(n)}$, for all non-negative integers n. Obviously, this map is surjective since $\rho(\nabla) = \nabla$, for all $\nabla \in \mathcal{RST}(n)$. Moreover, as detailed below, all the terms of $\rho(\nabla)$ can be expressed in function of these of ∇.

Proposition 3.6. *For all $(a_{i,j})_{1\leqslant i\leqslant j\leqslant n} \in \mathcal{ST}(n)$, we have*

$$\rho\left((a_{i,j})_{1\leqslant i\leqslant j\leqslant n}\right) = (a_{i,j} + a_{j-i+1,n-i+1} + a_{n-j+1,n+i-j} \bmod 2)_{1\leqslant i\leqslant j\leqslant n}\,.$$

Proof. First, by definition of r, we know that

$$r\left((a_{i,j})_{1\leqslant i\leqslant j\leqslant n}\right) = (a_{j-i+1,n-i+1})_{1\leqslant i\leqslant j\leqslant n}\,,$$

for all $(a_{i,j})_{1\leqslant i\leqslant j\leqslant n} \in \mathcal{ST}(n)$. Thus, we obtain

$$r^2\left((a_{i,j})_{1\leqslant i\leqslant j\leqslant n}\right) = r\left((a_{j-i+1,n-i+1})_{1\leqslant i\leqslant j\leqslant n}\right) = (a_{n-j+1,n+i-j})_{1\leqslant i\leqslant j\leqslant n}\,,$$

for all $(a_{i,j})_{1\leqslant i\leqslant j\leqslant n} \in \mathcal{ST}(n)$. The result follows. □

For any non-negative integer n, let U_n be the Steinhaus triangle of size n defined by

$$U_n = \rho\left(\nabla(1)_n\right).$$

It is clear that $U_0 = \emptyset$, $U_1 = \nabla(1)$, $U_2 = \nabla(00)$ and $U_n = \nabla(011\cdots110)$ for $n \geqslant 3$, since

$$U_n = \rho\left(\nabla(1)_n\right) = \nabla(1\cdots1) + \nabla(10\cdots0) + \nabla(0\cdots01) = \nabla((1\cdots1) + (10\cdots0) + (0\cdots01)),$$

for all positive integers n. The Steinhaus triangles U_n are depicted in Figure 8, for the first few values of n. Moreover, an explicit formula for the terms of U_n is given in the following

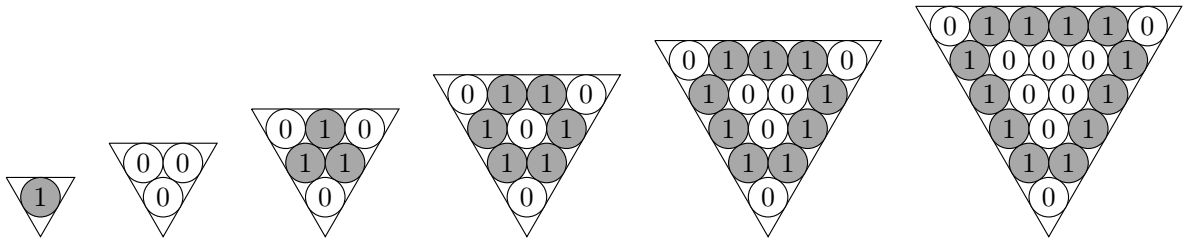


Figure 8: U_n for $n \in \{1,\ldots,6\}$

Proposition 3.7. *For any non-negative integer n, we have*

$$U_n = (\delta_{i,1} + \delta_{i,j} + \delta_{j,n} \bmod 2)_{1\leqslant i\leqslant j\leqslant n}\,.$$

Proof. First, if we denote $\nabla(1)_n = (a_{i,j})_{1\leqslant i\leqslant j\leqslant n}$ and $U_n = \rho(\nabla(1)_n) = (b_{i,j})_{1\leqslant i\leqslant j\leqslant n}$, we know from Proposition 3.6 that

$$b_{i,j} \equiv a_{i,j} + a_{j-i+1,n-i+1} + a_{n-j+1,n+i-j} \pmod 2,$$

for all integers i and j such that $1 \leqslant i \leqslant j \leqslant n$. Moreover, it is clear that $\partial^i(1)_n = (0)_{n-i}$, for all $i \in \{1, \ldots, n-1\}$. Therefore, $\nabla(1)_n = (\delta_{i,1})_{1\leqslant i\leqslant j\leqslant n}$. This leads to

$$b_{i,j} \equiv \delta_{i,1} + \delta_{j-i+1,1} + \delta_{n-j+1,1} \pmod 2,$$

for all integers i and j such that $1 \leqslant i \leqslant j \leqslant n$. Finally, since $\delta_{j-i+1,1} = \delta_{i,j}$ and $\delta_{n-j+1,1} = \delta_{j,n}$, the result follows. □

Corollary 3.8. $\mathrm{H}(U_n) = \nabla(0)_{n-3}$, *for all positive integers* $n \geqslant 3$.

Proof. Let $U_n = (a_{i,j})_{1\leqslant i\leqslant j\leqslant n}$. By definition of H and Proposition 3.7, we obtain that

$$\mathrm{H}(U_n) = (a_{i+1,j+2})_{1\leqslant i\leqslant j\leqslant n-3} = (\delta_{i+1,1} + \delta_{i+1,j+2} + \delta_{j+2,n} \bmod 2)_{1\leqslant i\leqslant j\leqslant n-3} = \nabla(0)_{n-3}.$$

□

For any non-negative integer k such that $3k \leqslant n$, we consider the iterated operator

$$\mathrm{H}^k = \underbrace{\mathrm{H}\mathrm{H}\cdots\mathrm{H}}_{k \text{ times}} : \begin{array}{ccc} \mathcal{ST}(n) & \longrightarrow & \mathcal{ST}(n-3k) \\ (a_{i,j})_{1\leqslant i\leqslant j\leqslant n} & \longmapsto & (a_{k+i,2k+j})_{1\leqslant i\leqslant j\leqslant n-3k} \end{array}$$

Using the operators H^k and the generating index set G_R, we obtain a family of bases of $\mathcal{RST}(n)$, for all non-negative integers n.

Theorem 3.9. *Let* n *and* m *be non-negative integers such that* $m = \left\lfloor \frac{n}{3} \right\rfloor + \delta_{1,(n \bmod 3)}$. *For every* $k \in \{0, \ldots, m-1\}$, *let* $\nabla_k \in \mathcal{RST}(n)$ *such that* $\mathrm{H}^k(\nabla_k) = U_{n-3k}$. *Then, the set* $\{\nabla_0, \ldots, \nabla_{m-1}\}$ *is a basis of* $\mathcal{RST}(n)$.

The proof is based on the following

Lemma 3.10. *Let* $\nabla = (a_{i,j})_{1\leqslant i\leqslant j\leqslant n} \in \mathcal{ST}(n)$ *such that* $\mathrm{H}^k(\nabla) = U_{n-3k}$, *for some* $k \in \left\{0, \ldots, \left\lfloor \frac{n}{3} \right\rfloor - 1\right\}$. *Then,*

$$a_{i,n-\left\lfloor \frac{n}{3} \right\rfloor} = \begin{cases} 1 & \text{for } i = k+1, \\ 0 & \text{for } i \in \left\{k+2, \ldots, \left\lfloor \frac{n}{3} \right\rfloor\right\}. \end{cases}$$

Moreover, when $n \equiv 1 \bmod 3$, *if* $\mathrm{H}^{\left\lfloor \frac{n}{3} \right\rfloor}(\nabla) = U_1 = (1)$, *then* $a_{\left\lfloor \frac{n}{3} \right\rfloor+1,n-\left\lfloor \frac{n}{3} \right\rfloor} = 1$.

Proof of Lemma 3.10. Let $k \in \left\{0, \ldots, \left\lfloor \frac{n}{3} \right\rfloor - 1\right\}$. By definition of H^k and Proposition 3.7, we deduce that

$$(a_{k+i,2k+j})_{1\leqslant i\leqslant j\leqslant n-3k} = \mathrm{H}^k(\nabla) = U_{n-3k} = (\delta_{i,1} + \delta_{i,j} + \delta_{j,n-3k} \bmod 2)_{1\leqslant i\leqslant j\leqslant n-3k}. \quad (4)$$

Since $k \leqslant \left\lfloor \frac{n}{3} \right\rfloor - 1$, we have

$$2k+1 \leqslant 2\left\lfloor \frac{n}{3} \right\rfloor - 1 < n - \left\lfloor \frac{n}{3} \right\rfloor < n-k.$$

It follows from (4) that

$$a_{i,n-\left\lfloor \frac{n}{3} \right\rfloor} = a_{k+(i-k),2k+(n-\left\lfloor \frac{n}{3} \right\rfloor-2k)} \equiv \delta_{i-k,1}+\delta_{i-k,n-\left\lfloor \frac{n}{3} \right\rfloor-2k}+\delta_{n-\left\lfloor \frac{n}{3} \right\rfloor-2k,n-3k} \pmod 2, \quad (5)$$

for all integers i such that $1 \leqslant i-k \leqslant n-\left\lfloor \frac{n}{3} \right\rfloor-2k$, i.e., $k+1 \leqslant i \leqslant n-\left\lfloor \frac{n}{3} \right\rfloor-k$. Since $k \leqslant \left\lfloor \frac{n}{3} \right\rfloor-1$, we obtain that

$$\delta_{n-\left\lfloor \frac{n}{3} \right\rfloor-2k,n-3k} = \delta_{k,\left\lfloor \frac{n}{3} \right\rfloor} = 0. \quad (6)$$

Moreover, this leads to

$$n-\left\lfloor \frac{n}{3} \right\rfloor-k \geqslant n-2\left\lfloor \frac{n}{3} \right\rfloor+1 \geqslant \left\lfloor \frac{n}{3} \right\rfloor+1$$

and

$$\delta_{i-k,n-\left\lfloor \frac{n}{3} \right\rfloor-2k} = \delta_{i,n-\left\lfloor \frac{n}{3} \right\rfloor-k} = 0, \quad (7)$$

for all integers $i \leqslant \left\lfloor \frac{n}{3} \right\rfloor$. Therefore, for all $i \in \left\{k+1,\dots,\left\lfloor \frac{n}{3} \right\rfloor\right\}$, we obtain from (5), (6) and (7) that

$$a_{i,n-\left\lfloor \frac{n}{3} \right\rfloor} = \delta_{i-k,1} = \delta_{i,k+1}.$$

This completes the proof when $k \in \left\{0,\dots,\left\lfloor \frac{n}{3} \right\rfloor-1\right\}$.

Finally, when $n \equiv 1 \bmod 3$, it is clear that

$$(1) = U_1 = \mathrm{H}^{\frac{n-1}{3}}(\nabla) = \left(a_{\frac{n+2}{3},\frac{2n+1}{3}}\right) = \left(a_{\left\lfloor \frac{n}{3} \right\rfloor+1,n-\left\lfloor \frac{n}{3} \right\rfloor}\right).$$

This completes the proof. □

Proof of Theorem 3.9. We consider the set

$$G_R := \left\{\left(i, n-\left\lfloor \frac{n}{3} \right\rfloor\right) \;\middle|\; i \in \{1,2,\dots,m\}\right\}.$$

For any $k \in \{0,\dots,m-1\}$, since $\mathrm{H}^k(\nabla_k) = U_{n-3k}$, it follows from Lemma 3.10 that

$$\pi_{G_R}(\nabla_k) = (\underbrace{*,\dots,*}_{k},1,\underbrace{0,\dots,0}_{m-k-1}).$$

Therefore, the set $\{\pi_{G_R}(\nabla_k) \mid k \in \{0,\dots,m-1\}\}$ is a basis of $\{0,1\}^{G_R}$. Finally, since G_R is a generating index set of $\mathcal{RST}(n)$ by Corollary 3.4, we conclude that the set $\{\nabla_k \mid k \in \{0,\dots,m-1\}\}$ is a basis of $\mathcal{RST}(n)$. □

Since $U_{n-3k} = \rho((1)_{n-3k})$ by definition, for all non-negative integers n and k such that $3k \leqslant n$, this leads to the following

Corollary 3.11. *Let n and m be non-negative integers such that $m = \left\lfloor \frac{n}{3} \right\rfloor + \delta_{1,(n \bmod 3)}$. For every $k \in \{0,\dots,m-1\}$, let S_k be a binary sequence of length n such that $\partial^k S_k = (1)_{n-k}$. Then, the set $\{\rho(\nabla S_0),\dots,\rho(\nabla S_{m-1})\}$ is a basis of $\mathcal{RST}(n)$.*

Proof. Let $k \in \{0,\dots,m-1\}$. First, by definition of the linear map ρ, we know that $\rho(\nabla S_k) \in \mathcal{RST}(n)$. Moreover, since $\partial^k S_k = (1)_{n-k}$, it follows that

$$\mathrm{H}^k(\rho(\nabla S_k)) = \rho\left(\mathrm{H}^k(\nabla S_k)\right) = \rho(\nabla(1)_{n-3k}) = U_{n-3k}.$$

Therefore, from Theorem 3.9, the set $\{\rho(\nabla S_k) \mid k \in \{0,\dots,m-1\}\}$ is a basis of $\mathcal{RST}(n)$. □

Binomial coefficients $\binom{a}{b}$ have been defined before for any non-negative integer a and any integer b. Now, we extend this definition for negative integers a. For any integers a and b, let $\binom{a}{b}$ denote the integers recursively defined by

- $\binom{a}{0} = 1$, for all $a \in \mathbb{Z}$,
- $\binom{0}{b} = 0$, for all $b \in \mathbb{N}^*$,
- $\binom{a}{b} = \binom{a-1}{b-1} + \binom{a-1}{b}$, for all $a, b \in \mathbb{Z}$.

When a is non-negative, it corresponds with the previous definition. Moreover, for any negative integer a, the following equality holds

$$\binom{a}{b} = \begin{cases} 0 & \text{for } b < 0, \\ (-1)^b \binom{b-a-1}{b} & \text{for } b \geqslant 0. \end{cases}$$

In this paper, we mainly consider the *infinite Pascal matrix modulo* 2, that is, the doubly indexed sequence $\left(\binom{a}{b}_2\right)_{(a,b)\in\mathbb{Z}^2}$, where $\binom{a}{b}_2$ is the value of $\binom{a}{b}$ mod 2. The first few values of this doubly infinite sequence are shown in Figure 9, where the terms $\binom{a}{0}_2$ are in blue, for all integers a, and the terms $\binom{0}{b}_2$ are in red, for all positive integers b.

For any integers k and l, let $\mathrm{S}_{k,l}^{(n)}$ be the subsequence of length n of the kth column of the infinite Pascal matrix modulo 2 defined by

$$\mathrm{S}_{k,l}^{(n)} = \left(\binom{l+j-1}{k}_2\right)_{1\leqslant j\leqslant n} = \left(\binom{l}{k}_2, \binom{l+1}{k}_2, \ldots\ldots, \binom{l+n-1}{k}_2\right).$$

For instance, the sequence $\mathrm{S}_{5,2}^{(7)} = \left(\binom{j+1}{5}_2\right)_{1\leqslant j\leqslant 7} = (0001010)$ appears in yellow in Figure 9. Since we retrieve the local rule (LR) in the infinite Pascal matrix modulo 2, it is straightforward to obtain the following

Proposition 3.12. *Let k and l be two integers and let n be a positive integer. Then,*

$$\partial^i \mathrm{S}_{k,l}^{(n)} = \mathrm{S}_{k-i,l}^{(n-i)} = \left(\binom{l+j-1}{k-i}_2\right)_{1\leqslant j\leqslant n-i},$$

for all $i \in \{0, \ldots, n-1\}$ and

$$\nabla \mathrm{S}_{k,l}^{(n)} = \left(\binom{l+j-i}{k+1-i}_2\right)_{1\leqslant i\leqslant j\leqslant n}.$$

For instance, the Steinhaus triangle $\nabla \mathrm{S}_{10,-10}^{(7)} = \nabla(0000110)$ appears in green in Figure 9.

We are now ready for giving explicit bases of $\mathcal{RST}(n)$, for every non-negative integer n, using Corollary 3.11 with binary sequences $\mathrm{S}_{k,l}^{(n)}$.

Theorem 3.13. *Let n and m be non-negative integers such that $m = \left\lfloor \frac{n}{3} \right\rfloor + \delta_{1,(n \bmod 3)}$. For any integers $l_0, \ldots, l_{m-1}$, the set $\left\{\rho\left(\nabla \mathrm{S}_{0,l_0}^{(n)}\right), \ldots, \rho\left(\nabla \mathrm{S}_{m-1,l_{m-1}}^{(n)}\right)\right\}$ is a basis of $\mathcal{RST}(n)$. Moreover, we have*

$$\rho\left(\nabla \mathrm{S}_{k,l_k}^{(n)}\right) = \left(\binom{l_k+j-i}{k+1-i} + \binom{l_k+n-j}{k+i-j} + \binom{l_k+i-1}{k+j-n} \bmod 2\right)_{1\leqslant i\leqslant j\leqslant n}, \quad (8)$$

for all $k \in \{0, \ldots, m-1\}$.

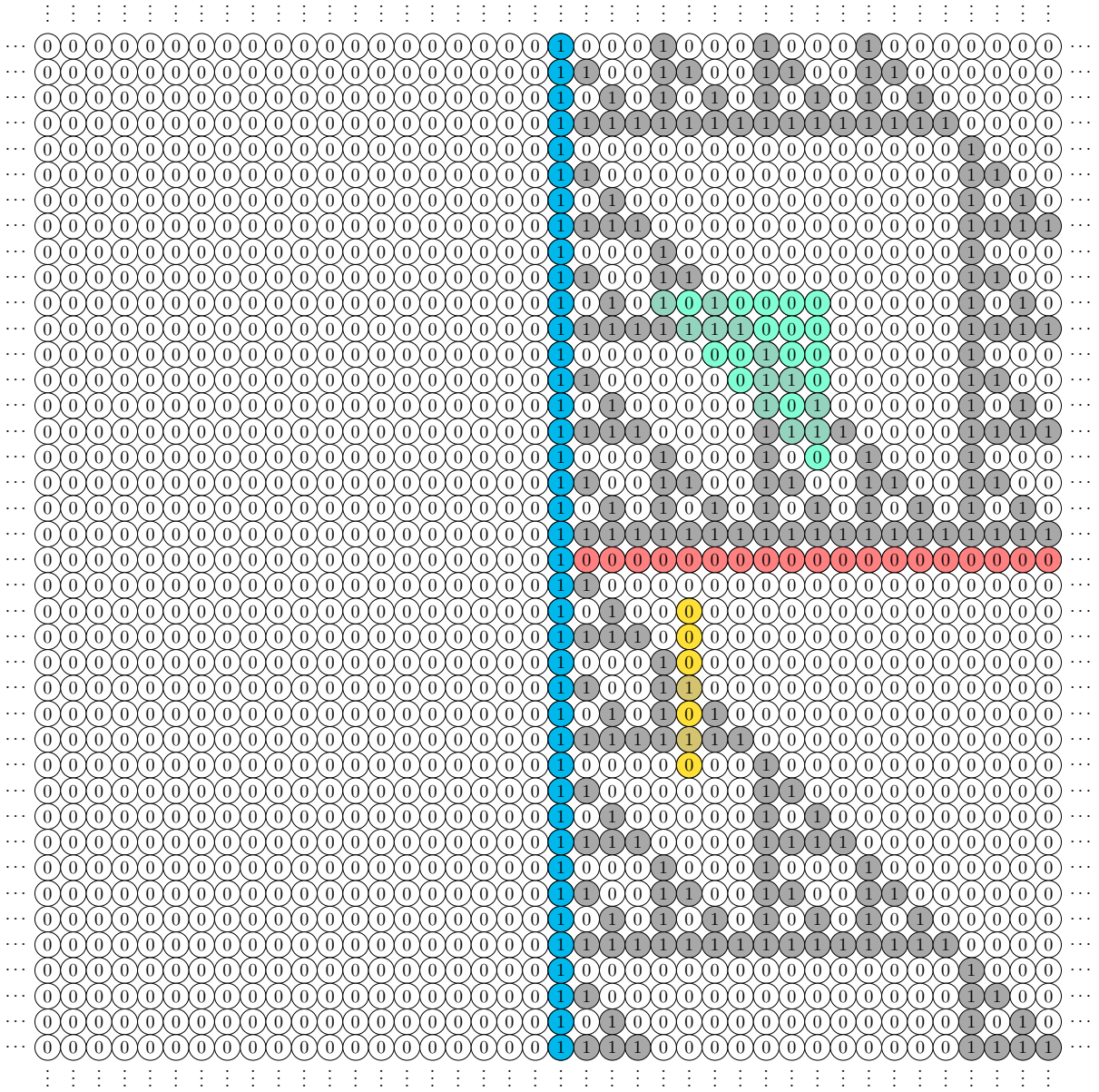


Figure 9: The infinite Pascal matrix modulo 2 with $\left(\binom{a}{0}_2\right)_{a\in\mathbb{Z}}$ in blue, $\left(\binom{0}{b}_2\right)_{b>0}$ in red, $\mathrm{S}^{(7)}_{5,2}$ in yellow and $\nabla\mathrm{S}^{(7)}_{10,-10}$ in green

Proof. By Proposition 3.12, we know that $\partial^k \mathrm{S}_{k,l_k}^{(n)} = \mathrm{S}_{0,l_k}^{(n-k)} = \left(\binom{l_k+j-1}{0}_2\right)_{1\leqslant j\leqslant n-k} = (1)_{n-k}$, for all $k \in \{0,\dots,m-1\}$. It follows from Corollary 3.11 that the set

$$\left\{\rho\left(\nabla \mathrm{S}_{k,l_k}^{(n)}\right) \;\middle|\; k \in \{0,\dots,m-1\}\right\}$$

is a basis of $\mathcal{RST}(n)$. Moreover, the formula of $\rho\left(\nabla \mathrm{S}_{k,l_k}^{(n)}\right)$ given in (8) directly comes from Proposition 3.12 and Proposition 3.6. □

Remark. For any integer l_0, we have $\rho\left(\nabla \mathrm{S}_{0,l_0}^{(n)}\right) = \rho\left(\nabla (1)_n\right) = U_n$.

For instance, for $n = 10$ and $l_0 = l_1 = l_2 = l_3 = 0$, we obtain

k	$\mathrm{S}_{k,0}^{(10)}$	$\rho\left(\nabla \mathrm{S}_{k,0}^{(10)}\right)$
0	(1111111111)	$\nabla_0 = \nabla(0111111110)$
1	(0101010101)	$\nabla_1 = \nabla(1001010111)$
2	(0011001100)	$\nabla_2 = \nabla(0001001000)$
3	(0001000100)	$\nabla_3 = \nabla(0010001100)$

All the rotationally symmetric Steinhaus triangles of size 10 are depicted in Figure 10, where the elements of the basis $\{\nabla_0, \nabla_1, \nabla_2, \nabla_3\}$ are in red and, for every $\nabla \in \mathcal{RST}(10)$, the coordinate vector (x_0, x_1, x_2, x_3) of $\nabla = x_0\nabla_0 + x_1\nabla_1 + x_2\nabla_2 + x_3\nabla_3$ is given.

4 Horizontally symmetric Steinhaus triangles

In this section, we characterize the horizontally symmetric Steinhaus triangles and we give a generating index set of $\mathcal{HST}(n)$. This permits us to obtain bases of $\mathcal{HST}(n)$, for all non-negative integers n.

4.1 Characterizations of $\mathcal{HST}(n)$

A binary sequence $S = (a_j)_{1\leqslant j\leqslant n}$ is said to be *symmetric* if $a_{n-j+1} = a_j$, for all $j \in \{1,\dots,n\}$. For instance, the sequence (010010010) is symmetric. As shown in the following result, the symmetry is preserved under the derivation process.

Proposition 4.1. *The binary sequence S is symmetric if and only if ∂S is symmetric and $\sigma_2(\partial S) = 0$.*

Proof. Let $S = (a_j)_{1\leqslant j\leqslant n}$ and $\partial S = (b_j)_{1\leqslant j\leqslant n-1}$. Then, by definition of ∂S, we have

$$\begin{aligned} b_i + b_{(n-1)-i+1} &= b_i + b_{n-i} \\ &\equiv (a_i + a_{i+1}) + (a_{n-i} + a_{n-i+1}) \pmod 2 \\ &= (a_i + a_{n-i+1}) + (a_{i+1} + a_{n-(i+1)+1}), \end{aligned} \tag{9}$$

for all $i \in \{1,\dots,n-1\}$. Moreover, the sum $\sigma(\partial S)$ satisfies

$$\sigma(\partial S) = \sum_{i=1}^{n-1} b_i \equiv \sum_{i=1}^{n-1} (a_i + a_{i+1}) \equiv a_1 + a_n \pmod 2. \tag{10}$$

First, if S is symmetric, we deduce from (9) that ∂S is also symmetric and from (10) that $\sigma(\partial S)$ is even. Conversely, if we suppose that ∂S is symmetric of even sum $\sigma(\partial S)$, we know from (10) that $a_1 = a_n$. Using this equality and (9), we can prove, by induction on i, that $a_i = a_{n-i+1}$, for all $i \in \{1,\dots,n\}$. This completes the proof. □

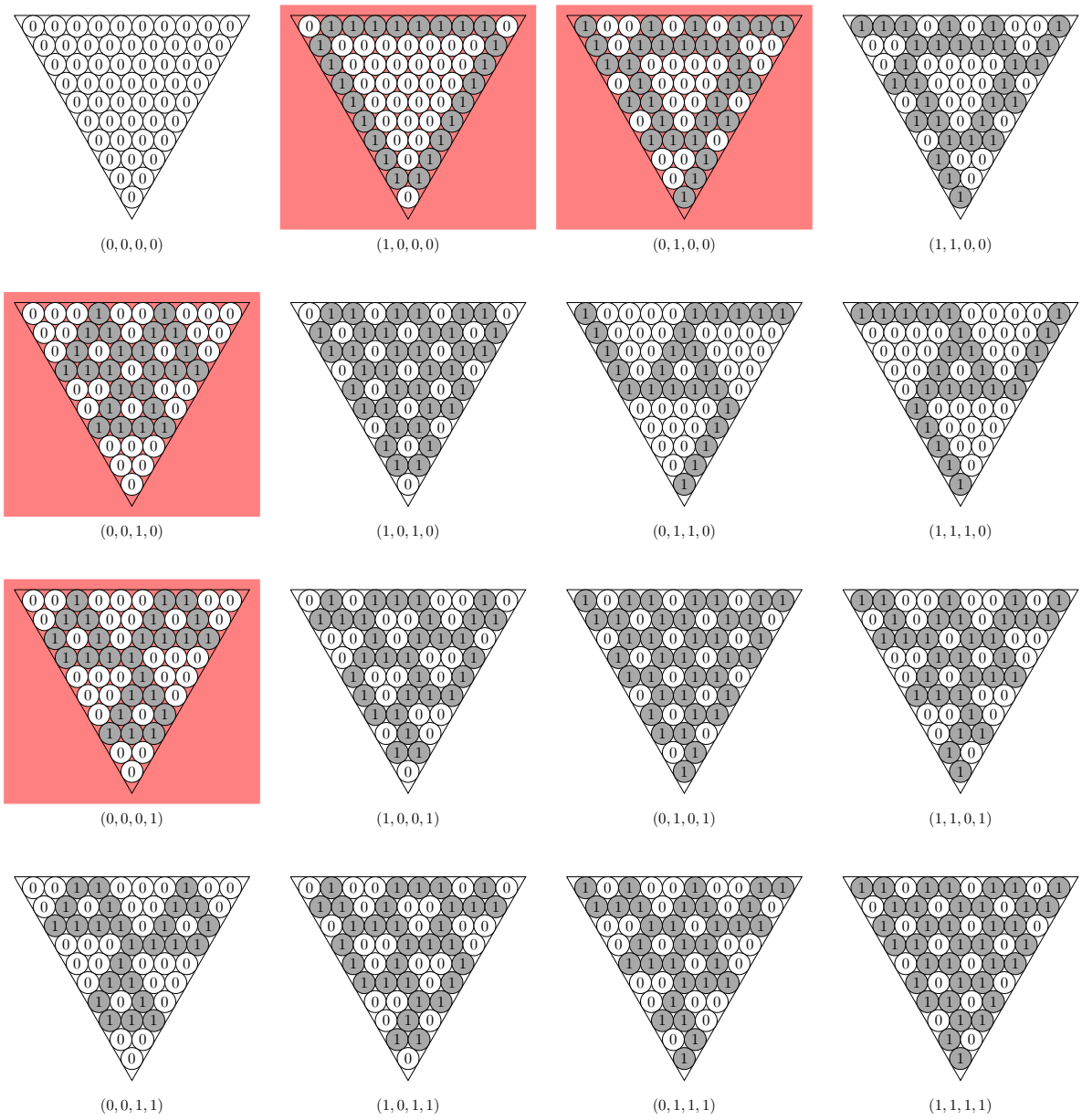


Figure 10: The 16 triangles of $\mathcal{RST}(10)$ where the 4 red triangles form a basis

Remark. It is natural to ask if there exists a similar result for the case where ∂S is symmetric and $\sigma(\partial S)$ is odd. It is known that the binary sequence $S = (a_j)_{1 \leqslant j \leqslant n}$ is antisymmetric, i.e., $a_{n-j+1} \equiv a_j + 1 \bmod 2$ for all $i \in \{1, \ldots, n\}$, if and only if ∂S is symmetric and $\sigma_2(\partial S) = 1$. Note that antisymmetric binary sequences only exist for odd lengths.

It follows that the horizontal symmetry of a Steinhaus triangle is only related to the symmetry of its first row.

Proposition 4.2. *The Steinhaus triangle ∇S is horizontally symmetric if and only if the sequence S is symmetric.*

Proof. First, it is clear that if ∇S is horizontally symmetric, then S is a symmetric sequence. Conversely, if we suppose that S is symmetric, we know from Proposition 4.1 that the iterated derived sequences $\partial^i S$ are symmetric, for all $i \in \{0, \ldots, n-1\}$. Therefore, the Steinhaus triangle ∇S is horizontally symmetric. □

Now, we show that the horizontal symmetry of a Steinhaus triangle only depends on the values of middle terms of its rows of odd lengths.

Proposition 4.3. *The Steinhaus triangle $(a_{i,j})_{1 \leqslant i \leqslant j \leqslant n}$, of size n, is horizontally symmetric if and only if $a_{n-2i,n-i} = 0$, for all $i \in \left\{0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1\right\}$.*

The proof is based on the following lemma which is straightforward from the definition of a symmetric sequence.

Lemma 4.4. *Let $S = (a_1, a_2, \ldots, a_n)$ be a symmetric binary sequence of length n. Then,*

$$\sigma_2(S) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ a_{\frac{n+1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

Proof of Proposition 4.3. First, suppose that the Steinhaus triangle $\nabla S = (a_{i,j})_{1 \leqslant i \leqslant j \leqslant n}$ of size n is horizontally symmetric. Then, since the iterated derived sequences $\partial^i S = (a_{i+1,j})_{i+1 \leqslant j \leqslant n}$ are symmetric, for all $i \in \{0, \ldots, n-1\}$, we know from Proposition 4.1 that $\sigma_2(\partial^i S) = 0$, for all $i \in \{1, \ldots, n-1\}$. Moreover, for any $i \in \{1, 2, \ldots, n\}$, the sequence $\partial^{n-i} S$ is of length i. It follows from Lemma 4.4 that $a_{n-2i,n-i} = \sigma_2\left(\partial^{n-(2i+1)} S\right) = 0$, for all $i \in \left\{0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1\right\}$.

Conversely, suppose that $a_{n-2i,n-i} = 0$, for all $i \in \left\{0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1\right\}$. We proceed by induction on n. For $n = 1$, the result is clear since any Steinhaus triangle of size $n = 1$ is horizontally symmetric. Suppose that the result is true for any Steinhaus triangle of size strictly lesser than n. We consider the subtriangle $\nabla \partial S = (a_{i+1,j+1})_{1 \leqslant i \leqslant j \leqslant n-1}$ of size $n-1$. Since the identities $a_{1+(n-1)-2i,1+(n-1)-i} = 0$ hold, for all $i \in \left\{0, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor - 1\right\}$, it follows that $\nabla \partial S$ is horizontally symmetric by induction hypothesis. Therefore, the sequence $\partial^i S$ is symmetric, for all $i \in \{1, \ldots, n-1\}$. If n is odd, then ∂S is symmetric and $\sigma_2(\partial S) = 0$ by Lemma 4.4. Otherwise, if n is even, then ∂S is symmetric and $\sigma_2(\partial S) = a_{2,\frac{n}{2}+1} = 0$ by Lemma 4.4 again. It follows from Proposition 4.1 that S symmetric. Therefore, in any case, the Steinhaus triangle ∇S is horizontally symmetric by Proposition 4.2. □

4.2 Generating index set of $\mathcal{HST}(n)$

Proposition 4.5. *Let n be a non-negative integer. The set*

$$G_H := \left\{(1,j) \;\middle|\; j \in \left\{1,\dots,\left\lceil \frac{n}{2} \right\rceil\right\}\right\}$$

is a generating index set of $\mathcal{HST}(n)$.

Proof. From Proposition 4.2, we deduce that $\mathcal{HST}(n)$ is isomorphic to the vector space of symmetric binary sequences of length n. Obviously, a symmetric sequence of length n is entirely determined by its $\left\lceil \frac{n}{2} \right\rceil$ first terms. □

Since the dimension of $\mathcal{HST}(n)$ corresponds to the cardinality of the generating index set G_H, it is straightforward to obtain the following

Corollary 4.6. $\dim \mathcal{HST}(n) = \left\lceil \frac{n}{2} \right\rceil$, *for all non-negative integers n.*

4.3 Bases of $\mathcal{HST}(n)$

Let n be a positive integer. For any positive integer $k \in \{1,\dots,n\}$, we denote by $\mathrm{E}_k^{(n)}$ the binary sequence of length n consisting only of zeroes, except at position k. In other words, we have

$$\mathrm{E}_k^{(n)} = \left(\binom{0}{j-k}_2\right)_{1\leqslant j\leqslant n},$$

for all $k \in \{1,\dots,n\}$. Since we retrieve the local rule (LR) of the infinite Pascal matrix modulo 2, we obtain that

$$\nabla \mathrm{E}_k^{(n)} = \left(\binom{i-1}{j-k}_2\right)_{1\leqslant i\leqslant j\leqslant n},$$

for all integers $k \in \{1,\dots,n\}$.

Proposition 4.7. *Let n be a positive integer. Then, the set $\left\{\nabla_1,\dots,\nabla_{\left\lceil \frac{n}{2} \right\rceil}\right\}$ is a basis of $\mathcal{HST}(n)$, where*

$$\nabla_k = \nabla\left(\mathrm{E}_k^{(n)} + \mathrm{E}_{n-k+1}^{(n)}\right) = \left(\binom{i-1}{j-k} + \binom{i-1}{j-n+k-1} \bmod 2\right)_{1\leqslant i\leqslant j\leqslant n},$$

for all $k \in \left\{1,\dots,\left\lfloor \frac{n}{2} \right\rfloor\right\}$, and

$$\nabla_{\frac{n+1}{2}} = \nabla \mathrm{E}_{\frac{n+1}{2}}^{(n)} = \left(\binom{i-1}{j-\frac{n+1}{2}}_2\right)_{1\leqslant i\leqslant j\leqslant n},$$

when n is odd.

Proof. First, by definition, it is clear that $\nabla_k \in \mathcal{HST}(n)$, for all $k \in \left\{1,\dots,\left\lceil \frac{n}{2} \right\rceil\right\}$. Now, we consider the set $G_H := \left\{(1,j) \;\middle|\; j \in \left\{1,\dots,\left\lceil \frac{n}{2} \right\rceil\right\}\right\}$. Since

$$\pi_{G_H}(\nabla_k) = \mathrm{E}_k^{\left(\left\lceil \frac{n}{2} \right\rceil\right)},$$

for all $k \in \left\{1,\dots,\left\lceil \frac{n}{2} \right\rceil\right\}$, it follows that the set $\left\{\pi_{G_H}(\nabla_1),\dots,\pi_{G_H}\left(\nabla_{\left\lceil \frac{n}{2} \right\rceil}\right)\right\}$ is a basis of $\{0,1\}^{G_H}$. Moreover, since G_H is a generating index set of $\mathcal{HST}(n)$ by Proposition 4.5, we conclude that the set $\left\{\nabla_1,\dots,\nabla_{\left\lceil \frac{n}{2} \right\rceil}\right\}$ is a basis of $\mathcal{HST}(n)$. □

In the end of this section, we show that the generating index set G_H also permits us to obtain a basis from the sequences $\mathrm{S}_{k,l}^{(n)} = \left(\binom{l+j-1}{k}_2\right)_{1\leqslant j\leqslant n}$ introduced in Section 3.

Lemma 4.8. *Let k and n be two positive integers of same parity. Then, the n-length sequence $\mathrm{S}_{k,l}^{(n)}$ is symmetric for $l = \frac{k-n}{2}$. Moreover, the $k+2$ middle terms of $\mathrm{S}_{k,\frac{k-n}{2}}^{(n)}$ are of the form $(*\cdots* 1 \underbrace{00\cdots00}_{k} 1 *\cdots*)$.*

Proof. Suppose that k and n are of same parity, i.e., $l = \frac{k-n}{2}$ is an integer. Let $\mathrm{S}_{k,l}^{(n)} = (a_j)_{1\leqslant j\leqslant n}$ with $a_j = \binom{l+j-1}{k}_2$, for all $j \in \{1,\ldots,n\}$. Since $\binom{a}{b}_2 = \binom{b-a-1}{b}_2$, for any integers a and b such that $b \geqslant 0$, it follows that

$$a_{n-j+1} = \binom{\frac{k-n}{2} + (n-j+1) - 1}{k}_2 = \binom{k - \left(\frac{k-n}{2} + n - j\right) - 1}{k}_2 = \binom{\frac{k-n}{2} + j - 1}{k}_2 = a_j,$$

for all $j \in \{1,\ldots,n\}$. Therefore, the sequence $\mathrm{S}_{k,l}^{(n)}$ is symmetric for $l = \frac{k-n}{2}$. Moreover, the $k+2$ middle terms of $\mathrm{S}_{k,l}^{(n)}$ are $\binom{j}{k}_2$, for $j \in \{-1,\ldots,k\}$, where $\binom{-1}{k}_2 = \binom{k}{k}_2 = 1$ and $\binom{j}{k}_2 = 0$, for all $j \in \{0,\ldots,k-1\}$, since $k \geqslant 1$. This completes the proof. □

Theorem 4.9. *Let n be a non-negative integer. The set $\left\{\nabla_1,\ldots,\nabla_{\lceil\frac{n}{2}\rceil}\right\}$ is a basis of $\mathcal{HST}(n)$, where*

$$\nabla_k = \nabla \mathrm{S}_{n-2k,-k}^{(n)} = \left(\binom{-k+j-i}{n-2k+1-i}_2\right)_{1\leqslant i\leqslant j\leqslant n},$$

for all $k \in \left\{1,\ldots,\lfloor\frac{n}{2}\rfloor\right\}$, and $\nabla_{\frac{n+1}{2}} = \nabla(1)_n$, when n is odd.

Proof. First, we know from Lemma 4.8 that the sequence $\mathrm{S}_{n-2k,-k}^{(n)}$ is symmetric, for all $k \in \left\{1,\ldots,\lfloor\frac{n}{2}\rfloor\right\}$. Obviously, when n is odd, the constant sequence $(1)_n$ is also symmetric. Therefore $\nabla_k \in \mathcal{HST}(n)$, for all $k \in \left\{1,\ldots,\lceil\frac{n}{2}\rceil\right\}$, by Proposition 4.2. Now, we consider the set $G_H := \left\{(1,j) \;\middle|\; j \in \left\{1,\ldots,\lceil\frac{n}{2}\rceil\right\}\right\}$. By Lemma 4.8 again, we know that the $n-2k+2$ middle terms of $\mathrm{S}_{n-2k,-k}^{(n)}$ are of the form $(\underbrace{*\cdots*}_{k-1} 1 \underbrace{00\cdots00}_{n-2k} 1 \underbrace{*\cdots*}_{k-1})$ and thus

$$\pi_{G_H}(\nabla_k) = (\underbrace{*,\cdots,*}_{k-1}, 1, \underbrace{0,\cdots,0}_{\lceil\frac{n}{2}\rceil-k}),$$

for all $k \in \left\{1,\ldots,\lfloor\frac{n}{2}\rfloor\right\}$. Moreover, $\pi_{G_H}\left(\nabla_{\frac{n+1}{2}}\right) = \pi_{G_H}(\nabla(1)_n) = (1,1,\ldots,1)$, when n is odd. Therefore, the set $\left\{\pi_{G_H}(\nabla_k) \;\middle|\; k \in \left\{1,\ldots,\lceil\frac{n}{2}\rceil\right\}\right\}$ is a basis of $\{0,1\}^{G_H}$. Finally, since G_H is a generating index set of $\mathcal{HST}(n)$ by Proposition 4.5, we conclude that the set $\left\{\nabla_1,\ldots,\nabla_{\lceil\frac{n}{2}\rceil}\right\}$ is a basis of $\mathcal{HST}(n)$. □

Remark. When n is even, we have $\nabla_{\frac{n}{2}} = \nabla \mathrm{S}_{0,-\frac{n}{2}}^{(n)} = \nabla(1)_n$. Therefore, $\nabla_{\lceil\frac{n}{2}\rceil} = \nabla(1)_n$, for all integers n.

For instance, for $n = 7$, we obtain

$$\nabla_1 = \nabla \mathrm{S}_{5,-1}^{(7)} = \nabla(1000001), \quad \nabla_2 = \nabla \mathrm{S}_{3,-2}^{(7)} = \nabla(0100010),$$
$$\nabla_3 = \nabla \mathrm{S}_{1,-3}^{(7)} = \nabla(1010101), \quad \nabla_4 = \nabla(1111111).$$

All the horizontally symmetric Steinhaus triangles of size 7 are depicted in Figure 11, where the elements of the basis $\{\nabla_1,\nabla_2,\nabla_3,\nabla_4\}$ are in red and, for every $\nabla \in \mathcal{HST}(7)$, the coordinate vector (x_1,x_2,x_3,x_4) of $\nabla = x_1\nabla_1 + x_2\nabla_2 + x_3\nabla_3 + x_4\nabla_4$ is given.

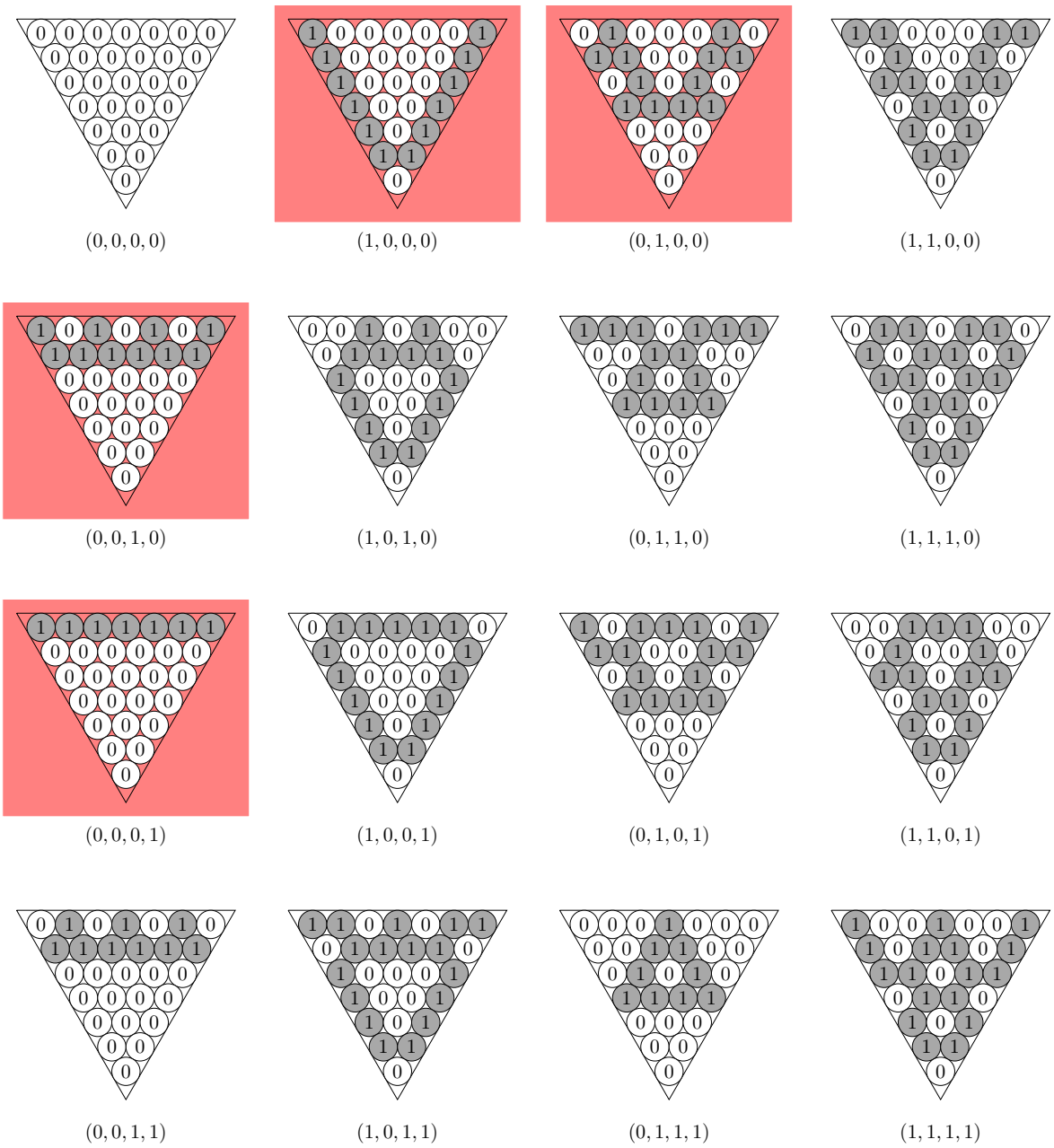


Figure 11: The 16 triangles of $\mathcal{HST}(7)$ where the 4 red triangles form a basis

5 Dihedrally symmetric Steinhaus triangles

In this section, after characterizing dihedrally symmetric Steinhaus triangles, we determine, for all non-negative integers n, generating index sets and a basis of $\mathcal{DST}(n)$.

5.1 Characterizations of $\mathcal{DST}(n)$

We begin by showing that the dihedral symmetry of a Steinhaus triangle is only related to the symmetry of its first row and of its right and left sides.

Proposition 5.1. *The Steinhaus triangle ∇ is dihedrally symmetric if and only if two of the three Steinhaus triangles ∇, $r(\nabla)$ and $r^2(\nabla)$ are horizontally symmetric.*

Proof. Directly comes from the fact that the automorphisms h and hr are generators of D_3. First, if ∇ is dihedrally symmetric, then $\nabla = r(\nabla) = r^2(\nabla)$. Since a dihedrally symmetric triangle is also horizontally symmetric, the result follows.

Suppose now that the Steinhaus triangles ∇ and $r(\nabla)$ are horizontally symmetric. It follows that $h(\nabla) = \nabla$ and $hr(\nabla) = r(\nabla)$. Combining these identities with the relations $r^3 = h^2 = hrhr = id_{\mathcal{ST}(n)}$, we obtain that

$$\begin{aligned}\nabla = r^2(r(\nabla)) &= r^2(hr(\nabla)) \\ &= r^2h(r(\nabla)) = hr(r(\nabla)) = hr^2(\nabla) = rh(\nabla) = r(h(\nabla)) = r(\nabla).\end{aligned}$$

We conclude that the Steinhaus triangle ∇ is dihedrally symmetric since it is horizontally symmetric and rotationally symmetric. If the two horizontally symmetric Steinhaus triangles are $r^2(\nabla)$ and ∇, or $r(\nabla)$ and $r^2(\nabla)$, we can show by the same way that the triangles $r^2(\nabla)$ or $r(\nabla)$ are dihedrally symmetric, since $r(r^2(\nabla)) = \nabla$ and $r(r(\nabla)) = r^2(\nabla)$, respectively. In any case, we obtain that the Steinhaus triangle $\nabla = r(\nabla) = r^2(\nabla)$ is dihedrally symmetric. □

Corollary 5.2. *The Steinhaus triangle $\nabla S = (a_{i,j})_{1\leqslant i\leqslant j\leqslant n}$ is dihedrally symmetric if and only if two of the three sequences, its first row $(a_{1,j})_{1\leqslant j\leqslant n}$, its right side $(a_{j,n})_{1\leqslant j\leqslant n}$ or its left side $(a_{j,j})_{1\leqslant j\leqslant n}$, are symmetric.*

Proof. Directly comes from Proposition 5.1, since, by Proposition 4.2, we know that the Steinhaus triangles ∇S, $r(\nabla S)$ and $r^2(\nabla S)$ are horizontally symmetric if and only if the sequences $(a_{1,j})_{1\leqslant j\leqslant n}$, $(a_{j,n})_{1\leqslant j\leqslant n}$ and $(a_{j,j})_{1\leqslant j\leqslant n}$ are symmetric, respectively. □

Proposition 5.1 also permits us to show that the dihedral symmetry of a Steinhaus triangle only depends on the values of middle terms of its rows, its columns or its diagonals of odd lengths.

Corollary 5.3. *The Steinhaus triangle $\nabla S = (a_{i,j})_{1\leqslant i\leqslant j\leqslant n}$ is dihedrally symmetric if and only if two of the three sets $\left\{a_{n-2i,n-i} \middle| i \in \left\{0,\ldots,\left\lfloor\frac{n}{2}\right\rfloor - 1\right\}\right\}$, $\left\{a_{i,2i-1} \middle| i \in \left\{1,\ldots,\left\lfloor\frac{n}{2}\right\rfloor\right\}\right\}$ and $\left\{a_{i,n-i+1} \middle| i \in \left\{1,\ldots,\left\lfloor\frac{n}{2}\right\rfloor\right\}\right\}$ are sets of zeroes.*

Proof. From Proposition 4.3, we know that the Steinhaus triangles ∇S, $r(\nabla S)$ and $r^2(\nabla S)$ are horizontally symmetric if and only if $a_{n-2i,n-i} = 0$ for all $i \in \left\{0,\ldots,\left\lfloor\frac{n}{2}\right\rfloor - 1\right\}$, $a_{i,2i-1} = 0$ for all $i \in \left\{1,\ldots,\left\lfloor\frac{n}{2}\right\rfloor\right\}$ and $a_{i,n-i+1} = 0$ for all $i \in \left\{1,\ldots,\left\lfloor\frac{n}{2}\right\rfloor\right\}$, respectively. Finally, by Proposition 5.1, the Steinhaus triangle ∇S is dihedrally symmetric if and only if two of the three Steinhaus triangles ∇S, $r(\nabla S)$ and $r^2(\nabla S)$ are horizontally symmetric. This completes the proof. □

For any positive integer $n \geqslant 3$, by definition of $\mathcal{DST}(n)$ and H, it is clear that $\mathrm{H}(\mathcal{DST}(n)) \subset \mathcal{DST}(n-3)$. The precise relationship between a dihedrally symmetric Steinhaus triangle ∇S and its subtriangle $\mathrm{H}(\nabla S)$ is given in the following

Proposition 5.4. *Let S be a binary sequence of length $n \geqslant 3$. The Steinhaus triangle ∇S is dihedrally symmetric if and only if $\mathrm{H}(\nabla S) = \nabla S'$ is dihedrally symmetric, $\sigma_2(S') = 0$ and $S = (0) \cdot \int_{i,x} S' \cdot (0)$, for some $i \in \{1, \ldots, n-2\}$ and some $x \in \{0, 1\}$.*

Proposition 5.4 appears in [4] in a more general context. For the convenience of the reader, a proof is given here.

Proof. Let $\nabla S = (a_{i,j})_{1 \leqslant i \leqslant j \leqslant n} \in \mathcal{ST}(n)$ such that $\nabla S' = \mathrm{H}(\nabla S) = (a_{1+i,2+j})_{1 \leqslant i \leqslant j \leqslant n-3} \in \mathcal{DST}(n-3)$. From Proposition 3.2, Proposition 4.2 and Proposition 4.1, we deduce that the Steinhaus triangle ∇S is dihedrally symmetric if and only if $a_{1,1} = a_{1,n} = \sigma_2(S')$, ∂S is symmetric and $\sigma_2(\partial S) = 0$. Since $\nabla S' \in \mathcal{DST}(n-3)$, the sequence S' is symmetric. Therefore the sequence ∂S is symmetric if and only if $a_{2,1} = a_{2,n}$. Moreover, it is clear that $\sigma(\partial S) \equiv \sigma(S') + a_{2,1} + a_{2,n} \bmod 2$. We claim that $a_{1,1} = a_{1,n} = \sigma_2(S')$, ∂S is symmetric and $\sigma_2(\partial S) = 0$ if and only if $a_{1,1} = a_{1,n} = \sigma_2(S') = 0$. First, suppose that $a_{1,1} = a_{1,n} = \sigma_2(S')$, ∂S is symmetric and $\sigma_2(\partial S) = 0$. Since ∂S is symmetric, it follows that $a_{2,1} = a_{2,n}$ and thus $\sigma_2(S') = \sigma_2(\partial S) = 0$. Conversely, suppose that $a_{1,1} = a_{1,n} = \sigma_2(S') = 0$. Since S' is symmetric and $\sigma_2(S') = 0$, we know from Proposition 4.1 that its antiderived sequence $\int_{i,x} S'$ is symmetric too. Moreover, we have $a_{1,1} = a_{1,n}$. It follows that the sequence S is symmetric and, by Proposition 4.1 again, we obtain that the sequence ∂S is symmetric and $\sigma_2(\partial S) = 0$. This completes the proof. □

For any non-negative integer n, the set of dihedrally symmetric Steinhaus triangles ∇S of size n with $\sigma(S)$ even is denoted by $\mathcal{DST}_0(n)$. It is clear that $\mathcal{DST}_0(n)$ is a linear subspace of $\mathcal{DST}(n)$. Moreover, the vector space $\mathcal{DST}(n)$ can be expressed in function of its linear subspace $\mathcal{DST}_0(n)$.

Proposition 5.5. *Let n be a non-negative integer. Then, we have*

$$\mathcal{DST}(n) = \begin{cases} \mathcal{DST}_0(n) & \text{for } n \text{ even,} \\ \mathcal{DST}_0(n) \sqcup (\mathcal{DST}_0(n) + U_n) & \text{for } n \text{ odd,} \end{cases}$$

where $\sqcup$ is the disjoint union of two sets.

Proof. For any symmetric sequence $S = (a_j)_{1 \leqslant j \leqslant n}$ of length n, we know from Lemma 4.4 that $\sigma_2(S) = 0$ when n is even and $\sigma_2(S) = a_{\frac{n+1}{2}}$ when n is odd. It follows that $\mathcal{DST}(n) = \mathcal{DST}_0(n)$, for n even. If n is odd, then we consider $U_n = \rho((1)_n)$. It is clear that $U_n \in \mathcal{DST}(n)$ and is generated from a sequence of odd sum, for all odd numbers n. Since $\nabla S \in \mathcal{DST}(n)$ with $\sigma_2(S) = 1$ if and only if $\nabla S + U_n \in \mathcal{DST}_0(n)$, it follows that $\mathcal{DST}(n) = \mathcal{DST}_0(n) \sqcup (\mathcal{DST}_0(n) + U_n)$, when n is odd. □

Using Lemma 4.4, it is straightforward to obtain from Propositon 5.4 the following

Corollary 5.6. *Let S be a binary sequence of length $n \geqslant 3$. For n even, the Steinhaus triangle ∇S is in $\mathcal{DST}_0(n)$ if and only if $\mathrm{H}(\nabla S) = \nabla S'$ is in $\mathcal{DST}_0(n-3)$ and $S = (0) \cdot \int_{i,x} S' \cdot (0)$, for some $i \in \{1, \ldots, n-2\}$ and some $x \in \{0, 1\}$. For n odd, the Steinhaus triangle ∇S is in $\mathcal{DST}_0(n)$ if and only if $\mathrm{H}(\nabla S) = \nabla S'$ is in $\mathcal{DST}_0(n-3)$ and $S = (0) \cdot \int_{\frac{n-1}{2},0} S' \cdot (0)$.*

5.2 Generating index sets of $\mathcal{DST}(n)$

We begin this subsection by giving, from Corollary 5.6, generating index sets of $\mathcal{DST}_0(n)$, for all non-negative integers n.

Theorem 5.7. *Let n and m be non-negative integers such that $m = \left\lfloor \frac{n}{6} \right\rfloor + \delta_{4,(n \bmod 6)}$. For every integer $i \in \left\{1, \ldots, \left\lfloor \frac{n}{3} \right\rfloor\right\}$, let $j_i \in \{2i, \ldots, n-i\}$. Then, the set*

$$G_{D_0} = \{(2i+1, j_{2i+1}) \mid i \in \{0, \ldots, m-1\}\},$$

when n is even, or

$$G_{D_0} = \{(2i, j_{2i}) \mid i \in \{1, \ldots, m\}\},$$

when n is odd, is a generating index set of $\mathcal{DST}_0(n)$.

Proof. By induction on n.

For $n \in \{0, 1, 2\}$, it is clear that $\nabla(0)_n$ is the only element of $\mathcal{DST}_0(n)$. Therefore the empty set $\emptyset$ is a generating set of $\mathcal{DST}_0(n)$, for these values of n.

Suppose now that the result is true for any size strictly lesser than $n \geqslant 3$. Let $m = \left\lfloor \frac{n}{6} \right\rfloor + \delta_{4,(n \bmod 6)}$. We distinguish two cases following the parity of n.

Case 1. Suppose first that n is odd. We consider the subset $\mathrm{H}(G_{D_0}) \subset \nabla(n-3)$ defined by

$$\mathrm{H}(G_{D_0}) = \{(2i-1, j_{2i}-2) \mid i \in \{1, \ldots, m\}\}$$

and the linear maps f_1 and f_2 defined by

$$\begin{array}{rccc} f_1 : & \mathcal{DST}_0(n) & \longrightarrow & \mathcal{DST}_0(n-3) \\ & \nabla S & \longmapsto & \mathrm{H}(\nabla S) \end{array}$$

and

$$\begin{array}{rccc} f_2 : & \mathcal{DST}_0(n-3) & \longrightarrow & \{0,1\}^m \\ & \nabla S' & \longmapsto & \pi_{\mathrm{H}(G_{D_0})}(\nabla S') \end{array}$$

Then, for any $(a_{i,j})_{1 \leqslant i \leqslant j \leqslant n} \in \mathcal{DST}_0(n)$, we have

$$\begin{aligned} f_2 f_1 \left((a_{i,j})_{1 \leqslant i \leqslant j \leqslant n}\right) &= f_2\left((a_{1+i,2+j})_{1 \leqslant i \leqslant j \leqslant n-3}\right) \\ &= \pi_{\mathrm{H}(G_{D_0})}\left((a_{1+i,2+j})_{1 \leqslant i \leqslant j \leqslant n-3}\right) \\ &= (a_{2i,j_{2i}})_{1 \leqslant i \leqslant m} = \pi_{G_{D_0}}\left((a_{i,j})_{1 \leqslant i \leqslant j \leqslant n}\right). \end{aligned}$$

Therefore $f_2 f_1 = \pi_{G_{D_0}}$. From Corollary 5.6, we know that f_1 is an isomorphism whose inverse is defined by ${f_1}^{-1}(\nabla S') = \nabla\Big((0) \cdot \int_{\frac{n-1}{2},0} S' \cdot (0)\Big)$, for all $\nabla S' \in \mathcal{DST}_0(n-3)$. Moreover, since we have

$$2(2i-1) \leqslant j_{2i} - 2 \leqslant (n-3) - (2i-1),$$

for all $i \in \{1, \ldots, m\}$, and $m = \left\lfloor \frac{n}{6} \right\rfloor = \left\lfloor \frac{n-3}{6} \right\rfloor + \delta_{4,(n-3 \bmod 6)}$ when n is odd, the set $\mathrm{H}(G_{D_0})$ is a generating index set of $\mathcal{DST}_0(n-3)$ by induction hypothesis. Therefore, f_2 is an isomorphism. Finally, since the linear map $\pi_{G_{D_0}} = f_2 f_1$ is an isomorphism, the set G_{D_0} is a generating index set of $\mathcal{DST}_0(n)$ in this case.

Case 2. Suppose now that n is even. We consider the subset $\mathrm{H}(G_{D_0}) \subset \nabla(n-3)$ defined by

$$\mathrm{H}(G_{D_0}) := \{(2i, j_{2i+1}-2) \mid i \in \{1,\dots,m-1\}\}$$

and the linear maps f_1 and f_2 defined by

$$\begin{array}{rccc} f_1: & \mathcal{DST}_0(n) & \longrightarrow & \{0,1\} \times \mathcal{DST}_0(n-3) \\ & \nabla S = (a_{i,j})_{1\leqslant i\leqslant j\leqslant n} & \longmapsto & (a_{1,j_1}, \mathrm{H}(\nabla S)) \end{array}$$

and

$$\begin{array}{rccc} f_2: & \{0,1\} \times \mathcal{DST}_0(n-3) & \longrightarrow & \{0,1\}^m \\ & (x, \nabla S') & \longmapsto & (x) \cdot \pi_{\mathrm{H}(G_{D_0})}(\nabla S') \end{array}$$

Then, for any $(a_{i,j})_{1\leqslant i\leqslant j\leqslant n} \in \mathcal{DST}_0(n)$, we have

$$\begin{aligned} f_2 f_1((a_{i,j})_{1\leqslant i\leqslant j\leqslant n}) &= f_2(a_{1,j_1}, (a_{1+i,2+j})_{1\leqslant i\leqslant j\leqslant n-3}) \\ &= (a_{1,j_1}) \cdot \pi_{\mathrm{H}(G_{D_0})}((a_{1+i,2+j})_{1\leqslant i\leqslant j\leqslant n-3}) \\ &= (a_{1,j_1}) \cdot \left(a_{2i+1,j_{2i+1}}\right)_{1\leqslant i\leqslant m-1} \\ &= \left(a_{2i+1,j_{2i+1}}\right)_{0\leqslant i\leqslant m-1} = \pi_{G_{D_0}}((a_{i,j})_{1\leqslant i\leqslant j\leqslant n}). \end{aligned}$$

Therefore $f_2 f_1 = \pi_{G_{D_0}}$. From Corollary 5.6, we know that f_1 is an isomorphism whose inverse is defined by ${f_1}^{-1}(x, \nabla S') = \nabla\Big((0) \cdot \int_{j_1-1,x} S' \cdot (0)\Big)$, for all $(x, \nabla S') \in \{0,1\} \times \mathcal{DST}_0(n-3)$. Moreover, since we have

$$4i \leqslant j_{2i+1} - 2 \leqslant (n-3) - 2i,$$

for all $i \in \{1,\dots,m-1\}$, and $m-1 = \left\lfloor \frac{n}{6} \right\rfloor + \delta_{4,(n \bmod 6)} - 1 = \left\lfloor \frac{n-3}{6} \right\rfloor$ when n is even, the set $\mathrm{H}(G_{D_0})$ is a generating index set of $\mathcal{DST}_0(n-3)$ by induction hypothesis. Therefore $\pi_{\mathrm{H}(G_{D_0})}$ and thus f_2 are isomorphisms. Finally, since the linear map $\pi_{G_{D_0}} = f_2 f_1$ is an isomorphism, the set G_{D_0} is a generating index set of $\mathcal{DST}_0(n)$ in this case.

This completes the proof. □

Since the dimension of $\mathcal{DST}_0(n)$ corresponds to the cardinality of the generating index set G_{D_0}, it is easy to obtain the following

Corollary 5.8. $\dim \mathcal{DST}_0(n) = \left\lfloor \frac{n}{6} \right\rfloor + \delta_{4,(n \bmod 6)}$, *for all non-negative integers* n.

Using Proposition 5.5 and Theorem 5.7, we are now ready to give a generating index set of $\mathcal{DST}(n)$, for all non-negative integers n.

Theorem 5.9. *Let* n *and* m *be non-negative integers such that* $m = \left\lfloor \frac{n+3}{6} \right\rfloor + \delta_{1,(n \bmod 6)}$. *For every integer* $i \in \left\{1,\dots,\left\lfloor \frac{n}{3} \right\rfloor\right\}$, *let* $j_i \in \{2i,\dots,n-i\}$. *Then, the set*

$$G_D = \{(2i+1, j_{2i+1}) \mid i \in \{0,\dots,m-1\}\},$$

when n *is even, or*

$$G_D = \{(1, j_1)\} \cup \{(2i, j_{2i}) \mid i \in \{1,\dots,m-1\}\},$$

when n *is odd, is a generating index set of* $\mathcal{DST}(n)$.

Proof. First, suppose that n is even. We know, from Proposition 5.5, that $\mathcal{DST}(n) = \mathcal{DST}_0(n)$. Moreover, since $m = \left\lfloor \frac{n+3}{6} \right\rfloor = \left\lfloor \frac{n}{6} \right\rfloor + \delta_{4,(n \bmod 6)}$ when n is even, it follows from Theorem 5.7 that

$$G_D = G_{D_0} = \{(2i+1, j_{2i+1}) \mid i \in \{0, \ldots, m-1\}\}$$

is a generating index set of $\mathcal{DST}(n)$ in this case.

Suppose now that n is odd. From Proposition 5.5, we know that $\mathcal{DST}(n) = \mathcal{DST}_0(n) \sqcup (\mathcal{DST}_0(n) + U_n)$. Therefore $\{(1, j_1)\} \cup G_{D_0}$ is a generating set of $\mathcal{DST}(n)$, where G_{D_0} is a generating index set of $\mathcal{DST}_0(n)$. Moreover, since $m - 1 = \left\lfloor \frac{n+3}{6} \right\rfloor + \delta_{1,(n \bmod 6)} - 1 = \left\lfloor \frac{n}{6} \right\rfloor$ when n is odd, it follows from Theorem 5.7 that

$$G_D = \{(1, j_1)\} \cup G_{D_0} = \{(1, j_1)\} \cup \{(2i, j_{2i}) \mid i \in \{1, \ldots, m-1\}\}$$

is a generating index set of $\mathcal{DST}(n)$ in this case. This completes the proof. □

Corollary 5.10. *Let n and m be non-negative integers such that $m = \left\lfloor \frac{n+3}{6} \right\rfloor + \delta_{1,(n \bmod 6)}$. The set*

$$G_D = \left\{ \left(2i+1, n - \left\lfloor \frac{n}{3} \right\rfloor\right) \;\middle|\; i \in \{0, \ldots, m-1\} \right\},$$

when n is even, or

$$G_D = \left\{ \left(1, n - \left\lfloor \frac{n}{3} \right\rfloor\right) \right\} \cup \left\{ \left(2i, n - \left\lfloor \frac{n}{3} \right\rfloor\right) \;\middle|\; i \in \{1, \ldots, m-1\} \right\},$$

when n is odd, is a generating index set of $\mathcal{DST}(n)$.

Proof. From Theorem 5.9, since $2i \leqslant n - \left\lfloor \frac{n}{3} \right\rfloor \leqslant n - i$, for all $i \in \left\{1, \ldots, \left\lfloor \frac{n}{3} \right\rfloor\right\}$. □

Since the dimension of $\mathcal{DST}(n)$ corresponds to the cardinality of the generating index set G_D, it is straightforward to obtain the following

Corollary 5.11. $\dim \mathcal{DST}(n) = \left\lfloor \frac{n+3}{6} \right\rfloor + \delta_{1,(n \bmod 6)}$, *for all non-negative integers n.*

5.3 Basis of $\mathcal{DST}(n)$

First, using the operators H^k and the generating index sets G_D introduced before, we obtain a family of bases of $\mathcal{DST}(n)$, for all non-negative integers n.

Theorem 5.12. *Let n and m be non-negative integers such that $m = \left\lfloor \frac{n+3}{6} \right\rfloor + \delta_{1,(n \bmod 6)}$. For every $k \in \left\{0, \ldots, \left\lfloor \frac{n}{3} \right\rfloor - 1\right\}$, let $\nabla_k \in \mathcal{DST}(n)$ such that $\mathrm{H}^k(\nabla_k) = U_{n-3k}$. Then, the set*

$$\{\nabla_{2k} \mid k \in \{0, \ldots, m-1\}\},$$

when n is even, or

$$\{\nabla_0\} \cup \{\nabla_{2k+1} \mid k \in \{0, \ldots, m-2\}\},$$

when n is odd, is a basis of $\mathcal{DST}(n)$.

Remark. $\nabla_0 = U_n$ in the previous result.

Proof. Suppose first that n is even. We consider the set

$$G_D = \left\{ \left(2i+1, n - \left\lfloor \frac{n}{3} \right\rfloor \right) \;\middle|\; i \in \{0, \ldots, m-1\} \right\}.$$

Let $k \in \{0, \ldots, m-1\}$. For $\nabla_{2k} = (a_{i,j})_{1 \leqslant i \leqslant j \leqslant n}$, since $\mathrm{H}^{2k}(\nabla_{2k}) = U_{n-6k}$, it follows from Lemma 3.10 that

$$a_{i,n-\left\lfloor \frac{n}{3} \right\rfloor} = \begin{cases} 1 & \text{for } i = 2k+1, \\ 0 & \text{for } i \in \left\{ 2k+2, \ldots, \left\lfloor \frac{n}{3} \right\rfloor \right\}. \end{cases}$$

Moreover, it is clear that $2m - 1 \leqslant \left\lfloor \frac{n}{3} \right\rfloor$ and thus

$$\pi_{G_D}(\nabla_{2k}) = (\underbrace{*, \ldots, *}_{k}, 1, \underbrace{0, \ldots, 0}_{m-k-1}).$$

Therefore the set $\{\pi_{G_D}(\nabla_{2k}) \mid k \in \{0, \ldots, m-1\}\}$ is a basis of $\{0,1\}^{G_D}$. Finally, since G_D is a generating index set of $\mathcal{DST}(n)$ by Corollary 5.10, we conclude that

$$\{\nabla_{2k} \mid k \in \{0, \ldots, m-1\}\}$$

is a basis of $\mathcal{DST}(n)$ in this case.

Suppose now that n is odd. The proof is similar to the even case by considering the generating index set

$$G_D = \left\{ \left(1, n - \left\lfloor \frac{n}{3} \right\rfloor \right) \right\} \cup \left\{ \left(2i, n - \left\lfloor \frac{n}{3} \right\rfloor \right) \;\middle|\; i \in \{1, \ldots, m-1\} \right\},$$

from Corollary 5.10 and since

$$\{\pi_{G_D}(\nabla_0)\} \cup \{\pi_{G_D}(\nabla_{2k+1}) \mid k \in \{0, \ldots, m-2\}\}$$

is a basis of $\{0,1\}^{G_D}$ using Lemma 3.10. This implies that the set

$$\{\nabla_0\} \cup \{\nabla_{2k+1} \mid k \in \{0, \ldots, m-2\}\}$$

is a basis of $\mathcal{DST}(n)$ in this case. This completes the proof. □

Now, we consider the restriction of the linear map ρ on the linear subspace $\mathcal{HST}(n)$, i.e., the linear map $\rho_{|\mathcal{HST}(n)} : \mathcal{HST}(n) \longrightarrow \mathcal{DST}(n)$ defined by $\rho = r^2 + r + id_n$, for all non-negative integers n. Obviously, this map is surjective since $\rho(\nabla) = \nabla$, for all $\nabla \in \mathcal{DST}(n)$. Since $U_{n-3k} = \rho((1)_{n-3k})$ by definition, for all non-negative integers n and k such that $3k \leqslant n$, this leads to the following

Corollary 5.13. *Let n and m be non-negative integers such that $m = \left\lfloor \frac{n+3}{6} \right\rfloor + \delta_{1,(n \bmod 6)}$. For every $k \in \left\{0, \ldots, \left\lfloor \frac{n}{3} \right\rfloor - 1\right\}$, let S_k be a symmetric binary sequence of length n such that $\partial^k S_k = (1)_{n-k}$. Then, the set*

$$\{\rho(\nabla S_{2k}) \mid k \in \{0, \ldots, m-1\}\},$$

when n is even, or

$$\{\rho(\nabla S_0)\} \cup \{\rho(\nabla S_{2k+1}) \mid k \in \{0, \ldots, m-2\}\},$$

when n is odd, is a basis of $\mathcal{DST}(n)$.

Remark. $\rho(\nabla S_0) = \rho(\nabla(1)_n) = U_n$ in the previous result.

Proof. Let $k \in \left\{0, \dots, \left\lfloor \frac{n}{3} \right\rfloor - 1\right\}$. First, since S_k is symmetric, we know from Proposition 4.2 that $\nabla S_k \in \mathcal{HST}(n)$ and thus $\rho(\nabla S_k) \in \mathcal{DST}(n)$, by definition of ρ. Moreover, since $\partial^k S_k = (1)_{n-k}$, it follows that

$$\mathrm{H}^k(\rho(\nabla S_k)) = \rho\left(\mathrm{H}^k(\nabla S_k)\right) = \rho(\nabla(1)_{n-3k}) = U_{n-3k}.$$

Therefore, the result directly comes from Theorem 5.12 by considering the dihedrally symmetric Steinhaus triangles $\nabla_k = \rho(\nabla S_k)$. □

We end this section by giving an explicit basis of $\mathcal{DST}(n)$ in terms of the n-length binary sequences

$$\mathrm{S}^{(n)}_{k,l} = \left(\binom{l+j-1}{k}_2\right)_{1 \leqslant j \leqslant n},$$

for all integers k and l.

Theorem 5.14. *Let n and m be non-negative integers such that $m = \left\lfloor \frac{n+3}{6} \right\rfloor + \delta_{1,(n \bmod 6)}$. For every $k \in \left\{0, \dots, \left\lfloor \frac{n}{3} \right\rfloor - 1\right\}$ of same parity than n, let*

$$\nabla_k = \rho\left(\nabla \mathrm{S}^{(n)}_{k,\frac{k-n}{2}}\right) = \left(\binom{\frac{k-n}{2}+j-i}{k+1-i} + \binom{\frac{k+n}{2}-j}{k+i-j} + \binom{\frac{k-n}{2}+i-1}{k+j-n} \bmod 2\right)_{1 \leqslant i \leqslant j \leqslant n}.$$

Then, the set

$$\{\nabla_{2k} \mid k \in \{0, \dots, m-1\}\},$$

when n is even, or

$$\{U_n\} \cup \{\nabla_{2k+1} \mid k \in \{0, \dots, m-2\}\},$$

when n is odd, is a basis of $\mathcal{DST}(n)$.

Proof. Let $k \in \left\{0, \dots, \left\lfloor \frac{n}{3} \right\rfloor - 1\right\}$ of same parity of n. First, we know from Lemma 4.8 that the sequence $\mathrm{S}^{(n)}_{k,\frac{k-n}{2}}$ is symmetric. Moreover, by Proposition 3.12, we have

$$\partial^k\left(\mathrm{S}^{(n)}_{k,\frac{k-n}{2}}\right) = \mathrm{S}^{(n-k)}_{0,\frac{k-n}{2}} = (1)_{n-k}.$$

We conclude the proof by using Corollary 5.13 with the sequences $S_k = \mathrm{S}^{(n)}_{k,\frac{k-n}{2}}$, for all $k \in \left\{0, \dots, \left\lfloor \frac{n}{3} \right\rfloor - 1\right\}$ of same parity of n, and $S_0 = (1)_n$, when n is odd. □

Corollary 5.15. *Let n and m be non-negative integers such that $m = \left\lfloor \frac{n}{6} \right\rfloor + \delta_{4,(n \bmod 6)}$. For every $k \in \left\{0, \dots, \left\lfloor \frac{n}{3} \right\rfloor - 1\right\}$ of same parity than n, let*

$$\nabla_k = \rho\left(\nabla \mathrm{S}^{(n)}_{k,\frac{k-n}{2}}\right) = \left(\binom{\frac{k-n}{2}+j-i}{k+1-i} + \binom{\frac{k+n}{2}-j}{k+i-j} + \binom{\frac{k-n}{2}+i-1}{k+j-n} \bmod 2\right)_{1 \leqslant i \leqslant j \leqslant n}.$$

Then, the set

$$\{\nabla_{2k} \mid k \in \{0, \dots, m-1\}\},$$

when n is even, or

$$\{\nabla_{2k+1} \mid k \in \{0, \dots, m-1\}\},$$

when n is odd, is a basis of $\mathcal{DST}_0(n)$.

Proof. From Theorem 5.14, since we have that $\mathcal{DST}(n) = \mathcal{DST}_0(n)$, when n is even, and $\mathcal{DST}(n) = \mathcal{DST}_0(n) \sqcup (\mathcal{DST}_0(n) + U_n)$, when n is odd, by Proposition 5.5. □

For instance, for $n = 22$, we obtain

k	$\mathrm{S}^{(22)}_{2k,k-11}$	$\rho\left(\nabla \mathrm{S}^{(22)}_{2k,k-11}\right)$
0	(1111111111111111111111)	$\nabla_0 = \nabla$(0111111111111111111110)
1	(1100110011001100110011)	$\nabla_2 = \nabla$(0110110011001100110110)
2	(1000011110000111100001)	$\nabla_4 = \nabla$(0111111110000111111110)
3	(0000001100000011000000)	$\nabla_6 = \nabla$(0000000100000010000000)

All the dihedrally symmetric Steinhaus triangles of size 22 are depicted in Figure 12, where the elements of the basis $\{\nabla_0, \nabla_2, \nabla_4, \nabla_6\}$ are in red and, for every $\nabla \in \mathcal{DST}(22)$, the coordinate vector (x_0, x_2, x_4, x_6) of $\nabla = x_0\nabla_0 + x_2\nabla_2 + x_4\nabla_4 + x_6\nabla_6$ is given.

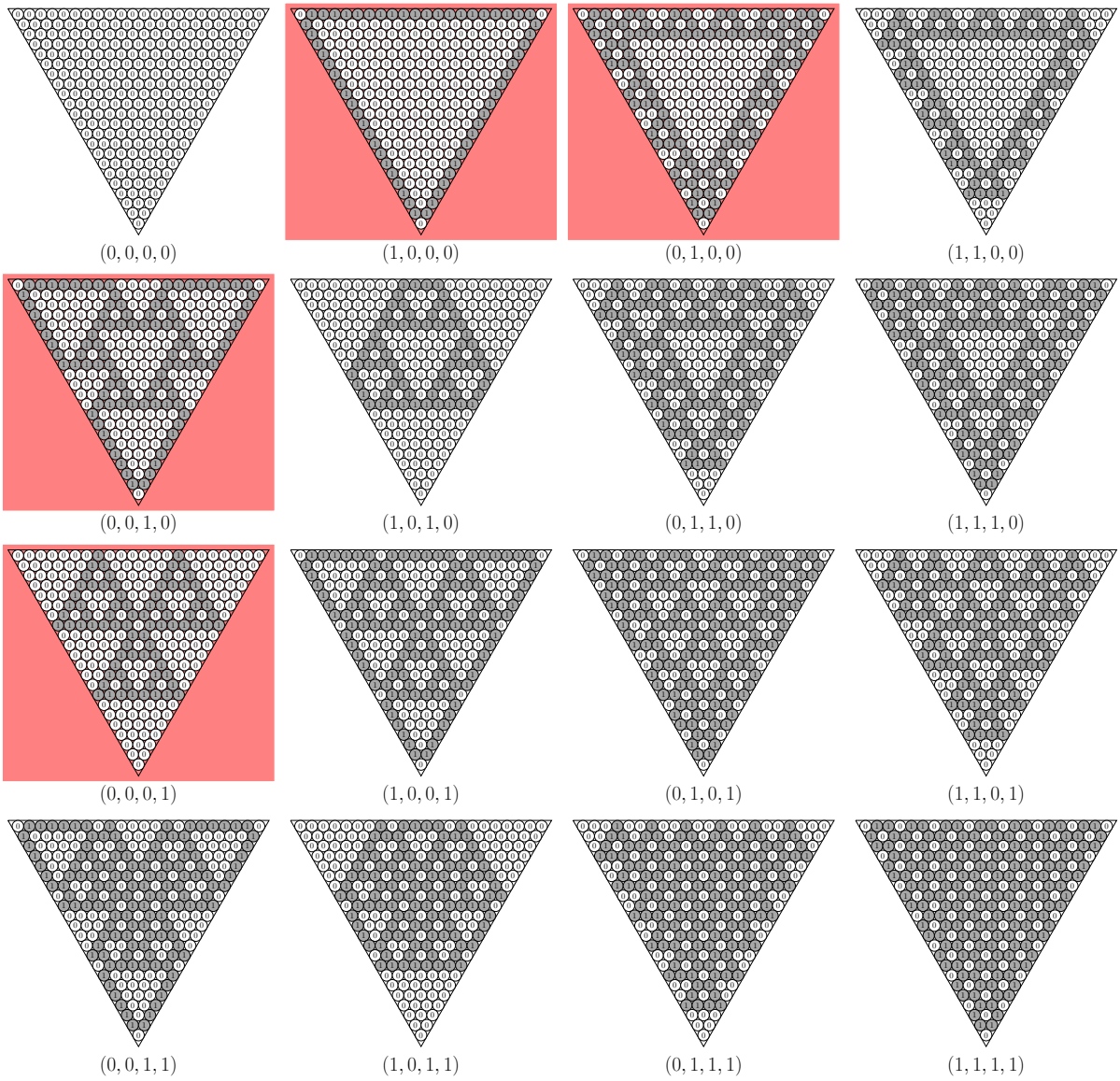


Figure 12: The 16 triangles of $\mathcal{DST}(22)$ where the 4 red triangles form a basis

6 Parity-regular Steinhaus graphs

For any binary sequence $S = (a_j)_{1\leqslant j\leqslant n}$ of length n, we denote by $ir(S)$ the *interlacing of the sequence S and its reversed sequence*, that is, the sequence $ir(S) = (b_j)_{1\leqslant j\leqslant 2n}$ of length $2n$ defined by

$$b_{2j-1} = a_j \quad \text{and} \quad b_{2j} = a_{n-j+1},$$

for all $j \in \{1, \ldots, n\}$. For instance, for $S = (101000)$, we have $ir(S) = (100010010001)$.

For any positive integer n, we consider the linear map

$$\begin{array}{rccc} \theta: & \mathcal{SG}(n) & \longrightarrow & \mathcal{ST}(2n-1) \\ & \mathrm{G}(S) & \longmapsto & \nabla\int_{n,0} ir(S) \end{array}$$

Note that the Steinhaus triangle $\nabla S \in \mathcal{ST}(n-1)$ is then a subtriangle of $\theta(\mathrm{G}(S)) \in \mathcal{ST}(2n-1)$. Indeed, for the sequence $S = (a_j)_{1\leqslant j\leqslant n-1}$ and the Steinhaus triangle $\theta(\mathrm{G}(S)) = \nabla\int_{n,0} ir(S) = (a_{i,j})_{1\leqslant i\leqslant j\leqslant 2n-1}$, the Steinhaus triangle ∇S is simply the subtriangle $(a_{2i,2j})_{1\leqslant i\leqslant j\leqslant n-1}$, since $a_{2,2j} = a_j$, for all $j \in \{1, \ldots, n-1\}$, by definition of θ, and, using the local rule (LR), we have

$$a_{2i,2j} \equiv a_{2i-1,2j-1} + a_{2i-1,2j} \equiv a_{2i-2,2j-2} + 2a_{2i-2,2j-1} + a_{2i-2,2j} \equiv a_{2i-2,2j-2} + a_{2i-2,2j} \pmod 2,$$

for all integers i and j such that $2 \leqslant i \leqslant j \leqslant n-1$. For instance, for the sequence $S = (101000)$, the Steinhaus triangle $\theta(\mathrm{G}(S))$ is depicted in Figure 13, where the subtriangle ∇S appears in red.

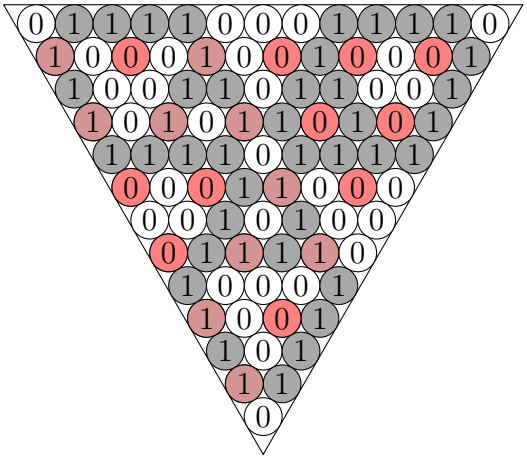


Figure 13: The Steinhaus triangle $\theta(\mathrm{G}(101000))$ where $\nabla(101000)$ appears in red

By definition of the linear map ir, we know that the sequence $ir(S)$ is symmetric and $\sigma_2(ir(S)) = 0$. It follows from Proposition 4.1 that the sequence $\int_{n,0} ir(S)$ is symmetric too. Therefore, using Proposition 4.2, we have

$$\theta(\mathrm{G}(S)) = \nabla\int_{n,0} ir(S) \in \mathcal{HST}(2n-1),$$

for all $\mathrm{G}(S) \in \mathcal{SG}(n)$. Moreover, using Lemma 4.4 and since the middle term of the sequence $\int_{n,0} ir(S)$ is 0 by definition, we obtain that

$$\sigma_2\left(\int_{n,0} ir(S)\right) = 0,$$

for any sequence S of length $n-1$.

The main result of this section is to show that the restriction of θ to the linear subspace of even Steinhaus graphs $\mathcal{ESG}(n)$ induces an isomorphism between $\mathcal{ESG}(n)$ and $\mathcal{DST}_0(2n-1)$, for all positive integers n.

Theorem 6.1. *Let S be a binary sequence of length $n-1 \geqslant 0$. Then, the Steinhaus graph $\mathrm{G}(S)$ is even if and only if the Steinhaus triangle $\theta(\mathrm{G}(S))$ is dihedrally symmetric.*

The proof is based on the following

Lemma 6.2. *Let S be a binary sequence of length $n-1 \geqslant 0$. Let v_i denote the ith vertex of the Steinhaus graph $\mathrm{G}(S)$, for all $i \in \{1,\ldots,n\}$, and let $\theta(\mathrm{G}(S)) = (b_{i,j})_{1\leqslant i\leqslant j\leqslant 2n-1} \in \mathcal{HST}(2n-1)$. Then,*

$$\deg(v_1) \equiv b_{1,1} \pmod 2$$

and, for any $i \in \{2,\ldots,n\}$, if the $(2i-2)$-th column $C_{2i-2} = (b_{j,2i-2})_{1\leqslant j\leqslant 2i-2}$ of $\theta(\mathrm{G}(S))$ is symmetric, then

$$\deg(v_i) \equiv b_{i,2i-1} \pmod 2.$$

Proof of Lemma 6.2. We consider the adjacency matrix $\mathrm{M}(S) = (a_{i,j})_{1\leqslant i,j\leqslant n}$ of the Steinhaus graph $\mathrm{G}(S)$. We already know that the upper-triangular part of $\mathrm{M}(S)$, i.e., the Steinhaus triangle $\nabla S = (a_{i,j})_{1\leqslant i<j\leqslant n}$, corresponds to the subtriangle $(b_{2i,2j})_{1\leqslant i\leqslant j\leqslant n-1}$ of $\theta(\mathrm{G}(S))$. In other words, we have $a_{i,j} = b_{2i,2j-2}$, for all integers i and j such that $1 \leqslant i < j \leqslant n$. Then,

$$\deg(v_1) = \sum_{j=2}^{n} a_{1,j} = \sum_{j=2}^{n} b_{2,2j-2} = \sum_{j=1}^{n-1} b_{2,2j}, \tag{11}$$

$$\deg(v_i) = \sum_{j=1}^{i-1} a_{j,i} + \sum_{j=i+1}^{n} a_{i,j} = \sum_{j=1}^{i-1} b_{2j,2i-2} + \sum_{j=i+1}^{n} b_{2i,2j-2} = \sum_{j=1}^{i-1} b_{2j,2i-2} + \sum_{j=i}^{n-1} b_{2i,2j}, \tag{12}$$

for all $i \in \{2,\ldots,n-1\}$, and

$$\deg(v_n) = \sum_{j=1}^{n-1} a_{j,n} = \sum_{j=1}^{n-1} b_{2j,2n-2}. \tag{13}$$

We claim that

$$\sum_{j=i}^{n-1} b_{2i,2j} = \sum_{j=2i}^{n+i-1} b_{2i,j}, \tag{14}$$

for all $i \in \{1,\ldots,n-1\}$. Since $\theta(\mathrm{G}(S))$ is horizontally symmetric, we know that its ith row $R_i = (b_{i,j})_{i\leqslant j\leqslant 2n-1}$ is symmetric, for all $i \in \{1,\ldots,2n-1\}$. Let $i \in \{1,\ldots,n-1\}$. Since the sequence R_{2i} is symmetric of even length $2(n-i)$ with $b_{2i,j} = b_{2i,2n-1+2i-j}$, for all $j \in \{2i,\ldots,2n-1\}$, we obtain the following identities by dividing in half $\sigma(R_{2i})$ in two different ways

$$\sigma(R_{2i}) = \sum_{j=2i}^{2n-1} b_{2i,j} = \sum_{j=i}^{n-1} b_{2i,2j} + \sum_{j=i}^{n-1} b_{2i,2j+1} = \sum_{j=i}^{n-1} b_{2i,2j} + \sum_{j=i}^{n-1} b_{2i,2n-1+2i-(2j+1)} = 2\sum_{j=i}^{n-1} b_{2i,2j},$$

and

$$\sigma(R_{2i}) = \sum_{j=2i}^{2n-1} b_{2i,j} = \sum_{j=2i}^{n+i-1} b_{2i,j} + \sum_{j=n+i}^{2n-1} b_{2i,j} = \sum_{j=2i}^{n+i-1} b_{2i,j} + \sum_{j=n+i}^{2n-1} b_{2i,2n-1+2i-j} = 2\sum_{j=2i}^{n+i-1} b_{2i,j}.$$

Combining these two identities, the claim (14) is proved.

Using (14) and the local rule, we deduce from (11) that

$$\deg(v_1) = \sum_{j=1}^{n-1} b_{2,2j} = \sum_{j=2}^{n} b_{2,j} \equiv \sum_{j=2}^{n}(b_{1,j-1}+b_{1,j}) = \sum_{j=1}^{n-1} b_{1,j}+\sum_{j=2}^{n} b_{1,j} \equiv b_{1,1}+b_{1,n} \pmod 2. \tag{15}$$

Now, let $i \in \{2,\dots,n\}$ and suppose that the column C_{2i-2} of even size $2i-2$ is symmetric. Then, as for (14), using a double counting of $\sigma(C_{2i-2})$, we obtain the following identity

$$\sum_{j=1}^{i-1} b_{2j,2i-2} = \sum_{j=i}^{2i-2} b_{j,2i-2}. \tag{16}$$

Using (14), (16) and the local rule, we deduce from (12) that

$$\begin{aligned}
\deg(v_i) &= \sum_{j=1}^{i-1} b_{2j,2i-2} + \sum_{j=i}^{n-1} b_{2i,2j} = \sum_{j=i}^{2i-2} b_{j,2i-2} + \sum_{j=2i}^{n+i-1} b_{2i,j} \\
&\equiv \sum_{j=i}^{2i-2}(b_{j,2i-1} + b_{j+1,2i-1}) + \sum_{j=2i}^{n+i-1}(b_{2i-1,j-1} + b_{2i-1,j}) \\
&= \sum_{j=i}^{2i-2} b_{j,2i-1} + \sum_{j=i+1}^{2i-1} b_{j,2i-1} + \sum_{j=2i-1}^{n+i-2} b_{2i-1,j} + \sum_{j=2i}^{n+i-1} b_{2i-1,j} \\
&\equiv b_{i,2i-1} + b_{2i-1,2i-1} + b_{2i-1,2i-1} + b_{2i-1,n+i-1} \equiv b_{i,2i-1} + b_{2i-1,n+i-1} \pmod 2,
\end{aligned} \tag{17}$$

if $i \in \{2,\dots,n-1\}$, and from (13) that

$$\begin{aligned}
\deg(v_n) &= \sum_{j=1}^{n-1} b_{2j,2n-2} = \sum_{j=n}^{2n-2} b_{j,2n-2} \equiv \sum_{j=n}^{2n-2}(b_{j,2n-1} + b_{j+1,2n-1}) \\
&= \sum_{j=n}^{2n-2} b_{j,2n-1} + \sum_{j=n+1}^{2n-1} b_{j,2n-1} \equiv b_{n,2n-1} + b_{2n-1,2n-1} \pmod 2.
\end{aligned} \tag{18}$$

Since $\theta(\mathrm{G}(S))$ is horizontally symmetric, we know that $b_{2n-1-2i,2n-1-i} = 0$, for all $i \in \{0,\dots,n-2\}$, by Proposition 4.3. Moreover, by definition of $\theta(\mathrm{G}(S)) = \nabla\int_{n,0} ir(S)$, we have that $b_{1,n} = 0$. Therefore, we have

$$b_{2i+1,n+i} = 0, \text{ for all } i \in \{0,\dots,n-1\}. \tag{19}$$

Finally, by combining (15), (17) and (18) with (19), the result of Lemma 6.2 is proved. □

Proof of Theorem 6.1. First, suppose that $\theta(\mathrm{G}(S))$ is dihedrally symmetric. Then, the Steinhaus triangle $r(\theta(\mathrm{G}(S))) = \theta(\mathrm{G}(S))$ is horizontally symmetric and the column C_i of $\theta(\mathrm{G}(S))$ is symmetric, for all $i \in \{1, \ldots, 2n-1\}$. It follows from Lemma 6.2 that

$$\deg(v_i) \equiv b_{i,2i-1} \pmod 2,$$

for all $i \in \{1, \ldots, n\}$. Since $\theta(\mathrm{G}(S))$ is dihedrally symmetric, we know from Corollary 5.3 that $b_{i,2i-1} = 0$, for all $i \in \{1, \ldots, n-1\}$. Moreover, since $\theta(\mathrm{G}(S)) \in \mathcal{DST}_0(2n-1)$, we have $b_{n,2n-1} = b_{1,n} = 0$. Therefore, for every $i \in \{1, \ldots, n\}$, the vertex v_i is of even degree and the Steinhaus graph $\mathrm{G}(S)$ is even.

Conversely, suppose that the Steinhaus graph $\mathrm{G}(S)$ is even. We prove, by induction on i, that all the columns C_i of $\theta(\mathrm{G}(S))$ are symmetric. From Lemma 6.2, we know that $b_{1,1} \equiv \deg(v_1) \bmod 2$. Since $\deg(v_1)$ is even, it follows that $b_{1,1} = 0$. Moreover, since $b_{2,2} \equiv b_{1,1} + b_{1,2} \equiv b_{1,2} \bmod 2$, we have that $b_{1,2} = b_{2,2}$. Therefore, the columns C_1 and C_2 are symmetric. Suppose now that the columns $C_1, C_2, \ldots, C_{2i}$ are symmetric, for some $i \in \{1, \ldots, n-1\}$. First, since C_{2i} is symmetric of even length, we know from Lemma 4.4 that $\sigma_2(C_{2i}) = 0$. Therefore, since $\partial C_{2i+1} = C_{2i}$, it follows from Proposition 4.1 that C_{2i+1} is symmetric. Moreover, since C_{2i} is symmetric and the vertex v_{i+1} is of even degree, we obtain by Lemma 6.2 that

$$b_{i+1,2i+1} \equiv \deg(v_{i+1}) \equiv 0 \pmod 2.$$

Since $b_{i+1,2i+1} = 0$, we obtain from Lemma 4.4 that $\sigma_2(C_{2i+1}) = 0$. Therefore, when $i < n-1$, since $\partial C_{2i+2} = C_{2i+1}$, using Proposition 4.1 again, we have that C_{2i+2} is symmetric. This concludes the proof that all the columns C_i of $\theta(\mathrm{G}(S))$ are symmetric. Obviously, it follows that the Steinhaus triangle $r(\theta(\mathrm{G}(S)))$ is horizontally symmetric. Finally, since the triangles $\theta(\mathrm{G}(S))$ and $r(\theta(\mathrm{G}(S)))$ are horizontally symmetric, we know from Proposition 5.1 that $\theta(\mathrm{G}(S))$ is dihedrally symmetric. □

Corollary 6.3. *For any positive integer n, the restriction*

$$\theta|_{\mathcal{ESG}(n)} : \mathcal{ESG}(n) \longrightarrow \mathcal{DST}_0(2n-1)$$

is an isomorphism.

Proof. Let n be a positive integer. We consider the linear map

$$\begin{array}{rccc} \psi: & \mathcal{DST}_0(2n-1) & \longrightarrow & \mathcal{ESG}(n) \\ & (a_{i,j})_{1\leqslant i\leqslant j\leqslant 2n-1} & \longmapsto & \mathrm{G}\left((a_{2,2j})_{1\leqslant j\leqslant n-1}\right) \end{array}$$

We know from Theorem 6.1 that the linear maps $\theta|_{\mathcal{ESG}(n)}$ and ψ are well defined. Moreover, it is easy to verify that $\psi \circ \theta|_{\mathcal{ESG}(n)} = id_{\mathcal{ESG}(n)}$ and $\theta|_{\mathcal{ESG}(n)} \circ \psi = id_{\mathcal{DST}_0(2n-1)}$. This completes the proof. □

This new result permits us to obtain the following two corollaries that were first proved in [16].

Corollary 6.4. $\dim \mathcal{ESG}(n) = \left\lfloor \frac{n-1}{3} \right\rfloor$, *for all positive integers n.*

Proof. Since the vector space $\mathcal{ESG}(n)$ is isomorphic to $\mathcal{DST}_0(2n-1)$ by Corollary 6.3, we deduce from Corollary 5.8 that

$$\dim \mathcal{ESG}(n) = \dim \mathcal{DST}_0(2n-1) = \left\lfloor \frac{2n-1}{6} \right\rfloor + \delta_{4,(2n-1 \bmod 6)} = \left\lfloor \frac{2n-1}{6} \right\rfloor = \left\lfloor \frac{n-1}{3} \right\rfloor,$$

for all positive integers n. □

Corollary 6.5. *The Steinhaus matrix* $\mathrm{M}(S)$ *associated to an even Steinhaus graph* $\mathrm{G}(S)$ *is doubly symmetric, i.e., all the diagonals of* $\mathrm{M}(S)$ *are symmetric.*

Remark. This was a key result in the first proof of the formula $\dim \mathcal{ESG}(n) = \left\lfloor \frac{n-1}{3} \right\rfloor$ in [16]. Another simple proof of this result can also be found in [10].

Proof. Let S be a binary sequence of length $n-1$ whose associated Steinhaus graph $\mathrm{G}(S)$ is even. In other words, we want to prove that the Steinhaus triangle $r^2(\nabla S)$ is horizontally symmetric. Let $\theta(\mathrm{G}(S)) = (b_{i,j})_{1 \leqslant i \leqslant j \leqslant 2n-1}$. Since $\theta(\mathrm{G}(S)) \in \mathcal{DST}_0(2n-1)$ by Theorem 6.1, it follows that $r^2(\theta(\mathrm{G}(S)))$ is horizontally symmetric. Therefore, the diagonal $D_i = (b_{j,i+j})_{1 \leqslant j \leqslant 2n-1-i}$ is symmetric, for all $i \in \{0, \ldots, 2n-2\}$. It follows that

$$b_{j,i+j} = b_{2n-i-j,2n-j}, \tag{20}$$

for all $j \in \{1, \ldots, 2n-1-i\}$ and for all $i \in \{0, \ldots, 2n-2\}$. Let $\nabla S = (a_{i,j})_{1 \leqslant i \leqslant j \leqslant n-1}$. As already seen, it corresponds to the subtriangle $\nabla S = (b_{2i,2j})_{1 \leqslant i \leqslant j \leqslant n-1}$ of $\theta(\mathrm{G}(S))$. Therefore, we have $a_{i,j} = b_{2i,2j}$, for all integers i and j such that $1 \leqslant i \leqslant j \leqslant n-1$. Let $i \in \{0, \ldots, n-2\}$. From (20), we obtain that

$$a_{j,i+j} = b_{2j,2i+2j} = b_{2n-2i-2j,2n-2j} = b_{2(n-i-j),2(n-j)} = a_{n-i-j,n-j},$$

for all $j \in \{1, \ldots, n-1-i\}$. We conclude that the diagonal $(a_{j,i+j})_{1 \leqslant j \leqslant n-1-i}$ is symmetric, for all $i \in \{0, \ldots, n-2\}$, and the Steinhaus triangle $r^2(\nabla S)$ is horizontally symmetric. □

Using Theorem 6.1 and the results of Section 5, we are now ready for giving a basis of $\mathcal{ESG}(n)$, for all positive integers n.

Theorem 6.6. *Let* n *be a positive integer. The set*

$$\left\{ \psi\rho\left(\nabla \mathrm{S}^{(2n-1)}_{2k+1,k-n+1}\right) \;\middle|\; k \in \left\{0, \ldots, \left\lfloor \frac{n-1}{3} \right\rfloor - 1\right\}\right\}$$

is a basis of $\mathcal{ESG}(n)$*, where* $\psi\rho\left(\nabla \mathrm{S}^{(2n-1)}_{2k+1,k-n+1}\right) = \mathrm{G}(S_k)$ *with*

$$S_k = \left(\binom{k-n+2j-1}{2k} + \binom{k+n-2j}{2k-2j+3} + \binom{k-n+2}{2k-2n+2j+2} \bmod 2\right)_{1 \leqslant j \leqslant n-1},$$

for all $k \in \left\{0, \ldots, \left\lfloor \frac{n-1}{3} \right\rfloor - 1\right\}$.

Proof. Let n be a positive integer. From Corollary 5.15, we know that

$$\left\{ \rho\left(\nabla \mathrm{S}^{(2n-1)}_{2k+1,k-n+1}\right) \;\middle|\; k \in \left\{0, \ldots, \left\lfloor \frac{n-1}{3} \right\rfloor - 1\right\}\right\}$$

is a basis of $\mathcal{DST}_0(2n-1)$, where $\rho\left(\nabla \mathrm{S}^{(2n-1)}_{2k+1,k-n+1}\right)$ is the triangle

$$\left(\binom{k-n+1+j-i}{2k+2-i} + \binom{k+n-j}{2k+1+i-j} + \binom{k-n+i}{2k+2+j-2n} \bmod 2\right)_{1 \leqslant i \leqslant j \leqslant 2n-1},$$

for all $k \in \left\{0, \ldots, \left\lfloor \frac{n-1}{3} \right\rfloor - 1\right\}$. It follows, by Corollary 6.3, that

$$\left\{ \psi\rho\left(\nabla \mathrm{S}^{(2n-1)}_{2k,k-n+1}\right) \;\middle|\; k \in \left\{0, \ldots, \left\lfloor \frac{n-1}{3} \right\rfloor - 1\right\}\right\}$$

is a basis of $\mathcal{ESG}(n)$. Since $\psi\left((a_{i,j})\right) = \mathrm{G}\left((a_{2,2j})_{1\leqslant j\leqslant n-1}\right)$, we conclude that

$$\psi\rho\left(\nabla \mathrm{S}^{(2n-1)}_{2k+1,k-n+1}\right) = \mathrm{G}\left(S_k\right),$$

with

$$S_k = \left(\binom{k-n+2j-1}{2k} + \binom{k+n-2j}{2k-2j+3} + \binom{k-n+2}{2k-2n+2j+2} \bmod 2\right)_{1\leqslant j\leqslant n-1},$$

for all $k \in \left\{0, 1, \ldots, \left\lfloor \frac{n-1}{3} \right\rfloor - 1\right\}$. □

For instance, for $n = 12$, we obtain

k	$\mathrm{S}^{(23)}_{2k+1,k-11}$	$\rho\left(\nabla \mathrm{S}^{(23)}_{2k+1,k-11}\right)$	$\psi\left(\rho\left(\nabla \mathrm{S}^{(23)}_{2k+1,k-11}\right)\right)$
0	(10101010101010101010)	∇(01101010101010101010)	$G_1 = \mathrm{G}\,(11111111110)$
1	(01000100010001000100)	∇(00010100010001000100)	$G_2 = \mathrm{G}\,(01101010110)$
2	(10000101000001010000)	∇(01111101000001011110)	$G_3 = \mathrm{G}\,(10011001000)$

All the even Steinhaus graphs of order 12 are depicted in Figure 14, where the elements of the basis $\{G_1, G_2, G_3\}$ are in red and, for every $G \in \mathcal{ESG}(12)$, the coordinate vector (x_1, x_2, x_3) of $G = x_1G_1 + x_2G_2 + x_3G_3$ is given.

We end this section by giving a basis of the linear subspace $\mathcal{PRSG}(n)$ of parity-regular Steinhaus graphs, for all positive integers n. By definition, we know that $\mathcal{PRSG}(n) = \mathcal{ESG}(n) \sqcup \mathcal{OSG}(n)$, where $\mathcal{OSG}(n)$ is the set of odd Steinhaus graphs of order n, for all positive integers n. As already remarked, it is clear that $\mathcal{OSG}(n) = \emptyset$, when n is odd. For n even, we obtain the following

Proposition 6.7. *For any positive integer n,*

$$\mathcal{OSG}(2n) = \mathcal{ESG}(2n) + \mathrm{G}\left((0)_{2n-2} \cdot (1)\right).$$

The proof is based on the following lemma, where the linear map ι is defined by

$$\begin{array}{rccc} \iota: & \mathcal{SG}(2n) & \longrightarrow & \mathcal{SG}(2n) \\ & \mathrm{G}\,(S) & \longmapsto & \mathrm{G}\,(S + (0)_{2n-2} \cdot (1)) \end{array}$$

for all positive integers n.

Lemma 6.8. *For any positive integer n, the Steinhaus graph $\mathrm{G}\,(S)$ of order $2n$ is even if and only if $\iota\left(\mathrm{G}\,(S)\right)$ is odd.*

Proof of Lemma 6.8. We consider the respective adjacency matrices $\mathrm{M}\,(S) = (a_{i,j})_{1\leqslant i,j\leqslant 2n}$ of $\mathrm{G}\,(S)$ and $\mathrm{M}\,(S + (0)_{2n-2} \cdot (1)) = (b_{i,j})_{1\leqslant i,j\leqslant 2n}$ of $\iota\left(\mathrm{G}\,(S)\right)$. Then ,it is easy to see that

$$\begin{cases} b_{i,j} = a_{i,j} & \text{for all } 1 \leqslant i < j \leqslant 2n-1, \\ b_{i,2n} \equiv a_{i,2n} + 1 \pmod 2 & \text{for all } 1 \leqslant i \leqslant 2n-1, \end{cases}$$

since the sequence $S + (0)_{2n-2} \cdot (1)$ only differs by the last term from the sequence S. It follows that

$$\deg_{\mathrm{G}(S)}(v_1) = \sum_{j=2}^{2n} a_{1,j} \equiv \sum_{j=2}^{2n} b_{1,n} + 1 = \deg_{\iota(\mathrm{G}(S))}(v_1) + 1 \pmod 2,$$

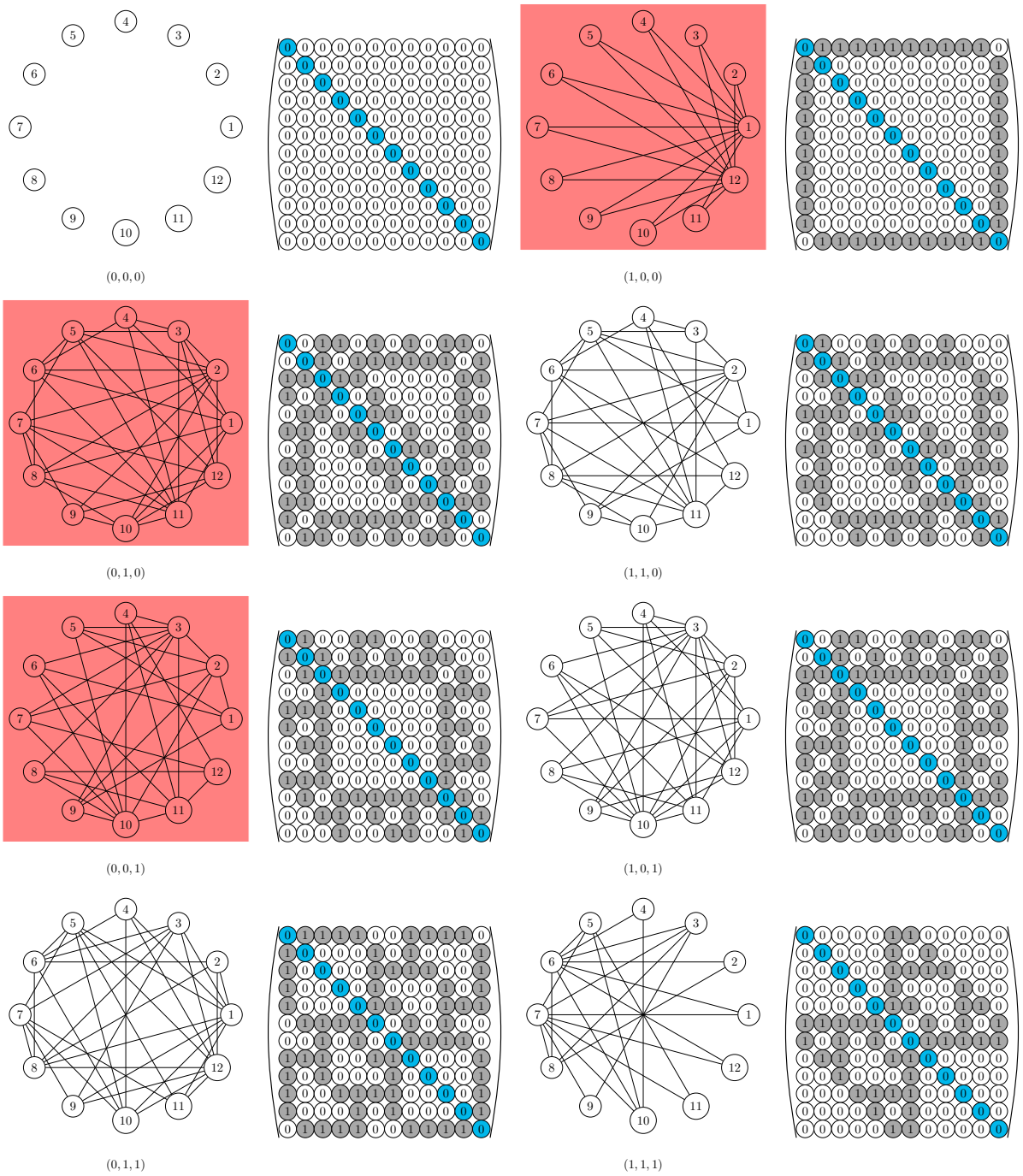


Figure 14: The 8 graphs of $\mathcal{ESG}(12)$ where the 3 red graphs form a basis

$$\deg_{\mathrm{G}(S)}(v_i) = \sum_{j=1}^{i-1} a_{j,i} + \sum_{j=i+1}^{2n} a_{i,j} \equiv \sum_{j=1}^{i-1} b_{j,i} + \sum_{j=i+1}^{2n} b_{i,j} + 1 = \deg_{\iota(\mathrm{G}(S))}(v_i) + 1 \pmod 2,$$

for all $i \in \{2, \ldots, 2n-1\}$, and

$$\deg_{\mathrm{G}(S)}(v_{2n}) = \sum_{i=1}^{2n-1} a_{i,2n} \equiv \sum_{i=1}^{2n-1} b_{i,2n} + 2n - 1 \equiv \deg_{\iota(\mathrm{G}(S))}(v_{2n}) + 1 \pmod 2,$$

where $\deg_{\mathrm{G}(S)}(v_i)$ and $\deg_{\iota(\mathrm{G}(S))}(v_i)$ are the degrees of the ith vertex of the Steinhaus graphs $\mathrm{G}(S)$ and $\iota(\mathrm{G}(S))$, respectively, for all $i \in \{1, \ldots, 2n\}$. Since $\deg_{\mathrm{G}(S)}(v_i) \equiv \deg_{\iota(\mathrm{G}(S))}(v_i) + 1 \bmod 2$, for all $i \in \{1, \ldots, 2n\}$, the result follows. □

Proof of Proposition 6.7. From Lemma 6.8, it is clear that ι induces an involution on $\mathcal{PRSG}(2n)$ with $\mathcal{OSG}(2n) = \iota(\mathcal{ESG}(2n)) = \mathcal{ESG}(2n) + \mathrm{G}((0)_{2n-2} \cdot (1))$, for all positive integers n. □

It immediately follows that, for any positive integer n, we have

$$\mathcal{PRSG}(2n-1) = \mathcal{ESG}(2n-1) \tag{21}$$

and

$$\mathcal{PRSG}(2n) = \mathcal{ESG}(2n) \sqcup (\mathcal{ESG}(2n) + \mathrm{G}((0)_{2n-2} \cdot (1))). \tag{22}$$

Therefore, we retrieve the following

Proposition 6.9. $\dim \mathcal{PRSG}(n) = \left\lfloor \frac{n-1}{3} \right\rfloor + \delta_{0,(n \bmod 2)}$, *for all positive integers* n.

Combining the identities (21) and (22) with Theorem 6.6, we obtain the following

Theorem 6.10. *Let* n *be a positive integer. The set*

$$\left\{ \psi\rho\left(\nabla \mathrm{S}^{(2n-1)}_{2k+1,k-n+1}\right) \;\middle|\; k \in \left\{0, \ldots, \left\lfloor \frac{n-1}{3} \right\rfloor - 1\right\}\right\},$$

when n *is odd, or the set*

$$\{\mathrm{G}((0)_{n-2} \cdot (1))\} \cup \left\{ \psi\rho\left(\nabla \mathrm{S}^{(2n-1)}_{2k+1,k-n+1}\right) \;\middle|\; k \in \left\{0, \ldots, \left\lfloor \frac{n-1}{3} \right\rfloor - 1\right\}\right\},$$

when n *is even, is a basis of* $\mathcal{PRSG}(n)$, *where* $\psi\rho\left(\nabla \mathrm{S}^{(2n-1)}_{2k+1,k-n+1}\right) = \mathrm{G}(S_k)$ *with*

$$S_k = \left(\binom{k-n+2j-1}{2k} + \binom{k+n-2j}{2k-2j+3} + \binom{k-n+2}{2k-2n+2j+2} \bmod 2 \right)_{1 \leqslant j \leqslant n-1},$$

for all $k \in \left\{0, \ldots, \left\lfloor \frac{n-1}{3} \right\rfloor - 1\right\}$.

For instance, for $n = 12$, we obtain

$$G_0 = \mathrm{G}(00000000001), \quad G_1 = \mathrm{G}(11111111110), \quad G_2 = \mathrm{G}(01101010110), \quad G_3 = \mathrm{G}(10011001000).$$

All the parity-regular Steinhaus graphs of order 12 are depicted in Figure 14 for even graphs and in Figure 15 for odd ones, where the elements of the basis $\{G_0, G_1, G_2, G_3\}$ are in red and, for every $G \in \mathcal{PRSG}(12)$, the coordinate vector (x_0, x_1, x_2, x_3) of $G = x_0G_0 + x_1G_1 + x_2G_2 + x_3G_3$ is given ((x_1, x_2, x_3) when $x_0 = 0$ in Figure 14).

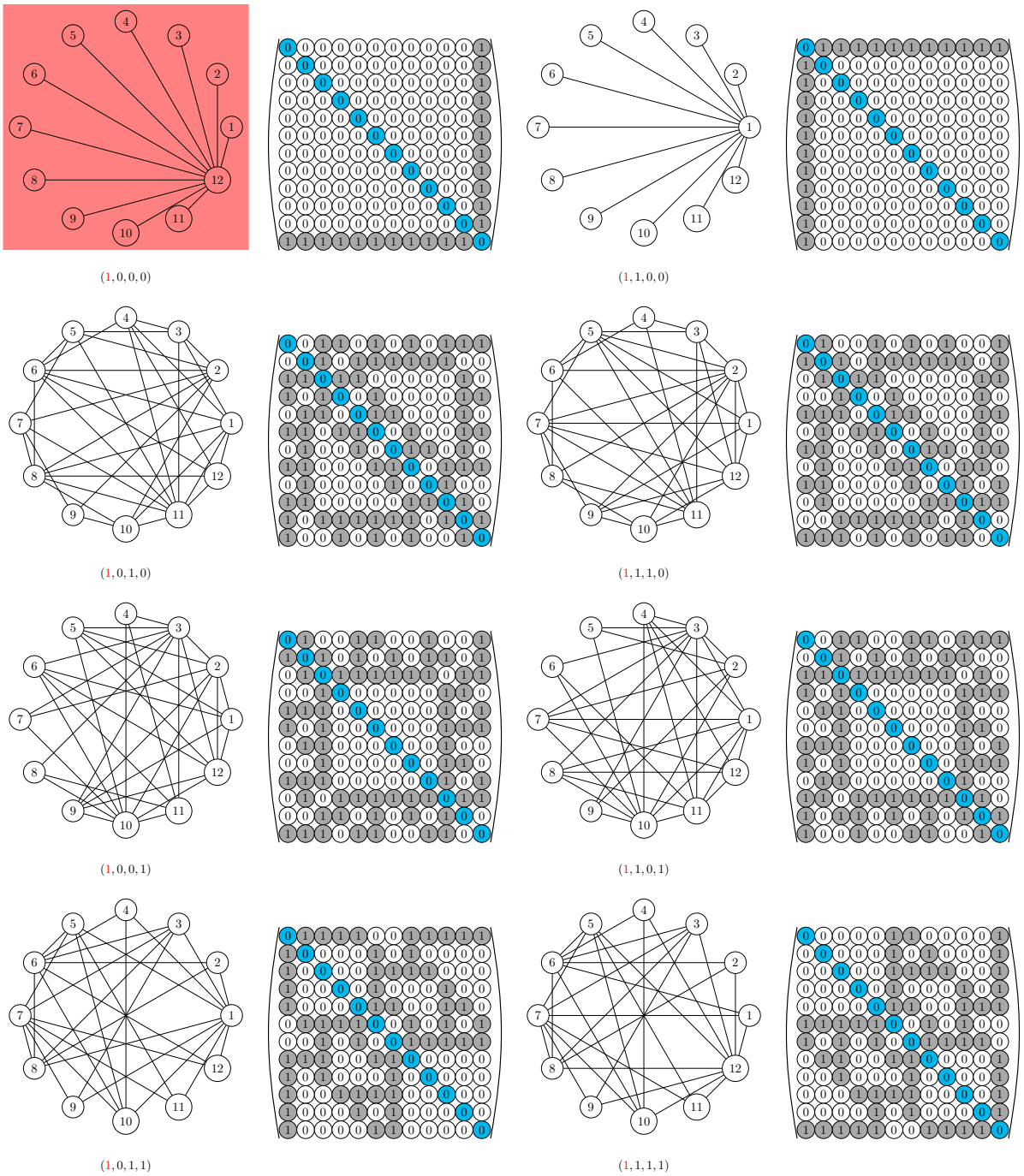


Figure 15: The 8 graphs of $\mathcal{OSG}(12)$

7 Symmetric generalized Pascal triangles

Another kind of binary triangles with a similar definition than Steinhaus triangles can be considered. A *generalized Pascal triangle of size* n is a triangle $\Delta = (a_{i,j})_{1 \leqslant j \leqslant i \leqslant n}$ of 0's and 1's verifying the local rule (LR), i.e., $a_{i,j} = a_{i-1,j-1} + a_{i-1,j} \bmod 2$, for all integers i, j such that $2 \leqslant j < i \leqslant n$. A generalized Pascal triangle $(a_{i,j})_{1 \leqslant j \leqslant i \leqslant n}$ is completely determined by its left side $L = (a_{i,1})_{1 \leqslant i \leqslant n}$ and its right side $R = (a_{i,i})_{1 \leqslant i \leqslant n}$. Therefore, we denote by $\Delta(L, R)$ the generalized Pascal triangle generated from the sequences L and R. The set of binary generalized Pascal triangles of size n is denoted by $\mathcal{PT}(n)$. Since the set of generalized Pascal triangles is closed under addition modulo 2, it follows that $\mathcal{PT}(n)$ is a vector space over $\mathbb{Z}/2\mathbb{Z}$. An example of generalized Pascal triangle of size 5 is depicted in Figure 16.

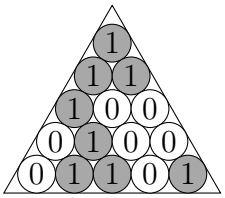


Figure 16: The generalized Pascal triangle $\Delta((11100), (11001))$

Since a generalized Pascal triangle is uniquely determined by its left and right sides, which have the same first term, the dimension of $\mathcal{PT}(n)$ is $2n-1$, for all positive integers n. Moreover, there exists a natural isomorphism between $\mathcal{PT}(n)$ and $\mathcal{ST}(2n-1)$, for all positive integers n. Indeed, as depicted in Figure 17, a generalized Pascal triangle of size n can be seen as a subtriangle of a Steinhaus triangle of size $2n-1$.

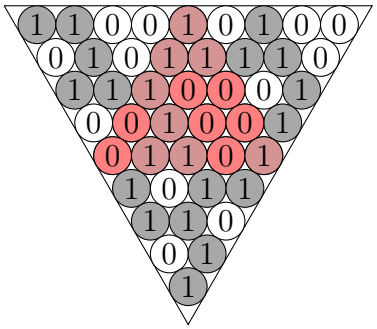


Figure 17: $\gamma(\nabla(110010100)) = \Delta((11100), (11001))$

For any positive integer n, let γ be the linear map defined by

$$\begin{array}{rccc} \gamma: & \mathcal{ST}(2n-1) & \longrightarrow & \mathcal{PT}(n) \\ & (a_{i,j})_{1 \leqslant i \leqslant j \leqslant 2n-1} & \longmapsto & (a_{i,n-1+j})_{1 \leqslant j \leqslant i \leqslant n} \end{array}$$

The linear map γ is well defined since the generalized Pascal triangles and the Steinhaus triangles share the same local rule (LR).

Proposition 7.1. *The linear map $\gamma : \mathcal{ST}(2n-1) \longrightarrow \mathcal{PT}(n)$ is an isomorphism.*

Proof. Let $\nabla = (a_{i,j})_{1 \leqslant i \leqslant j \leqslant 2n-1} \in \mathcal{ST}(2n-1)$ and $\Delta = \gamma(\nabla) = (a_{i,n-1+j})_{1 \leqslant j \leqslant i \leqslant n}$. The linear map γ is an isomorphism since the set G_{LR} of indices of the left and right sides of

$\Delta = (a_{i,n-1+j})_{1\leqslant j\leqslant i\leqslant n}$, i.e.,

$$G_{LR} = \{(i,n) \mid i \in \{1,\dots,n\}\} \cup \{(i,n-1+i) \mid i \in \{2,\dots,n\}\}$$

is a generating index set of $\mathcal{ST}(2n-1)$. First, all the terms of the first row $(a_{1,j})_{1\leqslant j\leqslant 2n-1}$ of ∇ can be expressed in function of the elements of the left side $(a_{i,n})_{1\leqslant i\leqslant n}$ and of the right side $(a_{i,n-1+i})_{1\leqslant i\leqslant n}$ of Δ. Indeed, for every $j \in \{1,\dots,n\}$, we know from Lemma 2.1 that

$$a_{1,j} \equiv \sum_{k=0}^{n-j} \binom{n-j}{k} a_{k+1,n} \pmod 2$$

and

$$a_{1,2n-j} \equiv \sum_{k=0}^{n-j} \binom{n-j}{k} a_{k+1,n+k} \pmod 2.$$

Since $G_1 = \{(1,j) \mid j \in \{1,\dots,2n-1\}\}$ is a generating index set of $\mathcal{ST}(2n-1)$, we conclude that G_{LR} is a generating index set of $\mathcal{ST}(2n-1)$ too. Therefore, the linear map γ is an isomorphism. □

As for Steinhaus triangles, the action of the dihedral group $D_3 = \langle r', h' \rangle$ on $\mathcal{PT}(n)$ can be considered, where the automorphisms r' and h' of $\mathcal{PT}(n)$ are defined by

$$\begin{array}{rccc} r' : & \mathcal{PT}(n) & \longrightarrow & \mathcal{PT}(n) \\ & (a_{i,j})_{1\leqslant j\leqslant i\leqslant n} & \longmapsto & (a_{n+j-i,n+1-i})_{1\leqslant j\leqslant i\leqslant n} \end{array}$$

and

$$\begin{array}{rccc} h' : & \mathcal{PT}(n) & \longrightarrow & \mathcal{PT}(n) \\ & (a_{i,j})_{1\leqslant j\leqslant i\leqslant n} & \longmapsto & (a_{i,1-j+i})_{1\leqslant j\leqslant i\leqslant n} \end{array}$$

For instance, for $L = (11100)$ and $R = (11001)$ and for all $g \in D_3$, the generalized Pascal triangles $g(\Delta(L,R))$ are depicted in Figure 18.

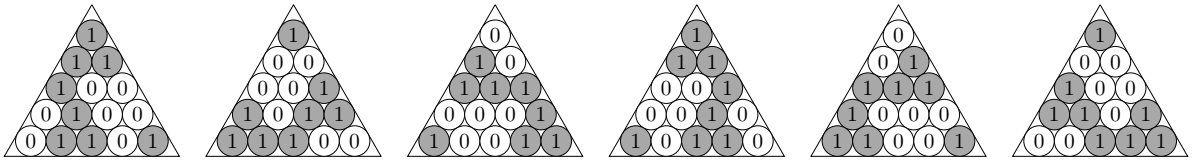


Figure 18: Action of D_3 on $\Delta((11100),(11001))$

Proposition 7.2. *For any positive integer n, we have*

$$\gamma r = r'\gamma \quad \textit{and} \quad \gamma h = h'\gamma.$$

Proof. First, we have

$$\gamma\left(r\left((a_{i,j})_{1\leqslant i\leqslant j\leqslant 2n-1}\right)\right) = \gamma\left((a_{j-i+1,2n-i})_{1\leqslant i\leqslant j\leqslant 2n-1}\right) = (a_{n+j-i,2n-i})_{1\leqslant j\leqslant i\leqslant n}$$

and

$$r'\left(\gamma\left((a_{i,j})_{1\leqslant i\leqslant j\leqslant 2n-1}\right)\right) = r'\left((a_{i,n-1+j})_{1\leqslant j\leqslant i\leqslant n}\right) = (a_{n+j-i,2n-i})_{1\leqslant j\leqslant i\leqslant n},$$

for all $(a_{i,j})_{1\leqslant i\leqslant j\leqslant 2n-1} \in \mathcal{ST}(2n-1)$. Moreover,

$$\gamma\left(h\left((a_{i,j})_{1\leqslant i\leqslant j\leqslant 2n-1}\right)\right) = \gamma\left((a_{i,2n-1-j+i})_{1\leqslant i\leqslant j\leqslant 2n-1}\right) = (a_{i,n+i-j})_{1\leqslant j\leqslant i\leqslant n}$$

and

$$h'\left(\gamma\left((a_{i,j})_{1\leqslant i\leqslant j\leqslant 2n-1}\right)\right) = h'\left((a_{i,n-1+j})_{1\leqslant j\leqslant i\leqslant n}\right) = (a_{i,n+i-j})_{1\leqslant j\leqslant i\leqslant n},$$

for all $(a_{i,j})_{1\leqslant i\leqslant j\leqslant 2n-1} \in \mathcal{ST}(2n-1)$. This completes the proof. □

A generalized Pascal triangle Δ of size n is said to be

- *rotationally symmetric* if $r'(\Delta) = \Delta$,
- *horizontally symmetric* if $h'(\Delta) = \Delta$,
- *dihedrally symmetric* if $r'(\Delta) = h'(\Delta) = \Delta$.

The sets of horizontally symmetric, rotationally symmetric and dihedrally symmetric generalized Pascal triangles of size n are denoted by $\mathcal{HPT}(n)$, $\mathcal{RPT}(n)$ and $\mathcal{DPT}(n)$, respectively, for all non-negative integers n. In other words, the sets $\mathcal{HPT}(n)$, $\mathcal{RPT}(n)$ and $\mathcal{DPT}(n)$ are simply the linear subspaces $\ker\left(h' - id_{\mathcal{PT}(n)}\right)$, $\ker\left(r' - id_{\mathcal{PT}(n)}\right)$ and $\ker\left(h' - id_{\mathcal{PT}(n)}\right) \cap \ker\left(r' - id_{\mathcal{PT}(n)}\right)$, respectively, where $id_{\mathcal{PT}(n)}$ is the identity map on $\mathcal{PT}(n)$, for all non-negative integers n. Examples of such triangles appear in Figure 19.

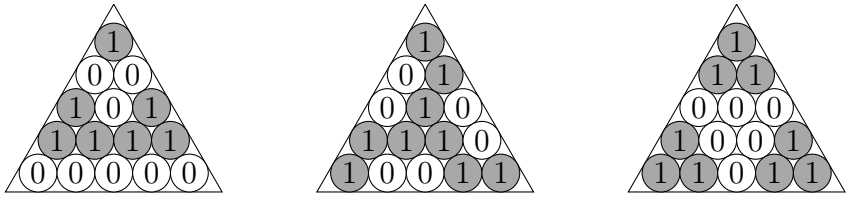


Figure 19: Triangles of $\mathcal{HPT}(5)$, $\mathcal{RPT}(5)$ and $\mathcal{DPT}(5)$.

It is now easy to see that a symmetric generalized Pascal triangles of size n corresponds to a symmetric Steinhaus triangle of size $2n-1$, for all positive integers n.

Proposition 7.3. *For any positive integer n, a steinhaus triangle ∇, of size $2n-1$, is horizontally, rotationally, or dihedrally symmetric if and only if the generalized Pascal triangle $\gamma(\nabla)$, of size n, is horizontally, rotationally, or dihedrally symmetric, respectively.*

Proof. From Propositions 7.1 and 7.2. □

Corollary 7.4. *The linear map γ induces isomorphisms of $\mathcal{HST}(2n-1)$ upon $\mathcal{HPT}(n)$, $\mathcal{RST}(2n-1)$ upon $\mathcal{RPT}(n)$ and $\mathcal{DST}(2n-1)$ upon $\mathcal{DPT}(n)$, respectively, for all positive integers n.*

Proof. From Proposition 7.3. □

Using the isomorphism γ and the results of the previous sections, we obtain the dimension and a basis for each linear subspace of symmetric generalized Pascal triangles of size n, for all positive integers n.

Proposition 7.5. *For any positive integer n, we have*

- $\dim \mathcal{HPT}(n) = n$,
- $\dim \mathcal{RPT}(n) = 2\left\lfloor \frac{n-1}{3} \right\rfloor + 1$,
- $\dim \mathcal{DPT}(n) = \left\lceil \frac{n}{3} \right\rceil$.

Proof. Let n be a positive integer. From Corollary 7.4 and Corollary 4.6, we obtain that

$$\dim \mathcal{HPT}(n) = \dim \mathcal{HST}(2n-1) = \left\lceil \frac{2n-1}{2} \right\rceil = n.$$

Moreover, from Corollary 7.4 and Corollary 3.5, we have

$$\begin{aligned} \dim \mathcal{RPT}(n) = \dim \mathcal{RST}(2n-1) &= \left\lfloor \frac{2n-1}{3} \right\rfloor + \delta_{1,(2n-1 \bmod 3)} \\ &= \left\lfloor \frac{2n-1}{3} \right\rfloor + \delta_{1,(n \bmod 3)} = 2\left\lfloor \frac{n-1}{3} \right\rfloor + 1. \end{aligned}$$

Finally, from Corollary 7.4 and Corollary 5.11, we obtain

$$\begin{aligned} \dim \mathcal{DPT}(n) = \dim \mathcal{DST}(2n-1) &= \left\lfloor \frac{2n+2}{6} \right\rfloor + \delta_{1,(2n-1 \bmod 6)} \\ &= \left\lfloor \frac{n+1}{3} \right\rfloor + \delta_{1,(n \bmod 3)} = \left\lceil \frac{n}{3} \right\rceil. \end{aligned}$$

This completes the proof. □

Theorem 7.6. *Let n and m be positive integers such that $m = 2\left\lfloor \frac{n-1}{3} \right\rfloor + 1$. For any integers $l_0, \ldots, l_{m-1}$, the set*

$$\left\{ \gamma\rho\left(\nabla \mathrm{S}^{(2n-1)}_{k,l_k}\right) \;\middle|\; k \in \{0, \ldots, m-1\} \right\}$$

is a basis of $\mathcal{RPT}(n)$, where

$$\gamma\rho\left(\nabla \mathrm{S}^{(2n-1)}_{k,l_k}\right) = \left(\binom{l_k + j - i + n - 1}{k + 1 - i} + \binom{l_k + n - j}{k + i - j - n + 1} + \binom{l_k + i - 1}{k + j - n} \bmod 2 \right)_{1 \leqslant j \leqslant i \leqslant n},$$

for all $k \in \{0, \ldots, m-1\}$.

Proof. From Theorem 3.13 and Corollary 7.4. □

Theorem 7.7. *Let n be a positive integer. The set*

$$\left\{ \gamma\left(\mathrm{S}^{(2n-1)}_{2(n-k)-1,-k}\right) \;\middle|\; k \in \{1, \ldots, n-1\} \right\} \cup \left\{ \gamma\left(\nabla(1)_{2n-1}\right) \right\}$$

is a basis of $\mathcal{HPT}(n)$, where

$$\gamma\left(\nabla \mathrm{S}^{(2n-1)}_{2(n-k)-1,-k}\right) = \left(\binom{-k + j - i + n - 1}{2(n-k) - i}_2 \right)_{1 \leqslant j \leqslant i \leqslant n},$$

for all $k \in \left\{1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$, and $\gamma\left(\nabla(1)_{2n-1}\right) = \Delta\left((1) \cdot (0)_{n-1}, (1) \cdot (0)_{n-1}\right)$.

Proof. From Theorem 4.9 and Corollary 7.4. □

Theorem 7.8. *Let n and m be positive integers such that $m = \left\lceil \frac{n}{3} \right\rceil$. Then, the set*

$$\{\gamma(U_{2n-1})\} \cup \left\{ \gamma\rho\left(\mathrm{S}^{(2n-1)}_{2k+1,k-n+1}\right) \;\middle|\; k \in \{0,\dots,m-2\} \right\}$$

is a basis of $\mathcal{DST}(n)$, where $\gamma(U_{2n-1}) = \Delta((1)\cdot(0)_{n-2}\cdot(1),(1)\cdot(0)_{n-2}\cdot(1))$ and

$$\gamma\rho\left(\mathrm{S}^{(2n-1)}_{2k+1,k-n+1}\right) = \left(\binom{k+j-i}{2k-i+2} + \binom{k-j+1}{i-j+2k-n+2} + \binom{k-n+i}{2k+j-n+1} \bmod 2\right)_{1\leqslant j\leqslant i\leqslant n},$$

for all $k \in \{0,\dots,m-2\}$.

Proof. From Theorem 5.14 and Corollary 7.4. □

We end this section by giving bases obtained from Theorem 7.6 for $\mathcal{RPT}(6)$, from Theorem 7.7 for $\mathcal{HPT}(4)$ and from Theorem 7.8 for $\mathcal{DPT}(11)$.

For $n = 7$ and $k_0 = k_1 = k_2 = k_3 = k_4 = 0$, we obtain the following basis

$$\left\{ \gamma\rho\left(\nabla\mathrm{S}^{(13)}_{k,0}\right) \;\middle|\; k \in \{0,1,2,3,4\} \right\}$$

of $\mathcal{RPT}(7)$, where

k	$\mathrm{S}^{(13)}_{k,0}$	$\rho\left(\nabla\mathrm{S}^{(13)}_{k,0}\right)$	$\gamma\rho\left(\nabla\mathrm{S}^{(13)}_{k,0}\right)$
0	(1111111111111)	∇(0111111111110)	$\Delta_0 = \Delta((1000001),(1000001))$
1	(0101010101010)	∇(0001010101000)	$\Delta_1 = \Delta((0100010),(0100010))$
2	(0011001100110)	∇(0101001100010)	$\Delta_2 = \Delta((1110101),(1010111))$
3	(0001000100010)	∇(0100000101010)	$\Delta_3 = \Delta((0000010),(0100000))$
4	(0000111100001)	∇(1111011110001)	$\Delta_4 = \Delta((1011001),(1001101))$

All the rotationnaly symmetric generalized Pascal triangles of size 7 are depicted in Figure 20, where the elements of the basis $\{\Delta_0,\Delta_1,\Delta_2,\Delta_3,\Delta_4\}$ are in red and, for every $\Delta \in \mathcal{RPT}(7)$, the coordinate vector (x_0,x_1,x_2,x_3,x_4) of $\Delta = x_0\Delta_0+x_1\Delta_1+x_2\Delta_2+x_3\Delta_3+x_4\Delta_4$ is given.

For $n = 4$, we obtain the following basis

$$\left\{ \gamma\left(\mathrm{S}^{(7)}_{7-2k,-k}\right) \;\middle|\; k \in \{1,2,3\} \right\} \cup \{\gamma(\nabla(1)_7)\}$$

of $\mathcal{HPT}(4)$, where

k	$\mathrm{S}^{(7)}_{7-2k,-k}$	$\gamma\rho\left(\nabla\mathrm{S}^{(7)}_{7-2k,-k}\right)$
1	(1000001)	$\Delta_1 = \Delta((0001),(0001))$
2	(0100010)	$\Delta_2 = \Delta((0011),(0011))$
3	(1010101)	$\Delta_3 = \Delta((0100),(0100))$

and $\Delta_4 = \gamma(\nabla(1)_7) = \Delta((1000),(1000))$. All the horizontally symmetric generalized Pascal triangles of size 4 are depicted in Figure 21, where the elements of the basis $\{\Delta_1,\Delta_2,\Delta_3,\Delta_4\}$ are in red and, for every $T \in \mathcal{HPT}(4)$, the coordinate vector (x_1,x_2,x_3,x_4) of $\Delta = x_1\Delta_1 + x_2\Delta_2 + x_3\Delta_3 + x_4\Delta_4$ is given.

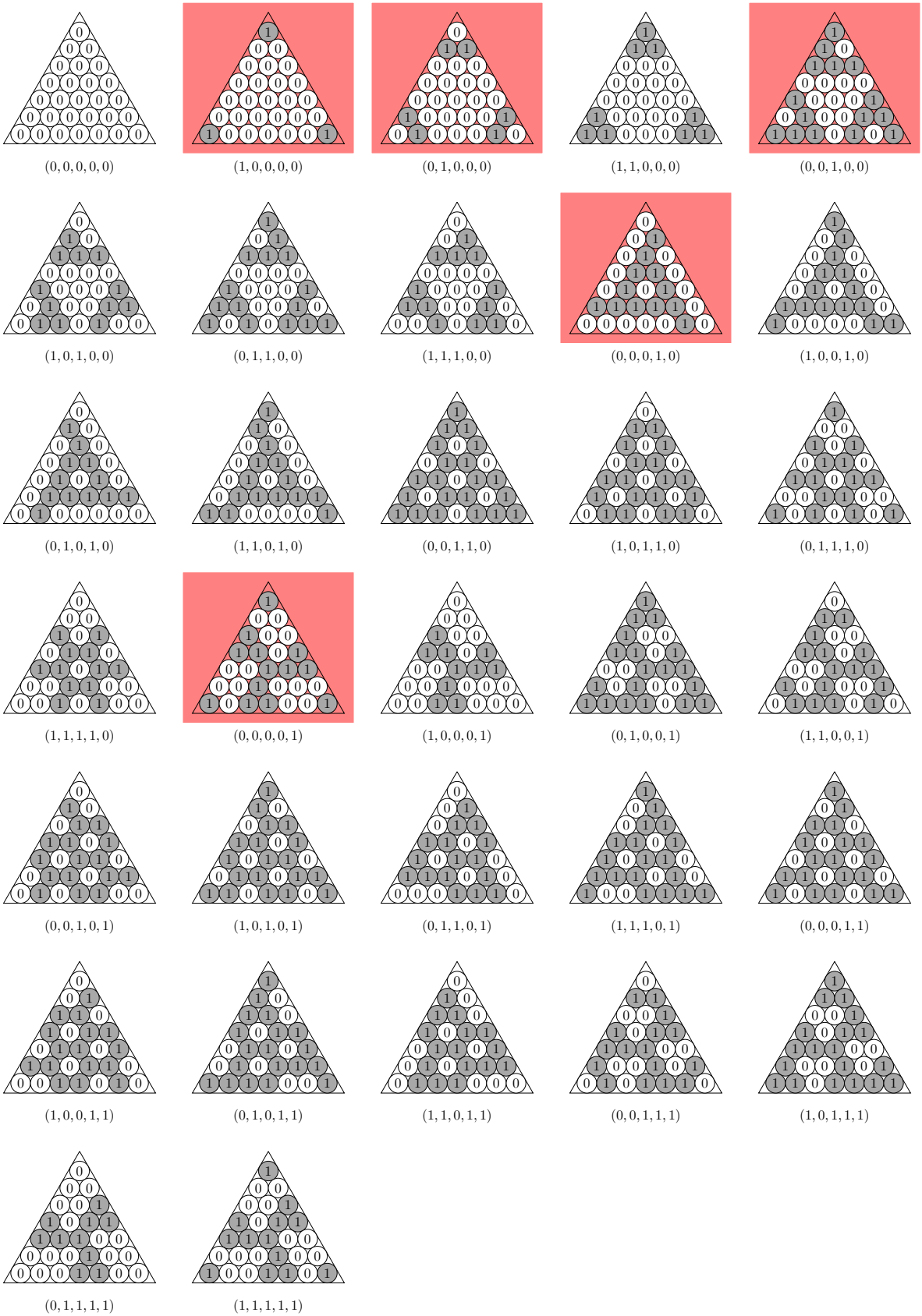


Figure 20: The 32 triangles of $\mathcal{RPT}(7)$ where the 5 red triangles form a basis

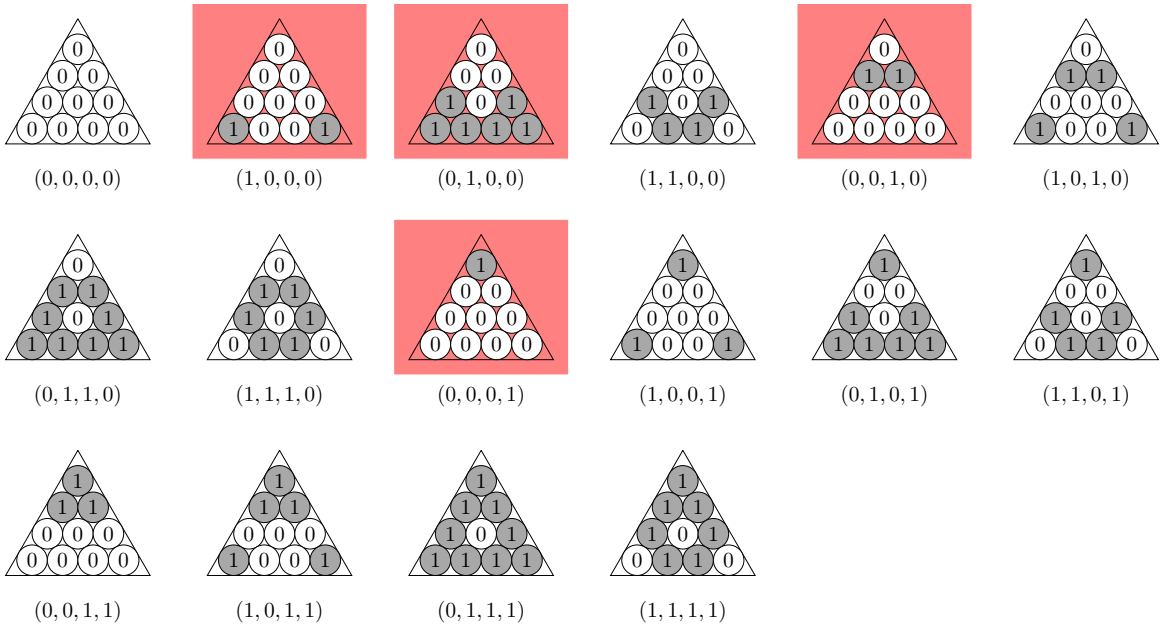


Figure 21: The 16 triangles of $\mathcal{HPT}(4)$ where the 4 red triangles form a basis

For $n = 11$, we obtain the following basis

$$\{\gamma(U_{21})\} \cup \left\{ \gamma\rho\left(\nabla \mathrm{S}^{(21)}_{2k+1,k-10}\right) \;\middle|\; k \in \{0,1,2\} \right\}$$

of $\mathcal{DPT}(11)$, where $\Delta_0 = \gamma(U_{21}) = \Delta((10000000001),(10000000001))$ and

k	$\mathrm{S}^{(21)}_{2k+1,k-10}$	$\rho\left(\nabla \mathrm{S}^{(21)}_{2k+1,k-10}\right)$	$\gamma\rho\left(\nabla \mathrm{S}^{(21)}_{2k+1,k-10}\right)$
0	(01010101010101010101)	∇(00010101010101010000)	$\Delta_1 = \Delta((01000000010),(01000000010))$
1	(10001000100010001001)	∇(01110001000100011110)	$\Delta_2 = \Delta((00110001100),(00110001100))$
2	(00000101000001010000)	∇(00000010000010000000)	$\Delta_3 = \Delta((00010001000),(00010001000))$

All the dihedrally symmetric generalized Pascal triangles of size 11 are depicted in Figure 22, where the elements of the basis $\{\Delta_0, \Delta_1, \Delta_2, \Delta_3\}$ are in red and, for every $\Delta \in \mathcal{DPT}(11)$, the coordinate vector (x_0, x_1, x_2, x_3) of $\Delta = x_0\Delta_0 + x_1\Delta_1 + x_2\Delta_2 + x_3\Delta_3$ is given.

8 Generalizations and open problems

In this section, we propose to extend the study of symmetric binary Steinhaus triangles in two directions: in higher dimension with the binary Steinhaus tetrahedra and for triangles of numbers in $\mathbb{Z}/m\mathbb{Z}$ with a local rule for which the set of triangles is closed under the action of the dihedral group D_3.

8.1 Symmetric binary tetrahedra

A *binary Steinhaus tetrahedron of size* n is a tetrahedron $(a_{i,j,k})_{1\leqslant i\leqslant j\leqslant k\leqslant n}$ of 0's and 1's verifying the local rule

$$a_{i,j,k} \equiv a_{i-1,j-1,k-1} + a_{i-1,j-1,k} + a_{i-1,j,k} \pmod 2, \tag{LR2}$$

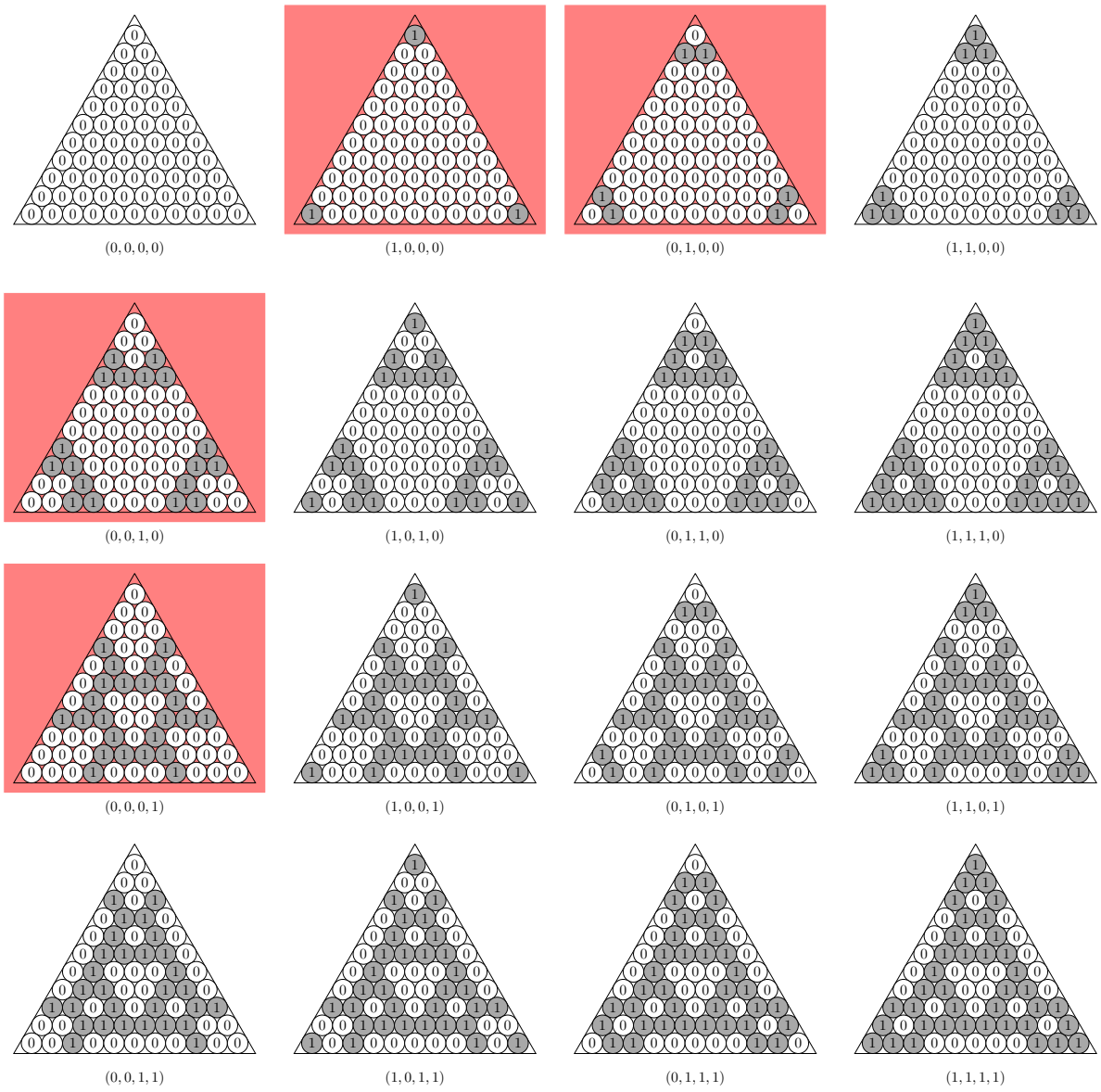


Figure 22: The 16 triangles of $\mathcal{DPT}(11)$ where the 4 red triangles form a basis

for all integers i, j, k such that $2 \leqslant i \leqslant j \leqslant k \leqslant n$. The set of Steinhaus tetrahedra of size n is denoted by $\mathcal{ST}_4(n)$. Since a Steinhaus tetrahedron $(a_{i,j,k})_{1\leqslant i\leqslant j\leqslant k\leqslant n}$ is uniquely determined by its first row $(a_{1,j,k})_{1\leqslant j\leqslant k\leqslant n}$, it follows that $\mathcal{ST}_4(n)$ is a vector space over $\mathbb{Z}/2\mathbb{Z}$ of dimension $\binom{n+1}{2}$. An example of Steinhaus tetrahedron of size 5 is depicted in Figure 23.

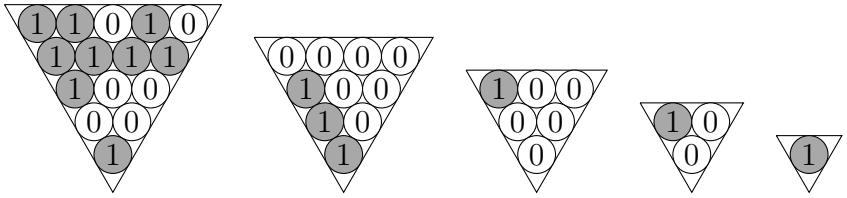


Figure 23: A Steinhaus tetrahedron of size 5

The symmetry group of a regular tetrahedron is constituted by the identity, 11 rotations (8 rotations by $\pm\frac{2\pi}{3}$ around an axis passing through a vertex and the middle of the opposite side and 3 rotations by $\frac{\pi}{2}$ around an axis passing through the middle of two opposite edges), 6 reflections and 6 rotoflections. This symmetry group is isomorphic to $\mathcal{S}_4$ and the subset of rotations is a subgroup of order 12 isomorphic to $\mathcal{A}_4$.

From the local rule (LR2), for any Steinhaus tetrahedron T and for all $g \in \mathcal{S}_4$, it is easy to see that $g(T)$ is also a Steinhaus tetrahedron, of the same size.

Problem 8.1. *For any subgroup G of $\mathcal{S}_4$ and any non-negative integer n, we consider the linear subspace of Steinhaus tetrahedra of size n defined by*

$$\mathcal{ST}_4(n)^G = \{T \in \mathcal{ST}_4(n) \mid \forall g \in G, g\,(T) = T\}\,.$$

The problem is then to characterize $\mathcal{ST}_4(n)^G$, to determine its dimension and a basis, for all integers n and all subgroups G of $\mathcal{S}_4$.

8.2 Symmetric triangles in $\mathbb{Z}/m\mathbb{Z}$

Triangles similar to binary Steinhaus triangles can be considered for other kinds of numbers. In [4], the authors study triangles defined from quasigroups. A *quasigroup* $(G, \star)$ is a finite set G with a binary operation $\star$ such that, for all $a, b \in G$, there exist unique $x, y \in G$ for which $a \star x = b$ and $y \star a = b$. A *$\nabla_\star$-configuration of size n* is a triangle $(a_{i,j})_{1\leqslant i\leqslant j\leqslant n}$ of elements in $(G, \star)$ verifying the local rule

$$a_{i,j} = a_{i-1,j-1} \star a_{i-1,j},$$

for all integers i, j such that $2 \leqslant i \leqslant j \leqslant n$. For $(G, \star) = (\mathbb{Z}/2\mathbb{Z}, +)$, a $\nabla_\star$-configuration is simply a binary Steinhaus triangle.

A quasigroup $(G, \star)$ is said to be *semisymmetric* if $y \star (x \star y) = x$, for all $x, y \in G$. It is easy to see that, for a semisymmetric quasigroup $(G, \star)$, the set of $\nabla_\star$-configurations is closed under the 120 degrees rotation. Moreover, if $(G, \star)$ is a commutative quasigroup, the set of $\nabla_\star$-configurations is also closed under the horizontal reflection.

A $\nabla_\star$-configuration is said to be *rotationally symmetric* if it is invariant under rotation and is said to be *dihedrally symmetric* if it is invariant under rotation and horizontal reflection (under the action of D_3).

The rotationally symmetric $\nabla_\star$-configurations, for semisymmetric quasigroups, and the dihedrally symmetric $\nabla_\star$-configurations, for commutative semisymmetric quasigroups, have been studied in [4]. Similar results than Propositions 3.2 and 5.4 are established and the cardinality of sets of rotationnaly and dihedrally $\nabla_\star$-configurations are given.

Theorem 8.2 (Theorem 3.4 in [4]). *Let $(G,\star)$ be a semisymmetric quasigroup. The number $\mathrm{conf}_{\mathcal{R}}(n)$ of rotationally symmetric $\nabla_\star$-configurations of size n is given by*

$$\mathrm{conf}_{\mathcal{R}}(n) = \begin{cases} |G|^k & \text{if } n = 3k, \\ |G|^{k+1} & \text{if } n = 3k+1, \\ |\mathrm{Fix}(\star)||G|^k & \text{if } n = 3k+2, \end{cases}$$

where $\mathrm{Fix}(\star) = \{x \in G \mid x \star x = x\}$.

Theorem 8.3 (Theorem 3.16 in [4]). *Let $(G,\star)$ be a commutative semisymmetric quasigroup. The number $\mathrm{conf}_{\mathcal{D}}(n)$ of dihedrally symmetric $\nabla_\star$-configurations of size n is given by*

$$\mathrm{conf}_{\mathcal{D}}(n) = \begin{cases} |G|^k & \text{if } n = 6k, \\ |G|^{k+1} & \text{if } n = 6k+1, \\ |\mathrm{Fix}(\star)||G|^k & \text{if } n = 6k+2, \\ |G|^{k+1} & \text{if } n = 6k+3, \\ |G|^{k+1} & \text{if } n = 6k+4, \\ |\mathrm{Fix}(\star)||G|^{k+1} & \text{if } n = 6k+5, \end{cases}$$

where $\mathrm{Fix}(\star) = \{x \in G \mid x \star x = x\}$.

Suppose now that $G = \mathbb{Z}/m\mathbb{Z}$, with $m \geqslant 2$, and let $\star$ be the binary operation defined by $x \star y = -(x+y)$, for all $x, y \in \mathbb{Z}/m\mathbb{Z}$. Then, the quasigroup $(\mathbb{Z}/m\mathbb{Z},\star)$ is commutative and semisymmetric. We are interested in the study of the sets of rotationally symmetric $\nabla_\star$-configurations and of dihedrally symmetric $\nabla_\star$-configurations of size n in $(\mathbb{Z}/m\mathbb{Z},\star)$, that we denote by $\mathcal{RSC}_{(\mathbb{Z}/m\mathbb{Z},\star)}(n)$ and $\mathcal{DSC}_{(\mathbb{Z}/m\mathbb{Z},\star)}(n)$, respectively. Examples of rotationally and dihedrally symmetric $\nabla_\star$-configurations of $\mathbb{Z}/6\mathbb{Z}$ are depicted in Figure 24.

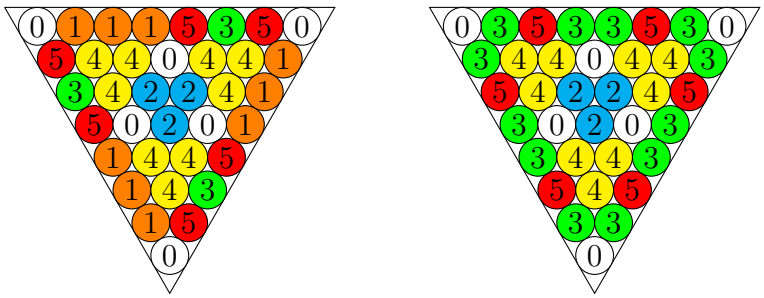


Figure 24: Triangles in $\mathcal{RSC}_{(\mathbb{Z}/6\mathbb{Z},\star)}(8)$ and $\mathcal{DSC}_{(\mathbb{Z}/6\mathbb{Z},\star)}(8)$

It is clear that $\mathcal{RSC}_{(\mathbb{Z}/m\mathbb{Z},\star)}(n)$ and $\mathcal{DSC}_{(\mathbb{Z}/m\mathbb{Z},\star)}(n)$ are submodule of the free $\mathbb{Z}/m\mathbb{Z}$-module of $\nabla_\star$-configurations.

Problem 8.4. *Let m be a positive integer. For any positive integer n and each submodule $\mathcal{RSC}_{(\mathbb{Z}/m\mathbb{Z},\star)}(n)$ and $\mathcal{DSC}_{(\mathbb{Z}/m\mathbb{Z},\star)}(n)$, determine its length and a generating set.*

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