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Unary profile of lambda terms with restricted De Bruijn indices*

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In this paper we present an average-case analysis of closed lambda terms with restricted values of De Bruijn indices in the model where each occurrence of a variable contributes one to the size. Given a fixed integer k , a lambda term in which all De Bruijn indices are bounded by k has the following shape: It starts with k De Bruijn levels, forming the so-called hat of the term, to which some number of k -colored Motzkin trees are attached. By means of analytic combinatorics, we show that the size of this hat is constant on average and that the average number of De Bruijn levels of k -colored Motzkin trees of size n is asymptotically $\Theta(\sqrt{n})$. Combining these two facts, we conclude that the maximal non-empty De Bruijn level in a lambda term with restrictions on De Bruijn indices and of size n is, on average, also of order \sqrt{n} . On this basis, we provide the average unary profile of such lambda terms.

Keywords: profile of lambda terms, singularity analysis, lambda terms with restrictions

1 Introduction

The last decade has seen an abundance of studies which focus on enumeration of objects originating from logic and computability theory. Seen from the combinatorial point of view, some of these objects provide intriguing problems related to counting them and an average-case analysis of their parameters. One of these are lambda terms, central objects of lambda calculus, which are investigated in this paper. Due to a simple combinatorial specification of lambda terms, no knowledge concerning lambda calculus is required to understand statements and proofs of the presented theorems. For more information on lambda calculus we refer a curious reader to Barendregt (1984).

Before we outline the results obtained hitherto, as well as our contribution, let us first introduce some basic notions.

Let \mathcal{V} be a countable set of variables. *Lambda terms* are defined by the following grammar:

$$\mathcal{T} ::= \mathcal{V} \mid (\lambda \mathcal{V}. \mathcal{T}) \mid (\mathcal{T} \mathcal{T}).$$

A term of the form $(\lambda x.M)$ is called an *abstraction*, while a term of the form $(M N)$ is called an *application*. For the sake of clarity, we omit some parentheses according to the standard convention, *i.e.*,

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outermost parentheses are dropped, an application is left- and an abstraction right-associative. By $\text{Var}(M)$ we denote the set of all variables occurring in M . The set $\text{FV}(M)$ of *free variables* in a term M is defined recursively as follows:

$$\text{FV}(x) = x, \quad \text{FV}(\lambda x.M) = \text{FV}(M) \setminus \{x\}, \quad \text{FV}(MN) = \text{FV}(M) \cup \text{FV}(N).$$

A term M is called *closed* if it contains no free variables, *i.e.*, when $\text{FV}(M) = \emptyset$.

Lambda terms have a natural representation by means of *enriched trees*, *i.e.*, rooted trees with additional directed edges (pointers). In order to construct the corresponding enriched tree for a given lambda term, first we construct a Motzkin tree, *i.e.*, a plane rooted tree with each node of out-degree 0, 1, or 2. In this tree each binary node corresponds to an application, each unary node to an abstraction, and each leaf to a variable. Now, for every occurrence of a bound variable x , we add a directed edge from the unary node corresponding to the particular abstraction, labelled with λx , to the variable. Therefore, each unary node of the Motzkin tree carries (zero, one, or more) pointers to leaves taken from the subtree rooted at this unary node; all leaves receiving a pointer from this unary node correspond to the same variable, and each leaf receives at most one pointer. By $\text{tree}(M)$ we denote the Motzkin tree obtained from a lambda term M by removing all pointers (see Grygiel et al. (2013) for a more detailed description).

In what follows, we will be interested only in closed terms. Moreover, terms that are equal up to α -conversion, *i.e.*, up to renaming of bound variables, are considered equivalent. This allows us to apply the De Bruijn notation for lambda terms, which consists in eliminating names of variables and replacing them by positive integers indicating, in the tree representation, the number of unary nodes on the path from a particular variable to its binder (see De Bruijn (1972)). In other words, instead of having the set \mathcal{V} of variables, we use the set $\{1, 2, 3, \dots\}$ of *De Bruijn indices*, where an index n occurring in a term M indicates that the n -th lambda lying on the path from the corresponding leaf to the root in $\text{tree}(M)$ points at this leaf. For every $k \in \mathbb{N}$, we say that a lambda term is *k-indexed* if all of its De Bruijn indices belong to the set $\{1, 2, \dots, k\}$.

Let M be a lambda term and v be a vertex in $\text{tree}(M)$. The *unary height of v in $\text{tree}(M)$* , denoted by $\mathfrak{h}(v)$, is defined as the number of unary nodes on the path connecting v with the root of $\text{tree}(M)$. For every $\ell \in \mathbb{N}$, the *ℓ -th De Bruijn level of $\text{tree}(M)$* is defined as the set of all vertices v in $\text{tree}(M)$ such that $\mathfrak{h}(v) = \ell$.

Remark. De Bruijn (1972) introduced the notion of a *level* exclusively for variables. We adopt this concept for all kinds of nodes in enriched trees as well as for Motzkin trees. However, to avoid any confusion, we want to point out that the name *De Bruijn level* has also been used in different contexts in previous papers so far.

As an example let us consider the term $\lambda a. ((\lambda b. (\lambda c. b)b)a) (\lambda d. ((\lambda e. d)d)(\lambda f. fa))$ (see Figure 1). Its De Bruijn notation reads as $\lambda ((\lambda(\lambda 2)1)1) (\lambda((\lambda 2)1)(\lambda 13))$, and it has three non-empty De Bruijn levels, which are not necessarily connected. Notice that the 0-th De Bruijn level is empty, as the term is in the form of an abstraction. Moreover, this term is *k-indexed* for every $k \geq 3$.

In this paper the size of a lambda term is defined as the total number of its variables, abstractions, and applications, *i.e.*, for any variable x and lambda terms M and N the size is defined as follows:

$$\begin{aligned} |x| &= 1, \\ |\lambda x.M| &= 1 + |M|, \\ |MN| &= 1 + |M| + |N|. \end{aligned}$$

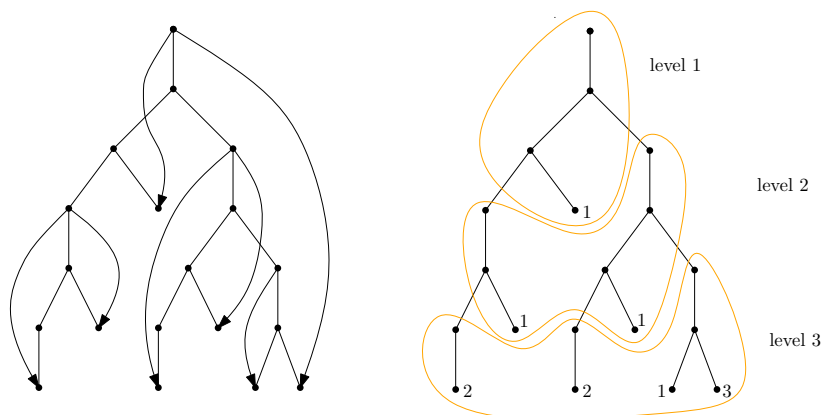


Fig. 1: The enriched tree of $\lambda a.((\lambda b.(\lambda c.b)b)a)(\lambda d.((\lambda e.d)d)(\lambda f.f a))$ and its decomposition into De Bruijn levels.

Therefore, the size of a lambda term is equal to the number of all vertices in the corresponding tree. The term in Figure 1 is of size 19. It has 7 variables, 6 applications, and 6 abstractions.

Different size models have been studied in the literature. The model studied within this paper provides a challenging and still open problem on the asymptotics of the sequence of the number of terms of a given size (for a thorough discussion see Bodini et al. (2013)). The encountered difficulties lead Bodini et al. (2018) to study restricted classes of terms, namely terms with a bounded number of De Bruijn levels and terms with bounded De Bruijn indices. Gittenberger and Larcher (2018) studied some statistical properties of lambda terms with bounded number of De Bruijn levels. Their results, exhibiting a change in the distribution of leaves within terms, shed some light on reasons for the strange behaviour of the counting sequences.

While the counting problem is also open for the model where variables do not contribute to the size, David et al. (2013) provided some results concerning typical parameters of closed lambda terms with no additional restrictions. The applied methods, however, seem not to work in the case of the size of each variable being one.

Another way of measuring terms consists in taking into account the depth of each variable, *i.e.*, the number of abstractions enclosing it. This approach was motivated by Tromp (2007) and further studied by Grygiel and Lescanne (2015); Bendkowski et al. (2016) (with the main focus on enumeration of terms) and Bendkowski et al. (accepted) (describing average values of several parameters in terms).

In this paper, in a similar vein to the work by Gittenberger and Larcher (2018), we perform an average-case analysis of lambda terms with bounded De Bruijn indices. Clearly, this class has a significantly weaker restriction compared to restricting the number of De Bruijn levels. On the basis of an empirical investigation on existing Haskell programs and their lambda calculus counterparts, we claim that imposing a bound on De Bruijn indices seems natural, as their values in the vast majority of programs remain small and rarely exceed 20 (Berger (2019)). Our research is hence motivated by getting a better understanding of the structure of lambda terms belonging to this class as well as explaining the structural discrepancies between terms from the two discussed classes.

2 Structure of lambda terms with bounded De Bruijn indices

In this section we want to illustrate the asymptotic shape of lambda terms with bounded De Bruijn indices in a very general way, while it will be investigated more thoroughly in the subsequent sections.

In order to set up the generating function for lambda terms with bounded De Bruijn indices we proceed analogously to Bodini et al. (2018). For every $k \geq 1$, let \mathcal{G}_k be the class of k -indexed lambda terms and let $G_k(z)$ be the corresponding generating function. Furthermore, for $i \in \{0, \dots, k\}$, let $\mathcal{G}_{k,i}$ be the class of unary-binary trees such that every leaf v can be labelled in $\min\{\mathbf{h}(v) + i, k\}$ ways. The classes $\mathcal{G}_{k,i}$ can be recursively specified, starting from a class \mathcal{Z} of atoms, in the following way:

$$\begin{aligned}\mathcal{G}_{k,k} &= k \mathcal{Z} \uplus (\mathcal{Z} \times \mathcal{G}_{k,k} \times \mathcal{G}_{k,k}) \uplus (\mathcal{Z} \times \mathcal{G}_{k,k}), \\ \mathcal{G}_{k,i} &= i \mathcal{Z} \uplus (\mathcal{Z} \times \mathcal{G}_{k,i} \times \mathcal{G}_{k,i}) \uplus (\mathcal{Z} \times \mathcal{G}_{k,i+1}).\end{aligned}$$

Then the classes \mathcal{G}_k and $\mathcal{G}_{k,0}$ are isomorphic and hence their generating functions coincide. Thus, by translating into generating functions, we directly get (cf. Bodini et al. (2018))

$$G_k(z) = \frac{1 - \sqrt{R_{k,k}(z)}}{2z},$$

where

$$R_{k,i}(z) = \begin{cases} 1 - 2z - (4k - 1)z^2, & i = 0, \\ 1 - 2z - (4k - 6)z^2 + 2z\sqrt{R_{k,0}(z)}, & i = 1, \\ 1 - 2z - 4(k - i)z^2 + 2z\sqrt{R_{k,i-1}(z)}, & i > 1. \end{cases}$$

Bodini et al. (2018, Lemma 5.4) proved that the dominant singularity of $G_k(z)$ comes from the innermost radicand, i.e., $R_{k,0}(z)$, and is equal to $\rho_k = \frac{1}{2\sqrt{k+1}}$. Furthermore, they provide an asymptotic estimate of the n -th coefficient of $G_k(z)$. Before showing this estimate let us define an auxiliary sequence $(c_i)_{i \geq 1}$:

$$c_1 = 5 \quad \text{and} \quad c_i = 4i - 5 + \sqrt{c_{i-1}} \text{ for } i \geq 2 \quad (1)$$

and constants $C_{i,k}$ with $i \geq 1$ and $k \geq i$:

$$C_{i,k} = \prod_{s=i}^k \frac{1}{\sqrt{c_s}}. \quad (2)$$

These numbers appear both in the announced estimate and throughout this paper. The first values of $(c_k)_{k \geq 1}$ and $(C_{i,k})_{i \geq k}$ are listed in Table 1.

Lemma 2.1 (Bodini et al. (2018, Theorem 5.6)). *For any fixed $k \geq 1$, let $G_k(z)$ be the generating function of the class of k -indexed lambda terms. Then*

$$[z^n]G_k(z) \sim \frac{C_{1,k}k^{1/4}}{2\sqrt{\pi\rho_k}}n^{-3/2}\rho_k^{-n},$$

where $C_{1,k}$ is defined as in (2).

k	c_k	$C_{1,k}$	$C_{2,k}$	$C_{3,k}$	$C_{4,k}$
1	5.000000000	0.4472135954			
2	5.236067977	0.1954395076	0.4370160245		
3	9.288245611	0.0641276779	0.1433938471	0.3281203412	
4	14.04766232	0.0171097429	0.0382585483	0.0875449552	0.2668074612
5	18.74802112	0.0039515344	0.0088358996	0.0202187084	0.0616197956
6	23.32989851	0.0008181055	0.0018293396	0.0041859784	0.0127574486
7	27.83010336	0.0001550786	0.0003467663	0.0007934866	0.0024182791
8	32.27542448	0.0000272970	0.0000610381	0.0001396701	0.0004256674
9	36.68114640	0.0000045070	0.0000100781	0.0000230612	0.0000702828
10	41.05649622	0.0000007034	0.0000015729	0.0000035991	0.0000109688

Tab. 1: First values of the sequences $(c_k)_{k \geq 1}$ and $(C_{i,k})_{k \geq i}$ for $i \in \{1, 2, 3, 4\}$.

This result already gives us a hint that lambda terms with bounded De Bruijn indices behave somewhat treelike. However, in order to get a better intuition why this is the case and how exactly these terms look like, we set up the generating function $G_k(z)$ in a different way: Instead of interpreting a lambda term belonging to that class as a structure that involves iterated unary-binary trees, we consider it to be built of leaf-labelled binary trees that are glued together via unary nodes (cf. Figure 2). Thereby, the labels of the leaves correspond to the respective De Bruijn indices. Obviously, this implies that within the whole tree each of the labels belongs to the set $\{1, \dots, k\}$. However, in the first $k - 1$ De Bruijn levels (excluding the 0-th level, which contains no variables) we have a stronger restriction. Since we consider only closed terms, no label (i.e., no De Bruijn index) can exceed the De Bruijn level the respective leaf is located in. Thus, with $B(z, w)$ denoting the bivariate generating function of binary trees where z marks the size (i.e., the total number of nodes) and w marks the number of leaves, and with $M_k(z)$ denoting the generating function of Motzkin trees where each leaf can be labelled in k ways (k -colored Motzkin trees in short), we get

$$G_k(z) = B(z, B(z, 1 + B(z, 2 + \dots + B(z, k - 1 + M_k(z)) \dots))), \quad (3)$$

where

$$B(z, w) = \frac{1 - \sqrt{1 - 4wz^2}}{2z} \quad \text{and} \quad M_k(z) = \frac{1 - z - \sqrt{(1 - z)^2 - 4kz^2}}{2z}. \quad (4)$$

Now, let us give an interpretation of Equation (3). Each tree representing a lambda term starts with a binary tree, in which all the leaves are replaced by unary nodes to which we add further binary trees. This is necessary for a lambda term to be closed. These newly added binary trees represent the first De Bruijn level. Next, there are two possibilities for each leaf in this level: Either it receives the label 1 or, alternatively, it is replaced with a unary node with a new binary tree attached, which belongs to the next De Bruijn level. In this level the leaves can already be labelled with two different labels (namely 1 or 2), or they can be replaced with unary nodes with new binary trees attached. Starting from the k -th De Bruijn level, the number of possible labellings for the leaves does not increase anymore. Thus, we finally get

$B(z, k + B(z, k + B(z, k + \dots)))$, which is exactly the generating function $M_k(z)$ of k -colored Motzkin trees given in (4).

Therefore, the enriched tree corresponding to a k -indexed lambda term is constructed as follows (cf. Figure 2):

- It starts with the *hat* consisting of all De Bruijn levels from 0 to $k - 1$ along with all unary nodes from the k -th level;
- To this hat structure we attach k -colored Motzkin trees via unary nodes.

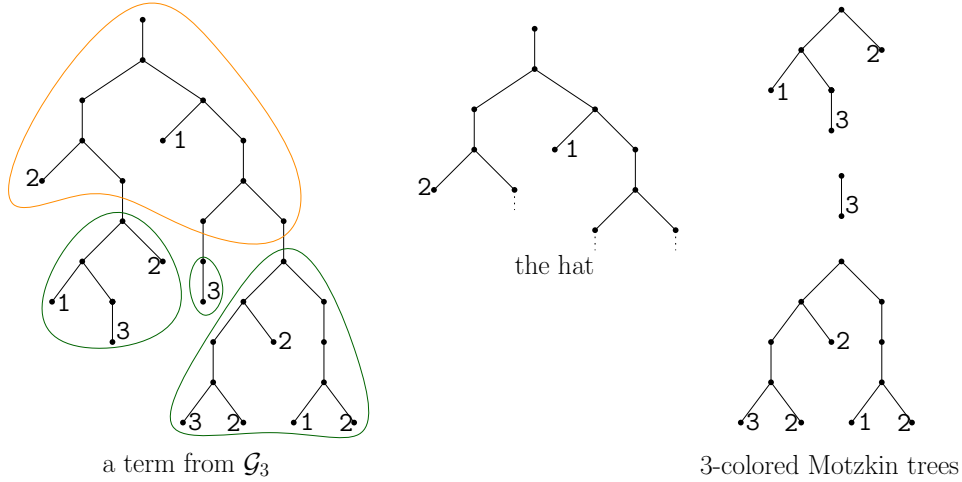


Fig. 2: A lambda term from \mathcal{G}_3 decomposed into the hat and three subterms represented by 3-colored Motzkin trees.

Remark. Note that the glued binary trees in Equation (3) constitute the hat of the structure, to which we attach the (comparatively large) k -colored Motzkin trees.

In the subsequent sections we investigate the structure of these terms in more detail. We prove that, for a fixed $k \geq 1$, the hat of a k -indexed lambda term is on average of constant size and that the average number of De Bruijn levels of a term of size n is asymptotically \sqrt{n} . Finally, we provide its unary profile.

3 Average size of a hat

In this section we focus on the size of the hat of k -indexed lambda terms. We prove that the average size of a hat is asymptotically constant, *i.e.*, it does not depend on the size of a term. This implies that on average the number of k -colored Motzkin trees in the decomposition described in the previous section is also constant.

Theorem 3.1. *For $k \geq 1$, let χ_k be the size of the hat of a lambda term where all De Bruijn indices are at most k . Then, as n tends to infinity,*

$$\mathbb{E}_{\mathcal{G}_{k,n}}(\chi_k) = k + 4(k + \sqrt{k} - 1) \sum_{i=1}^k \frac{C_{1,i}}{\sqrt{c_i}} + \sum_{i=0}^{k-2} \left(1 + 2\sqrt{k} + 4i - \sqrt{c_{k-i-1}}\right) \sum_{j=k-i}^k \frac{C_{k-i,j}}{\sqrt{c_j}} + o(1)$$

with $(c_i)_{i \geq 1}$ and $C_{i,k}$ defined in (1) and (2).

Proof: Let $G_k(z, u)$ be the generating function for k -indexed lambda terms with z marking the size of terms and u marking the size of their hats. The average size of a hat is hence given by

$$\mathbb{E}_{\mathcal{G}_{k,n}}(\chi_k) = \frac{[z^n] \frac{\partial G_k(z, u)}{\partial u} \Big|_{u=1}}{[z^n] G_k(z)}.$$

Since we want to mark by u all the nodes that belong to the hat, we get

$$G_k(z, u) = B(zu, B(zu, 1 + B(zu, 2 + \dots + B(zu, k-1 + M_k(z)) \dots))),$$

where $B(z, w)$ is the function defined in (4). This gives

$$G_k(z, u) = \frac{1 - \sqrt{R_{k,k}(z, u)}}{2zu}$$

where

$$R_{k,i}(z, u) = \begin{cases} 1 - 2z - (4k-1)z^2, & i = 0, \\ 1 - 2zu^2 - (4k-6)z^2u^2 + 2zu^2\sqrt{R_{k,0}(z, u)}, & i = 1, \\ 1 - 2zu - 4(k-i)z^2u^2 + 2zu\sqrt{R_{k,i-1}(z, u)}, & i > 1. \end{cases}$$

Therefore, the derivatives can also be recursively defined via

$$\frac{\partial R_{k,i}(z, u)}{\partial u} \Big|_{u=1} = \begin{cases} 0, & i = 0, \\ -4z - 4(2k-3)z^2 + 4z\sqrt{R_{k,0}(z, 1)} + \frac{z}{\sqrt{R_{k,0}(z, 1)}} \frac{\partial R_{k,0}(z, u)}{\partial u} \Big|_{u=1}, & i = 1, \\ -2z - 8(k-i)z^2 + 2z\sqrt{R_{k,i-1}(z, 1)} + \frac{z}{\sqrt{R_{k,i-1}(z, 1)}} \frac{\partial R_{k,i-1}(z, u)}{\partial u} \Big|_{u=1}, & i > 1, \end{cases}$$

and we get

$$\begin{aligned} \frac{\partial G_k(z, u)}{\partial u} \Big|_{u=1} &= \frac{\sqrt{R_{k,k}(z, 1)} - 1}{2z} - \frac{1}{4z\sqrt{R_{k,k}(z, 1)}} \cdot \frac{\partial R_{k,k}(z, u)}{\partial u} \Big|_{u=1} \\ &= \frac{\sqrt{R_{k,k}(z, 1)} - 1}{2z} + \sum_{i=0}^{k-2} \frac{z^i(1 + 4iz - \sqrt{R_{k,k-i-1}(z, 1)})}{2 \prod_{j=k-i}^k \sqrt{R_{k,j}(z, 1)}} \\ &\quad + \frac{z^{k-1}(1 + (2k-3)z - \sqrt{R_{k,0}(z, 1)})}{\prod_{j=1}^k \sqrt{R_{k,j}(z, 1)}}. \end{aligned} \tag{5}$$

Expanding the radicands $R_{k,i}$ around $z = \rho_k$ yields

$$R_{k,i}(\rho_k(1 - \varepsilon), 1) = \begin{cases} 4\rho_k\sqrt{k}\varepsilon + \mathcal{O}(|\varepsilon|^2), & i = 0, \\ c_i\rho_k^2 + d_{k,i}\sqrt{\varepsilon} + \mathcal{O}(|\varepsilon|), & i > 0 \end{cases}$$

where $\varepsilon \in \mathbb{C} \setminus \mathbb{R}_-$ and $|\varepsilon| \rightarrow 0$ and with $d_{k,m} = 4C_{1,m-1}\rho_k^{3/2}k^{1/4}$.

Hence, we have

$$\sqrt{R_{k,i}(\rho_k(1-\varepsilon), 1)} = \begin{cases} 2\sqrt{\rho_k}k^{1/4}\sqrt{\varepsilon} + \mathcal{O}(|\varepsilon|), & i = 0, \\ \sqrt{c_i}\rho_k + b_{k,i}\sqrt{\varepsilon} + \mathcal{O}(|\varepsilon|), & i > 0 \end{cases}$$

with

$$b_{k,i} = \frac{d_{k,i}}{2\rho_k\sqrt{c_i}} = 2C_{1,m}\sqrt{\rho_k}k^{1/4}. \quad (6)$$

Plugging this into Equation (5) gives

$$\begin{aligned} \left. \frac{\partial G_k(\rho_k(1-\varepsilon), u)}{\partial u} \right|_{u=1} &= \frac{\sqrt{c_k}\rho_k + b_{k,k}\sqrt{\varepsilon} - 1}{2\rho_k} + \sum_{i=0}^{k-2} \frac{\rho_k^i (1 + 4i\rho_k - \sqrt{c_{k-i-1}}\rho_k - b_{k-i-1}\sqrt{\varepsilon})}{2\prod_{j=k-i}^k (\sqrt{c_j}\rho_k + b_{k,j}\sqrt{\varepsilon})} \\ &\quad + \frac{\rho_k^{k-1} (1 + (2k-3)\rho_k - 2\sqrt{\rho_k}k^{1/4}\sqrt{\varepsilon})}{\prod_{j=1}^k (\sqrt{c_j}\rho_k + b_{k,j}\sqrt{\varepsilon})} + \mathcal{O}(|\varepsilon|) \\ &= A_k - B_k\sqrt{\varepsilon} + \mathcal{O}(|\varepsilon|), \end{aligned}$$

where A_k and B_k are constants depending on k with

$$\begin{aligned} B_k &= C_{1,k} \left(k^{1/4}\rho_k^{-1/2} + \frac{1 + (2k-3)\rho_k}{\rho_k^2} \sum_{j=1}^k \frac{b_{k,j}}{\sqrt{c_j}} \right) \\ &\quad + \frac{1}{2\rho_k^2} \sum_{i=0}^{k-2} C_{k-i,k} \left(b_{k,k-i}\sqrt{c_{k-i}}\rho_k + (1 + 4i\rho_k - \rho_k\sqrt{c_{k-i-1}}) \sum_{j=k-i}^k \frac{b_{k,j}}{\sqrt{c_j}} \right). \end{aligned}$$

Since A_k is not important for the result, we skip computing its exact value. By inserting (6) into the formula for B_k and after some simplifications, we obtain

$$B_k = \frac{C_{1,k}k^{1/4}}{\sqrt{\rho_k}} \left(k + 4(k + \sqrt{k} - 1) \sum_{i=1}^k \frac{C_{1,i}}{\sqrt{c_i}} + \sum_{i=0}^{k-2} \left(1 + 2\sqrt{k} + 4i - \sqrt{c_{k-i-1}} \right) \sum_{j=k-i}^k \frac{C_{k-i,j}}{\sqrt{c_j}} \right). \quad (7)$$

By singularity analysis applied to

$$\left. \frac{\partial G_k(z, u)}{\partial u} \right|_{u=1} = A_k - B_k\sqrt{1 - \frac{z}{\rho_k}} + \mathcal{O}\left(\left|1 - \frac{z}{\rho_k}\right|\right),$$

we immediately obtain, as n tends to infinity,

$$[z^n] \left. \frac{\partial G_k(z, u)}{\partial u} \right|_{u=1} \sim \frac{B_k}{2\sqrt{\pi}} \rho_k^n n^{-3/2}.$$

Finally, by (7) and Lemma 2.1, we get

$$\mathbb{E}_{\mathcal{G}_{k,n}}(\chi_k) = k + 4(k + \sqrt{k} - 1) \sum_{i=1}^k \frac{C_{1,i}}{\sqrt{c_i}} + \sum_{i=0}^{k-2} \left(1 + 2\sqrt{k} + 4i - \sqrt{c_{k-i-1}} \right) \sum_{j=k-i}^k \frac{C_{k-i,j}}{\sqrt{c_j}} + o(1).$$

□

In Figure 3 we list average sizes of hats of k -indexed lambda terms for k greater than or equal to 10 and plot the corresponding values for k up to 100.

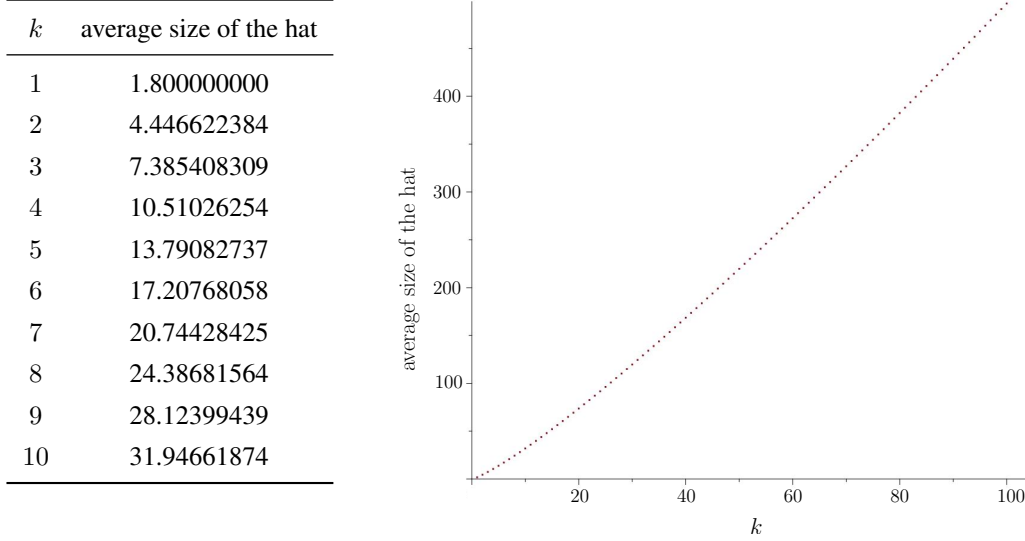


Fig. 3: Average size of the hat for $k \in \{1, \dots, 10\}$ (table) and for $k \in \{1, \dots, 100\}$ (plot).

Since the hat of a k -indexed lambda term, for any $k \geq 1$, is constant on average, such a term has on average a finite number of unary nodes in the k -th De Bruijn level. Therefore, we can conclude what follows.

Corollary 3.2. *For every $k \geq 1$, the average number of k -colored Motzkin trees in the decomposition (see page 6) of k -indexed lambda terms is constant.*

4 Average number of De Bruijn levels

In order to determine the average number of De Bruijn levels of k -indexed lambda terms, we first compute the average number of De Bruijn levels of k -colored Motzkin trees. To this end, we use the following result by Drmota et al. (2014). The notation $A(z) \preceq B(z)$ used therein means that $[z^n]A(z) \leq [z^n]B(z)$ for every $n \geq 0$.

Lemma 4.1 (Drmota et al. (2014, Lemma 1.4)). *Suppose that $F(z, t)$ is an analytic function at $(z, t) = (0, 0)$ such that the equation $T(z) = F(z, T(z))$ has a solution $T(z)$ that is analytic at $z = 0$ and has non-negative Taylor coefficients. Suppose that $T(z)$ has a square-root singularity at $z = z_0$ and can be continued to a region $\{z \in \mathbb{C}: |z| < z_0 + \varepsilon\} \setminus [z_0, \infty)$ for some $\varepsilon > 0$, such that $F_t(z_0, t_0) = 1$, $F_z(z_0, t_0) \neq 0$, and $F_{tt}(z_0, t_0) \neq 0$, where $t_0 = T(z_0)$.*

Let $T^{[0]}(z)$ be a power series with $0 \preceq T^{[0]}(z) \preceq T(z)$ such that $T^{[0]}(z)$ is analytic at $z = z_0$, and let $T^{[k]}(z)$, $k \geq 1$ be iteratively defined by

$$T^{[k]}(z) = F(z, T^{[k-1]}(z)).$$

Assume that $T^{[k-1]}(z) \preceq T^{[k]}(z) \preceq T(z)$.

Let H_n be an integer valued random variable that is defined by

$$\mathbb{P}\{H_n \leq k\} = \frac{[z^n]T^{[k]}(z)}{[z^n]T(z)}$$

for those n with $[z^n]T(z) > 0$. Then

$$\mathbb{E}H_n \sim \sqrt{\frac{2\pi n}{z_0 F_z(z_0, t_0) F_{tt}(z_0, t_0)}}.$$

Lemma 4.2. *The average number of De Bruijn levels of a k -colored Motzkin tree of size n is asymptotically equal to*

$$\sqrt{\frac{\pi n}{2k + \sqrt{k}}}.$$

Proof: For $k \geq 1$ and $h \geq 0$, the generating function $M_k^{[h]}(z)$ of k -colored Motzkin trees with at most h De Bruijn levels fulfills

$$M_k^{[h+1]}(z) = kz + zM_k^{[h]}(z) + z(M_k^{[h+1]}(z))^2,$$

and hence

$$M_k^{[h+1]}(z) = \frac{1 - \sqrt{1 - 4kz^2 - 4z^2 M_k^{[h]}(z)}}{2z}.$$

Let us fix $k \geq 1$ and define $F_k(z, t) := \frac{1 - \sqrt{1 - 4kz^2 - 4z^2 t}}{2z}$. Let us notice that $F_k(z, t)$ satisfies the assumptions of Lemma 4.1. Indeed, the function $M_k(z)$, with a square-root singularity at $z = \rho_k = \frac{1}{1+2\sqrt{k}}$, is a solution of $F_k(z, M_k(z)) = M_k(z)$ fulfilling all necessary conditions. Furthermore, the function $M_k^{[0]}(z)$ enumerates all k -colored Motzkin trees with only one (the 0-th) De Bruijn level. These trees are binary trees with k possible labels for each node, thus $M_k^{[0]}(z) = \frac{1 - \sqrt{1 - 4kz^2}}{2z}$. As $M_k^{[0]}(z)$ has its dominant singularity at $z = \frac{1}{2\sqrt{k}}$, it is analytic at $z = \frac{1}{1+2\sqrt{k}}$. Moreover, by a purely combinatorial argument, $M_k^{[h]}(z) \preceq M_k^{[h+1]}(z) \preceq M_k(z)$ for every $h \geq 0$. Finally, since $F(z, M_k^{[h]}(z)) = M_k^{[h+1]}(z)$, we can apply Lemma 4.1. We have $M_k(\rho_k) = \sqrt{k}$ and

$$\left. \frac{\partial F_k(z, t)}{\partial z} \right|_{(z,t)=(\rho_k, \sqrt{k})} = (1 + 2\sqrt{k})^2 \sqrt{k} \quad \text{and} \quad \left. \frac{\partial^2 F_k(z, t)}{\partial t^2} \right|_{(z,t)=(\rho_k, \sqrt{k})} = 2,$$

thus the average number of De Bruijn levels of k -colored Motzkin trees is asymptotically equal to

$$\sqrt{\frac{2\pi n}{\frac{1}{1+2\sqrt{k}} \cdot (1 + 2\sqrt{k})^2 \sqrt{k} \cdot 2}} = \sqrt{\frac{\pi n}{2k + \sqrt{k}}}.$$

□

Corollary 4.3. *For every $k \geq 1$, the average number of De Bruijn levels of a k -indexed lambda term of size n is $\Theta(\sqrt{n})$.*

Proof: By Corollary 3.2, the number of k -colored Motzkin trees in the decomposition of lambda terms is constant on average. Therefore, the size of a largest such a tree in the decomposition of a k -indexed lambda term of size n is asymptotically $\Theta(n)$. Since the average number of De Bruijn levels of k -colored Motzkin trees of size asymptotic to n is $\Theta(\sqrt{n})$, the same is true for k -indexed lambda terms, which have just k levels more than a longest (in terms of De Bruijn levels) k -colored Motzkin tree in their decomposition. □

5 Unary profile

By the *unary profile* of a lambda term we mean the sequence of numbers of variables in each De Bruijn level of the term. In this section, we determine the mean unary profile of a random lambda term of some large size.

In the forthcoming proof, we will make use of the following technical results.

Lemma 5.1 (Gittenberger (1999, Lemma 3.4)). *Let γ be a Hankel contour truncated at K . Then we have, for $\alpha, \beta > 0$,*

$$\frac{1}{2\pi i} \int_{\gamma} e^{-\alpha\sqrt{-t}-\beta t} dt = \frac{\alpha\beta^{-\frac{3}{2}}}{2\sqrt{\pi}} \exp\left(-\frac{\alpha^2}{4\beta}\right) + \mathcal{O}\left(\frac{1}{\beta} e^{-K\beta}\right).$$

Lemma 5.2. *Let $\varepsilon > 0$ and $\gamma = \left\{ \rho_k \left(1 + \frac{t+i}{n}\right) : t \in [\log^2 n, n\varepsilon] \right\}$. Then*

$$\max_{z \in \gamma} \left| \frac{\sqrt{1-2z-(4k-1)z^2}}{z} \right| = \mathcal{O}\left(\frac{\log n}{\sqrt{n}}\right).$$

Proof: The closer to ρ_k an argument of the function $\gamma \ni z \mapsto \frac{\sqrt{1-2z-(4k-1)z^2}}{z}$ is, the greater its modulus gets. Thus, we set $z = \rho_k \left(1 + \frac{\log^2 n}{n} + \frac{i}{n}\right)$, which is the closest point to ρ_k on γ , and we get

$$\frac{\sqrt{1-2z-(4k-1)z^2}}{z} = \frac{\sqrt{a+ib}}{\rho_k \left(1 + \frac{\log^2 n}{n} + \frac{i}{n}\right)}$$

with $a \sim -4\sqrt{k}\rho_k \frac{\log^2 n}{n}$ and $b \sim -4\sqrt{k}\rho_k \frac{1}{n}$. Plugging in the asymptotic formulas for a and b directly yields the desired result. □

Now we are in the position to prove the main theorem of this section.

Theorem 5.3. *Let $\kappa > 0$ be a fixed real number. The expected number of variables at De Bruijn level $\lfloor \kappa\sqrt{n} \rfloor$ in a k -indexed lambda term of size n is asymptotically equal to*

$$2\kappa \exp\left(-\kappa^2(2k + \sqrt{k})\right) \sqrt{n}.$$

Proof: Let $U_{k,\ell}(z, u)$ be the bivariate generating function for k -indexed terms with z marking the size and u marking the number of leaves in the $(k + \ell)$ -th De Bruijn level, where $\ell \geq 1$. Then we have

$$U_{k,\ell}(z, u) = B\left(z, B\left(z, 1+B\left(z, 2+\dots+B\left(z, k + \underbrace{B(z, k + B(\dots B(z, k+B(z, ku+M_k(z))))}_{\ell \text{ occurrences of } B}}\right)\right)\right)\right)\right).$$

Applying formulas for $B(z, w)$ and $M_k(z)$ given in (4) yields

$$U_{k,\ell}(z, u) = \frac{1 - \sqrt{R_{k,k+\ell}(z, u)}}{2z}$$

where

$$R_{k,i}(z, u) = \begin{cases} 1 - 2z - (4k - 1)z^2, & i = 0, \\ 1 - 2z - (4ku - 2)z^2 + 2z\sqrt{R_{k,0}(z, u)}, & i = 1, \\ 1 - 2z - 4kz^2 + 2z\sqrt{R_{k,i-1}(z, u)}, & i \in \{2, \dots, \ell\}, \\ 1 - 2z - 4(k - i + \ell)z^2 + 2z\sqrt{R_{k,i-1}(z, u)}, & i \in \{\ell + 1, \dots, \ell + k\}. \end{cases}$$

Furthermore, we have

$$\frac{\partial R_{k,i}(z, u)}{\partial u} = \begin{cases} 0, & i = 0, \\ -4kz^2, & i = 1, \\ \frac{-4kz^{i+1}}{\prod_{j=1}^{i-1} \sqrt{R_{k,j}(z, u)}}, & i > 1, \end{cases}$$

and hence

$$\frac{\partial U_{k,\ell}(z, u)}{\partial u} = \frac{z^{k+\ell}}{\prod_{j=1}^{k+\ell} \sqrt{R_{k,j}(z, u)}}.$$

Given the De Bruijn level $\ell = \lfloor \kappa\sqrt{n} \rfloor$ with $\kappa > 0$, we are interested in estimating

$$\frac{[z^n] \frac{\partial U_{k,\ell}(z, u)}{\partial u} \Big|_{u=1}}{[z^n] G_k(z)}.$$

In order to make further computations easier, let us notice that $|\sqrt{R_{k,j}(z, 1)}| = |z + \sqrt{R_{k,0}(z, 1)}|$ for $j \in \{1, \dots, \ell\}$, *i.e.*, all these radicands describe the same function. Indeed, let us first notice that the above holds for $j = 1$, since

$$R_{k,1}(z, 1) = R_{k,0}(z, 1) + z^2 + 2z\sqrt{R_{k,0}(z, 1)} = \left(\sqrt{R_{k,0}(z, 1)} + z\right)^2.$$

Next, by (4), we can notice that $x = M_k(z)$ is a solution of the equation $x = B(z, k + x)$. Therefore, in particular,

$$B(z, k + B(z, k + M_k(z))) = B(z, k + M_k(z)),$$

which gives us $\sqrt{R_{k,1}(z, 1)} = \sqrt{R_{k,2}(z, 1)}$. By iteration we obtain the result for $j \in \{3, \dots, \ell\}$.

For $z = \rho_k \left(1 + \frac{t}{n}\right)$ we get the expansions

$$\sqrt{R_{k,i}(z, 1)} = \begin{cases} 2k^{1/4} \rho_k^{1/2} \sqrt{-t/n} + \mathcal{O}(|t|/n), & i = 0, \\ \rho_k + 2k^{1/4} \rho_k^{1/2} \sqrt{-t/n} + \mathcal{O}(|t|/n), & i \in \{1, \dots, \ell\}, \\ \sqrt{c_{i-\ell}} \rho_k + b_{i-\ell} \sqrt{-t/n} + \mathcal{O}(|t|/n), & i \in \{\ell + 1, \dots, \ell + k\}, \end{cases} \quad (8)$$

where $(c_i)_{i \geq 1}$ and $(b_{k,i})_{k,i \geq 1}$ are as before (see (1) and (6)).

Let $\varepsilon > 0$. We have

$$[z^n] \frac{\partial U_{k,\ell}(z, u)}{\partial u} \Big|_{u=1} = \frac{1}{2\pi i} \int_{\gamma} \frac{z^{k+\ell-n-1}}{\prod_{j=1}^{k+\ell} \sqrt{R_{k,j}(z, 1)}} dz,$$

where as an integration path we choose a truncated Hankel contour $\gamma_1 \cup \gamma_2 \cup \gamma_3$ encircling the dominant singularity ρ_k and a circular arc γ_4 :

$$\begin{aligned} \gamma_1 &= \left\{ z = \rho_k \left(1 + \frac{t}{n}\right) : t = e^{-i\theta}, \theta \in [-\pi/2, \pi/2] \right\} \cup \left\{ z = \rho_k \left(1 + \frac{t \pm i}{n}\right) : t \in (0, \log^2 n) \right\}, \\ \gamma_2 &= \left\{ z = \rho_k \left(1 + \frac{t+i}{n}\right) : t \in [\log^2 n, n\varepsilon] \right\}, \\ \gamma_3 &= \left\{ z = \rho_k \left(1 + \frac{t-i}{n}\right) : t \in [\log^2 n, n\varepsilon] \right\}, \\ \gamma_4 &= \left\{ z : |z| = \rho_k \left|1 + \varepsilon + \frac{i}{n}\right|, \Re(z) \leq \rho_k (1 + \varepsilon) \right\}. \end{aligned}$$

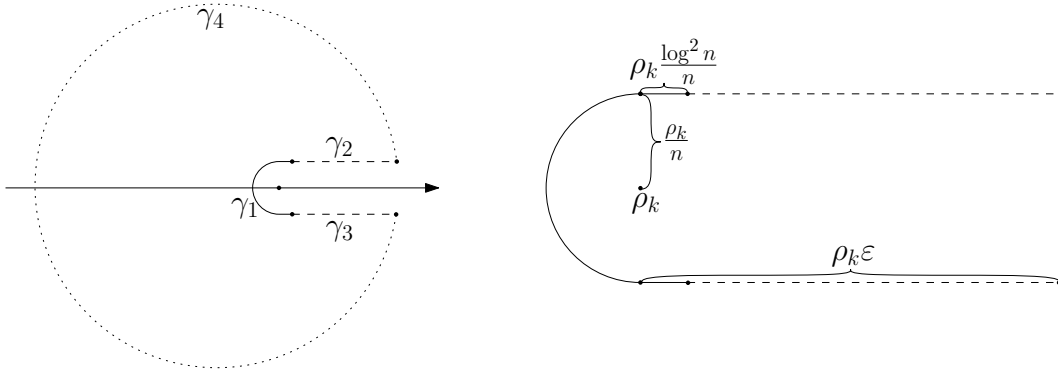


Fig. 4: Contour of integration: γ_1 plotted with a solid line, γ_2 and γ_3 with dashed lines, and γ_4 with a dotted line.

We start by estimating the integral along γ_1 . To this end, we apply the substitution $z = \rho_k(1 + t/n)$,

where $\tilde{\gamma}_1$ denotes the transformed curve and we use the expansions given in (8):

$$\begin{aligned}
& \int_{\tilde{\gamma}_1} \frac{z^{k+\ell-n-1}}{\prod_{j=1}^{k+\ell} \sqrt{R_{k,j}(z,1)}} dz \\
&= \frac{\rho_k^{k+\ell-n}}{n} \int_{\tilde{\gamma}_1} \left(1 + \frac{t}{n}\right)^{-n+k+\ell} \left(\frac{1}{\rho_k + 2k^{1/4} \rho_k^{1/2} \sqrt{-t/n} + \mathcal{O}(|t|/n)} \right)^\ell \prod_{j=1}^k \frac{1}{\sqrt{c_j} \rho_k + b_{k,j} \sqrt{-t/n} + \mathcal{O}(|t|/n)} dt \\
&= \frac{\rho_k^{k+\ell-n}}{n} \int_{\tilde{\gamma}_1} e^{-t} \left(1 + \frac{t}{n}\right)^{k+\ell} \left(\frac{1}{\rho_k + 2k^{1/4} \rho_k^{1/2} \sqrt{-t/n}} \right)^\ell \prod_{j=1}^k \frac{1}{\sqrt{c_j} \rho_k + b_{k,j} \sqrt{-t/n}} \left(1 + \mathcal{O}\left(\frac{|t|}{n}\right)\right) dt \\
&= \frac{\rho_k^{k-n}}{n} \int_{\tilde{\gamma}_1} e^{-t} \left(1 + \frac{t}{n}\right)^{k+\kappa\sqrt{n}} \left(\frac{1}{1 + 2k^{1/4} \rho_k^{-1/2} \sqrt{-t/n}} \right)^{\kappa\sqrt{n}} \prod_{j=1}^k \frac{1}{\sqrt{c_j} \rho_k + b_{k,j} \sqrt{-t/n}} \left(1 + \mathcal{O}\left(\frac{|t|}{n}\right)\right) dt \\
&= \frac{\rho_k^{k-n}}{n} \int_{\tilde{\gamma}_1} e^{-t - \frac{2\kappa k^{1/4}}{\sqrt{\rho_k}} \sqrt{-t}} \left(1 + \frac{\kappa t}{\sqrt{n}}\right) \left(1 - \frac{2\kappa\sqrt{kt}}{\rho_k\sqrt{n}}\right) \prod_{j=1}^k \frac{1}{\sqrt{c_j} \rho_k + b_{k,j} \sqrt{-t/n}} \left(1 + \mathcal{O}\left(\frac{|t|}{n}\right)\right) dt \\
&= \frac{C_{1,k}}{n} \rho_k^{-n} \int_{\tilde{\gamma}_1} e^{-t - \frac{2\kappa k^{1/4}}{\sqrt{\rho_k}} \sqrt{-t}} \left(1 + \frac{\kappa t}{\sqrt{n}}\right) \left(1 - \frac{2\kappa\sqrt{kt}}{\rho_k\sqrt{n}}\right) \left(1 - \frac{\sqrt{-t}}{\rho_k\sqrt{n}} \sum_{j=1}^k \frac{b_{k,j}}{\sqrt{c_j}}\right) \left(1 + \mathcal{O}\left(\frac{|t|}{n}\right)\right) dt \\
&= \frac{C_{1,k}}{n} \rho_k^{-n} \int_{\tilde{\gamma}_1} e^{-t - \frac{2\kappa k^{1/4}}{\sqrt{\rho_k}} \sqrt{-t}} \left(1 + \frac{1}{\rho_k\sqrt{n}} \left(\kappa t(\rho_k - 2\sqrt{k}) - \sqrt{-t} \sum_{j=1}^k \frac{b_{k,j}}{\sqrt{c_j}}\right)\right) \left(1 + \mathcal{O}\left(\frac{|t|}{n}\right)\right) dt \\
&= \frac{C_{1,k}}{n} \rho_k^{-n} \int_{\tilde{\gamma}_1} e^{-t - \frac{2\kappa k^{1/4}}{\sqrt{\rho_k}} \sqrt{-t}} \left(1 + \mathcal{O}\left(\frac{|t|}{\sqrt{n}}\right)\right) dt.
\end{aligned}$$

Now, by applying Lemma 5.1, we get that the integral above can be further estimated to result in

$$\begin{aligned}
\int_{\tilde{\gamma}_1} \frac{z^{k+\ell-n-1}}{\prod_{j=1}^{k+\ell} \sqrt{R_{k,j}(z,1)}} dz &= \frac{C_{1,k}}{n} \rho_k^{-n} \int_{\tilde{\gamma}_1} e^{-t - \frac{2\kappa k^{1/4}}{\sqrt{\rho_k}} \sqrt{-t}} dt + \mathcal{O}\left(\frac{\rho_k^{-n}}{n^{3/2}}\right) \\
&= \frac{\kappa k^{1/4} C_{1,k}}{\sqrt{\pi} \rho_k n} \rho_k^{-n} \exp\left(-\kappa^2(2k + \sqrt{k})\right) + \mathcal{O}\left(\frac{\rho_k^{-n}}{n^{3/2}}\right).
\end{aligned} \tag{9}$$

Next, we show that the integrals along γ_j for $j \in \{2, 3, 4\}$ are all of order $o(\rho_k^{-n} n^{-3/2})$ and hence the whole asymptotic contribution comes from integration along γ_1 .

First, let us consider the integral along γ_4 :

$$\begin{aligned}
\left| \int_{\gamma_4} \frac{z^{k+\ell-n-1}}{\prod_{j=1}^{k+\ell} \sqrt{R_{k,j}(z,1)}} dz \right| &\leq (\rho_k(1+\varepsilon))^{k+[\kappa\sqrt{n}]-n-1} |\gamma_4| \max_{z \in \gamma_4} \left| \frac{1}{\prod_{j=1}^{k+[\kappa\sqrt{n}]} \sqrt{R_{k,j}(z,1)}} \right| \\
&\leq C \rho_k^{-n} (1+\varepsilon)^{-n} (\rho_k(1+\varepsilon))^{[\kappa\sqrt{n}]} \min_{z \in \gamma_4} \left| \sqrt{R_{k,1}(z,1)} \right|^{-[\kappa\sqrt{n}]},
\end{aligned}$$

where C is some positive constant. Here, $(1+\varepsilon)^{-n}$ contributes an exponential factor e^{-Dn} with a positive constant D , which compensates the factor $\min_{z \in \gamma_4} \left| \sqrt{R_{k,1}(z,1)} \right|^{-\lfloor \kappa\sqrt{n} \rfloor}$ and thus guarantees

$$\int_{\gamma_4} \frac{z^{k+\ell-n-1}}{\prod_{j=1}^{k+\ell} \sqrt{R_{k,j}(z,1)}} dz = \mathcal{O}((\rho_k(1-\varepsilon)^{-n})) = o(\rho_k^{-n} n^{-3/2}).$$

Now, we estimate the integral along γ_2 . For some constant $C > 0$, we have

$$\begin{aligned} \left| \int_{\gamma_2} \frac{z^{k+\ell-n-1}}{\prod_{j=1}^{k+\ell} \sqrt{R_{k,j}(z,1)}} dz \right| &\leq C \left| \int_{\log^2 n}^{\varepsilon n} \frac{\rho_k^{\lfloor \kappa\sqrt{n} \rfloor - n} \left(1 + \frac{t}{n} + \frac{i}{n}\right)^{\lfloor \kappa\sqrt{n} \rfloor - n}}{\sqrt{R_{k,1}(\rho_k(1 + \frac{t}{n} + \frac{i}{n}), 1)}^{\lfloor \kappa\sqrt{n} \rfloor}} \frac{1}{n} dt \right| \\ &\leq C \rho_k^{-n} \frac{1}{n} \rho_k^{\lfloor \kappa\sqrt{n} \rfloor} \max_{\gamma_2} \left| \frac{z}{\sqrt{R_{k,1}(z,1)}} \right|^{\lfloor \kappa\sqrt{n} \rfloor} \int_{\log^2 n}^{\varepsilon n} \left(1 + \frac{t}{n} + \frac{i}{n}\right)^{-n} dt. \end{aligned}$$

Using the fact that $\left| \sqrt{R_{k,1}(z,1)} \right| = \left| z + \sqrt{R_{k,0}(z,1)} \right|$ and by Lemma 5.2, we get that the maximum contributes a factor

$$\max_{z \in \gamma_2} \left| \frac{z}{\sqrt{R_{k,1}(z,1)}} \right|^{\lfloor \kappa\sqrt{n} \rfloor} = \max_{z \in \gamma_2} \left| \frac{1}{1 + \frac{1}{z} \sqrt{R_{k,0}(z,1)}} \right|^{\lfloor \kappa\sqrt{n} \rfloor} = \left(1 + \tilde{C} \frac{\log n}{\sqrt{n}}\right)^{\lfloor \kappa\sqrt{n} \rfloor} \sim e^{\bar{C} \log n}$$

for some positive constants \tilde{C} and $\bar{C} > 0$. The remaining integral can be estimated by

$$\int_{\log^2 n}^{\varepsilon n} \left(1 + \frac{t}{n} + \frac{i}{n}\right)^{-n} dt = \mathcal{O}\left(e^{-\log^2 n}\right),$$

which finally gives us

$$\left| \int_{\gamma_2} \frac{z^{k+\ell-n-1}}{\prod_{j=1}^{k+\ell} \sqrt{R_{k,j}(z,1)}} dz \right| = \mathcal{O}\left(\rho_k^{-n} \frac{1}{n} e^{-\log^2 n + \bar{C} \log n}\right) = o(\rho_k^{-n} n^{-3/2}).$$

The estimate of the integral along γ_3 works analogously.

Therefore, by (9), we get

$$[z^n] \frac{\partial U_{k,\ell}(z,u)}{\partial u} \Big|_{u=1} = \frac{\kappa k^{1/4} C_{1,k}}{\sqrt{\pi} \rho_k n} \rho_k^{-n} \exp\left(-\kappa^2(2k + \sqrt{k})\right) + \mathcal{O}\left(\frac{\rho_k^{-n}}{n^{3/2}}\right).$$

Combining this result and the asymptotic behavior of the sequence enumerating all k -indexed terms, we finally obtain that the expected number of leaves at the level $\lfloor \kappa\sqrt{n} \rfloor$ is given by

$$\frac{[z^n] \frac{\partial U_{k,\ell}(z,u)}{\partial u} \Big|_{u=1}}{[z^n] G_k(z)} \sim \frac{\frac{\kappa k^{1/4} C_{1,k}}{\sqrt{\pi} \rho_k n} \rho_k^{-n} \exp\left(-\kappa^2(2k + \sqrt{k})\right)}{\frac{k^{1/4} C_{1,k}}{2\sqrt{\pi} \rho_k} n^{-3/2} \rho_k^{-n}} = 2\kappa \exp\left(-\kappa^2(2k + \sqrt{k})\right) \sqrt{n}.$$

□

6 Final remarks

By Theorem 5.3, we can observe that the expected unary profile of lambda terms with De Bruijn indices at most k looks like the density of a Rayleigh distribution.

So far, the distribution of the number of leaves in each De Bruijn level has not been investigated, however, the total number of leaves within these terms, as well as their distribution, have been studied asymptotically by Gittenberger and Larcher (2018).

Theorem 6.1 (Gittenberger and Larcher (2018), Theorem 1). *Let X_n be the total number of variables in a random closed lambda term of size n where the De Bruijn index of each variable is at most k . Then X_n is asymptotically normally distributed with*

$$\mathbb{E}X_n \sim \frac{\sqrt{k}}{1 + 2\sqrt{k}}n \quad \text{and} \quad \mathbb{V}X_n \sim \frac{\sqrt{k}}{2(1 + 2\sqrt{k})^2}n \quad \text{as } n \rightarrow \infty.$$

Since the number of binary nodes differs only by 1 from the number of leaves, and the remaining nodes (that are neither binary nodes nor leaves) have to be unary nodes, we can state the following corollary.

Corollary 6.2. *Let Y_n be the total number of binary nodes in a random closed lambda term of size n with De Bruijn index at most k , and let Z_n be the total number of unary nodes, respectively. Then*

$$\mathbb{E}Y_n = \mathbb{E}X_n \sim \sqrt{k}\rho_k n \quad \text{and} \quad \mathbb{E}Z_n \sim \rho_k n \quad \text{as } n \rightarrow \infty,$$

with X_n being defined as in Theorem 6.1.

Remark. Thus, it is an immediate observation that on average each lambda binds \sqrt{k} leaves in lambda terms with De Bruijn indices being at most k .

By calculating the asymptotic number of individual constructors that occur in k -colored Motzkin trees, we get exactly the same results as in Theorem 6.1 (and therefore also as in Corollary 6.2). Furthermore, the height and the profile of these k -colored Motzkin trees is also very similar to that of lambda terms with De Bruijn indices at most k . Thus, k -indexed lambda terms are very much alike k -colored Motzkin trees. However, their counting sequences differ significantly (by a factor $C_{1,k/2}$) due to the restrictions on labelling leaves in hats of the terms. So, there are way more k -colored Motzkin trees than k -indexed lambda terms. Nevertheless the great majority of them exhibits the same structural properties.

This leads to the conjecture that the problem of generating random lambda terms could be solved by means of generating random k -colored Motzkin trees and finding a suitable algorithm for *repairing their hats*. The resulting generation would not be perfectly uniform, but potentially very close to the uniform one and it would definitively be an interesting future topic to investigate.

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