

A Piecewise-Linear NDF

Microfacet BSDFs Generated from NDFs and Explicit Microgeometry

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This document contains the mathematical justifications concerning the *Piecewise-Linear NDF* presented in the paper. We provide all the mathematical details for the normalization factor, its corresponding importance sampling procedure (Section 2) as well as the Smith-Bourlier GAF computation (Section 3) thus giving practical information on its implementation.

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1 A Piecewise-Linear normal distribution function

1.1 General anisotropic formulation

We define our anisotropic Normal Distribution Function (NDF from now on) $D(m)$ (with φ and θ the polar coordinates of the normal m) as an interpolation over φ from two normalized isotropic distributions D_x when $\varphi = 0$ and D_y when $\varphi = \frac{\pi}{2}$:

$$D(m) = \frac{\cos^2 \varphi D_x(\theta) D_y(0) + \sin^2 \varphi D_y(\theta) D_x(0)}{c} \quad (1)$$

with $c = \frac{1}{2}(D_y(0) + D_x(0))$ the normalization factor. This latter is computed in the following manner:

$$\begin{aligned} c &= \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} D(\theta, \varphi) \cos \theta \sin \theta \, d\theta \, d\varphi \\ &= 4 \int_{\varphi=0}^{\frac{\pi}{2}} \int_{\theta=0}^{\frac{\pi}{2}} D(\theta, \varphi) \cos \theta \sin \theta \, d\theta \, d\varphi \\ &= 4 \int_{\varphi=0}^{\frac{\pi}{2}} \int_{\theta=0}^{\frac{\pi}{2}} (\cos^2 \varphi D_x(\theta) D_y(0) + \sin^2 \varphi D_y(\theta) D_x(0)) \cos \theta \sin \theta \, d\theta \, d\varphi \\ &= 4 \int_{\varphi=0}^{\frac{\pi}{2}} D_y(0) \cos^2 \varphi \int_{\theta=0}^{\frac{\pi}{2}} D_x(\theta) \cos \theta \sin \theta \, d\theta \, d\varphi \\ &\quad + 4 \int_{\varphi=0}^{\frac{\pi}{2}} D_x(0) \sin^2 \varphi \int_{\theta=0}^{\frac{\pi}{2}} D_y(\theta) \cos \theta \sin \theta \, d\theta \, d\varphi. \end{aligned} \quad (2)$$

$D_x(\theta)$ and $D_y(\theta)$ have to be normalized, then $\int_{\theta=0}^{\frac{\pi}{2}} D_{iso}(\theta) \cos(\theta) \sin(\theta) \, d\theta = \frac{1}{2\pi}$ and finally

$$\begin{aligned} c &= \frac{4D_y(0)}{2\pi} \int_{\varphi=0}^{\frac{\pi}{2}} \cos^2(\varphi) \, d\varphi + \frac{4D_x(0)}{2\pi} \int_{\varphi=0}^{\frac{\pi}{2}} \sin^2(\varphi) \, d\varphi \\ &= \frac{2D_y(0)}{\pi} \frac{\pi}{4} + \frac{2D_x(0)}{\pi} \frac{\pi}{4} \\ &= \frac{1}{2}(D_y(0) + D_x(0)) \end{aligned} \quad (3)$$

Note that the formulation proposed in Equation 1 is different from common anisotropic distributions (*e.g* GGX or Beckmann). For example, if we choose D_x and D_y as isotropic beckmann distributions, Equation 1 gives an alternative anisotropic distribution slightly different from the conventional anisotropic Beckmann formulation.

In the case where $D_x(0) = D_y(0) = 0$, the normalization factor is null and Equation 1 is undefined. Hopefully, in such a case, the distribution can be simply expressed as $D(m) = \cos^2 \varphi D_x(\theta) + \sin^2 \varphi D_y(\theta)$ because it is already normalized.

1.2 Controllable Isotropic NDF

Our anisotropic NDF formulation (Equation 1) allows to blend two different isotropic NDFs: for example one could be defined from GGX formulation and the second from Beckmann, both with eventually different roughness parameters. This give more flexibility in NDF shape design. Unfortunately, Smith-Bourlier GAF derivation [9, 3] is not trivial.

To give more user NDF control, and to overcome the intractable GAF computation issue, we build a user controllable NDF upon a piecewise continuous isotropic NDF $D_{iso}(\theta)$ defined as a combination of n

linear functions

$$D_{iso}(\theta) = \begin{cases} a_1\theta + b_1 & \theta_0 = 0 \leq \theta \leq \theta_1 \\ a_2\theta + b_2 & \theta_1 \leq \theta \leq \theta_2 \\ \dots & \\ a_n\theta + b_n & \theta_{n-1} \leq \theta \leq \theta_n = \frac{\pi}{2} \end{cases} \quad (4)$$

This distribution have to be normalized which means $\int_{\Omega^+} D_{iso}(m)(m \cdot n)dm = 1$. The normalization factor c_{iso} is given by

$$\begin{aligned} c_{iso} &= \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\frac{\pi}{2}} D_{iso}(\theta) \cos \theta \sin \theta \, d\theta \, d\varphi \\ &= 2\pi \int_{\theta=0}^{\frac{\pi}{2}} D_{iso}(\theta) \cos \theta \sin \theta \, d\theta \\ &= 2\pi \sum_{i=1}^n \int_{\theta=\theta_{i-1}}^{\theta_i} (a_i\theta + b_i) \cos \theta \sin \theta \, d\theta \\ &= \frac{\pi}{4} \sum_{i=1}^n 2(a_i\theta_{i-1} + b_i) \cos(2\theta_{i-1}) - 2(a_i\theta_i + b_i) \cos(2\theta_i) \\ &\quad + a_i(\sin(2\theta_i) - \sin(2\theta_{i-1})) \end{aligned} \quad (5)$$

In the following, we always consider D_{iso} as normalized by default (already divided by c_{iso}). Now, the parameters D_x and D_y can be inserted in Equation 1. We obtain finally our full Piecewise-Linear NDF, where D_x and D_y can be discretized from any common NDFs (Beckmann [2], GGX [10], or more generally STD [7]) or directly controlled by the user by moving up or down the values of the distribution in discrete θ .

2 Importance sampling

The microfacet normal m is sampled according to the probability density function $pdf(m) = D(m)|m \cdot n|$ and its associated cumulative distribution function:

$$cdf(m) = \frac{1}{c} \int_{\varphi=0}^{\varphi_m} \int_{\theta=0}^{\theta_m} (\cos^2(\varphi)D_x(\theta)D_y(0) + \sin^2(\varphi)D_y(\theta)D_x(0)) \cos(\theta) \sin(\theta) \, d\theta \, d\varphi \quad (6)$$

The sampling of the microfacet azimuthal angle φ_m and the elevation angle θ_m is not trivial. We first describe how to choose φ_m independently of θ_m and then explain how to sample θ_m .

2.1 φ_m importance sampling

As said before, the choice of φ_m is independent of θ_m . Thus, the *cdf* becomes:

$$\begin{aligned}
cdf(\varphi_m) &= \frac{1}{c} \int_{\varphi=0}^{\varphi_m} \int_{\theta=0}^{\frac{\pi}{2}} (\cos^2(\varphi)D_x(\theta)D_y(0) + \sin^2(\varphi)D_y(\theta)D_x(0)) \cos(\theta) \sin(\theta) d\theta d\varphi \\
&= \frac{1}{c} \left(\int_{\varphi=0}^{\varphi_m} \cos^2(\varphi)D_y(0) \int_{\theta=0}^{\frac{\pi}{2}} D_x(\theta) \cos(\theta) \sin(\theta) d\theta d\varphi \right. \\
&\quad \left. + \int_{\varphi=0}^{\varphi_m} \sin^2(\varphi)D_x(0) \int_{\theta=0}^{\frac{\pi}{2}} D_y(\theta) \cos(\theta) \sin(\theta) d\theta d\varphi \right) \\
&= \frac{1}{c} \left(\frac{D_y(0)}{2\pi} \int_{\varphi=0}^{\varphi_m} \cos^2(\varphi) d\varphi + \frac{D_x(0)}{2\pi} \int_{\varphi=0}^{\varphi_m} \sin^2(\varphi) d\varphi \right) \\
&= \frac{D_y(0)}{\pi(D_y(0) + D_x(0))} \left(\frac{\varphi_m}{2} + \frac{1}{4} \sin(2\varphi_m) \right) \\
&\quad + \frac{D_x(0)}{\pi(D_y(0) + D_x(0))} \left(\frac{\varphi_m}{2} - \frac{1}{4} \sin(2\varphi_m) \right) \\
cdf(\varphi_m) &= \frac{\varphi_m}{2\pi} + \frac{D_y(0) - D_x(0)}{4\pi(D_y(0) + D_x(0))} \sin(2\varphi_m). \tag{7}
\end{aligned}$$

Unfortunately, this CDF is not directly invertible. As shown in Figure 1, the CDF is equal to the linear function $\frac{\varphi}{2\pi}$ for $\varphi = \{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$. If we limit the domain to $[0, \frac{\pi}{2}]$, we can observe that the shape of

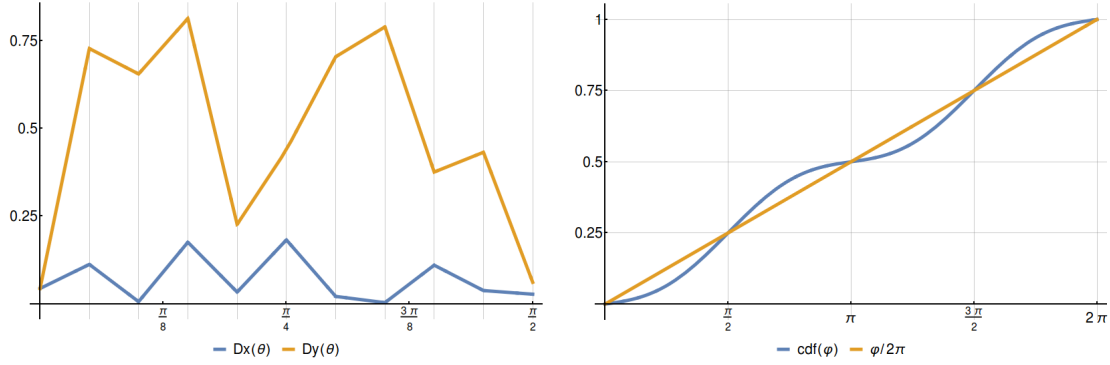


Figure 1: Two isotropic randomly generated piecewise continuous distributions (first plot) are mixed to create an anisotropic distribution. The cumulative distribution function $cdf(\varphi)$ (second plot) oscillates around the linear function $\frac{\varphi_m}{2\pi}$.

the CDF is smooth and have a close-to-linear nature (exactly linear if $D_x(0) = D_y(0)$). In the manner of [6], we can combine a bracketing interval and Newton's method (alg. 1) to invert the CDF given ξ_1 , a uniform random number in $[0, 1)$. For that, we need the derivative of the CDF:

$$cdf'(\varphi_m) = \frac{1}{2\pi} + \frac{D_y(0) - D_x(0)}{2\pi(D_y(0) + D_x(0))} \cos(2\varphi_m).$$

2.2 θ_m importance sampling

Knowing the azimuthal angle φ_m , the *cdf* to sample θ_m is:

$$cdf(\theta_m|\varphi_m) = \frac{1}{c} \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\theta_m} (\cos^2(\varphi)D_x(\theta)D_y(0) + \sin^2(\varphi)D_y(\theta)D_x(0)) \cos \theta \sin \theta d\theta d\varphi. \tag{8}$$

Algorithm 1 Importance Sampling of φ_m .

Require: $\xi_1 \in [0, 1)$

```
1: if  $\xi_1 \geq 0.75$  then
2:    $a \leftarrow \frac{3\pi}{2}$ 
3:    $c \leftarrow 2\pi$ 
4: else if  $\xi_1 \geq 0.5$  then
5:    $a \leftarrow \pi$ 
6:    $c \leftarrow \frac{3\pi}{2}$ 
7: else if  $\xi_1 \geq 0.25$  then
8:    $a \leftarrow \frac{\pi}{2}$ 
9:    $c \leftarrow \pi$ 
10: else
11:    $a \leftarrow 0$ 
12:    $c \leftarrow \frac{\pi}{2}$ 
13: end if
14:  $\varphi_m \leftarrow \xi_1 2\pi$  // The starting point is given by the linear function  $\frac{\varphi_m}{2\pi}$ 
15: while true do
16:   if  $!(\varphi_m \geq a \ \&\& \ \varphi_m \leq c)$  then
17:      $\varphi_m \leftarrow 0.5 * (a + c)$ 
18:   end if
19:    $value \leftarrow (\frac{\varphi_m}{2\pi} + \frac{D_y(0) - D_x(0)}{4\pi(D_y(0) + D_x(0))} \sin(2\varphi_m)) - \xi_1$ 
20:   if  $|value| < 1e - 5$  then
21:     break
22:   end if
23:   if  $value > 0$  then
24:      $c \leftarrow \varphi_m$ 
25:   else
26:      $a \leftarrow \varphi_m$ 
27:   end if
28:    $derivative \leftarrow \frac{1}{2\pi} + \frac{D_y(0) - D_x(0)}{2\pi(D_y(0) + D_x(0))} \cos(2\varphi_m)$  // The Newton step
29:    $\varphi_m - = value / derivative$ 
30: end while
31: return  $\varphi_m$ 
```

Note that $cdf(\frac{\pi}{2}) = \frac{D_y(0)\cos^2(\varphi_m)+D_x(0)\sin^2(\varphi_m)}{c}$ which is in fact the normalization factor of the cdf . Thus, we obtain

$$cdf(\theta_m|\varphi_m) = \xi_2 * \frac{D_y(0)\cos^2(\varphi_m) + D_x(0)\sin^2(\varphi_m)}{c}$$

with $\xi_2 \in [0, 1)$ a second uniform random number. The problem here again is to invert the cdf and given a uniform random value of ξ_2 , get the value of θ_m :

$$cdf^{-1}(\xi_2') = \theta_m$$

with $\xi_2' = \xi_2 * \frac{D_y(0)\cos^2(\varphi_m)+D_x(0)\sin^2(\varphi_m)}{c}$. Our distribution is a continuous piecewise distribution and the cdf can be written as a sum of integrals:

$$\begin{aligned} cdf(\theta_m|\varphi_m) &= \cos^2(\varphi_m) \sum_{i=1}^k \int_{\theta=\theta_i}^{\theta_{i+1}} \frac{2\pi D_x(\theta)D_y(0)\cos(\theta)\sin(\theta)}{c} d\theta \\ &+ \sin^2(\varphi_m) \sum_{i=1}^k \int_{\theta=\theta_i}^{\theta_{i+1}} \frac{2\pi D_y(\theta)D_x(0)\cos(\theta)\sin(\theta)}{c} d\theta \\ &+ \frac{2\pi}{c} \int_{\theta=\theta_{k+1}}^{\theta_m} (\cos^2(\varphi_m)D_x(\theta)D_y(0) + \sin^2(\varphi_m)D_y(\theta)D_x(0)) \cos(\theta)\sin(\theta) d\theta. \end{aligned} \quad (9)$$

The terms in red color can be precomputed and tabulated and we call it T_x^k and T_y^k respectively. The sum of these two first terms is also the $cdf(\theta_{k+1}|\varphi_m)$ of the elevation angle θ_{k+1} for a given azimuthal angle φ_m . Moreover, given a uniform number ξ_2 , the corresponding elevation angle is $\theta_m \in [\theta_{k+1}, \theta_{k+2}]$. Thus, we have

$$\begin{aligned} \frac{2\pi}{c} \int_{\theta=\theta_{k+1}}^{\theta_m} (\cos^2(\varphi_m)D_x(\theta)D_y(0) + \sin^2(\varphi_m)D_y(\theta)D_x(0)) \cos(\theta)\sin(\theta) d\theta &= \xi_2' - cdf(\theta_{k+1}|\varphi_m) \\ \int_{\theta=\theta_{k+1}}^{\theta_m} (\cos^2(\varphi_m)D_x(\theta)D_y(0) + \sin^2(\varphi_m)D_y(\theta)D_x(0)) \cos(\theta)\sin(\theta) d\theta &= \frac{c(\xi_2' - cdf(\theta_{k+1}|\varphi_m))}{2\pi} \\ \int_{\theta=\theta_{k+1}}^{\theta_m} ((\cos^2(\varphi_m)D_y(0)a_{k+1}^x + \sin^2(\varphi_m)D_x(0)a_{k+1}^y) \theta + \\ \cos^2(\varphi_m)D_y(0)b_{k+1}^x + \sin^2(\varphi_m)D_x(0)b_{k+1}^y) \cos(\theta)\sin(\theta) d\theta &= \frac{c(\xi_2' - cdf(\theta_{k+1}|\varphi_m))}{2\pi}. \end{aligned}$$

with a_k^j and b_k^j parameters of linear functions for the interval k and axis j in our piecewise isotropic NDF definition (Equation 4). To simplify the notations, we note

$$\begin{aligned} A^{aniso} &= \cos^2(\varphi_m)D_y(0)a_{k+1}^x + \sin^2(\varphi_m)D_x(0)a_{k+1}^y \\ B^{aniso} &= \cos^2(\varphi_m)D_y(0)b_{k+1}^x + \sin^2(\varphi_m)D_x(0)b_{k+1}^y \\ C &= \frac{b(\xi_2' - cdf(\theta_{k+1}|\varphi_m))}{2\pi}. \end{aligned}$$

Now we have:

$$\begin{aligned} \int_{\theta=\theta_{k+1}}^{\theta_m} (A^{aniso}\theta + B) \cos(\theta)\sin(\theta) d\theta &= C \\ \frac{1}{8} (2(A^{aniso}\theta_{k+1} + B^{aniso}) \cos(2\theta_{k+1}) - 2(A^{aniso}\theta_m + B^{aniso}) \cos(2\theta_m) + \\ A^{aniso}(\sin(2\theta_m) - \sin(2\theta_{k+1}))) &= C. \end{aligned}$$

This function is smooth enough to adopt the same strategy as for the sampling of φ_m . Knowing the θ interval where is the desired θ_m (for a given ξ_2), we propose the algorithm 2. The starting point is given by the linear function $\xi_2' = a\theta_m + shift$ with $a = \frac{cdf(\theta_{k+2}|\varphi_m) - cdf(\theta_{k+1}|\varphi_m)}{\theta_{k+2} - \theta_{k+1}}$ and $shift = cdf(\theta_{k+1}|\varphi_m) - a\theta_{k+1}$.

Algorithm 2 Importance Sampling of θ_m .

Require: $\varphi_m, \xi_2 \in [0, 1)$ and $k + 1$ the index of the minimum plausible θ_m

```
1:  $A^{aniso} \leftarrow \cos^2(\varphi_m)D_y(0)a_{k+1}^x + \sin^2(\varphi_m)D_x(0)a_{k+1}^y$ 
2:  $B^{aniso} \leftarrow \cos^2(\varphi_m)D_y(0)b_{k+1}^x + \sin^2(\varphi_m)D_x(0)b_{k+1}^y$ 
3:  $\xi_2' \leftarrow \xi_2 * \frac{D_y(0)\cos^2(\varphi_m)+D_x(0)\sin^2(\varphi_m)}{b}$ 
4:  $C \leftarrow \frac{b(\xi_2'-cdf(\theta_{k+1}|\varphi_m))}{2\pi}$ 
5:  $bmin \leftarrow \theta_{k+1}$ 
6:  $bmax \leftarrow \theta_{k+2}$ 
   // The starting point is given by the linear function  $\frac{cdf(\theta_{k+2}|\varphi_m)-cdf(\theta_{k+1}|\varphi_m)}{\theta_{k+2}-\theta_{k+1}}\theta_m -$ 
    $\frac{cdf(\theta_{k+2}|\varphi_m)-cdf(\theta_{k+1}|\varphi_m)}{\theta_{k+2}-\theta_{k+1}}\theta_{k+1}$ 
7:  $a \leftarrow \frac{cdf(\theta_{k+2}|\varphi_m)-cdf(\theta_{k+1}|\varphi_m)}{\theta_{k+2}-\theta_{k+1}}$ 
8:  $shift \leftarrow cdf(\theta_{k+1}|\varphi_m) - a\theta_{k+1}$ 
9:  $\theta_m \leftarrow \frac{\xi_2'-shift}{a}$ 
10: while true do
11:   if  $!(\theta_m \geq bmin \ \&\& \ \theta_m \leq bmax)$  then
12:      $\theta_m \leftarrow 0.5 * (bmin + bmax)$ 
13:   end if
14:    $value \leftarrow 2(A^{aniso}\theta_{k+1} + B^{aniso})\cos(2\theta_{k+1})$ 
15:    $value \leftarrow value - 2(A^{aniso}\theta_m + B^{aniso})\cos(2\theta_m) + A^{aniso}(\sin(2\theta_m) - \sin(2\theta_{k+1}))$ 
16:    $value \leftarrow \frac{value}{8} - C$ 
17:   if  $|value| < 1e - 5$  then
18:     break
19:   end if
20:   if  $value > 0$  then
21:      $bmax \leftarrow \theta_m$ 
22:   else
23:      $bmin \leftarrow \theta_m$ 
24:   end if
25:    $derivative \leftarrow \frac{A^{aniso}\theta_m+B^{aniso}}{2} \sin 2\theta_m$  // The Newton step
26:    $\theta_m - = value/derivative$ 
27: end while
28: return  $\theta_m$ 
```

3 GAF computation

Following the Smith-Bourlier GAF assumptions, Ashikhmin et al. [1, 5] propose the expression

$$G_1(o, m) = \frac{\cos \theta_o}{\int_{\Omega} \langle o \cdot m \rangle D(m) d m} \quad (10)$$

where the integral is the most complicated part to compute. A first solution is to pre-compute the GAF for different values of (θ_o, φ_o) (the spherical coordinates of the vector o) and store them in a 2D-table. We propose here an alternative solution by trying to solve the integral by decomposing the integration domain in sub-domains.

$$\begin{aligned} \int_{\Omega} \langle o \cdot m \rangle D(m) d m &= \int_{\varphi_m=0}^{2\pi} \int_{\theta_m=0}^{\frac{\pi}{2}} \chi^+(o \cdot m) (o \cdot m) D(m) \sin \theta_m d\theta_m d\varphi_m \\ &= \int_{\varphi_m=0}^{2\pi} \int_{\theta_m=0}^{\frac{\pi}{2}} \chi^+(x_o \sin \theta_m \cos \varphi_m + y_o \sin \theta_m \sin \varphi_m + z_o \cos \theta_m) \\ &\quad (x_o \sin \theta_m \cos \varphi_m + y_o \sin \theta_m \sin \varphi_m + z_o \cos \theta_m) D(m) \sin \theta_m d\theta_m d\varphi_m \end{aligned} \quad (11)$$

with $\chi^+(a)$ the Heaviside function (1 if $a > 0$ and 0 if $a \leq 0$). When $\varphi_m \in [\varphi_o - \frac{\pi}{2}, \varphi_o + \frac{\pi}{2}]$, the dot product is positive so the Heaviside function can be dropped on this half integration domain

$$\begin{aligned} \int_{\Omega} \langle o \cdot m \rangle D(m) d m &= \int_{\varphi_m=\varphi_o-\frac{\pi}{2}}^{\varphi_o+\frac{\pi}{2}} \int_{\theta_m=0}^{\frac{\pi}{2}} (o \cdot m) D(m) \sin \theta_m d\theta_m d\varphi_m \\ &\quad + \int_{\varphi_m=\varphi_o+\frac{\pi}{2}}^{\varphi_o+\frac{3\pi}{2}} \int_{\theta_m=0}^{\frac{\pi}{2}} \chi^+(x_o \sin \theta_m \cos \varphi_m + y_o \sin \theta_m \sin \varphi_m + z_o \cos \theta_m) \\ &\quad (x_o \sin \theta_m \cos \varphi_m + y_o \sin \theta_m \sin \varphi_m + z_o \cos \theta_m) D(m) \sin \theta_m d\theta_m d\varphi_m. \end{aligned} \quad (12)$$

Using a trigonometric transformation, we can observe that $(o \cdot m) = \sin \theta_o \sin \theta_m \cos(\varphi_m - \varphi_o) + \cos \theta_o \cos \theta_m$ and so

$$\begin{aligned} \int_{\Omega} \langle o \cdot m \rangle D(m) d m &= \int_{\theta_m=0}^{\frac{\pi}{2}} \int_{\varphi_m=\varphi_o-\frac{\pi}{2}}^{\varphi_o+\frac{\pi}{2}} (\sin \theta_o \sin \theta_m \cos(\varphi_m - \varphi_o) + \cos \theta_o \cos \theta_m) \\ &\quad D(\theta_m, \varphi_m) \sin \theta_m d\varphi_m d\theta_m \\ &\quad + \int_{\theta_m=0}^{\frac{\pi}{2}} \int_{\varphi_m=\varphi_o+\frac{\pi}{2}}^{\varphi_o+\frac{3\pi}{2}} \chi^+(\sin \theta_o \sin \theta_m \cos(\varphi_m - \varphi_o) + \cos \theta_o \cos \theta_m) \\ &\quad (\sin \theta_o \sin \theta_m \cos(\varphi_m - \varphi_o) + \cos \theta_o \cos \theta_m) D(\theta_m, \varphi_m) \sin \theta_m d\varphi_m d\theta_m. \end{aligned} \quad (13)$$

The integrals of this equation are noted I_1 and I_2 respectively. I_1 can be analytically computed (Sections 3.1). I_2 is more complicated and we explain in Section 3.2 how to handle it.

3.1 Computation of I_1

$$\begin{aligned}
I_1 &= \int_{\theta_m=0}^{\frac{\pi}{2}} \int_{\varphi_m=\varphi_o-\frac{\pi}{2}}^{\varphi_o+\frac{\pi}{2}} (\sin \theta_o \sin \theta_m \cos(\varphi_m - \varphi_o) + \cos \theta_o \cos \theta_m) \\
&\quad D(\theta_m, \varphi_m) \sin \theta_m d\varphi_m d\theta_m \\
&= \frac{\sin \theta_o D_y(0)}{c} \int_{\theta_m=0}^{\frac{\pi}{2}} D_x(\theta_m) \sin^2 \theta_m \int_{\varphi_m=\varphi_o-\frac{\pi}{2}}^{\varphi_o+\frac{\pi}{2}} \cos(\varphi_m - \varphi_o) \cos^2 \varphi_m d\varphi_m d\theta_m \\
&\quad + \frac{\sin \theta_o D_x(0)}{c} \int_{\theta_m=0}^{\frac{\pi}{2}} D_y(\theta_m) \sin^2 \theta_m \int_{\varphi_m=\varphi_o-\frac{\pi}{2}}^{\varphi_o+\frac{\pi}{2}} \cos(\varphi_m - \varphi_o) \sin^2 \varphi_m d\varphi_m d\theta_m \\
&\quad + \frac{\cos \theta_o D_y(0)}{c} \int_{\theta_m=0}^{\frac{\pi}{2}} D_x(\theta_m) \cos \theta_m \sin \theta_m \int_{\varphi_m=\varphi_o-\frac{\pi}{2}}^{\varphi_o+\frac{\pi}{2}} \cos^2 \varphi_m d\varphi_m d\theta_m \\
&\quad + \frac{\cos \theta_o D_x(0)}{c} \int_{\theta_m=0}^{\frac{\pi}{2}} D_y(\theta_m) \cos \theta_m \sin \theta_m \int_{\varphi_m=\varphi_o-\frac{\pi}{2}}^{\varphi_o+\frac{\pi}{2}} \sin^2 \varphi_m d\varphi_m d\theta_m \\
&= \frac{\sin \theta_o D_y(0)(3 + \cos 2\varphi_o)}{3c} \int_{\theta_m=0}^{\frac{\pi}{2}} D_x(\theta_m) \sin^2 \theta_m d\theta_m \\
&\quad + \frac{\sin \theta_o D_x(0)(3 - \cos 2\varphi_o)}{3c} \int_{\theta_m=0}^{\frac{\pi}{2}} D_y(\theta_m) \sin^2 \theta_m d\theta_m \\
&\quad + \frac{\sin \theta_o D_x(0)\pi}{2c} \int_{\theta_m=0}^{\frac{\pi}{2}} D_y(\theta_m) \cos \theta_m \sin \theta_m d\theta_m \\
&\quad + \frac{\sin \theta_o D_y(0)\pi}{2c} \int_{\theta_m=0}^{\frac{\pi}{2}} D_x(\theta_m) \cos \theta_m \sin \theta_m d\theta_m \\
&= \frac{\sin \theta_o D_y(0)(3 + \cos 2\varphi_o)}{3c} \int_{\theta_m=0}^{\frac{\pi}{2}} D_x(\theta_m) \sin^2 \theta_m d\theta_m \\
&\quad + \frac{\sin \theta_o D_x(0)(3 - \cos 2\varphi_o)}{3c} \int_{\theta_m=0}^{\frac{\pi}{2}} D_y(\theta_m) \sin^2 \theta_m d\theta_m \\
&\quad + \frac{\cos \theta_o}{2}
\end{aligned} \tag{14}$$

For the isotropic distribution, the integral over the θ_m domain can be written as

$$\begin{aligned}
\int_{\theta_m=0}^{\frac{\pi}{2}} D_{iso}(\theta_m) \sin^2 \theta_m d\theta_m &= \sum_{i=1}^n \int_{\theta_m=\theta_i}^{\theta_{i+1}} (a_i \theta_m + b_i) \sin^2 \theta_m d\theta_m \\
&= \frac{1}{8} \sum_{i=1}^n a_i \cos(2\theta_i) - a_i \cos(2\theta_{i+1}) \\
&\quad + 2(\theta_{i+1} - \theta_i)(2b_i + a_i(\theta_i + \theta_{i+1})) \\
&\quad + 2(b_i + a_i \theta_i) \sin(2\theta_i) - 2(b_i + a_i \theta_{i+1}) \sin(2\theta_{i+1}) \\
&= \mathbf{A}^n
\end{aligned} \tag{15}$$

and noted \mathbf{A}_x^n and \mathbf{A}_y^n in order to simplify the notations. Inserting this notation in equation 14, we have

$$I_1 = \frac{\sin \theta_o D_y(0)(3 + \cos 2\varphi_o)}{3c} A_x^n + \frac{\sin \theta_o D_x(0)(3 - \cos 2\varphi_o)}{3c} A_y^n + \frac{\cos \theta_o}{2}. \tag{16}$$

3.2 Computation of I_2

The second double integral I_2 in Equation 12 is the most complicated part to compute due to the Heaviside function:

$$I_2 = \int_{\theta_m=0}^{\frac{\pi}{2}} \int_{\varphi_m=\varphi_o+\frac{\pi}{2}}^{\varphi_o+\frac{3\pi}{2}} \chi^+(\sin \theta_o \sin \theta_m \cos(\varphi_m - \varphi_o) + \cos \theta_o \cos \theta_m) (\sin \theta_o \sin \theta_m \cos(\varphi_m - \varphi_o) + \cos \theta_o \cos \theta_m) D(\theta_m, \varphi_m) \sin \theta_m d\varphi_m d\theta_m. \quad (17)$$

Here again the Heaviside function can be discarded if we restrict the integration domain:

$$\begin{aligned} & \chi^+(o \cdot m) > 0 \\ \Leftrightarrow & o \cdot m > 0 \\ \Leftrightarrow & \sin \theta_o \sin \theta_m \cos \varphi_m + \cos \theta_o \cos \theta_m > 0 \\ \Leftrightarrow & \sin \theta_o \sin \theta_m \cos \varphi_m > -\cos \theta_o \cos \theta_m \\ \Leftrightarrow & \cos \varphi_m > -\frac{\cos \theta_o \cos \theta_m}{\sin \theta_o \sin \theta_m} \\ \Leftrightarrow & \cos \varphi_m > -\frac{1}{\tan \theta_o \tan \theta_m} \\ \Leftrightarrow & \varphi_m \leq \arccos\left(-\frac{1}{\tan \theta_o \tan \theta_m}\right) \\ \Leftrightarrow & \varphi_m \leq \arccos\left(-\frac{1}{\tan \theta_o \tan \theta_m}\right) \\ \Leftrightarrow & \varphi_m \leq C \end{aligned} \quad (18)$$

with $C = \arccos\left(-\frac{1}{\tan \theta_o \tan \theta_m}\right)$. Note that $\varphi_o + \frac{\pi}{2} < \varphi_m < \varphi_o + \frac{3\pi}{2}$ and thus the integral domain for φ_m is split in two sub-domains $[\varphi_o + \frac{\pi}{2}, \varphi_o + C]$ and $[\varphi_o - C(+2\pi), \varphi_o + \frac{3\pi}{2}]$. On the other side, C is defined only if

$$\begin{aligned} \Leftrightarrow & -1 \leq \frac{1}{\tan \theta_o \tan \theta_m} \leq 1 \\ \Leftrightarrow & -1 \leq 1 \leq \tan \theta_o \tan \theta_m \\ \Leftrightarrow & \tan \theta_m \geq \frac{1}{\tan \theta_o} \\ \Leftrightarrow & \theta_m \geq \arctan(\cot \theta_o) \\ \Leftrightarrow & \theta_m \geq \frac{\pi}{2} - \theta_o \end{aligned}$$

The integral I_2 becomes

$$\begin{aligned} I_2 &= \int_{\theta_m=0}^{\frac{\pi}{2}-\theta_o} \int_{\varphi_m=\frac{\pi}{2}+\varphi_o}^{\frac{3\pi}{2}+\varphi_o} (\sin \theta_o \sin \theta_m \cos(\varphi_m - \varphi_o) + \cos \theta_o \cos \theta_m) D(\theta_m, \varphi_m) \sin \theta_m d\varphi_m d\theta_m \\ &+ \int_{\theta_m=\frac{\pi}{2}-\theta_o}^{\frac{\pi}{2}} \left(\int_{\varphi_m=\varphi_o+\frac{\pi}{2}}^{\varphi_o+C} (\sin \theta_o \sin \theta_m \cos(\varphi_m - \varphi_o) + \cos \theta_o \cos \theta_m) D(\theta_m, \varphi_m) \sin \theta_m d\varphi_m + \right. \\ &\quad \left. \int_{\varphi_m=\varphi_o+2\pi-C}^{\varphi_o+\frac{3\pi}{2}} (\sin \theta_o \sin \theta_m \cos(\varphi_m - \varphi_o) + \cos \theta_o \cos \theta_m) D(\theta_m, \varphi_m) \sin \theta_m d\varphi_m \right) d\theta_m \end{aligned} \quad (19)$$

with the first double integral without the Heaviside function because $o \cdot m > 0$ when $\theta_m \in [0, \frac{\pi}{2} - \theta_o]$ and $\varphi_m \in [\frac{\pi}{2} + \varphi_o, \frac{3\pi}{2} + \varphi_o]$ (Figure 2). We note respectively I_2^1 and I_2^2 the two integrals in Equation 19.

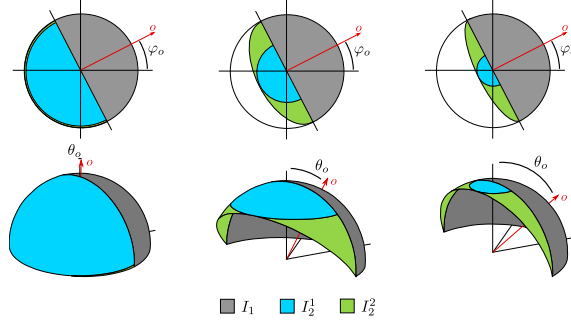


Figure 2: GAF computation $G_1(\mathbf{o}, \mathbf{m})$: the integration domain Ω is split in three sub-domains dependently on the \mathbf{o} direction (shown here for three θ_o angle configurations).

Computation of I_2^1

We call $E = \frac{\pi}{2} - \theta_o$ an angle in $[\theta_k, \theta_{k+1}]$ in our piecewise distribution.

$$\begin{aligned}
I_2^1 &= \int_{\theta_m=0}^E \int_{\varphi_m=\frac{\pi}{2}+\varphi_o}^{\frac{3\pi}{2}+\varphi_o} (\sin \theta_o \sin \theta_m \cos(\varphi_m - \varphi_o) + \cos \theta_o \cos \theta_m) D(\theta_m, \varphi_m) \sin \theta_m \, d\varphi_m \, d\theta_m \\
&= \sin \theta_o \int_{\theta_m=0}^E \sin^2 \theta_m \int_{\varphi_m=\frac{\pi}{2}+\varphi_o}^{\frac{3\pi}{2}+\varphi_o} D(\theta_m, \varphi_m) \cos(\varphi_m - \varphi_o) \, d\varphi_m \, d\theta_m \\
&\quad + \cos \theta_o \int_{\theta_m=0}^E \cos \theta_m \sin \theta_m \int_{\varphi_m=\frac{\pi}{2}+\varphi_o}^{\frac{3\pi}{2}+\varphi_o} D(\theta_m, \varphi_m) \, d\varphi_m \, d\theta_m \\
&= \frac{\sin \theta_o D_y(0)}{c} \int_{\theta_m=0}^E D_x(\theta_m) \sin^2 \theta_m \int_{\varphi_m=\frac{\pi}{2}+\varphi_o}^{\frac{3\pi}{2}+\varphi_o} \cos(\varphi_m - \varphi_o) \cos^2 \varphi_m \, d\varphi_m \, d\theta_m \\
&\quad + \frac{\sin \theta_o D_x(0)}{c} \int_{\theta_m=0}^E D_y(\theta_m) \sin^2 \theta_m \int_{\varphi_m=\frac{\pi}{2}+\varphi_o}^{\frac{3\pi}{2}+\varphi_o} \cos(\varphi_m - \varphi_o) \sin^2 \varphi_m \, d\varphi_m \, d\theta_m \\
&\quad + \frac{\cos \theta_o D_y(0)}{c} \int_{\theta_m=0}^E D_x(\theta_m) \cos \theta_m \sin \theta_m \int_{\varphi_m=\frac{\pi}{2}+\varphi_o}^{\frac{3\pi}{2}+\varphi_o} \cos^2 \varphi_m \, d\varphi_m \, d\theta_m \\
&\quad + \frac{\cos \theta_o D_x(0)}{c} \int_{\theta_m=0}^E D_y(\theta_m) \cos \theta_m \sin \theta_m \int_{\varphi_m=\frac{\pi}{2}+\varphi_o}^{\frac{3\pi}{2}+\varphi_o} \sin^2 \varphi_m \, d\varphi_m \, d\theta_m \\
&= \frac{\sin \theta_o D_y(0)(-3 - \cos 2\varphi_o)}{3c} \int_{\theta_m=0}^E D_x(\theta_m) \sin^2 \theta_m \, d\theta_m \\
&\quad + \frac{\sin \theta_o D_x(0)(-3 + \cos 2\varphi_o)}{3c} \int_{\theta_m=0}^E D_y(\theta_m) \sin^2 \theta_m \, d\theta_m \\
&\quad + \frac{\cos \theta_o D_y(0)\pi}{2c} \int_{\theta_m=0}^E D_x(\theta_m) \cos \theta_m \sin \theta_m \, d\theta_m \\
&\quad + \frac{\cos \theta_o D_x(0)\pi}{2c} \int_{\theta_m=0}^E D_y(\theta_m) \cos \theta_m \sin \theta_m \, d\theta_m \tag{20}
\end{aligned}$$

Using the fact that our distribution is a piecewise function, we can separate the integrals in the sum of integrals:

$$\int_{\theta_m=0}^E f(\theta_m) \, d\theta_m = \int_{\theta_m=0}^{\theta_k} f(\theta_m) \, d\theta_m + \int_{\theta_m=\theta_k}^E f(\theta_m) \, d\theta_m = \sum_{i=1}^{k-1} \int_{\theta_m=\theta_i}^{\theta_{i+1}} f(\theta_m) \, d\theta_m + \int_{\theta_m=\theta_k}^E f(\theta_m) \, d\theta_m.$$

$$\begin{aligned}
I_2^1 &= \frac{\sin \theta_o D_y(0)(-3 - \cos 2\varphi_o)}{3c} \left(\sum_{i=1}^{k-1} \int_{\theta_m=\theta_i}^{\theta_{i+1}} D_x(\theta_m) \sin^2 \theta_m d\theta_m + \right. \\
&\quad \left. \int_{\theta_m=\theta_k}^E D_x(\theta_m) \sin^2 \theta_m d\theta_m \right) \\
&+ \frac{\sin \theta_o D_x(0)(-3 + \cos 2\varphi_o)}{3c} \left(\sum_{i=1}^{k-1} \int_{\theta_m=\theta_i}^{\theta_{i+1}} D_y(\theta_m) \sin^2 \theta_m d\theta_m + \right. \\
&\quad \left. \int_{\theta_m=\theta_k}^E D_y(\theta_m) \sin^2 \theta_m d\theta_m \right) \\
&+ \frac{\cos \theta_o}{4} \left(\sum_{i=1}^{k-1} \int_{\theta_m=\theta_i}^{\theta_{i+1}} \frac{2\pi D_x(\theta_m) D_y(0) \cos \theta_m \sin \theta_m}{c} d\theta_m + \right. \\
&\quad \left. \int_{\theta_m=\theta_k}^E \frac{2\pi D_x(\theta_m) D_y(0) \cos \theta_m \sin \theta_m}{c} d\theta_m \right) \\
&+ \frac{\cos \theta_o}{4} \left(\sum_{i=1}^{k-1} \int_{\theta_m=\theta_i}^{\theta_{i+1}} \frac{2\pi D_y(\theta_m) D_x(0) \cos \theta_m \sin \theta_m}{c} d\theta_m + \right. \\
&\quad \left. \int_{\theta_m=\theta_k}^E \frac{2\pi D_y(\theta_m) D_x(0) \cos \theta_m \sin \theta_m}{c} d\theta_m \right)
\end{aligned} \tag{21}$$

We can see that the two terms in red correspond to the coefficients A_x^{k-1} and A_y^{k-1} respectively and the two terms in blue correspond to the coefficients T_x^{k-1} and T_y^{k-1} .

$$\begin{aligned}
I_2^1 &= \frac{\sin \theta_o D_y(0)(-3 - \cos 2\varphi_o)}{3c} \left(A_x^{k-1} + \int_{\theta_m=\theta_k}^E D_x(\theta_m) \sin^2 \theta_m d\theta_m \right) \\
&+ \frac{\sin \theta_o D_x(0)(-3 + \cos 2\varphi_o)}{3c} \left(A_y^{k-1} + \int_{\theta_m=\theta_k}^E D_y(\theta_m) \sin^2 \theta_m d\theta_m \right) \\
&+ \frac{\cos \theta_o}{4} \left(T_x^{k-1} + \int_{\theta_m=\theta_k}^E \frac{2\pi D_x(\theta_m) D_y(0) \cos \theta_m \sin \theta_m}{c} d\theta_m \right) \\
&+ \frac{\cos \theta_o}{4} \left(T_y^{k-1} + \int_{\theta_m=\theta_k}^E \frac{2\pi D_y(\theta_m) D_x(0) \cos \theta_m \sin \theta_m}{c} d\theta_m \right).
\end{aligned} \tag{22}$$

The other integrals can also be computed analytically:

$$\begin{aligned}
I_2^1 &= \frac{\sin \theta_o D_y(0)(-3 - \cos 2\varphi_o)}{3c} \left(A_x^{k-1} + \right. \\
&\quad \left. \frac{1}{8} \left(a_x^k (\cos 2\theta_k + \cos 2\theta_o) + 2(E - \theta_k)(a_x^k(\theta_k + E) + 2b_x^k) + 2(a_x^k \theta_k + b_x^k) \sin 2\theta_k - 2(a_x^k E + b_x^k) \sin 2\theta_o \right) \right) \\
&+ \frac{\sin \theta_o D_x(0)(-3 + \cos 2\varphi_o)}{3c} \left(A_y^{k-1} + \right. \\
&\quad \left. \frac{1}{8} \left(a_y^k (\cos 2\theta_k + \cos 2\theta_o) + 2(E - \theta_k)(a_y^k(\theta_k + E) + 2b_y^k) + 2(a_y^k \theta_k + b_y^k) \sin 2\theta_k - 2(a_y^k E + b_y^k) \sin 2\theta_o \right) \right) \\
&+ \frac{\cos \theta_o}{4} \left(T_x^{k-1} + \frac{2\pi D_y(0)}{8c} \left(2(a_x^k \theta_k + b_x^k) \cos 2\theta_k + 2(a_x^k E + b_x^k) \cos 2\theta_o + a_x^k (\sin 2\theta_o - \sin 2\theta_k) \right) \right) \\
&+ \frac{\cos \theta_o}{4} \left(T_y^{k-1} + \frac{2\pi D_x(0)}{8c} \left(2(a_y^k \theta_k + b_y^k) \cos 2\theta_k + 2(a_y^k E + b_y^k) \cos 2\theta_o + a_y^k (\sin 2\theta_o - \sin 2\theta_k) \right) \right).
\end{aligned} \tag{23}$$

Computation of I_2^2

We call $E = \frac{\pi}{2} - \theta_o$ an angle in $[\theta_k, \theta_{k+1}]$ in our piecewise distribution. We have

$$\begin{aligned}
I_2^2 &= \int_{\theta_m=E}^{\frac{\pi}{2}} \int_{\varphi_m=\varphi_o+\frac{\pi}{2}}^{\varphi_o+C} (\sin \theta_o \sin \theta_m \cos(\varphi_m - \varphi_o) + \cos \theta_o \cos \theta_m) D(\theta_m, \varphi_m) \sin \theta_m d\varphi_m \\
&\quad + \int_{\varphi_m=\varphi_o+2\pi-C}^{\frac{3\pi}{2}+\varphi_o} (\sin \theta_o \sin \theta_m \cos(\varphi_m - \varphi_o) + \cos \theta_o \cos \theta_m) D(\theta_m, \varphi_m) \sin \theta_m d\varphi_m d\theta_m \\
&= \sin \theta_o \int_{\theta_m=E}^{\frac{\pi}{2}} \sin^2 \theta_m \left(\int_{\varphi_m=\varphi_o+\frac{\pi}{2}}^{\varphi_o+C} D(\theta_m, \varphi_m) \cos(\varphi_m - \varphi_o) d\varphi_m + \right. \\
&\quad \left. \int_{\varphi_m=\varphi_o+2\pi-C}^{\frac{3\pi}{2}+\varphi_o} D(\theta_m, \varphi_m) \cos(\varphi_m - \varphi_o) d\varphi_m \right) d\theta_m \\
&\quad + \cos \theta_o \int_{\theta_m=E}^{\frac{\pi}{2}} \cos \theta_m \sin \theta_m \left(\int_{\varphi_m=\varphi_o+\frac{\pi}{2}}^{\varphi_o+C} D(\theta_m, \varphi_m) d\varphi_m + \int_{\varphi_m=\varphi_o+2\pi-C}^{\frac{3\pi}{2}+\varphi_o} D(\theta_m, \varphi_m) d\varphi_m \right) d\theta_m \\
&= \frac{\sin \theta_o D_y(0)}{c} \int_{\theta_m=E}^{\frac{\pi}{2}} D_x(\theta_m) \sin^2 \theta_m \left(\int_{\varphi_m=\varphi_o+\frac{\pi}{2}}^{\varphi_o+C} \cos(\varphi_m - \varphi_o) \cos^2 \varphi_m d\varphi_m + \right. \\
&\quad \left. \int_{\varphi_m=\varphi_o+2\pi-C}^{\frac{3\pi}{2}+\varphi_o} \cos(\varphi_m - \varphi_o) \cos^2 \varphi_m d\varphi_m \right) d\theta_m \\
&\quad + \frac{\sin \theta_o D_x(0)}{c} \int_{\theta_m=E}^{\frac{\pi}{2}} D_y(\theta_m) \sin^2 \theta_m \left(\int_{\varphi_m=\varphi_o+\frac{\pi}{2}}^{\varphi_o+C} \cos(\varphi_m - \varphi_o) \sin^2 \varphi_m d\varphi_m + \right. \\
&\quad \left. \int_{\varphi_m=\varphi_o+2\pi-C}^{\frac{3\pi}{2}+\varphi_o} \cos(\varphi_m - \varphi_o) \sin^2 \varphi_m d\varphi_m \right) d\theta_m \\
&\quad + \frac{\cos \theta_o D_y(0)}{c} \int_{\theta_m=E}^{\frac{\pi}{2}} D_x(\theta_m) \cos \theta_m \sin \theta_m \left(\int_{\varphi_m=\varphi_o+\frac{\pi}{2}}^{\varphi_o+C} \cos^2 \varphi_m d\varphi_m + \int_{\varphi_m=\varphi_o+2\pi-C}^{\frac{3\pi}{2}+\varphi_o} \cos^2 \varphi_m d\varphi_m \right) d\theta_m \\
&\quad + \frac{\cos \theta_o D_x(0)}{c} \int_{\theta_m=E}^{\frac{\pi}{2}} D_y(\theta_m) \cos \theta_m \sin \theta_m \left(\int_{\varphi_m=\varphi_o+\frac{\pi}{2}}^{\varphi_o+C} \sin^2 \varphi_m d\varphi_m + \int_{\varphi_m=\varphi_o+2\pi-C}^{\frac{3\pi}{2}+\varphi_o} \sin^2 \varphi_m d\varphi_m \right) d\theta_m
\end{aligned} \tag{24}$$

$$\begin{aligned}
I_2^2 &= \frac{\sin \theta_o D_y(0)}{3c} \int_{\theta_m=E}^{\frac{\pi}{2}} D_x(\theta_m) \sin^2 \theta_m (3 + \cos 2\varphi_o(\cos 2C - 2 \sin C)) (\sin C - 1) d\theta_m \\
&+ \frac{\sin \theta_o D_x(0)}{3c} \int_{\theta_m=E}^{\frac{\pi}{2}} D_y(\theta_m) \sin^2 \theta_m (3 - \cos 2\varphi_o(\cos 2C - 2 \sin C)) (\sin C - 1) d\theta_m \\
&+ \frac{\cos \theta_o D_y(0)}{c} \int_{\theta_m=E}^{\frac{\pi}{2}} D_x(\theta_m) \cos \theta_m \sin \theta_m \left(C - \frac{\pi}{2} + \frac{1}{2} \sin 2C \cos 2\varphi_o \right) d\theta_m \\
&+ \frac{\cos \theta_o D_x(0)}{c} \int_{\theta_m=E}^{\frac{\pi}{2}} D_y(\theta_m) \cos \theta_m \sin \theta_m \left(C - \frac{\pi}{2} - \frac{1}{2} \sin 2C \cos 2\varphi_o \right) d\theta_m. \\
I_2^2 &= \frac{\sin \theta_o D_y(0)}{3c} \int_{\theta_m=E}^{\frac{\pi}{2}} D_x(\theta_m) \sin^2 \theta_m (3 + \cos 2\varphi_o(\cos 2C - 2 \sin C)) (\sin C - 1) d\theta_m \\
&+ \frac{\sin \theta_o D_x(0)}{3c} \int_{\theta_m=E}^{\frac{\pi}{2}} D_y(\theta_m) \sin^2 \theta_m (3 - \cos 2\varphi_o(\cos 2C - 2 \sin C)) (\sin C - 1) d\theta_m \\
&+ \cos \theta_o \left(-\frac{1}{4} \left(\int_{\theta_{k+1}}^{\frac{\pi}{2}} \frac{2\pi D_x(\theta_m) D_y(0) \cos \theta_m \sin \theta_m}{c} d\theta_m + \int_{\theta_m=E}^{\theta_{k+1}} \frac{2\pi D_x(\theta_m) D_y(0) \cos \theta_m \sin \theta_m}{c} d\theta_m \right) \right. \\
&+ \left. \frac{D_y(0)}{c} \int_{\theta_m=E}^{\frac{\pi}{2}} D_x(\theta_m) \cos \theta_m \sin \theta_m \left(C + \frac{1}{2} \sin 2C \cos 2\varphi_o \right) d\theta_m \right) \\
&+ \cos \theta_o \left(-\frac{1}{4} \left(\int_{\theta_{k+1}}^{\frac{\pi}{2}} \frac{2\pi D_y(\theta_m) D_x(0) \cos \theta_m \sin \theta_m}{c} d\theta_m + \int_{\theta_m=E}^{\theta_{k+1}} \frac{2\pi D_y(\theta_m) D_x(0) \cos \theta_m \sin \theta_m}{c} d\theta_m \right) \right. \\
&+ \left. \frac{D_x(0)}{c} \int_{\theta_m=E}^{\frac{\pi}{2}} D_y(\theta_m) \cos \theta_m \sin \theta_m \left(C - \frac{1}{2} \sin 2C \cos 2\varphi_o \right) d\theta_m \right). \tag{25}
\end{aligned}$$

We can see that the two terms in blue correspond to the coefficients $(T_x^n - T_x^k)$ and $(T_y^n - T_y^k)$. The two integrals over $\theta_m \in [E, \theta_{k+1}]$ can be also computed analytically. The other integrals are grouped as a sum in a function called Λ :

$$\begin{aligned}
I_2^2 &= \Lambda \\
&- \frac{\cos \theta_o}{4} \left((T_x^n - T_x^k) + \frac{2\pi D_y(0)}{8c} (-2(a_x^k E + b_x^k) \cos 2\theta_o - 2(a_x^k \theta_{k+1} + b_x^k) \cos 2\theta_{k+1} + a_x^k (\sin 2\theta_{k+1} - \sin 2\theta_o)) \right) \\
&- \frac{\cos \theta_o}{4} \left((T_y^n - T_y^k) + \frac{2\pi D_x(0)}{8c} (-2(a_y^k E + b_y^k) \cos 2\theta_o - 2(a_y^k \theta_{k+1} + b_y^k) \cos 2\theta_{k+1} + a_y^k (\sin 2\theta_{k+1} - \sin 2\theta_o)) \right) \tag{26}
\end{aligned}$$

Λ is

$$\begin{aligned}
\Lambda &= \frac{\sin \theta_o}{3} \left(\int_{\theta_m=E}^{\frac{\pi}{2}} D(\theta_m, 0) \sin^2 \theta_m (3 + \cos 2\varphi_o(\cos 2C - 2 \sin C)) (\sin C - 1) \right. \\
&+ \left. D(\theta_m, \frac{\pi}{2}) \sin^2 \theta_m (3 - \cos 2\varphi_o(\cos 2C - 2 \sin C)) (\sin C - 1) d\theta_m \right) \\
&+ \cos \theta_o \left(\int_{\theta_m=E}^{\frac{\pi}{2}} D(\theta_m, 0) \cos \theta_m \sin \theta_m \left(C + \frac{1}{2} \sin 2C \cos 2\varphi_o \right) \right. \\
&+ \left. D(\theta_m, \frac{\pi}{2}) \cos \theta_m \sin \theta_m \left(C - \frac{1}{2} \sin 2C \cos 2\varphi_o \right) d\theta_m \right) \tag{27}
\end{aligned}$$

The two integral cannot be computed analytically and we propose to use a numerical approach to evaluate it. As our distribution is a piecewise function, we use a trapezoidal rule to approximate it:

$$\begin{aligned}
\Lambda &= \frac{\sin \theta_o}{3} \left(\int_{\theta_m=E}^{\theta_{k+1}} \text{part1}(\theta_m) d\theta_m + \sum_{i=k+1}^n \int_{\theta_m=\theta_i}^{\theta_{i+1}} \text{part1}(\theta_m) d\theta_m \right) \\
&+ \cos \theta_o \left(\int_{\theta_m=E}^{\theta_{k+1}} \text{part2}(\theta_m) d\theta_m + \sum_{i=k+1}^n \int_{\theta_m=\theta_i}^{\theta_{i+1}} \text{part2}(\theta_m) d\theta_m \right) \\
\Lambda &\approx \frac{\sin \theta_o}{6} \left((\theta_{k+1} - E)(\text{part1}(\theta_{k+1}) + \text{part1}(E)) + \Delta_\theta \sum_{i=k+1}^n \text{part1}(\theta_{i+1}) + \text{part1}(\theta_i) \right) \\
&+ \frac{\cos \theta_o}{2} \left((\theta_{k+1} - E)(\text{part2}(\theta_{k+1}) + \text{part2}(E)) \Delta_\theta \sum_{i=k+1}^n \text{part2}(\theta_{i+1}) + \text{part2}(\theta_i) \right)
\end{aligned} \tag{28}$$

with Δ_θ a constant if each sector of the decomposition over θ is equal and part1 and part2 the integrals over θ_m in Equation 27.

3.3 GAF plots

Finally, the GAF expression is

$$G_1(o, m) = \frac{\cos \theta_o}{\int_{\Omega} \langle o \cdot m \rangle D(m) d m} \approx \frac{\cos \theta_o}{I_1 + I_2^1 + I_2^2}. \tag{29}$$

Figure 3 shown two examples of the GAF computed for two random configurations of our controllable distribution.

3.4 Correlated GAF

Previous computations are based on the Ashikhmin et al. [1, 5] expression (Equation 10). Other more convenient expressions of this function can be obtained starting from the work of Bourlier [3]. Walter et al. [10] and more recently Heitz [5] express the normal distribution in the slopes space $P_{22}(p, q)$:

$$P_{22}(p, q) = \cos^4 \theta_m D(m), \tag{30}$$

where p and q correspond to the surface slopes in the local coordinate system, defining the normal m , such that $p^2 + q^2 = \tan^2 \theta_m$. The one dimensional distribution of slopes is given by:

$$P_2(q) = \int_{-\infty}^{+\infty} P_{22}(p, q) dp. \tag{31}$$

Finally, $G_1(v)$ is obtained by the integration of P_2 :

$$G_1(v) = \frac{1}{1 + \Lambda(v)}, \tag{32}$$

where $\Lambda(v) = \frac{1}{\mu} \int_{-\infty}^{+\infty} (q - \mu) P_2(q) dq$ and $\mu = \cot \theta_v$. A formal proof of these derivations is given by Heitz in [5].

Note that this GAF version is used by most rendering systems, though the discussion proposed by Ross et al. [8] or Heitz et al. [4] expresses correlated versions between masking and shadowing, considered

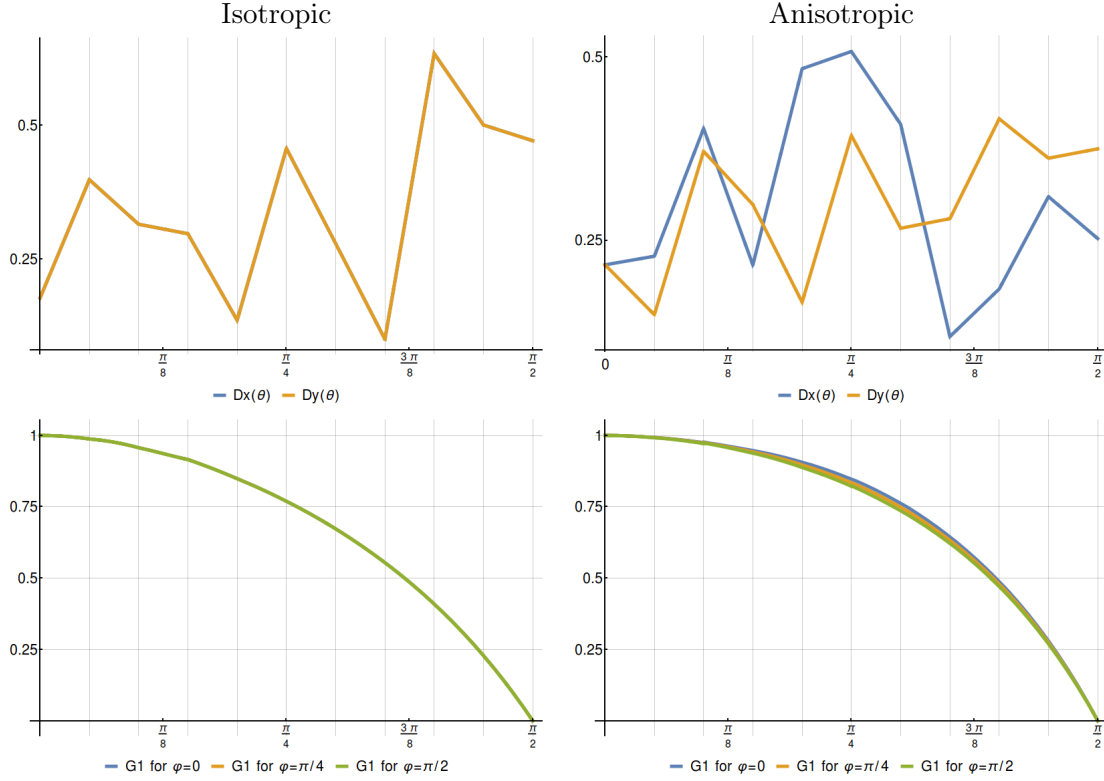


Figure 3: Two examples of the GAF with our controllable distribution (randomly generated here): isotropic on the left and anisotropic on the right.

as physically more plausible. One more accurate formulation, equivalently computational costly, is the heighth-correlated masking and shadowing expression:

$$G(i, o, m) = \frac{1}{1 + \Lambda(i) + \Lambda(o)}. \quad (33)$$

Note that Equations 10 and 32 are equivalent and thus the expression of the function Λ is given by:

$$\begin{aligned} \frac{\cos \theta_o}{\int_{\Omega} \langle \mathbf{o} \cdot \mathbf{m} \rangle D(\mathbf{m}) d \mathbf{m}} &= \frac{1}{1 + \Lambda(\mathbf{o})} \\ \Rightarrow \Lambda(\mathbf{o}) &= \frac{\int_{\Omega} \langle \mathbf{o} \cdot \mathbf{m} \rangle D(\mathbf{m}) d \mathbf{m}}{\cos \theta_o} - 1 \end{aligned}$$

In Equation 16, $I_1 = \frac{\cos \theta_o}{2} + I_{11}$ and thus

$$\Lambda(\mathbf{o}) = \frac{I_{11} + I_2 + I_3}{\cos \theta_o} - \frac{1}{2}$$

with

$$I_1 = \frac{\sin \theta_o D_y(0)(3 + \cos 2\varphi_o)}{3c} A_x^n + \frac{\sin \theta_o D_x(0)(3 - \cos 2\varphi_o)}{3c} A_y^n. \quad (34)$$

Knowing the function Λ allows to compute the heighth-correlated masking and shadowing expression of the Smith GAF (Equation 33) considered as physically more plausible.

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