Skewed Generalized T-Distribution

Microfacet BSDFs Generated from NDFs and Explicit Microgeometry

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Controlling the NDF shape is mandatory to represent more complex micro-surface distributions, as pointed out by several authors [8, 1, 5, 3, 4, 10, 9].

Moreover, to be efficiently included in rendering systems, a NDF has to come with an analytical Smith GAF derivation and importance sampling schemes. This latter should be ideally implemented with visible normal importance sampling as proposed in [7]. To our knowledge, only Beckmann [2] and GGX [12] distributions offer all these features. These distributions are also the most used in rendering engines. In the following, we first propose a classification of NDFs which allows to derive new general distributions with more control on the shape (skewness, kurtosis, lobe shift and width) called *Skewed Generalized T-Distribution* (SGTD).

The second part of this document contains the mathematical justifications concerning the SGTD and its sub-configurations. We provide all the mathematical details for the normalization factor, GAF computation and importance sampling when these features can be analytically computed with conventional methods. When analytical computations are not available, some approximate solutions can be found: we voluntary let this in future work.

Contents

	Isotropic	Anisotropic	Norm. Factor A	Imp. Sampl.	Smith GAF (Λ)
SGTD	Eq. 1	–	Eq. 5 and 6	–	–
GTD	Eq. 13	Eq. 14	Eq. 15 and 16	–	–
SGND	Eq. 22	–	– (Eq. 23, $\mu = 0$)	– (Eq. 34, $\mu = 0$)	–
STD	Eq. 25	–	Eq. 26 and 27	–	–
GND	Eq. 28	Eq. 29	– (Eq. 30, $\mu = 0$)	–	–
SND	Eq. 31	–	Eq. 32 and 33		–
Student's TD	Eq. 35	Eq. 36	Eq. 37	– (see [10], $\mu = 0$)	– (see [10], $\mu = 0$)
Shifted Beck.	Eq. 38	Eq. 39	$A = 1$	– (see [12], $\mu = 0$)	– (see [12], $\mu = 0$)
Shifted GGX	Eq. 40	Eq. 41	Eq. 42	– (see [12], $\mu = 0$)	– (see [12], $\mu = 0$)

Table 1: SGTD and sub-configurations with references to equations for each feature. Symbols – stand for unresolved (*i.e.* with no direct analytical expression) features. Note that for some configurations, a feature could be resolved for a restricted version (*e.g.* for SGND with the normalization factor which can be computed for unshifted version $\mu = 0$).

1 NDFs classification using SGTD

Beckmann [2] and especially GGX [12] distributions are currently considered as the standard in computer graphics. Their shape is only controlled by one roughness parameter σ which drives the lobe width. Unfortunately, real material normal distributions exhibit more complex shapes as mentioned by Holzschuch et al. [9], who also propose a generalization of Beckmann's distribution, called *Generalized Normal Distribution* (GND). GND includes an additional parameter γ which characterizes the lobe's flattening (called here *tailedness*), defining the drop at the origin. Ribardiere *et al.* [10] proposed to use an even more general distribution, *Student's T-Distribution* (*Student's* TD), with an equivalent *tailedness* parameter γ. *Student's* TD encompasses *GGX* when $\gamma = 2$ and tends to Beckmann when $\gamma \to \infty$. They also derive a more general formulation which allows to include the *Hyper-Cauchy Distribution* (HCD) as a specific configuration of *Student's* TD. HCD was first introduced by Wellems et al. [13] and used to fit measured BRDF by Butler and Marciniak [4]. All these configurations control the shape through two parameters: σ for the lobe width and γ for its *tailedness*. Contrary to GND, *Student's* TD comes with the analytic Λ function to derive the Smith GAF expression. An open question concerns the degree of control required on the NDF shape for fitting parameters, designing and rendering materials, and if such a control can be reached by an existing distribution. All the previous distributions are in fact sub-families of a more general distribution, the *Skewed Generalized T-Distribution* (SGTD), introduced in financial statistics by Theodossiou [11]. We propose a 5 parameters distribution dedicated to NDFs representation, with the following controls:

$$\begin{aligned} \sigma > 0 &\longmapsto \text{lobe width} \\ \gamma > \frac{3}{2} &\longmapsto \text{tailedness} \\ p > 1 &\longmapsto \text{kurtosis} \\ \mu > 0 &\longmapsto \text{lobe shift} \\ -1 < \lambda < 1 &\longmapsto \text{skewness} \end{aligned}$$

Note that all the lobe characteristics (shifting, width, kurtosis, tailedness) are defined by the above parameters. The isotropic representation is thus given by the following expression:

$$D^{SGTD}(\mathbf{m}) = \frac{A}{\pi\ \sigma^2 \cos^4\theta \left(1 + \frac{(|\tan\theta - \mu|)^{2p}}{(\gamma-1)\sigma^{2p} C}\right)^{\gamma}},$$

with A, the normalization factor (its analytical form is given in supplemental material) and $C = (\lambda \text{ Sign}(\tan\theta - \mu) + 1)^{2p}$.

A complete classification can be proposed from *Beckmann* and GGX to SGTD, including *Student's* TD or GND. Figure 1 illustrates this classification and the relationship from different interesting configurations. Note that GND and *Student's* TD are presented here in their shifted version ($\mu \neq 0$) even if they are usually used in their unshifted version ($\mu = 0$) in [9] and [10] respectively.

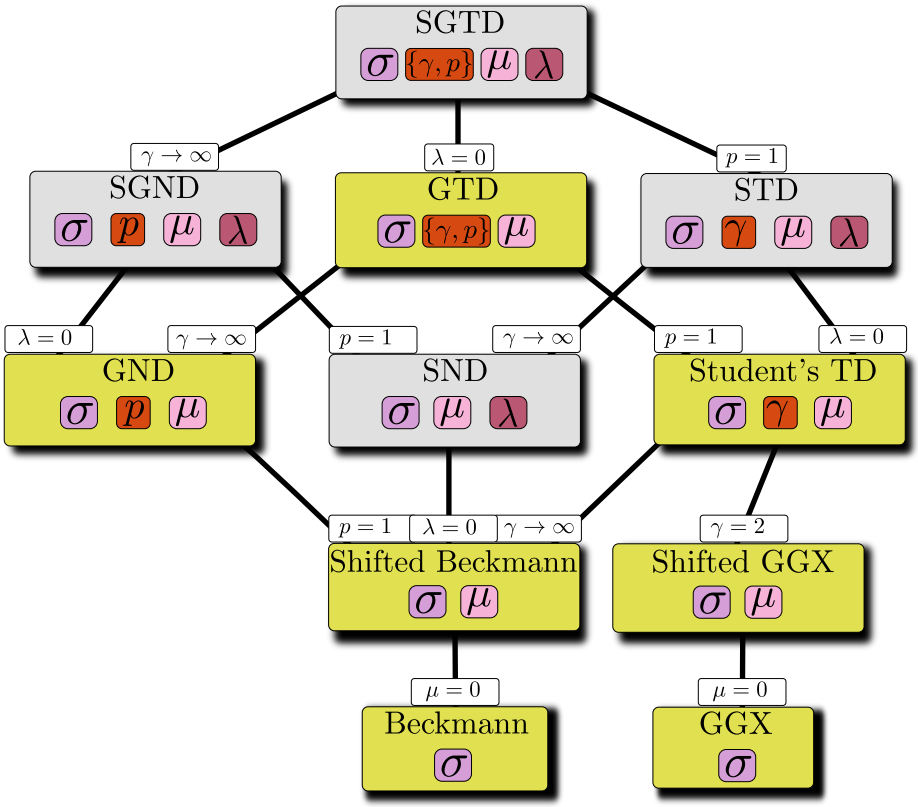

Figure 1: The *Skewed Generalized T-Distribution* [14] (SGTD) tree: Yellow blocks represent shape invariant distributions coming with anisotropy. For the sake of readability, some configurations (Laplacian distributions and Cauchy ones) are not mentioned.

The *shape invariance* property [6, 8] defines the invariance of both the NDF shape and the masking function when the surface configuration is stretched. A shape invariant NDF can be written as follows:

$$\frac{f\left(\frac{\tan\theta}{\sigma}\right)}{\sigma^2 \cos^4\theta}.$$

Such a formulation also helps to derive anisotropic derivations of the distribution.

Unfortunately, SGTD does not come with importance sampling and analytical Smith GAF, required for path tracing applications. Though, interesting sub-configurations can be produced by fixing some of its five parameters. SGTD is not shape invariant due to the skewness parameter λ in factor C, and anisotropy cannot be derived using this property. Fixing λ to 0 corresponds to a pruning in the classification tree (Figure 1), so as to only select all sub-configurations starting from GTD: All the selected distributions are shape invariant and thus anisotropy can be easily derived. Expressions for each configuration and mathematical development for GAF computations and importance sampling (which cannot be analytically obtained in most cases) are provided in the next sections of this document.

2 Skewed Generalized T-Distribution (SGTD)

As presented in the paper, SGTD expression is given by

$$D^{SGTD}(m) = \frac{A}{\pi\ \sigma^2 \cos^4\theta \left(1 + \frac{(|\tan\theta - \mu|)^{2p}}{(\gamma-1)\sigma^{2p}C}\right)^{\gamma}} \tag{1}$$

with

$$\begin{aligned} A &= \text{the normalization factor} \\ C &= (\lambda\ \text{Sign}\,(\tan\theta - \mu) + 1)^{2p}\,. \end{aligned}$$

and

$$\begin{aligned} \sigma > 0 &\longmapsto \text{controls the lobe width,} \\ \gamma > \frac{3}{2}\ ,\ p > 1 &\longmapsto \text{the kurtosis,} \\ \mu > 0 &\longmapsto \text{the lobe shift,} \\ -1 < \lambda < 1 &\longmapsto \text{the skewness.} \end{aligned}$$

This NDF is not *shape invariant* [6, 8] and thus cannot be given in an anisotropic form.

2.1 Normalization factor

The distribution have to normalized *i.e.*

$$\frac{A}{\pi\sigma^2}\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\frac{\pi}{2}} \frac{\tan\theta}{\cos^2\theta}\left(1 + \frac{|\tan\theta - \mu|^{2p}}{(\gamma-1)\sigma^{2p}\left(\lambda\ \text{Sign}(\tan\theta - \mu) + 1\right)^{2p}}\right)^{-\gamma} d\theta\ d\varphi = 1. \tag{2}$$

With $\cos^2\theta = \frac{1}{1+\tan^2\theta}$ and the substitution $t = \tan\theta$, we obtain

$$\frac{A}{\pi\sigma^2}\int_{t=0}^{\infty} t\left(1 + \frac{|t-\mu|^{2p}}{(\gamma-1)\sigma^{2p}\left(\lambda\ \text{Sign}(t-\mu) + 1\right)^{2p}}\right)^{-\gamma} dt = 1. \tag{3}$$

Note that $t - \mu > 0$ if $t > \mu$ and the integral can be split in two parts, removing the absolute and the *sign* components

$$A\left(\int_{t=0}^{\mu} t\left(1 + \frac{(\mu-t)^{2p}}{\sigma^{2p}(\gamma-1)(1-\lambda)^{2p}}\right)^{-\gamma} dt + \int_{t=\mu}^{\infty} t\left(1 + \frac{(t-\mu)^{2p}}{\sigma^{2p}(\gamma-1)(1+\lambda)^{2p}}\right)^{-\gamma} dt\right) = \frac{\sigma^2}{2}. \tag{4}$$

Let's denote I_1 the first integral and I_2 the second one, we have

$$A = \frac{\sigma^2/2}{I_1 + I_2} \tag{5}$$

with

$$\begin{aligned} I_1 &= \mu^2\left({}_2F_1\left(\gamma, \frac{1}{2p}, 1 + \frac{1}{2p}, \frac{-\mu^{2p}}{\sigma^{2p}(\gamma-1)(1-\lambda)^{2p}}\right) - \frac{1}{2}{}_2F_1\left(\gamma, \frac{1}{p}, 1 + \frac{1}{p}, \frac{-\mu^{2p}}{\sigma^{2p}(\gamma-1)(1-\lambda)^{2p}}\right)\right) \\ I_2 &= \frac{(\gamma-1)^{1/p}(\lambda+1)^2\sigma^2\left(\frac{2\mu}{\sigma(\gamma-1)^{1/2p}(1+\lambda)}\Gamma(\gamma - 1/2p)\Gamma(1+1/2p) + \Gamma(\gamma-1/p)\Gamma(1+1/p)\right)}{2\Gamma(\gamma)} \end{aligned} \tag{6}$$

with $\Gamma(x =)$ the Eulerian gamma function and ${}_2F_1$ the Gauss hypergeometric function.

2.2 GAF computation

The GAF is computed using the same process as previously proposed in [?, 6, 12]. The normal distribution is first expressed in the slopes space:

$$P_{22}^{SGTD}(r,q) = D^{SGTD}(\theta_m) * \cos^4\theta_m = \frac{A}{\pi\ \sigma^2\theta\left(1+\frac{(|\sqrt{r^2+q^2}-\mu|)^{2p}}{(\gamma-1)\sigma^{2p}\left(\lambda\ \mathrm{Sign}(\sqrt{r^2+q^2}-\mu)+1\right)^{2p}}\right)^{\gamma}}, \tag{7}$$

with $r^2+q^2=\tan^2\theta_m$. The one dimensional distribution of slopes in the incidence plane is given by:

$$\begin{aligned} P_2^{SGTD}(q) &= \int_{-\infty}^{\infty} P_{22}^{SGTD}(r,q)\ dr \\ &= \frac{A(\gamma-1)^{\gamma}\sigma^{2p\gamma-2}}{\pi}\int_{-\infty}^{\infty}\left((\gamma-1)\sigma^{2p}+\frac{|\sqrt{r^2+q^2}-\mu|^{2p}}{\left(\lambda\ \mathrm{Sign}(\sqrt{r^2+q^2}-\mu)+1\right)^{2p}}\right)^{-\gamma} dr. \end{aligned} \tag{8}$$

Finally, the GAF $G_1(\theta)$ is defined as:

$$G_1^{STD}(\theta) = \frac{1}{1+\Lambda^{SGTD}(\theta)}, \tag{9}$$

where

$$\Lambda^{SGTD}(\theta) = \frac{1}{\cot\theta}\int_{\cot\theta}^{+\infty}(q-\cot\theta)P_2^{SGTD}(q)\ dq. \tag{10}$$

Equation 8 and consequently Equation 10 cannot be computed analytically. A solution is to precompute Λ and store it in a table indexed by the SGTD parameters and θ. Due to the number of parameter, this table can be large in the case of SGTD.

2.3 Importance sampling

The microfacet normal m is sampled according to the probability density function $pdf^{SGTD}(m) = D^{SGTD}(m)|m\cdot n|$ and its associated cumulative distribution function:

$$cdf^{SGTD}(m) = \int_{\varphi=0}^{\varphi_m}\int_{\theta=0}^{\theta_m} D^{SGTD}(m)\cos(\theta)\sin(\theta)\ d\theta\ d\varphi, \tag{11}$$

We first sample the microfacet azimuthal angle φ_m; the elevation angle θ_m is sampled secondly. The choice of φ_m is independent of θ_m and SGTD is isotropic, thus

$$\varphi_m = 2\pi\xi_1$$

with ξ_1 is a uniform random number in $[0,1)$.

Knowing the azimuthal angle φ_m, the *cdf* to sample θ_m is:

$$cdf^{SGTD}(\theta_m|\varphi_m) = \frac{2A}{\sigma^2}\int_{\theta=0}^{\theta_m}\frac{\tan(\theta)}{\cos^2(\theta)}\left(1+\frac{|\tan\theta-\mu|^{2p}}{(\gamma-1)\sigma^{2p}\left(\lambda\ \mathrm{Sign}(\tan\theta-\mu)+1\right)^{2p}}\right)^{-\gamma} d\theta = \xi_2 \tag{12}$$

with, here again, ξ_2 a second uniform random number in $[0,1)$. Analytically invert this *cdf* is not trivial: the computation result exhibits a sum of Gauss hypergeometric functions ${}_2F_1$ which is not invertible.

3 Generalized T-Distribution (GTD): $\lambda = 0$

The isotropic expression of GTD is:

$$D^{GTD}(m) = \frac{A}{\pi\ \sigma^2 \cos^4\theta \left(1 + \frac{(|\tan\theta - \mu|)^{2p}}{(\gamma-1)\sigma^{2p}}\right)^{\gamma}}. \tag{13}$$

This function is *shape invariant* and its anisotropic form is:

$$D^{GTD}(m) = \frac{A}{\pi\ \sigma_x\sigma_y \cos^4\theta \left(1 + \frac{B(\varphi)^p(|\tan\theta - \mu|)^{2p}}{(\gamma-1)}\right)^{\gamma}} \tag{14}$$

avec $B(\varphi) = \frac{\cos^2\varphi}{\sigma_x^2} + \frac{\sin^2\varphi}{\sigma_y^2}$.

3.1 Normalization factor

With the same mathematical development as for SGTD, we have

$$A = \frac{\sigma^2/2}{I_1 + I_2} \tag{15}$$

with

$$\begin{aligned} I_1 &= \mu^2 \left({}_2F_1\left(\gamma, \frac{1}{2p}, 1 + \frac{1}{2p}, \frac{-\mu^{2p}}{\sigma^{2p}(\gamma-1)}\right) - \frac{1}{2}{}_2F_1\left(\gamma, \frac{1}{p}, 1 + \frac{1}{p}, \frac{-\mu^{2p}}{\sigma^{2p}(\gamma-1)}\right)\right) \\ I_2 &= \frac{(\gamma-1)^{1/p}\sigma^2 \left(\frac{2\mu}{\sigma(\gamma-1)^{1/2p}}\Gamma(\gamma - 1/2p)\Gamma(1 + 1/2p) + \Gamma(\gamma - 1/p)\Gamma(1 + 1/p)\right)}{2\Gamma(\gamma)} \end{aligned} \tag{16}$$

For the anisotropic version, $\sigma^2 = \sigma_x\sigma_y$.

3.2 GAF computation

The normal distribution in the slopes space is

$$P_{22}^{SGTD}(r,q) = D^{SGTD}(\theta_m) * \cos^4\theta_m = \frac{A}{\pi\ \sigma^2\theta \left(1 + \frac{(|\sqrt{r^2+q^2} - \mu|)^{2p}}{(\gamma-1)\sigma^{2p}}\right)^{\gamma}}, \tag{17}$$

with $r^2 + q^2 = \tan^2\theta_m$. The one dimensional distribution of slopes in the incidence plane is given by:

$$\begin{aligned} P_2^{GTD}(q) &= \int_{-\infty}^{\infty} P_{22}^{SGTD}(r,q)\ dr \\ &= \frac{A(\gamma-1)^{\gamma}\sigma^{2p\gamma-2}}{\pi} \int_{-\infty}^{\infty} \left((\gamma-1)\sigma^{2p} + |\sqrt{r^2+q^2} - \mu|^{2p}\right)^{-\gamma}\ dr. \end{aligned} \tag{18}$$

Finally, the GAF $G_1(\theta)$ is defined as:

$$G_1^{STD}(\theta) = \frac{1}{1 + \Lambda^{GTD}(\theta)}, \tag{19}$$

where

$$\Lambda^{GTD}(\theta) = \frac{1}{\cot\theta} \int_{\cot\theta}^{+\infty} (q - \cot\theta) P_2^{GTD}(q)\ dq. \tag{20}$$

Equation 18 and consequently Equation 20 cannot be computed analytically.

3.3 Importance sampling

Here again, computation leads to invertible *cdf*.

4 Skewed Generalized Normal Distribution (SGND): $\gamma \to \infty$

SGND can be derived from SGTD using Taylor series expansion:

$$\begin{aligned} D^{SGTD}(m) &= \frac{A}{\pi\ \sigma^2 \cos^4\theta \left(1 + \frac{|\tan\theta - \mu|^{2p}}{(\gamma-1)\sigma^{2p}C}\right)^{\gamma}} \\ &= \frac{A}{\pi\ \sigma^2 \cos^4\theta} \left(\sum_{k=0}^{+\infty} \binom{-\gamma}{k} \left(\frac{|\tan\theta - \mu|^{2p}}{C\sigma^{2p}}\right)^k\right) \\ &= \frac{A}{\pi\ \sigma^2 \cos^4\theta} \left(\sum_{k=0}^{+\infty} \frac{\gamma(\gamma+1)\cdots(\gamma+k-1)}{(\gamma-1)^k k!} \left(\frac{-|\tan\theta - \mu|^{2p}}{C\sigma^{2p}}\right)^k\right) \end{aligned} \tag{21}$$

with

$$C = (\lambda\ \text{Sign}\,(\tan\theta - \mu) + 1)^{2p}\,.$$

In the case of SGND where $\lim_{\gamma\to+\infty} \frac{\gamma(\gamma+1)\cdots(\gamma+k-1)}{(\gamma-1)^k} = 1$, we obtain

$$D^{SGND}(m) = \frac{A}{\pi\ \sigma^2 \cos^4\theta} \exp\left(\frac{-|\tan\theta - \mu|^{2p}}{C\sigma^{2p}}\right) \tag{22}$$

This distribution is not *shape invariant* and thus anisotropy cannot be derived directly.

4.1 Normalization factor

The normalization factor leads intractable integral but can be computed for unshifted version with $\mu = 0$. In that case, the normalization factor is given by:

$$A = \frac{p}{(1+\lambda)^2 \Gamma(\frac{1}{p})} \tag{23}$$

4.2 GAF computation

The function λ cannot be analytically obtained with SGND.

4.3 Importance sampling

Importance sampling cannot be derived for the shifted version. However, it can be computed when $\mu = 0$. As SNGD is isotropic, $\varphi_m = 2\pi\xi_1$.

$$\begin{aligned} cdf^{SGTD}(\theta_m) &= \frac{2A}{\sigma^2} \int_{t=0}^{\tan\theta_m} t \exp\left(\frac{-|\tan\theta - \mu|^{2p}}{(1+\lambda)^{2p}\sigma^{2p}}\right)\ dt = \xi_2 \\ &\Rightarrow \frac{-(1+\lambda)^2 A}{p} \left[\Gamma\left(\frac{1}{p}, \frac{t^{2p}}{(1+\lambda)^{2p}\sigma^{2p}}\right)\right]_0^{\tan\theta_m} = \xi_2 \\ &\Rightarrow \Gamma\left(\frac{1}{p}, \frac{\tan^{2p}\theta_m}{(1+\lambda)^{2p}\sigma^{2p}}\right) = \Gamma(1/p)\xi_2 \\ &\Rightarrow \frac{\tan^{2p}\theta_m}{(1+\lambda)^{2p}\sigma^{2p}} = \Gamma_u^{-1}(1/p, \xi_2) \\ &\Rightarrow \theta_m = \arctan\left((1+\lambda)\sigma\Gamma_u^{-1}(1/p, \xi_2)^{\frac{1}{2p}}\right) \end{aligned} \tag{24}$$

with Γ_u^{-1} the inverse of the normalized upper incomplete gamma function.

5 Skewed T-Distribution (STD): $p = 2$

Its expression for the isotropic case (STD is not *shape invariant*) is given by

$$D^{SGTD}(m) = \frac{A}{\pi\ \sigma^2 \cos^4\theta \left(1 + \frac{(|\tan\theta - \mu|)^2}{(\gamma-1)\sigma^2 C}\right)^{\gamma}} \tag{25}$$

with

$$C \quad = \quad (\lambda\ \mathrm{Sign}\,(\tan\theta - \mu) + 1)^2\,.$$

5.1 Normalization factor

Starting from Equations 5 and 6 for SGTD, we have

$$A = \frac{\sigma^2/2}{I_1 + I_2} \tag{26}$$

with

$$\begin{aligned} I_1 &= \mu^2 \left({}_2F_1\left(\gamma, \frac{1}{2}, \frac{3}{2}, \frac{-\mu^2}{\sigma^2(\gamma-1)(1-\lambda)^2}\right) - \frac{1}{2}{}_2F_1\left(\gamma, 1, 2, \frac{-\mu^2}{\sigma^2(\gamma-1)(1-\lambda)^2}\right)\right) \\ I_2 &= \frac{(\gamma-1)(\lambda+1)^2\sigma^2\left(\frac{2\mu}{\sigma\sqrt{\gamma-1}(1+\lambda)}\Gamma(\gamma-1/2)\frac{\sqrt{\pi}}{2} + \Gamma(\gamma-1)\right)}{2\Gamma(\gamma)} \end{aligned} \tag{27}$$

5.2 GAF computation

The function λ cannot be analytically obtained with STD.

5.3 Importance sampling

Computation leads to invertible *cdf*.

6 Generalized Normal Distribution (GND): $\gamma \to \infty$, $\lambda = 0$

For unshifted version of GND, we let the reader to the paper [9] for GAF computation and importance sampling. Its anisotropic expression is

$$D^{GND}(m) = \frac{A}{\pi\ \sigma^2 \cos^4\theta} \exp\left(\frac{-|\tan\theta - \mu|^{2p}}{\sigma^{2p}}\right). \tag{28}$$

This function is *shape invariant* and its anisotropic form is:

$$D^{GND}(m) = \frac{A}{\pi\ \sigma_x\sigma_y \cos^4\theta} \exp\left(-(B(\varphi)^p|\tan\theta - \mu|)^{2p}\right) \tag{29}$$

avec $B(\varphi) = \frac{\cos^2\varphi}{\sigma_x^2} + \frac{\sin^2\varphi}{\sigma_y^2}$.

6.1 Normalization factor

The normalization factor leads intractable integral but can be computed for unshifted version with $\mu = 0$. In that case, the normalization factor is given by:

$$A = \frac{p}{\Gamma(1/p)}. \tag{30}$$

6.2 GAF computation

No analytical GAF computation.

6.3 Importance sampling

Importance sampling cannot be derived directly for the shifted case. As shown in [9] for the unshifted case, ϕ_m can be sampled uniformly over 2π (the obtained *cdf* is not invertible) and the choice of θ_m can be done by importance.

7 Skewed Normal Distribution (SND): $\gamma \to \infty$, $p = 1$

$$D^{SND}(m) = \frac{A}{\pi\ \sigma^2 \cos^4\theta} \exp\left(\frac{-|\tan\theta - \mu|^2}{C\sigma^2}\right) \tag{31}$$

with

$$C = (\lambda\ \text{Sign}\,(\tan\theta - \mu) + 1)^2\,.$$

This distribution is not *shape invariant* in its shifted version and thus anisotropy cannot be derived directly.

7.1 Normalization factor

The normalization factor is given by:

$$A = \frac{\sigma^2}{I} \tag{32}$$

with

$$\begin{aligned} I &= (1+\lambda)^2\sigma^2 + (1-\lambda)^2\sigma^2\left(\exp\left(\frac{-\mu^2}{\sigma^2(1*\lambda)}\right) - 1\right) + (1+\lambda)\sigma\mu\sqrt{\pi} \\ &\quad +(1-\lambda)\sigma\mu\sqrt{\pi}\text{erf}\left(\frac{\mu}{(1-\lambda)\sigma}\right) \end{aligned} \tag{33}$$

7.2 GAF computation

No analytical GAF computation.

7.3 Importance sampling

Importance sampling cannot be derived for the shifted version. However, it can be computed when $\mu = 0$. As SND is isotropic, $\varphi_m = 2\pi\xi_1$.

$$\begin{aligned} cdf^{SND}(\theta_m) &= \frac{2A}{\sigma^2}\int_{t=0}^{\tan\theta_m} t\exp\left(\frac{-|\tan\theta - \mu|^2}{(1+\lambda)^2\sigma^2}\right)\ dt = \xi_2 \\ &= A(1+\lambda)^2 - A(1+\lambda)^2\exp\left(\frac{\tan^2\theta}{\sigma^2(1+\lambda)^2}\right) = \xi_2 \\ &\Rightarrow\ \theta_m = \arctan\left(\frac{-\log(1-\xi_2)}{\sigma^2(1+\lambda)^2}\right). \end{aligned} \tag{34}$$

8 Student's T-Distribution (*Student's* TD): $p = 1$, $\lambda = 0$

For unshifted version of *Student's* TD, we let the reader to the paper [10] for GAF computation and importance sampling.

The isotropic expression of *Student's* TD is:

$$D^{StudTD}(m) = \frac{A}{\pi\ \sigma^2 \cos^4\theta \left(1 + \frac{(|\tan\theta - \mu|)^2}{(\gamma-1)\sigma^2}\right)^{\gamma}}. \tag{35}$$

This function is *shape invariant* and its anisotropic form is:

$$D^{StudTD}(m) = \frac{A}{\pi\ \sigma_x\sigma_y \cos^4\theta \left(1 + \frac{(B(\varphi)|\tan\theta - \mu|)^2}{(\gamma-1)}\right)^{\gamma}} \tag{36}$$

avec $B(\varphi) = \frac{\cos^2\varphi}{\sigma_x^2} + \frac{\sin^2\varphi}{\sigma_y^2}$.

8.1 Normalization factor

$$\begin{aligned}
&\frac{2A}{\sigma^2}\int_{\theta=0}^{\pi/2} \frac{\tan\theta(1+\tan^2\theta)}{\theta\left(1 + \frac{(|\tan\theta-\mu|)^2}{(\gamma-1)\sigma^2}\right)^{\gamma}}\ d\theta = 1 \\
\Rightarrow\quad & A\int_{T=-\mu}^{+\infty} \frac{T-\mu}{\left(1 + \frac{T^2}{(\gamma-1)\sigma^2}\right)^{\gamma}}\ dT = \frac{\sigma^2}{2} \\
\Rightarrow\quad & A = \frac{\sigma^2}{\frac{((\gamma-1)\sigma^2+\mu^2)^{1-\gamma}}{(\gamma-1)^{\gamma+1}\sigma^{2\gamma}} - \frac{\sqrt{\gamma-1}\sigma\mu\Gamma(\gamma-1/2)}{\Gamma(\gamma)} - 2\mu^2\,{}_2F_1(\frac{1}{2},\gamma,\frac{3}{2},\frac{-\mu^2}{(\gamma-1)\sigma^2})}
\end{aligned} \tag{37}$$

For the anisotropic version, $\sigma^2 = \sigma_x\sigma_y$.

8.2 GAF computation

No analytical GAF computation but the λ function can be found in [10] for the unshifted version.

8.3 Importance sampling

Computation leads to invertible *cdf* with the shifted version of *Student's* TD. However, importance sampling can be derived for the unshifted version [10].

9 Shifted Beckmann: $\gamma \to \infty$, $\lambda = 0$, $p = 1$

The isotropic expression of Shifted Beckmann is:

$$D^{ShiftBeck}(m) = \frac{1}{\pi\ \sigma^2 \cos^4} \exp\left(\frac{-|\tan\theta - \mu|^2}{\sigma^2}\right). \tag{38}$$

This function is *shape invariant* and its anisotropic form is:

$$D^{ShiftBeck}(m) = \frac{1}{\pi\ \sigma^2 \cos^4} \exp\left(-B(\varphi)|\tan\theta - \mu|^2\right). \tag{39}$$

avec $B(\varphi) = \frac{\cos^2\varphi}{\sigma_x^2} + \frac{\sin^2\varphi}{\sigma_y^2}$.

9.1 Normalization factor

Starting from the GND expression and fixing $p = 1$, the shifted Beckmann distribution is already normalized.

9.2 GAF computation

No analytical GAF computation but the λ function can be found in [12] for the unshifted version.

9.3 Importance sampling

Computation leads to non invertible cumulative distribution functions whether for $cdf(\varphi_m)$ or $cdf(\theta_m|\varphi_m)$.

10 Shifted GGX: $\gamma = 2$, $\lambda = 0$, $p = 1$

The isotropic expression of Shifted GGX is:

$$D^{ShiftGGX}(m) = \frac{A}{\pi\ \sigma^2 \cos^4\theta \left(1 + \frac{|\tan\theta - \mu|^2}{\sigma^2}\right)^2}. \tag{40}$$

This function is *shape invariant* and its anisotropic form is:

$$D^{ShiftGGX}(m) = \frac{A}{\pi\ \sigma_x\sigma_y \cos^4\theta \left(1 + B(\varphi)|\tan\theta - \mu|^2\right)^2} \tag{41}$$

avec $B(\varphi) = \frac{\cos^2\varphi}{\sigma_x^2} + \frac{\sin^2\varphi}{\sigma_y^2}$.

10.1 Normalization factor

$$\begin{aligned}
& \frac{2A}{\sigma^2} \int_{\theta=0}^{\pi/2} \frac{\tan\theta(1+\tan^2\theta)}{\theta \left(1 + \frac{(|\tan\theta - \mu|)^2}{\sigma^2}\right)^2}\ d\theta = 1 \\
\Rightarrow\quad & A \int_{T=-\mu}^{+\infty} \frac{T+\mu}{\left(1 + \frac{T^2}{\sigma^2}\right)^2}\ dT = \frac{\sigma^2}{2} \\
\Rightarrow\quad & A = \frac{2\sigma^2}{\mu\pi\sigma + 2\sigma^2 + 2\mu\sigma \arctan\left(\frac{\mu}{\sigma}\right)}
\end{aligned} \tag{42}$$

For the anisotropic version, $\sigma^2 = \sigma_x\sigma_y$.

10.2 GAF computation

No analytical GAF computation but the λ function can be found in [12] for the unshifted version.

10.3 Importance sampling

Importance sampling is well known for unshifted GGX [12]. For shifted version, the cumulative distribution function for θ, knowing the φ value is given by:

$$\begin{aligned}
cdf(\theta|\varphi) \quad = \quad & \frac{2A}{\sigma} \int_{T=-\mu}^{\tan\theta - \mu} \frac{T+\mu}{\left(1 + \frac{T^2}{\sigma^2}\right)^2}\ dT = \xi \\
\Rightarrow\quad & \frac{2A}{\sigma} \int_{X=-\mu/\sigma}^{\frac{\tan\theta-\mu}{\sigma}} \frac{\sigma X + \mu}{(1+X^2)^2}\ dX = \xi \\
\Rightarrow\quad & \frac{2A}{\sigma} \left(\frac{\mu\left(\frac{\tan\theta-\mu}{\sigma}\right) - \sigma}{1 + \left(\frac{\tan\theta-\mu}{\sigma}\right)^2} + \mu \arctan\left(\frac{\tan\theta - \mu}{\sigma}\right) + \frac{\mu^2/\sigma + \sigma}{1 + \mu^2/\sigma^2} - \mu \arctan\left(\frac{-\mu}{\sigma}\right) \right) = \xi \\
\Rightarrow\quad & \frac{Y - \frac{\sigma}{\mu}}{1+Y^2} + \arctan Y = \frac{\sigma\xi}{\mu A} - \frac{\sigma}{\mu} + \arctan\frac{\mu}{\sigma}
\end{aligned} \tag{43}$$

with $Y = \frac{\tan\theta - \mu}{\sigma}$. Even if this *cdf* is not directly invertible, efficient strategies (*e.g.* Newton based method with bracketing interval) can be adopted.

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