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Vorticity, Helicity, Intrinsic geometry for Navier-Stokes equations

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Abstract

We will consider the Navier-Stokes equation on a Riemannian manifold M with Ricci tensor bounded below, the involved Laplacian operator is De Rham-Hodge Laplacian. The novelty of this work is to introduce a family of connections which are related to solutions of the Navier-Stokes equation, so that vorticity and helicity can be linked through the associated time-dependent Ricci tensor in intrinsic way in the case where $\dim(M) = 3$.

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Keywords: Vorticity, helicity, intrinsic Ricci tensor, De Rham-Hodge Laplacian, Navier-Stokes equations

1 Introduction

The Navier-Stokes equation in a domain of \mathbb{R}^n is a system of partial differential equations

$$\partial_t u_t + (u_t \cdot \nabla) u_t - \nu \Delta u_t + \nabla p_t = 0, \quad \nabla \cdot u_t = 0, \quad u|_{t=0} = u_0, \quad (1.1)$$

which describes the evolution of the velocity u_t and the pressure p_t of an incompressible viscous fluid with kinematic viscosity $\nu > 0$. The model of periodic boundary conditions for (1.1) on a torus \mathbb{T}^n has been introduced to simplify mathematical considerations. In [14], Navier-Stokes equations on a compact Riemannian manifold M have been considered using the framework of the group of diffeomorphisms of M initiated by V. Arnold in [5]; where the Laplace operator involved in the text of [14] is de Rham-Hodge Laplacian \square , however, the authors said in the note added in proof that the convenient Laplace operator comes from deformation tensor.

In this article, we would like to explore the rich geometry coded in the Navier-Stokes equation on a manifold.

Let ∇ be the Levi-Civita connection on M . For a vector field A on M , the deformation tensor $\text{Def}(A)$ is a symmetric tensor of type $(0, 2)$ defined by

$$(\text{Def } A)(X, Y) = \frac{1}{2} \left(\langle \nabla_X A, Y \rangle + \langle \nabla_Y A, X \rangle \right), \quad X, Y \in \mathcal{X}(M), \quad (1.2)$$

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where $\mathcal{X}(M)$ is the space of vector fields on M . Then $\text{Def} : \text{TM} \rightarrow \text{S}^2\text{T}^*\text{M}$ maps a vector field to a symmetric tensor of type $(0, 2)$. Let $\text{Def}^* : \text{S}^2\text{T}^*\text{M} \rightarrow \text{TM}$ be the adjoint operator. In [32] or in [36] (see page 493), the authors considered the following Laplacian

$$\hat{\square} = 2\text{Def}^* \text{Def}. \quad (1.3)$$

They considered the Navier-Stokes equation with viscosity described by $\hat{\square}$, namely

$$\partial_t u_t + \nabla_{u_t} u_t + \nu \hat{\square} u_t = -\nabla p_t, \quad \text{div}(u_t) = 0, \quad u|_{t=0} = u_0, \quad (1.4)$$

The reader may also refer to [33] in which the author considered the same equation as (1.4) on a complete Riemannian manifold with negative curvature. Variational principles in the class of incompressible Brownian martingales in the spirit of [5] were established recently in [10, 2, 3, 4] for the Navier-Stokes equation (1.4).

In this work, we will be concerned with a complete Riemannian manifold M of dimension n , with Ricci curvature bounded from below. We are interested in the following Navier-Stokes equation on M defined with the De Rham-Hodge Laplacian \square ,

$$\begin{cases} \partial_t u_t + \nabla_{u_t} u_t + \nu \square u_t = -\nabla p_t, \\ \text{div}(u_t) = 0, \quad u|_{t=0} = u_0, \end{cases} \quad (1.5)$$

where $u(x, t)$ denotes the velocity vector field at time t , and $p(x, t)$ models the pressure. If no confusion may arise, we will use u_t (resp. p_t) to denote the vector field $u(\cdot, t)$ (resp. $p(\cdot, t)$) for each t .

There are a few works [26, 38] which support this choice of \square . The probabilistic representation formulae behave better with Navier-Stokes equation (1.5) (see [11, 20, 19]). Our preference here for \square is motivated by its good geometric behavior and its deep links with Stochastic Analysis. See for example [6, 7, 8, 12, 13, 15, 18, 17, 22, 25, 27, 31, 34]. From the view of kinetic mechanics, the viscosity effect of a non-homogeneous fluid should be mathematically described by the Bochner Laplacian of the velocity vector field, where the metric tensor describes the local viscosity distribution. On the other hand, the de Rham-Hodge Laplacian operating on one forms is mathematically more appealing. By invoking de Rham-Hodge Laplacian in the model, according to the Bochner identity, one then produces a no-physical additional term which is however linear in the velocity. An additional linear term in the Navier-Stokes equation will not alter the fundamental difficulty, nor to alter the physics of the fluid flows, which justify the use of de Rham-Hodge Laplacian. There is also a good reason too to consider Navier-Stokes equations on manifolds, if one wants to model the global behavior of the pacific ocean climate for example.

Let's first say a few words on the definition of \square on vector fields. There is a one-to-one correspondence between the space of vector fields $\mathcal{X}(M)$ and that of differential 1-forms $\Lambda^1(M)$. Given a vector field A (resp. differential 1-form ω), we shall denote by \tilde{A} (resp. ω^\sharp) the corresponding differential 1-form (resp. vector field). To see more intuitively these correspondences, let's explain on a local chart U : as usual, we denote by $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ the basis of the tangent space $T_x M$ and by $\{dx^1, \dots, dx^n\}$ the dual basis of $T_x^* M$, called the co-tangent space at x , that is, $dx^i(\frac{\partial}{\partial x_j}) = \delta_{ij}$. The inner product in $T_x M$ as well as the one in the dual space $T_x^* M$ will be denoted by $\langle \cdot, \cdot \rangle$, while the duality between $T_x^* M$ and $T_x M$ will be denoted by (\cdot, \cdot) . Set $g_{ij} = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle$. Let u be a vector field on M , on U , $u = \sum_{i=1}^n u_i \frac{\partial}{\partial x_i}$,

then \tilde{u} admits the expression

$$\tilde{u} = \sum_{i=1}^n \left(\sum_{j=1}^n g_{ij} u_j \right) dx^i.$$

Let $g^{ij} = \langle dx^i, dx^j \rangle$. Then the matrix (g^{ij}) is the inverse matrix of (g_{ij}) . For a differential 1-form $\omega = \sum_{j=1}^n \omega_j dx^j$, the associated vector field $\omega^\#$ has the expression

$$\omega^\# = \sum_{i=1}^n \left(\sum_{\ell=1}^n g^{i\ell} \omega_\ell \right) \frac{\partial}{\partial x_i}.$$

Concisely

$$(\omega, A) = \langle \omega^\#, A \rangle = \langle \omega, \tilde{A} \rangle, \quad A \in \mathcal{X}(M), \quad \omega \in \Lambda^1(M).$$

Now for $A \in \mathcal{X}(M)$, the De-Rham Hodge Laplacian $\square A$ is defined by

$$\square A = (\square \tilde{A})^\#, \quad \square = dd^* + d^*d,$$

where d^* is adjoint operator of exterior derivative d . Then we have the following relation

$$\int_M (\square \omega, A) dx = \int_M \langle \square \omega, \tilde{A} \rangle dx = \int_M \langle \omega, \square \tilde{A} \rangle dx = \int_M (\omega, \square A) dx$$

where dx denotes the Riemannian measure on M . The classical Bochner-Weitzenböck reads as

$$\square A = -\Delta A + \text{Ric}(A), \quad A \in \mathcal{X}(M), \quad (1.6)$$

where Ric is the Ricci tensor on M and $\Delta A = \text{Trace}(\nabla \nabla A)$, characterized by

$$-\int_M \langle \Delta A, A \rangle dx = \int_M |\nabla A|^2 dx. \quad (1.7)$$

Let $T : \mathcal{X}(M) \rightarrow \mathcal{X}(M)$ be a tensor of type $(1, 1)$, and denote by $T^\# : \Lambda^1(M) \rightarrow \Lambda^1(M)$ its adjoint defined by

$$(T^\# \omega, A) = (\omega, T(A)), \quad A \in \mathcal{X}(M), \quad (1.8)$$

where we used notation $\Lambda^p(M)$ to denote the space of differential p -forms on M .

In the space of \mathbb{R}^3 , the inner product between two vectors u, v will be noted by $u \cdot v$. The vorticity ξ_t of a velocity u_t is a vector field defined as $\xi_t = \nabla \times u_t$. When u_t is a solution to Navier-Stokes equation (1.1), then ξ_t satisfies the following heat equation

$$\frac{d\xi_t}{dt} + \nabla_{u_t} \xi_t - \nu \Delta \xi_t = \nabla_{\xi_t}^s u_t \quad (1.9)$$

where $\nabla^s u_t$ is the symmetric part of ∇u_t , such that $\nabla_{\xi_t}^s u_t \cdot v = \text{Def } u_t(\xi_t, v)$ with Def introduced in (1.2). How to interpret the term $\nabla_{\xi_t}^s u_t$? From (1.9), a formal computation leads to

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\xi_t|^2 dx + \nu \int_{\mathbb{R}^3} |\nabla \xi_t|^2 dx = \int_{\mathbb{R}^3} \text{Def}(u_t)(\xi_t, \xi_t) dx. \quad (1.10)$$

Since K. Itô introduced the tool of stochastic parallel translations along paths of Brownian motion on a Riemannian manifold, especially after the works by Eells, Elworthy, Malliavin and Bismut (see for example [31, 16, 8]), there are profound involvements of Stochastic Analysis in the study of linear second order partial differential equations and in Riemannian geometry [6, 34, 25, 29]. The purpose of this work is to geometrically explain the right hand side of (1.10). To this end, we will consider Navier-Stokes equation in a geometric framework in order that suitable geometric meaning could be found.

In what follows, we present the organisation of the paper and main results. In Section 2, first we follow more or less the exposition of [36]. To a solution u_t to Navier-Sokes equaion (1.5), we associate a differential 2-form $\tilde{\omega}_t$ which is the exterior derivative of \tilde{u}_t ; a heat equation for $\tilde{\omega}_t$ will be established with involvement of $\nabla^s u_t$. When M is of dimension 3, the Hodge star $*$ operator sends $\tilde{\omega}_t$ to a differential 1-form ω_t . In flat case of \mathbb{R}^3 , $\omega_t = \widetilde{\nabla \times u_t}$. We call such ω_t the vorticity of u_t ; a heat equation for ω_t is also obtained in Section 2. In second part of Section 2, the a priori evolution equation for ω_t is established. Using heat semi-group $e^{-t\Delta}$ on differential forms as well as Bismut formulae, the existence of weak solutions in the sense of Leray to Navier-Stokes equation (1.5) over any intervall $[0, T]$ is proved under suitable hypothesis on boundedness of Ricci tensor : to our knowledge, these results are new while comparing to recent results obtained in [33]. In Section 3, we give an exposition of the involvement of Stochastic Analysis on Riemannian manifolds; stochastic differential equations on M , defining the Brownian motion with drift $u \in L^2([0, T], H^1(M))$ of divergence free is proved to be stochastic complete; then ω_t admits a probabilistic representation. By introducing a suitable metric compatible affine connection on M , a Brownian motion with drift u on M can be obtained by rolling without friction flat Brownian motion of \mathbb{R}^n on M with respect to it : it was a main idea in [31, 16], and well developed in [25]. So to a velocity u_t , we associate a metric compatible connection ∇^t on M , which admits the following global expression

$$\nabla_X^t Y = \nabla_X Y - \frac{2}{n-1} K_t(X, Y), \quad X, Y \in \mathcal{X}(M)$$

where $K_t(X, Y) = \langle Y, u_t \rangle X - \langle X, Y \rangle u_t$: it gives rise to a connection with torsion T^t which is not of skew-symmetric. Section 4 is devoted to compute the associated intrinsic Ricci tensor $\widehat{\text{Ric}}^t$ which was first introduced by B. Driver in [12] as follows:

$$\widehat{\text{Ric}}^t(X) = \text{Ric}^t(X) + \sum_{i=1}^n (\nabla_{e_i}^t T^t)(X, e_i),$$

where Ric^t is the Ricci tensor associated to ∇^t and $\{e_1, \dots, e_n\}$ is an orthonormal basis at tangent spaces. The formula (1.10) has the following geometric counterpart for 3D Riemannian manifold M ,

$$\frac{1}{2} \frac{d}{dt} \int_M |\omega_t|^2 dx + \nu \int_M |\nabla \omega_t|^2 dx = \frac{1}{2\nu} \int_M (\omega_t, u_t)^2 dx - \nu \int_M (\widehat{\text{Ric}}^{t, \#} \omega_t, \omega_t) dx. \quad (1.11)$$

As well as vorticity ω_t is not orthogonal to velocity u_t , a phenomenon of helicity (ω_t, u_t) will appear. Formula (1.11) says how helicity and intrinsic Ricci tensor fit into the evolution of vorticity in time and in space. Section 5 is devoted to interpretation of main results obtained in Section 4 in the framework of vector calculus. Finally in Section 6, we collect and prove technical results used previously.

2 Vorticity, Helicity and their evolution equations

Let u_t be a (smooth) solution to the Navier-Stokes equation on M ,

$$\partial_t u_t + \nabla_{u_t} u_t + \nu \square u_t = -\nabla p_t, \quad \operatorname{div}(u_t) = 0, \quad u|_{t=0} = u_0. \quad (2.1)$$

Transforming Equation (2.1) into differential forms, \tilde{u}_t satisfies

$$\begin{cases} \partial_t \tilde{u}_t + \nabla_{u_t} \tilde{u}_t + \nu \square \tilde{u}_t = -dp_t, \\ d^* \tilde{u}_t = 0, \quad \tilde{u}|_{t=0} = \tilde{u}_0. \end{cases} \quad (2.2)$$

Let

$$\tilde{\omega}_t = d\tilde{u}_t, \quad (2.3)$$

which is a differential 2-form. For vector fields X, v on M , Lie derivative \mathcal{L} satisfies the product rule, that is,

$$\mathcal{L}_v(\tilde{u}, X) = (\mathcal{L}_v \tilde{u}, X) + (\tilde{u}, \mathcal{L}_v X),$$

where

$$\mathcal{L}_v(\tilde{u}, X) = (\nabla_v \tilde{u}, X) + (\tilde{u}, \nabla_v X).$$

By taking $v = u$, we get

$$(\mathcal{L}_u \tilde{u} - \nabla_u \tilde{u}, X) = (\tilde{u}, \nabla_u X - \mathcal{L}_u X) = (\tilde{u}, \nabla_X u) = \langle u, \nabla_X u \rangle = \frac{1}{2}(d|u|^2, X)$$

which yields that

$$\mathcal{L}_u \tilde{u} - \nabla_u \tilde{u} = \frac{1}{2}d|u|^2. \quad (2.4)$$

By definition $\mathcal{L}_u = i_u d + di_u$ where i_u denotes the interior product by u , so the exterior derivative d commutes with \mathcal{L}_u since $d\mathcal{L}_u = \mathcal{L}_u d = di_u d$, and therefore by using (2.4),

$$d\nabla_u \tilde{u} = d\mathcal{L}_u \tilde{u} = \mathcal{L}_u d\tilde{u}.$$

It is obvious that $\square d = d\square$. Then by acting d on the two sides of (2.2), we get

$$\begin{cases} \partial_t \tilde{\omega}_t + \mathcal{L}_{u_t} \tilde{\omega}_t + \nu \square \tilde{\omega}_t = 0, \\ \tilde{\omega}|_{t=0} = \tilde{\omega}_0. \end{cases} \quad (2.5)$$

Remark 2.1. Since $d^* \tilde{u} = 0$, by definition (2.3), $d^* \tilde{\omega} = d^* d\tilde{u} = \square \tilde{u}$, and therefore, as \square admits a spectral gap, \tilde{u} can be solved by

$$\tilde{u} = \square^{-1}(d^* \tilde{\omega}).$$

It is sometimes more convenient to use covariant derivatives. To do this, let β be a differential p -form and $T : \mathcal{X}(M) \rightarrow \mathcal{X}(M)$ be a tensor of type $(1, 1)$. Define for X_1, \dots, X_p ,

$$(\beta \triangleleft T)(X_1, \dots, X_p) = \beta(T(X_1), X_2, \dots, X_p) + \dots + \beta(X_1, \dots, X_{p-1}, T(X_p)). \quad (2.6)$$

If β is a 2-form and $T = \nabla u$, then for $X, Y \in \mathcal{X}(M)$,

$$(\beta \triangleleft \nabla u)(X, Y) = \beta(\nabla_X u, Y) + \beta(X, \nabla_Y u). \quad (2.7)$$

In the same way as for proving (2.4), we have

$$(\mathcal{L}_v\beta - \nabla_v\beta)(X, Y) = \beta(\nabla_X v, Y) + \beta(X, \nabla_Y v) = (\beta \triangleleft \nabla v)(X, Y).$$

Now replacing $\mathcal{L}_u\tilde{\omega}$ by $\nabla_u\tilde{\omega} + \tilde{\omega} \triangleleft \nabla u$ in (2.5), we obtain the following form

$$\begin{cases} \partial_t\tilde{\omega}_t + \nabla_{u_t}\tilde{\omega}_t + \nu\Box\tilde{\omega}_t = -\tilde{\omega}_t \triangleleft \nabla u_t, \\ \tilde{\omega}|_{t=0} = \tilde{\omega}_0. \end{cases} \quad (2.8)$$

Proposition 2.2. *Let $\nabla^{sk}u$ be the skew-symmetric part of ∇u , that is,*

$$\langle \nabla^{sk}u, X \otimes Y \rangle = \frac{1}{2}(\langle \nabla_X u, Y \rangle - \langle \nabla_Y u, X \rangle).$$

Then $\tilde{\omega} \triangleleft \nabla^{sk}u = 0$.

Proof. Fix $x \in M$ and let $\{e_1, \dots, e_n\}$ be an orthonormal basis of $T_x M$. Then

$$\begin{aligned} \nabla_X^{sk}u &= \sum_{i,j=1}^n \langle \nabla_{e_i}^{sk}u, e_j \rangle \langle X, e_i \rangle e_j \\ &= \sum_{i,j=1}^n d\tilde{u}(e_i, e_j) \langle X, e_i \rangle e_j = \sum_{j=1}^n \tilde{\omega}(X, e_j) e_j, \end{aligned}$$

so that

$$\tilde{\omega}(\nabla_X^{sk}u, Y) = \sum_{j=1}^n \tilde{\omega}(X, e_j) \tilde{\omega}(e_j, Y) = \tilde{\omega}(\nabla_Y^{sk}u, X).$$

Combing these relations and Definition (2.6), we have

$$(\tilde{\omega} \triangleleft \nabla^{sk}u)(X, Y) = \tilde{\omega}(\nabla_X^{sk}v, Y) + \tilde{\omega}(X, \nabla_Y^{sk}v) = 0.$$

□

Let $\nabla^s u$ be the symmetric part of ∇u , that is

$$\langle \nabla^s u, X \otimes Y \rangle = \frac{1}{2}(\langle \nabla_X u, Y \rangle + \langle \nabla_Y u, X \rangle).$$

$\nabla^s u$ is called the rate of strain tensor in the literature on fluid dynamics. Therefore Equation (2.8) can be written in the following form:

$$\begin{cases} \partial_t\tilde{\omega}_t + \nabla_{u_t}\tilde{\omega}_t + \nu\Box\tilde{\omega}_t = -\tilde{\omega}_t \triangleleft \nabla^s u_t, \\ \tilde{\omega}|_{t=0} = \tilde{\omega}_0. \end{cases} \quad (2.9)$$

In the case where $\dim(M) = 2$ or 3 , Equation (2.9) can be simplified using Hodge star operator $*$. Assume that M is oriented and ω_n is the n -form of Riemannian volume, let $\omega = *\tilde{\omega}$, which is a $(n-2)$ form such that

$$\tilde{\omega} \wedge \alpha = \langle \omega, \alpha \rangle_{\Lambda^{n-2}} \omega_n, \quad \text{for any } \alpha \in \Lambda^{n-2}(M),$$

or

$$\beta \wedge *\tilde{\omega} = \langle \tilde{\omega}, \beta \rangle_{\Lambda^2} \omega_n, \quad \text{for any } \beta \in \Lambda^2(M).$$

Proposition 2.3. *Let ω be a p -form on M and $\operatorname{div}(u) = 0$. Then $\nabla_u(*\omega) = *(\nabla_u\omega)$.*

Proof. Let β be a p -form. Then $\beta \wedge *\omega = \langle \beta, \omega \rangle \omega_n$. Taking the covariant derivative with respect to u , the left hand side gives

$$\nabla_u \beta \wedge (*\omega) + \beta \wedge \nabla_u(*\omega) = \langle \nabla_u \beta, \omega \rangle \omega_n + \beta \wedge \nabla_u(*\omega),$$

while the right hand side gives

$$\langle \nabla_u \beta, \omega \rangle \omega_n + \langle \beta, \nabla_u \omega \rangle \omega_n = \langle \nabla_u \beta, \omega \rangle \omega_n + \beta \wedge *\nabla_u \omega$$

as $\nabla_u \omega_n = 0$. Therefore $\beta \wedge \nabla_u(*\omega) = \beta \wedge (*\nabla_u \omega)$ holds for any p -form β , the result follows. \square

Proposition 2.4. *Assume $\dim(M) = 3$. Then*

$$*(\tilde{\omega}_t \lrcorner \nabla^s u) = -(*\tilde{\omega}_t) \lrcorner \nabla^s u. \quad (2.10)$$

Proof. Fix $x \in M$; let $\{e_1, e_2, e_3\}$ be an orthonormal basis of $T_x M$, $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3\}$ be the dual basis of $T_x^* M$. Let $\{i_1, i_2, i_3\}$ be a direct permutation of $\{1, 2, 3\}$, and $\omega = \tilde{e}_{i_1} \wedge \tilde{e}_{i_2}$. Then

$$\begin{aligned} (\omega \lrcorner \nabla^s u)(X, Y) &= \left((\nabla_X^s u)_{i_1} Y_{i_2} - Y_{i_1} (\nabla_X^s u)_{i_2} \right) + \left((\nabla_Y^s u)_{i_2} X_{i_1} - X_{i_2} (\nabla_Y^s u)_{i_1} \right) \\ &= \sum_{j=1}^3 \left[(\nabla_{e_j}^s u)_{i_1} X_j Y_{i_2} - (\nabla_{e_j}^s u)_{i_2} X_j Y_{i_1} + (\nabla_{e_j}^s u)_{i_2} X_{i_1} Y_j - (\nabla_{e_j}^s u)_{i_1} X_{i_2} Y_j \right] \\ &= \sum_{j=1}^3 (\nabla_{e_j}^s u)_{i_1} (X_j Y_{i_2} - X_{i_2} Y_j) + \sum_{j=1}^3 (\nabla_{e_j}^s u)_{i_2} (X_{i_1} Y_j - X_j Y_{i_1}). \end{aligned}$$

It follows that

$$\omega \lrcorner \nabla^s u = \sum_{j=1}^3 (\nabla_{e_j}^s u)_{i_1} \tilde{e}_j \wedge \tilde{e}_{i_2} + \sum_{j=1}^3 (\nabla_{e_j}^s u)_{i_2} \tilde{e}_{i_1} \wedge \tilde{e}_j.$$

More precisely

$$\begin{aligned} \omega \lrcorner \nabla^s u &= (\nabla_{e_{i_1}}^s u)_{i_1} \tilde{e}_{i_1} \wedge \tilde{e}_{i_2} + (\nabla_{e_{i_2}}^s u)_{i_2} \tilde{e}_{i_1} \wedge \tilde{e}_{i_2} \\ &\quad + (\nabla_{e_{i_3}}^s u)_{i_1} \tilde{e}_{i_3} \wedge \tilde{e}_{i_2} + (\nabla_{e_{i_3}}^s u)_{i_2} \tilde{e}_{i_1} \wedge \tilde{e}_{i_3}. \end{aligned}$$

Since $\sum_{j=1}^3 (\nabla_{e_{i_j}}^s u)_{i_j} = \operatorname{Trace}(\nabla u) = \operatorname{div}(u) = 0$, therefore finally we get

$$*(\omega \lrcorner \nabla^s u) = -(\nabla_{e_{i_3}}^s u)_{i_1} \tilde{e}_{i_1} - (\nabla_{e_{i_3}}^s u)_{i_2} \tilde{e}_{i_2} - (\nabla_{e_{i_3}}^s u)_{i_3} \tilde{e}_{i_3}. \quad (2.11)$$

On the other hand, $*\omega = \tilde{e}_{i_3}$, so that

$$(*\omega) \lrcorner (\nabla^s u)(X) = (*\omega)(\nabla_X^s u) = \sum_{j=1}^3 (\nabla_{e_j}^s u)_{i_3} X_j.$$

It follows that

$$(*\omega) \lrcorner (\nabla^s u) = (\nabla_{e_{i_1}}^s u)_{i_3} \tilde{e}_{i_1} + (\nabla_{e_{i_2}}^s u)_{i_3} \tilde{e}_{i_2} + (\nabla_{e_{i_3}}^s u)_{i_3} \tilde{e}_{i_3} \quad (2.12)$$

Now combing (2.11), (2.12), and by symmetry of $\nabla^s u$, we get (2.10). \square

Corollary 2.5. *Let $\dim(M) = 3$ and $\omega_t = *\tilde{\omega}_t$. Then*

$$\partial_t \omega_t + \nabla_u \omega_t + \nu \square \omega_t = \omega_t \lrcorner (\nabla^s u_t). \quad (2.13)$$

Proof. First note that $\square* = *\square$ (see [40], p. 221), so (2.13) follows from Proposition 2.3 and Proposition 2.4. \square

Remark 2.6. *Since $** = (-1)^{p(n-p)}$ on p -form, so for $n = 3$, $\tilde{\omega}_t = *\omega_t$ and in the case where \square admits a spectral gap, the following relation holds*

$$\tilde{u}_t = \square^{-1}(d^*(\omega_t)). \quad (2.14)$$

Proposition 2.7. *In the smooth case, it holds*

$$\frac{1}{2} \frac{d}{dt} \int_M |u_t|^2 dx + \nu \int_M |\nabla u_t|^2 dx = -\nu \int_M \langle \text{Ric } u_t, u_t \rangle dx. \quad (2.15)$$

Proof. Remark first that $\int_M \langle \nabla_{u_t} u_t, u_t \rangle dx = \frac{1}{2} \int_M \mathcal{L}_{u_t} |u_t|^2 dx = 0$ and $\int_M \langle \nabla p, u_t \rangle dx = 0$. Then using equation (2.1), we get

$$\frac{1}{2} \frac{d}{dt} \int_M |u_t|^2 dx + \nu \int_M \langle \square u_t, u_t \rangle dx = 0.$$

Now using Bochner-Weitzenböck formula (1.6) and (1.7) yields (2.15). \square

Proposition 2.8. *Assume that there exists a constant $\kappa \in \mathbb{R}$ such that*

$$\text{Ric} \geq -\kappa. \quad (2.16)$$

Then the following a priori estimate holds

$$\frac{1}{2} \|u_t\|_2^2 + \nu \int_0^t \|\nabla u_s\|_2^2 ds \leq \frac{1}{2} \|u_0\|_2^2 \exp(2\nu t \kappa^+), \quad (2.17)$$

where $\kappa^+ = \sup\{\kappa, 0\}$.

Proof. Using (2.16) and (2.15), we get inequality

$$\frac{1}{2} \frac{d}{dt} \int_M |u_t|^2 dx + \nu \int_M |\nabla u_t|^2 dx \leq \nu \kappa \int_M |u_t|^2 dx \leq \nu \kappa^+ \int_M |u_t|^2 dx.$$

Let $\psi(t) = \frac{1}{2} \|u_t\|_2^2 + \nu \int_0^t \|\nabla u_s\|_2^2 ds$. Then ψ satisfies inequality

$$\psi(t) \leq \frac{1}{2} \|u_0\|_2^2 + 2\nu \kappa^+ \int_0^t \psi(s) ds.$$

Gronwall lemma yields (2.17). \square

In what follows, we will establish the existence of weak solutions in Leray sense over any $[0, T]$ and

$$u \in L^2([0, T], H^1(M)) \cap L^\infty([0, T], L^2(M)).$$

To this end, we will use the heat semi-group $\mathbf{T}_t = e^{-t\square/2}$ to regularize vector fields. Let v be a continuous vector field on M with compact support and define $\mathbf{T}_t v = (\mathbf{T}_t \tilde{v})^\#$. Then $\mathbf{T}_t v$ solves the heat equation

$$\left(\frac{\partial}{\partial t} + \frac{1}{2}\square\right)(\mathbf{T}_t v) = 0.$$

By ellipticity of \square (see for example [40]), $(t, x) \rightarrow (\mathbf{T}_t v)(x)$ is smooth. It was shown in [21] that

$$\operatorname{div}(\mathbf{T}_t v) = \mathbf{T}_t^M(\operatorname{div}(v)),$$

where \mathbf{T}_t^M denotes heat semi-group on functions. Hence \mathbf{T}_t preserves the space of divergence free vector fields. By (6.14) in Section 6 it holds true that

$$|\mathbf{T}_t v| \leq e^{t\kappa+/2} \mathbf{T}_t^M |v|. \quad (2.18)$$

It follows that for $1 \leq p \leq +\infty$, $\|\mathbf{T}_t v\|_p \leq e^{t\kappa+/2} \|v\|_p$, and for $1 \leq p < +\infty$, $\mathbf{T}_t v \rightarrow v$ in L^p .

Consider a family of smooth functions $\varphi_\varepsilon \in C_c^\infty(M)$ with compact support such that

$$0 \leq \varphi_\varepsilon \leq 1, \quad \varphi_\varepsilon(x) = 1 \text{ for } x \in B(x_M, 1/\varepsilon) \quad \text{and} \quad \sup_{\varepsilon > 0} \|\nabla \varphi_\varepsilon\|_\infty < +\infty, \quad (2.19)$$

where x_M is a fixed point of M . For $\varepsilon > 0$, we define

$$F_\varepsilon(u) = -\mathbf{T}_\varepsilon \mathbf{P}(\varphi_\varepsilon \nabla_{\mathbf{T}_\varepsilon u}(\varphi_\varepsilon \mathbf{T}_\varepsilon u)) - \nu \mathbf{T}_\varepsilon \square \mathbf{T}_\varepsilon u, \quad u \in L^2(M)$$

where \mathbf{P} is the orthogonal projection from $L^2(M)$ to the subspace of vector fields of divergence free. We have

$$\|\mathbf{T}_\varepsilon \mathbf{P}(\varphi_\varepsilon \nabla_{\mathbf{T}_\varepsilon u}(\varphi_\varepsilon \mathbf{T}_\varepsilon u))\|_2 \leq e^{\varepsilon\kappa+ / 2} \|\mathbf{P}(\varphi_\varepsilon \nabla_{\mathbf{T}_\varepsilon u}(\varphi_\varepsilon \mathbf{T}_\varepsilon u))\|_2 \leq e^{\varepsilon\kappa+ / 2} \|\nabla_{\varphi_\varepsilon \mathbf{T}_\varepsilon u}(\varphi_\varepsilon \mathbf{T}_\varepsilon u)\|_2.$$

Since φ_ε is of compact support, we have

$$\|\nabla_{\varphi_\varepsilon \mathbf{T}_\varepsilon u}(\varphi_\varepsilon \mathbf{T}_\varepsilon u)\|_2 \leq \|\varphi_\varepsilon \mathbf{T}_\varepsilon u\|_\infty \|\nabla(\varphi_\varepsilon \mathbf{T}_\varepsilon u)\|_2. \quad (2.20)$$

Again due to compact support of φ_ε , when $n = 3$, by Sobolev's embedding theorem, there is a constant $\beta(\varepsilon) > 0$ such that

$$\|\varphi_\varepsilon \mathbf{T}_\varepsilon u\|_\infty \leq \beta(\varepsilon) \|\varphi_\varepsilon \mathbf{T}_\varepsilon u\|_{H^2}.$$

For the general case, it is sufficient to bound the uniform norm by the norm of H^m with $m > \frac{n}{2}$.

Proposition 2.9. *For any $T > 0$, there are constants β_1, β_2 such that*

$$\|\square \mathbf{T}_\varepsilon u\|_2 \leq \frac{\beta_1}{\varepsilon} \|u\|_2, \quad \|\nabla \mathbf{T}_\varepsilon u\|_2 \leq \frac{\beta_2}{\sqrt{\varepsilon}}, \quad \varepsilon > 0. \quad (2.21)$$

Proof. We will restate, in Section 6, (2.21) with more precise coefficients dependent of curvatures of M and give a proof based on Bismut formulae obtained in [18, 13]. \square

By Proposition 2.9, there are constants $\beta(\varepsilon) > 0, \tilde{\beta}(\varepsilon) > 0$ such that

$$\|\varphi_\varepsilon \mathbf{T}_\varepsilon u\|_\infty \leq \beta(\varepsilon) \|u\|_2, \quad \|\mathbf{T}_\varepsilon \square \mathbf{T}_\varepsilon u\|_2 \leq \tilde{\beta}(\varepsilon) \|u\|_2. \quad (2.22)$$

Combining (2.20) and (2.22), there are two constants $\beta_1(\varepsilon) > 0$ and $\beta_2(\varepsilon) > 0$ such that

$$\|F_\varepsilon(u)\|_2 \leq \beta_1(\varepsilon) \|u\|_2^2 + \beta_2(\varepsilon) \|u\|_2,$$

and F_ε is locally Lipschitz. By theory of ordinary differential equation, there is a unique solution u^ε to

$$\frac{du_t^\varepsilon}{dt} = F_\varepsilon(u_t^\varepsilon), \quad u_0^\varepsilon = u_0 \in L^2, \quad \operatorname{div}(u_t^\varepsilon) = 0, \quad (2.23)$$

up to the explosion time τ .

Theorem 2.10. *Assume that $\|\operatorname{Ric}\|_\infty < +\infty$ and that \mathcal{R}_2 is bounded below. Then for any $T > 0$, there is a weak solution $u \in L^2([0, T], H^1)$ to Navier-Stokes equation (2.1) such that*

$$\frac{1}{2}\|u_t\|_2^2 + \nu \int_0^t \|\nabla u_s\|_2^2 ds \leq \frac{1}{2}\|u_0\|_2^2 \exp(2\nu t \kappa^+),$$

where κ is lower bound of Ric and \mathcal{R}_2 is the Weitzenböck curvature on 2-differential forms defined in (6.8).

Proof. Rewriting Equation (2.23) in the following explicit form, for $t < \tau$,

$$\frac{du_t^\varepsilon}{dt} + \mathbf{T}_\varepsilon \mathbf{P}(\varphi_\varepsilon \nabla_{\mathbf{T}_\varepsilon u_t^\varepsilon}(\varphi_\varepsilon \mathbf{T}_\varepsilon u_t^\varepsilon)) + \nu \mathbf{T}_\varepsilon \square \mathbf{T}_\varepsilon u_t^\varepsilon = 0.$$

Note that

$$\begin{aligned} \int_M \langle \mathbf{T}_\varepsilon \mathbf{P}(\varphi_\varepsilon \nabla_{\mathbf{T}_\varepsilon u_t^\varepsilon}(\varphi_\varepsilon \mathbf{T}_\varepsilon u_t^\varepsilon)), u_t^\varepsilon \rangle dx &= \int_M \langle \nabla_{\mathbf{T}_\varepsilon u_t^\varepsilon}(\varphi_\varepsilon \mathbf{T}_\varepsilon u_t^\varepsilon), \varphi_\varepsilon \mathbf{T}_\varepsilon u_t^\varepsilon \rangle dx \\ &= \int_M \mathcal{L}_{\mathbf{T}_\varepsilon u_t^\varepsilon} |\varphi_\varepsilon \mathbf{T}_\varepsilon u_t^\varepsilon|^2 dx = 0, \end{aligned}$$

since $\operatorname{div}(\mathbf{T}_\varepsilon u_t^\varepsilon) = 0$, and

$$\int_M \langle \mathbf{T}_\varepsilon \square \mathbf{T}_\varepsilon u_t^\varepsilon, u_t^\varepsilon \rangle dx = \int_M |\nabla \mathbf{T}_\varepsilon u_t^\varepsilon|^2 dx + \int_M \langle \operatorname{Ric}(\mathbf{T}_\varepsilon u_t^\varepsilon), \mathbf{T}_\varepsilon u_t^\varepsilon \rangle dx.$$

Hence

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_M |u_t^\varepsilon|^2 dx + \nu \int_M |\nabla \mathbf{T}_\varepsilon u_t^\varepsilon|^2 dx &= -\nu \int_M \langle \operatorname{Ric}(\mathbf{T}_\varepsilon u_t^\varepsilon), \mathbf{T}_\varepsilon u_t^\varepsilon \rangle dx \\ &\leq -\nu \kappa \int_M |\mathbf{T}_\varepsilon u_t^\varepsilon|^2 dx, \end{aligned}$$

or in the form

$$\frac{1}{2}\|u_t^\varepsilon\|_2^2 + \nu \int_0^t \|\nabla \mathbf{T}_\varepsilon u_s^\varepsilon\|_2^2 ds \leq \frac{1}{2}\|u_0\|_2^2 + \nu \kappa^+ \int_0^t \|\mathbf{T}_\varepsilon u_s^\varepsilon\|_2^2 ds. \quad (2.24)$$

According to (2.18), above inequality implies that

$$\frac{1}{2}\|u_t^\varepsilon\|_2^2 \leq \frac{1}{2}\|u_0\|_2^2 + \nu \kappa^+ e^{\varepsilon \kappa^+} \int_0^t \|u_s^\varepsilon\|_2^2 ds.$$

Gronwall lemma implies that for $t < \tau$

$$\frac{1}{2}\|u_t^\varepsilon\|_2^2 \leq \frac{1}{2}\|u_0\|_2^2 \exp(t\nu \kappa^+ e^{\varepsilon \kappa^+}).$$

It follows that $\tau = +\infty$. Now again by (2.18) and (2.24), we get

$$\frac{1}{2}\|\mathbf{T}_\varepsilon u_t^\varepsilon\|_2^2 + \nu e^{\varepsilon \kappa^+} \int_0^t \|\nabla \mathbf{T}_\varepsilon u_s^\varepsilon\|_2^2 ds \leq \frac{1}{2} e^{\varepsilon \kappa^+} \|u_0\|_2^2 + \nu \kappa^+ e^{\varepsilon \kappa^+} \int_0^t \|\mathbf{T}_\varepsilon u_s^\varepsilon\|_2^2 ds.$$

Gronwall lemma yields, for $\varepsilon \leq 1$, that

$$\frac{1}{2} \|\mathbf{T}_\varepsilon u_t^\varepsilon\|_2^2 + \nu e^{\varepsilon\kappa^+} \int_0^t \|\nabla \mathbf{T}_\varepsilon u_s^\varepsilon\|_2^2 ds \leq \frac{e^{\kappa^+}}{2} \|u_0\|_2^2 \exp(t\nu\kappa^+ e^{\kappa^+}). \quad (2.25)$$

Let $T > 0$. By (2.25), the family $\{\mathbf{T}_\varepsilon u^\varepsilon; \varepsilon \in (0, 1]\}$ is bounded in $L^2([0, T], H^1)$ as well in $L^\infty([0, T], L^2)$. Then there is a sequence ε_n and a $u \in L^2([0, T], H^1) \cap L^\infty([0, T], L^2)$ such that $\mathbf{T}_{\varepsilon_n} u^{\varepsilon_n}$ converges weakly to u in $L^2([0, T], H^1)$ and $*$ -weakly in $L^\infty([0, T], L^2)$. Now standard arguments allow to prove that u is a weak solution (2.1). The boundedness of Ric is needed while passing to the limit of the term $\int_M \langle \text{Ric}(\mathbf{T}_\varepsilon u_t^\varepsilon), v_t \rangle dx$. \square

Proposition 2.11. *Let $\dim(M) = 3$. The vorticity ω_t satisfies a priori identity:*

$$\frac{1}{2} \frac{d}{dt} \int_M |\omega_t|^2 dx + \nu \int_M |\nabla \omega_t|^2 dx = -\nu \int_M \langle \text{Ric} \omega_t, \omega_t \rangle dx + \int_M \langle \omega_t \lrcorner \nabla^s u_t, \omega_t \rangle dx. \quad (2.26)$$

Proof. Using Equation (2.13) and the same as proving (2.15) yields (2.26). \square

The term $H_t := \int_M (\omega_t, u_t) dx$ is called helicity in theory of the fluid mechanics.

Proposition 2.12. *Let $\dim(M) = 3$. Then*

$$\begin{aligned} \frac{d}{dt} \int_M (\omega_t, u_t) dx &= -\nu \int_M \langle d\omega_t, *\omega_t \rangle_{\Lambda^2} dx - \nu \int_M (\nabla \omega_t, \nabla u_t) dx \\ &\quad - \nu \int_M (\omega_t, \text{Ric} u_t) dx + \int_M (\omega_t, \nabla_{u_t}^s u_t) dx. \end{aligned} \quad (2.27)$$

Proof. Using Equation (2.1) and Equation (2.13), we have

$$\begin{aligned} \frac{d}{dt} (\omega_t, u_t) &= -(\nabla_{u_t} \omega_t, u_t) - \nu (\square \omega_t, u_t) + (\omega_t \lrcorner \nabla^s u_t, u_t) \\ &\quad - (\omega_t, \nabla_{u_t} u_t) - \nu (\omega_t, \square u_t) - (\omega_t, \nabla p). \end{aligned}$$

It is obvious that

$$\int_M [(\nabla_{u_t} \omega_t, u_t) + (\omega_t, \nabla_{u_t} u_t)] dx = \int_M \mathcal{L}_{u_t} (\omega_t, u_t) dx = 0.$$

In addition, by ([40], page 220), $d^* = (-1)^{n(p+1)+1} * d^*$ and $** = (-1)^{p(n-p)}$ on p -forms. Then $d^* * = \pm * d$, so that

$$\int_M \langle \omega_t, dp \rangle dx = \int_M \langle *\tilde{\omega}_t, dp \rangle dx = \int_M d^* (*\tilde{\omega}_t) p dx = \pm \int_M *(d\tilde{\omega}_t) p dx = 0.$$

On one hand, using Hodge star operator,

$$\int_M (\omega_t, \square u_t) dx = \int_M \langle \omega_t, d^* d\tilde{u}_t \rangle dx = \int_M \langle d\omega_t, \tilde{\omega}_t \rangle dx = \int_M \langle d\omega_t, *\omega_t \rangle dx.$$

On the other hand, using Bochner-Weitzenböck formula,

$$\int_M (\omega_t, \square u_t) dx = \int_M (\nabla \omega_t, \nabla u_t) dx + \int_M (\omega_t, \text{Ric} u_t) dx.$$

By putting these terms together we conclude that

$$\begin{aligned} \frac{d}{dt} \int_M \langle \omega_t, u_t \rangle dx &= -\nu \int_M \langle d\omega_t, * \omega_t \rangle dx - \nu \int_M \langle \nabla \omega_t, \nabla u_t \rangle dx \\ &\quad - \nu \int_M \langle \omega_t, \text{Ric } u_t \rangle dx + \int_M \langle \omega_t, \nabla_{u_t}^s u_t \rangle dx, \end{aligned}$$

since $(\omega_t \triangleleft \nabla^s u_t, u_t) = (\omega_t, \nabla_{u_t}^s u_t)$. We get (2.27). \square

3 Heat equations on differential forms

We will express solutions to equation (2.13) by means of principal bundle of orthonormal frames $O(M)$. An element $r \in O(M)$ is an isometry from \mathbb{R}^n onto $T_{\pi(r)}M$ where $\pi : O(M) \rightarrow M$ is the canonical projection. More precisely, an element of $O(M)$ is composed of (x, r) , where $x = \pi(x, r)$ and r is an orthonormal frame at x , that is, an isometry from \mathbb{R}^n onto $T_x M$. For the sake of simplicity, we read r as $(\pi(r), r)$, but we sometimes have to distinguish them. The Levi-Civita connection on M gives rise to n canonical horizontal vector fields $\{A_1, \dots, A_n\}$ on $O(M)$, which are such that $d\pi(r) \cdot A_r = r\varepsilon_i$, where $\{\varepsilon_1, \dots, \varepsilon_n\}$ is the canonical basis of \mathbb{R}^n . A vector field v on M can be lift to a horizontal vector field V on $O(M)$ such that $d\pi(r)V_r = v_{\pi(r)}$. Let ω be a differential 1-form. Following Malliavin [31], we define

$$F_\omega^i(r) = (\omega_{\pi(r)}, r\varepsilon_i) = (\pi^* \omega, A_i)_r, \quad i = 1, \dots, n, \quad (3.1)$$

where $\pi^* \omega$ is the pull-back of ω by $\pi : O(M) \rightarrow M$. We have

$$(\mathcal{L}_{A_j} F_\omega^i)(r) = (\nabla_{r\varepsilon_j} \omega, r\varepsilon_i) = (\nabla \omega, r\varepsilon_j \otimes r\varepsilon_i), \quad (3.2)$$

where the second duality makes sense in $T_{\pi(r)}M \otimes T_{\pi(r)}M$. In fact, let $t \rightarrow r(t) \in O(M)$ be the smooth curve such that $r(0) = r, r'(0) = A_j(r)$. Let $\xi_t = \pi(r(t))$. Then $//_t^{-1} := r \circ r(t)^{-1}$ is the parallel translation from $T_{\xi_t}M$ onto $T_x M$ along ξ . and

$$F_\omega^i(r(t)) = (\omega_{\xi_t}, r(t)\varepsilon_i) = (//_t^{-1} \omega_{\xi_t}, r\varepsilon_i).$$

Taking the derivative with respect to t at $t = 0$ yields (3.2). In the same way, we get $(\mathcal{L}_{A_j}^2 F_\omega^i)(r) = (\nabla_{r\varepsilon_j} \nabla \omega, r\varepsilon_j \otimes r\varepsilon_i)$. Therefore

$$\Delta_{O(M)} F_\omega^i := \sum_{j=1}^n \mathcal{L}_{A_j}^2 F_\omega^i = (\Delta \omega, r\varepsilon_i) = F_{\Delta \omega}^i(r).$$

Let U_t be the horizontal lift of u_t to $O(M)$. Then $U_t(r) = \sum_{j=1}^n \langle u_t(x), r\varepsilon_j \rangle A_j(r)$, where $x = \pi(r)$ and according to (3.2),

$$(\mathcal{L}_{U_t} F_\omega^i)(r) = \sum_{j=1}^n \langle u_t, r\varepsilon_j \rangle (\mathcal{L}_{A_j} F_\omega^i)(r) = \langle \nabla_{u_t} \omega, r\varepsilon_i \rangle = F_{\nabla_{u_t} \omega}^i(r).$$

Let $\phi_t = \omega_t \triangleleft \nabla^s u$; then

$$F_{\phi_t}^i(r) = (\phi_t, r\varepsilon_i) = \omega_t(\nabla_{r\varepsilon_i}^s u_t) = \sum_{j=1}^n \langle \nabla_{r\varepsilon_i}^s u_t, r\varepsilon_j \rangle (\omega_t, r\varepsilon_j) = \sum_{j=1}^n \langle \nabla_{r\varepsilon_i}^s u_t, r\varepsilon_j \rangle F_{\omega_t}^j.$$

Define $K_{ij}(t, r) = \langle \nabla_{r\varepsilon_i}^s u_t(\pi(r)), r\varepsilon_j \rangle$ and $K(t, r) = (K_{ij}(t, r))$. Then $F_{\phi_t}(r) = K(t, r)F_{\omega_t}(r)$. By applying Bochner-Weitzenböck formula (see (1.6)) to 1-form ω , $\square\omega = -\Delta\omega + \text{Ric}^\#\omega$. Let $\text{ric}_r = r^{-1}\text{Ric}_{\pi(r)}r$ denote the equivariant representation of Ric on $O(M)$. Then $F_{\text{Ric}^\#\omega} = \text{ric}F_\omega$, since ric is symmetric. Now applying F on two sides of Equation (2.13), we get the following heat equation defined on $O(M)$, but taking values in flat space \mathbb{R}^n :

$$\frac{d}{dt}F_{\omega_t} = \nu\Delta_{O(M)}F_{\omega_t} - \mathcal{L}_{U_t}F_{\omega_t} + (K(t, \cdot) - \nu \text{ric})F_{\omega_t}. \quad (3.3)$$

This equation was extensively studied in the field of Stochastic analysis, see [6, 7, 8, 15, 17, 25, 27, 31, 34] for example. However the situation becomes more complicated when the vector field is time-dependent (see [35]).

In what follows, we will derive a stochastic representation formula for the solution to (3.3). First of all, we have to prove that the concerned diffusion processes do not explode at a finite time. For this purpose, consider a family of vector fields $\{v_t(x); t \geq 0\}$ on M . We assume here that $(t, x) \rightarrow v_t(x)$ is continuous and for each $t \geq 0$, $v_t \in C^{1+\alpha}$ with $\alpha > 0$, and $\text{div}(v_t) = 0$. Let V_t be the horizontal lift of v_t to $O(M)$. Then $\text{div}(V_t) = \text{div}(v_t) \circ \pi$ (see [21], 595) and therefore $\text{div}(V_t) = 0$.

Consider the following Stratonovich stochastic differential equation (SDE)

$$dr_t = \sum_{k=1}^n A_k(r_t) \circ dW_t^k + V_t(r_t)dt, \quad r|_{t=0} = r_0 \quad (3.4)$$

where $W_t = (W_t^1, \dots, W_t^n)$ is a standard Brownian motion on \mathbb{R}^n . Denote by $r_t(w, r_0)$ the solution to (3.4). Let $\zeta(w, r_0)$ be the explosion time of SDE (3.4). Let

$$\Sigma(t, w) = \{r_0 \in O(M); \zeta(w, r_0) > t\}.$$

Then for each $t > 0$ given, almost surely $\Sigma(t, w)$ is an open subset of $O(M)$ and $r_0 \rightarrow r_t(w, r_0)$ is a local diffeomorphism on $\Sigma(t, w)$ (see [27]). To be short, set $r_t(r_0) = r_t(w, r_0)$. The Jacobian J_{r_t} of $r_0 \rightarrow r_t(r_0)$ is equal to 1, since by [27], the Jacobian $J_{r_t^{-1}}$ of inverse map r_t^{-1} admits expression

$$J_{r_t^{-1}} = \exp\left(-\int_0^t \sum_{k=1}^n \text{div}(H_k)(r_s(r_0)) \circ dW_s^k - \int_0^t \text{div}(V_s)(r_s(r_0)) ds\right) = 1.$$

Then for any $\varphi \in C_c(O(M))$, almost surely,

$$\int_{O(M)} \varphi(r_t(r_0)) \mathbf{1}_{\Sigma(t, w)}(r_0) dr_0 = \int_{O(M)} \varphi(r_0) \mathbf{1}_{r_t(\Sigma(t, w))}(r_0) dr_0, \quad (3.5)$$

where dr_0 is the Liouville measure on $O(M)$ (see [34], page 185) such that $\pi_\#(dr_0) = dx_0$.

Let $d_M(x, y)$ be the Riemannian distance on M between x and y . Fix a reference point $x_M \in M$, consider

$$\rho(r) = d_M(\pi(r), x_M).$$

It is known that for each x_0 given, $x \rightarrow d_M(x, x_0)$ is smooth out of $C_{x_0} \cup \{x_0\}$, where C_{x_0} is the cut-locus of x_0 . It is known that C_{x_0} is negligible with respect to dx . Then ρ is smooth out of $\pi^{-1}(C_{x_M} \cup \{x_M\})$. By [34], p. 197, out of $\pi^{-1}(C_{x_0} \cup \{x_0\})$,

$$\frac{1}{2}\Delta_{O(M)}d_M(\pi(\cdot), x_0) \leq \frac{n-1}{2d_M(\pi(\cdot), x_0)} + \frac{1}{2}\sqrt{n\kappa}. \quad (3.6)$$

It is known that out of $C_{x_0} \cup \{x_0\}$, $|\nabla_x d_M(x, x_0)| = 1$. Therefore out of $\pi^{-1}(C_{x_0} \cup \{x_0\})$,

$$|\mathcal{L}_{V_t} d_M(\pi(\cdot), x_0)| \leq |V_t|. \quad (3.7)$$

The lower bound of $\frac{1}{2}\Delta_{O(M)}\rho$ is more delicate. According to [24], page 90,

$$\frac{1}{2}\Delta_{O(M)}d_M(\pi(\cdot), x_0) \geq \frac{n-1}{2\rho} - \frac{1}{2}\sqrt{n(n-1)K_R}, \quad \text{quad } \pi(r) \in B(x_M, R) \setminus (C_{x_0} \cup \{x_0\}). \quad (3.8)$$

where K_R is the upper bound of sectional curvature on the big ball $B(x_M, R)$.

Proposition 3.1. *Assume furthermore that*

$$\int_0^T \int_M |v_s(x)|^2 dx ds < +\infty. \quad (3.9)$$

Then there is a non-decreasing process $\hat{L}_t \geq 0$ and a Brownian motion $\{\beta_t; t \geq 0\}$ on \mathbb{R} such that for almost surely initial r_0 ,

$$\rho(r_t) - \rho(r_0) = \beta_t + \int_0^t \left(\frac{1}{2}\Delta_{O(M)} + \mathcal{L}_{V_s} \right) \rho(r_s) ds - \hat{L}_t, \quad t < \zeta(w, r_0). \quad (3.10)$$

Proof. The proof will be given in Section 6. \square

Theorem 3.2. *Assume $\text{Ric} \geq -\kappa$ and (3.9) holds. Then for almost all r_0 , $\zeta(w, r_0) = +\infty$ almost surely.*

Proof. We have, by (3.10),

$$\rho(r_{t \wedge \zeta})^2 \leq \rho(r_0)^2 + t \wedge \zeta + 2 \int_0^{t \wedge \zeta} \rho(r_s) d\beta_s + 2 \int_0^{t \wedge \zeta} \rho(r_s) (L_s \rho)(r_s) ds,$$

where $\mathcal{L}_s = \frac{1}{2}\Delta_{O(M)} + \mathcal{L}_{V_s}$. Using (3.6) and (3.7), there is constants $C > 0$ such that

$$\begin{aligned} \mathbb{E}(\rho(r_{t \wedge \zeta})^2) &\leq \rho(r_0)^2 + C \int_0^t \mathbb{E} \left((2\rho(r_s)(L_s \rho)(r_s) + 1) \mathbf{1}_{(s < \zeta)} \right) ds \\ &\leq \rho(r_0)^2 + 2C \int_0^t \mathbb{E} \left((1 + \rho(r_s))(1 + |V_s(r_s)|) \mathbf{1}_{(s < \zeta)} \right) ds. \end{aligned}$$

Let μ be the probability measure on $O(M)$ defined in (6.6). Then

$$\begin{aligned} \int_{O(M)} \mathbb{E}(\rho(r_{t \wedge \zeta})^2) d\mu &\leq \int_{O(M)} \rho(r_0)^2 d\mu + 2C \int_0^t \int_{O(M)} \mathbb{E} \left((1 + \rho(r_s))(1 + |V_s(r_s)|) \mathbf{1}_{(s < \zeta)} \right) d\mu ds \\ &\leq \int_{O(M)} \rho(r_0)^2 d\mu + 4C \left(\int_0^t \int_{O(M)} \mathbb{E} \left((1 + \rho(r_{s \wedge \zeta})^2) \right) d\mu ds \right)^{1/2} \times \\ &\quad \times \left(\int_0^t \int_{O(M)} \mathbb{E} \left((1 + |V_s(r_s)|)^2 \mathbf{1}_{(s < \zeta)} \right) d\mu ds \right)^{1/2}. \end{aligned}$$

Note that

$$\int_0^t \int_{O(M)} \mathbb{E} \left((1 + |V_s(r_s)|)^2 \mathbf{1}_{(s < \zeta)} \right) d\mu ds \leq 2 \left(T + \int_0^T \int_M |v_s(x)|^2 dx ds \right).$$

Set $\psi(t) = \int_{O(M)} \mathbb{E}(\rho(r_{t \wedge \zeta})^2) d\mu$ and

$$C(T, v) = 4C\sqrt{2}\sqrt{T + \|v\|_{L^2([0, T] \times M)}^2}. \quad (3.11)$$

Remarking that $\sqrt{\xi} \leq 1 + \xi$ for $\xi \geq 0$, above two inequalities imply that

$$\psi(t) \leq \left(\int_{O(M)} \rho(r_0)^2 d\mu + C(T, v) \right) + C(T, v) \int_0^t \psi(s) ds.$$

The Gronwall lemma then yields

$$\int_{O(M)} \mathbb{E}(\rho(r_{t \wedge \zeta})^2) d\mu \leq \left(\int_{O(M)} \rho(r_0)^2 d\mu + C(T, v) \right) \exp(C(T, v)).$$

The result follows. \square

Now we are going to obtain a probabilistic representation for solution to the heat equation (3.3). To this end, set $F(t, r) = F_{\omega_t}(r)$. Let $T > 0$ be fixed. Assume that u_t is a solution to (2.1) such that $(t, x) \rightarrow u_t(x)$ is continuous and for each $t \geq 0$, $u_t \in C^{1+\alpha}$ with $\alpha > 0$. Consider the following SDE on $O(M)$,

$$\begin{cases} dr_{s,t}(r, w) = \sqrt{2\nu} \sum_{i=1}^n A_i(r_{s,t}(r, w)) \circ dW_t^i - U_{T-t}(r_{s,t}(r, w)) dt, & s < t < T, \\ r_{s,s}(r, w) = r. \end{cases} \quad (3.12)$$

Let $v_t(x) = u_{T-t}(x)$. Then by Theorem 3.2, SDE (3.12) is stochastic complete. Let $Q_{s,t}(w)$ be solution to the resolvent equation

$$\frac{d}{dt} Q_{s,t}(w) = Q_{s,t}(w) J_{T-t}(r_{s,t}(r, w)), \quad s < t < T, \quad Q_{s,s}(w) = Id \quad (3.13)$$

where

$$J_t(r) = K(t, r) - \nu \operatorname{ric}_r. \quad (3.14)$$

For the sake of simplicity, we denote $r_{s,t} = r_{s,t}(r, w)$. Applying Itô formula to $Q_{s,t}F(T-t, r_{s,t})$ for d_t with $t \in (s, T)$, we have

$$\begin{aligned} d_t \left(Q_{s,t} F(T-t, r_{s,t}) \right) &= d_t Q_{s,t} F(T-t, r_{s,t}) + Q_{s,t} d_t (F(T-t, r_{s,t})) \\ &= Q_{s,t} J_{T-t}(r_{s,t}) F(T-t, r_{s,t}) + \sqrt{2\nu} Q_{s,t} \sum_{i=1}^n (\mathcal{L}_{A_i} F)(T-t, r_{s,t}) dW_t^i \\ &\quad + Q_{s,t} \left(-(\partial_t F)(T-t, r_{s,t}) + \nu (\Delta_{O(M)} F)(T-t, r_{s,t}) - (\mathcal{L}_{U_{T-t}} F)(T-t, r_{s,t}) \right) dt \\ &= \sqrt{2\nu} Q_{s,t} \sum_{i=1}^n (\mathcal{L}_{A_i} F)(T-t, r_{s,t}) dW_t^i, \end{aligned}$$

where the last equality is due to Equation (3.3). It follows that

$$Q_{s,t} F(T-t, r_{s,t}) - F(T-s, r) = \sqrt{2\nu} \sum_{i=1}^n \int_s^t Q_{s,\tau} (\mathcal{L}_{A_i} F)(T-\tau, r_{s,\tau}) dW_\tau^i.$$

Taking expectation on the two sides gives $\mathbb{E}\left(Q_{s,t} F(T-t, r_{s,t})\right) = F(T-s, r)$. Let $t = T$. Then $\mathbb{E}\left(Q_{s,T} F(0, r_{s,T})\right) = F(T-s, r)$. Replacing s by $T-t$, we get the following representation formula to (3.3):

$$F_{\omega_t} = \mathbb{E}\left(Q_{T-t,T} F_{\omega_0}(r_{T-t,T})\right). \quad (3.15)$$

In what follows, we will explain how a vector field v on M gives rise to a metric compatible connection Γ^v . For a time-independent vector field v on M , the diffusion processes $\{x_t, t \geq 0\}$ associated to the generator $\frac{1}{2}\Delta_M + v$ can be constructed in the following way:

$$dr_t = \sum_{i=1}^n A_i(r_t) \circ dW_t^i + V(r_t) dt \quad (3.16)$$

where V is the horizontal lift of v to $O(M)$, and let $x_t = \pi(r_t)$. We assume that the lift-time $\zeta = +\infty$ almost surely.

In Chapter V of [25], Ikeda and Watanabe introduced a metric compatible connection Γ^v so that the diffusion process of generator $\frac{1}{2}\Delta_M + v$ can be constructed by rolling without friction Brownian motion on \mathbb{R}^n with respect to the connection Γ^v . More precisely let $\{B_1, \dots, B_n\}$ be the canonical horizontal vector fields on $O(M)$ with respect to Γ^v , consider SDE on $O(M)$:

$$dr_w(t) = \sum_{i=1}^n B_i(r_w(t)) \circ dW_t^i, \quad r_w(0) = r.$$

Then the generator of diffusion process $t \rightarrow x_t(w) = \pi(r_w(t))$ is $\frac{1}{2}\Delta_M + v$. In fact, it holds

$$\frac{1}{2} \sum_{j=1}^n \mathcal{L}_{B_j}^2(f \circ \pi) = \left(\frac{1}{2}\Delta_M + v\right)f \circ \pi. \quad (3.17)$$

This connection Γ^v was defined locally in [25]. On a local chart U , $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ is a local basis of tangent spaces $T_x M$ with $x \in U$, and $v = \sum_{i=1}^n v^i \frac{\partial}{\partial x_i}$. Let $\Gamma_{ij}^{0,k}$ be the Christoffel coefficients of Levi-Civita connection. According to ([25], p.271), the Christoffel coefficients Γ_{ij}^k of Γ^v is defined by (see also [1]),

$$\Gamma_{ij}^k = \Gamma_{ij}^{0,k} - \frac{2}{n-1} \left(\delta_{ki} \sum_{\ell=1}^n g_{j\ell} v^\ell - g_{ij} v^k \right). \quad (3.18)$$

Proposition 3.3. *Let ∇^v be the covariant derivative with respect to the connection Γ^v , and ∇^0 with respect to the Levi-Civita connection. Then for two vector fields X, Y on M ,*

$$\nabla_X^v Y = \nabla_X^0 Y - \frac{2}{n-1} K_v(X, Y), \quad (3.19)$$

where

$$K_v(X, Y) = \langle Y, v \rangle X - \langle X, Y \rangle v. \quad (3.20)$$

Proof. We have, using (3.18),

$$\begin{aligned}\nabla_X^v Y &= \sum_{k=1}^n \left[\sum_{i,j=1}^n X^i Y^j \Gamma_{ij}^k + \sum_{i=1}^n X^i \frac{\partial}{\partial x_i} Y^k \right] \frac{\partial}{\partial x_k} \\ &= \sum_{k=1}^n \left[\sum_{i,j=1}^n X^i Y^j \Gamma_{ij}^{0,k} + \sum_{i=1}^n X^i \frac{\partial}{\partial x_i} Y^k \right] \frac{\partial}{\partial x_k} - \frac{2}{n-1} I_2,\end{aligned}$$

where

$$I_2 = \sum_{i,j,k=1}^n X^i Y^j \delta_{ki} \left\langle \frac{\partial}{\partial x_j}, v \right\rangle \frac{\partial}{\partial x_k} - \sum_{i,j,k=1}^n X^i Y^j \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle v^k \frac{\partial}{\partial x_k},$$

since $\sum_{\ell=1}^n g_{j\ell} v^\ell = \left\langle \frac{\partial}{\partial x_j}, v \right\rangle$. It is obvious to see that the first sum in I_2 is equal to $\langle Y, v \rangle X$, while the second sum yields $\langle X, Y \rangle v$. The relation (3.19) and (3.20) follow. \square

Having this explicit expression, we will compute the associated torsion tensor T^v .

Proposition 3.4. $T^v(X, Y)$ admits the expression:

$$T^v(X, Y) = \frac{-2}{n-1} \left(\langle Y, v \rangle X - \langle X, v \rangle Y \right). \quad (3.21)$$

Moreover, T^v is skew-symmetric (TSS), that is $\langle T^v(X, Y), Z \rangle = -\langle T^v(Z, Y), X \rangle$ holds for all $X, Y, Z \in \mathcal{X}(M)$ if and only if $v = 0$.

Proof. Using (3.19) and the fact $\nabla_X^0 Y - \nabla_Y^0 X - [X, Y] = 0$, we have

$$T^v(X, Y) = -\frac{2}{n-1} (K_v(X, Y) - K_v(Y, X)) = \frac{-2}{n-1} \left(\langle Y, v \rangle X - \langle X, v \rangle Y \right),$$

that is nothing but (3.21). Now if for any $X, Y, Z \in \mathcal{X}(M)$, $\langle T^v(X, Y), Z \rangle + \langle T^v(Z, Y), X \rangle = 0$, then this equality yields

$$2\langle Y, v \rangle \langle X, Z \rangle = \langle X, v \rangle \langle Y, Z \rangle + \langle Z, v \rangle \langle Y, X \rangle.$$

Taking $Y = v$ and $X = Z$ in above equality, we get

$$|v|^2 |X|^2 = \langle X, v \rangle^2.$$

If $v \neq 0$, taking X orthogonal to v yields a contradiction. \square

4 Intrinsic Ricci tensors for Navier-Stokes equations

In what follows, we will denote Levi-Civita covariant derivative by ∇^0 . We first compute the Ricci tensor associated to the connection ∇^v .

Proposition 4.1. Let Ric^0 be the Ricci curvature associated to ∇^0 , and Ric^v to ∇^v . Then

$$\text{Ric}^v(X) = \text{Ric}^0(X) - \frac{4(n-2)}{(n-1)^2} K_v(X, v) + \frac{2(n-2)}{n-1} \nabla_X^0 v + \frac{2}{n-1} \text{div}(v) X. \quad (4.1)$$

Proof. For the sake of simplicity, put $\nabla_Y^v Z = \nabla_Y^0 Z + S(Y, Z)$, where S is a $(1, 2)$ type tensor on M . Then

$$\begin{aligned}\nabla_X^v \nabla_Y^v Z &= \nabla_X^0 \nabla_Y^v Z + S(X, \nabla_Y^v Z) = \nabla_X^0 (\nabla_Y^0 Z + S(Y, Z)) + S(X, \nabla_Y^v Z) \\ &= \nabla_X^0 \nabla_Y^0 Z + (\nabla_X^0 S)(Y, Z) + S(\nabla_X^0 Y, Z) + S(Y, \nabla_X^0 Z) + S(X, \nabla_Y^v Z).\end{aligned}$$

Changing role between X and Y yields

$$\nabla_Y^v \nabla_X^v Z = \nabla_Y^0 \nabla_X^0 Z + (\nabla_Y^0 S)(X, Z) + S(\nabla_Y^0 X, Z) + S(X, \nabla_Y^0 Z) + S(Y, \nabla_X^v Z).$$

Also

$$\nabla_{[X, Y]}^v Z = \nabla_{[X, Y]}^0 Z + S([X, Y], Z).$$

Combining above equations, the curvature tensor

$$R^v(X, Y)Z = \nabla_X^v \nabla_Y^v Z - \nabla_Y^v \nabla_X^v Z - \nabla_{[X, Y]}^v Z$$

which admits the following expression

$$\begin{aligned}R^0(X, Y)Z + (\nabla_X^0 S)(Y, Z) - (\nabla_Y^0 S)(X, Z) + S(\nabla_X^0 Y - \nabla_Y^0 X, Z) \\ - S(Y, S(X, Z)) + S(X, S(Y, Z)) - S([X, Y], Z).\end{aligned}$$

Let $x \in M$ and $\{e_1, \dots, e_n\}$ an orthonormal basis of $T_x M$. Then $\text{Ric}^v(X) = \sum_{i=1}^n R^v(X, e_i)e_i$.

Note that $S(X, Y) = -\frac{2}{n-1}K_v(X, Y)$. Put

$$I_1 = \sum_{i=1}^n S(X, S(e_i, e_i)), \quad I_2 = \sum_{i=1}^n S(e_i, S(X, e_i)).$$

$$I_3 = \sum_{i=1}^n (\nabla_X^0 S)(e_i, e_i), \quad I_4 = \sum_{i=1}^n (\nabla_{e_i}^0 S)(X, e_i).$$

Then

$$\text{Ric}^v(X) = \text{Ric}^0(X) + I_1 - I_2 + I_3 - I_4.$$

By a completely elementary computation, we find

$$I_1 = \frac{4}{(n-1)^2} \sum_{i=1}^n K_v(X, K_v(e_i, e_i)) = -\frac{4(n-1)}{(n-1)^2} K_v(X, v)$$

and

$$I_2 = \frac{4}{(n-1)^2} \sum_{i=1}^n K_v(e_i, K_v(X, e_i)) = -\frac{4}{(n-1)^2} K_v(X, v).$$

For two other terms,

$$(\nabla_X^0 S)(Y, Z) = -\frac{2}{n-1} K_{\nabla_X^0 v}(Y, Z)$$

and

$$(\nabla_Y^0 S)(X, Z) = -\frac{2}{n-1} K_{\nabla_Y^0 v}(X, Z).$$

Therefore

$$I_3 = -\frac{2}{n-1} \sum_{i=1}^n K_{\nabla_X^0 v}(e_i, e_i) = 2\nabla_X^0 v.$$

$$I_4 = -\frac{2}{n-1} \sum_{i=1}^n K_{\nabla_{e_i}^0 v}(X, e_i) = -\frac{2}{n-1} \operatorname{div}(v) X + \frac{2}{n-1} \nabla_X^0 v.$$

Finally

$$\operatorname{Ric}^v(X) = \operatorname{Ric}^0(X) - \frac{4(n-2)}{(n-1)^2} K_v(X, v) + \frac{2(n-2)}{n-1} \nabla_X^0 v + \frac{2}{n-1} \operatorname{div}(v) X$$

and the computations are complete. \square

Since the connection ∇^v has torsion, we have to take account of torsion tensor into Ricci tensor in a suitable way. A Weitzenböck formula for a connection which is not of torsion skew-symmetric was established in [17]. Since the dual connection of Γ^v is not metric, we prefer here avoid to use it. We will define the so-called *Intrinsic Ricci tensor*, which was firstly introduced by B. Driver in [12], in the framework of stochastic analysis on the path space of Riemannian manifolds (see also [8, 22, 24, 30]). Such a connection was also used in [1] to obtain an integration by parts formula for second order differential operators on Riemannian path spaces.

Definition 4.2. *The intrinsic Ricci tensor is given by*

$$\widehat{\operatorname{Ric}}^v(X) = \operatorname{Ric}^v(X) + \sum_{i=1}^n (\nabla_{e_i}^v T^v)(X, e_i). \quad (4.2)$$

where (e_i) is a local orthonormal frame field of the tangent bundle.

Theorem 4.3. *Assume that $\dim(M) = 3$. Then $\widehat{\operatorname{Ric}}^v$ admits the following simple expression:*

$$\widehat{\operatorname{Ric}}^v = \operatorname{Ric}^0 + 2v \otimes v + 2\nabla^{0,s} v, \quad (4.3)$$

where $\nabla^{0,s} v$ denotes the symmetric part of $\nabla^0 v$.

Proof. By (3.21),

$$\begin{aligned} (\nabla_{e_i}^v T)^v(X, e_i) &= \frac{-2}{n-1} \left(\langle e_i, \nabla_{e_i}^v v \rangle X - \langle X, \nabla_{e_i}^v v \rangle e_i \right) \\ &= \frac{-2}{n-1} \left(\langle e_i, \nabla_{e_i}^0 v \rangle X - \langle X, \nabla_{e_i}^0 v \rangle e_i \right) + J_i, \end{aligned}$$

where

$$J_i = \frac{4}{(n-1)^2} \left(\langle e_i, K_v(e_i, v) \rangle X - \langle X, K_v(e_i, v) \rangle e_i \right).$$

Then

$$\sum_{i=1}^n J_i = \frac{4}{(n-1)^2} \left((n-1)|v|^2 X - K_v(X, v) \right).$$

Therefore the sum $\sum_{i=1}^n (\nabla_{e_i}^v T^v)(X, e_i)$ is equal to

$$\frac{-2}{n-1} \left(\operatorname{div}(v) X - \sum_{i=1}^n \langle X, \nabla_{e_i}^0 v \rangle e_i \right) + \frac{4}{(n-1)^2} \left((n-1)|v|^2 X - K_v(X, v) \right).$$

When $n = 3$, the above formula yields that

$$\sum_{i=1}^3 (\nabla_{e_i}^v T^v)(X, e_i) = -\operatorname{div}(v) X + \sum_{i=1}^3 \langle X, \nabla_{e_i}^0 v \rangle e_i + 2|v|^2 X - K_v(X, v). \quad (4.4)$$

On the other hand, by (4.1), for $n = 3$,

$$\operatorname{Ric}^v(X) = \operatorname{Ric}^0(X) - K_v(X, v) + \nabla_X^0 v + \operatorname{div}(v) X. \quad (4.5)$$

Note that

$$\sum_{i=1}^3 \langle X, \nabla_{e_i}^0 v \rangle e_i + \nabla_X^0 v = \sum_{i=1}^3 \left(\langle X, \nabla_{e_i}^0 v \rangle + \langle \nabla_X^0 v, e_i \rangle \right) e_i = 2\nabla_X^{0,s} v.$$

According to this and summing up (4.4) and (4.5), we then obtain

$$\widehat{\operatorname{Ric}}^v(X) = \operatorname{Ric}^0(X) + 2|v|^2 X - 2K_v(X, v) + 2\nabla_X^{0,s} v.$$

Now remarking that $|v|^2 X - K_v(X, v) = \langle X, v \rangle v$, we deduce that

$$\widehat{\operatorname{Ric}}^v(X) = \operatorname{Ric}^0(X) + 2\langle X, v \rangle v + 2\nabla_X^{0,s} v$$

for any vector field X and therefore (4.3) holds. \square

Remark 4.4. Consider the following SDE on $O(M)$:

$$dr_w(t) = \sqrt{2\nu} \sum_{i=1}^n B_i(r_w(t)) \circ dW_t^i, \quad r_w(0) = r,$$

which has its infinitesimal generator

$$\nu \sum_{i=1}^n \mathcal{L}_{B_i}^2(f \circ \pi) = \left((\nu \Delta_M + 2\nu v) f \right) \circ \pi.$$

According to Equation (3.12), we have to choose $v = -\frac{1}{2\nu} u_t$. The term $\operatorname{Ric}^0 - \frac{1}{\nu} \nabla^{0,s} u_t$ has already appeared in resolvent equation (3.13). In this case, we denote Ric^t instead of $\operatorname{Ric}^{-u_t/2\nu}$ and we have

$$\widehat{\operatorname{Ric}}^t = \operatorname{Ric}^0 + \frac{1}{2\nu^2} u_t \otimes u_t - \frac{1}{\nu} \nabla^{0,s} u_t. \quad (4.6)$$

Proposition 4.5. Assume that $\dim(M) = 3$. Then

(i) The following holds:

$$\operatorname{div}(\widehat{\operatorname{Ric}}^t) = \operatorname{div}(\operatorname{Ric}^0) + \frac{1}{2\nu^2} \nabla_{u_t} u_t - \frac{1}{\nu} \operatorname{Ric}^0 u_t. \quad (4.7)$$

(ii) Let $\widehat{\operatorname{Scal}}^t$ be the associated scalar curvature, that is $\widehat{\operatorname{Scal}}^t = \sum_{i=1}^n \langle \widehat{\operatorname{Ric}}^t e_i, e_i \rangle$ for any orthonormal basis (e_i) of $T_x M$. Then

$$\widehat{\operatorname{Scal}}^t = \operatorname{Scal}^0 + \frac{1}{2\nu^2} |u_t|^2. \quad (4.8)$$

Proof. (i) Since $\operatorname{div}(u_t) = 0$, we have $\operatorname{div}(u_t \otimes u_t) = \nabla_{u_t} u_t$, and

$$\nabla u_t = \nabla^s u_t + \nabla^{sk} u_t.$$

We claim that

$$\operatorname{div}(\nabla^{sk} u_t) = -\square u_t.$$

In fact, let $X \in \mathcal{X}(M)$, we have

$$\begin{aligned} \int_M \langle \operatorname{div}(\nabla^{sk} u_t), X \rangle dx &= - \int_M \langle \nabla^{sk} u_t, \nabla X \rangle dx = - \int_M \langle \nabla^{sk} u_t, \nabla^{sk} X \rangle dx \\ &= - \int_M \langle d\tilde{u}_t, d\tilde{X} \rangle dx = - \int_M \langle d^* d\tilde{u}_t, \tilde{X} \rangle dx = - \int_M \langle \square \tilde{u}_t, \tilde{X} \rangle dx. \end{aligned}$$

Therefore

$$\operatorname{div}(\nabla^s u_t) = \Delta u_t + \square u_t = \operatorname{Ric}^0 u_t.$$

The result (4.7) follows.

(ii) Concerning (4.8), by (4.6), it is enough to remark that

$$\sum_{i=1}^n \langle \nabla_{e_i}^{0,s} u_t, e_i \rangle = \operatorname{div}(u_t) = 0.$$

□

Theorem 4.6. *Let $\dim(M) = 3$, and (u_t, ω_t) be a regular solution to Equation (2.13). Then the following identity holds ,*

$$\frac{1}{2} \frac{d}{dt} \int_M |\omega_t|^2 dx + \nu \int_M |\nabla^0 \omega_t|^2 dx = \frac{1}{2\nu} \int_M (\omega_t, u_t)^2 dx - \nu \int_M (\widehat{\operatorname{Ric}}^t \# \omega_t, \omega_t) dx. \quad (4.9)$$

where $(\widehat{\operatorname{Ric}}^t \# \omega_t, A) = (\omega_t, \widehat{\operatorname{Ric}}^t A)$ for $A \in \mathcal{X}(M)$.

Proof. Using (4.6),

$$(\widehat{\operatorname{Ric}}^t \# \omega_t, A) = (\omega_t, \operatorname{Ric}^0 A) + \frac{1}{2\nu^2} (\omega_t, u_t) \langle u_t, A \rangle - \frac{1}{\nu} (\omega_t, \nabla_A^{0,s} u_t).$$

Note that according to Definition (2.7), $(\omega_t, \nabla_A^{0,s} u_t) = (\omega_t \triangleleft \nabla^{0,s} u_t)(A)$. It follows that

$$\widehat{\operatorname{Ric}}^t \# \omega_t = \operatorname{Ric}^{0,\#} \omega_t + \frac{1}{2\nu^2} (\omega_t, u_t) \tilde{u}_t - \frac{1}{\nu} \omega_t \triangleleft \nabla^{0,s} u_t. \quad (4.10)$$

We shall express the right hand side of (2.26) in term of $\widehat{\operatorname{Ric}}^t \#$. By (4.10),

$$\langle \widehat{\operatorname{Ric}}^t \# \omega_t, \omega_t \rangle = \langle \operatorname{Ric}^0 \omega_t, \omega_t \rangle + \frac{1}{2\nu^2} (\omega_t, u_t)^2 - \frac{1}{\nu} \langle \omega_t \triangleleft \nabla^{0,s} u_t, \omega_t \rangle.$$

Then

$$-\nu \langle \operatorname{Ric}^0 \omega_t, \omega_t \rangle + \langle \omega_t \triangleleft \nabla^{0,s} u_t, \omega_t \rangle = -\nu \langle \widehat{\operatorname{Ric}}^t \# \omega_t, \omega_t \rangle + \frac{1}{2\nu} (\omega_t, u_t)^2.$$

Substituting this term in the right hand side of (2.26), we get (4.9). □

Remark 4.7. The term (ω_t, u_t) in the right hand side of (4.9) is called helical density, which involves explicitly in the evolution of vorticity in time and in space.

Theorem 4.8. Let $\dim(M) = 3$. Then

$$\begin{aligned} \frac{d}{dt} \int_M (\omega_t, u_t) dx &= -\nu \int_M \langle d\omega_t, *\omega_t \rangle_{\Lambda^2} dx - \nu \int_M (\nabla \omega_t, \nabla u_t) dx \\ &\quad - \nu \int_M (\omega_t, \widehat{\text{Ric}}^t u_t) dx + \frac{1}{2\nu} \int_M (\omega_t, u_t) |u_t|^2 dx. \end{aligned} \quad (4.11)$$

Proof. By (4.6),

$$\widehat{\text{Ric}}^t u_t = \text{Ric}^0 u_t + \frac{1}{2\nu^2} |u_t|^2 u_t - \frac{1}{\nu} \nabla_{u_t}^{0,s} u_t.$$

Hence

$$-\nu \text{Ric}^0 u_t + \nabla_{u_t}^{0,s} u_t = -\nu \widehat{\text{Ric}}^t u_t + \frac{1}{2\nu} |u_t|^2 u_t.$$

Substituting this term in the right hand of (2.27), we get (4.11). \square

5 Case of \mathbb{R}^3

We will specify results obtained in Section 4 on \mathbb{R}^n . There are an ocean of publications on Navier-Stokes equations on \mathbb{R}^n . We only refer to [23, 28] for nice expositions and to [9] for wellposedness of global solutions. We keep notations used in Section 2 for correspondences between vector fields and differential forms. In this case, $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ form an orthonormal basis at each tangent space of \mathbb{R}^3 , and $\{dx, dy, dz\}$ an orthonormal basis at each co-tangent space. Let u be a vector field on \mathbb{R}^3 : $u = u_1 \frac{\partial}{\partial x} + u_2 \frac{\partial}{\partial y} + u_3 \frac{\partial}{\partial z}$, then $\tilde{u} = u_1 dx + u_2 dy + u_3 dz$ and

$$\tilde{\omega} = d\tilde{u} = \left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x} \right) dz \wedge dx + \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right) dx \wedge dy + \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z} \right) dy \wedge dz.$$

Hodge star operator gives an isomorphism between $\Lambda^2(\mathbb{R}^3)$ and $\Lambda^1(\mathbb{R}^3)$, we have

$$\omega = *\tilde{\omega} = \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z} \right) dx + \left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x} \right) dy + \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right) dz.$$

In this case $\omega = \widetilde{\text{curl}} u$, where $\text{curl}(u)$ is the curl of u , denoted sometimes by $\nabla \times u$. We have the following relations

$$\omega = \widetilde{\nabla \times u}, \quad \nabla \times (\nabla \times u) = (d^* d\tilde{u})^\# = (d^* \tilde{\omega})^\#. \quad (5.1)$$

By (5.1),

$$\int_{\mathbb{R}^3} \langle d\omega_t, *\omega_t \rangle_{\Lambda^2} dx = \int_{\mathbb{R}^3} \langle \omega_t, d^*(\tilde{\omega}) \rangle dx = \int_{\mathbb{R}^3} \nabla \times (\nabla \times u) \cdot (\nabla \times u) dx.$$

In what follows, we denote $\xi_t = \nabla \times u_t$. In this flat case, the intrinsic Ricci tensor $\widehat{\text{Ric}}^t$ defined in Formula (4.6) has expression

$$\widehat{\text{Ric}}^t = \frac{1}{2\nu^2} u_t \otimes u_t - \frac{1}{\nu} \nabla^s u_t, \quad (5.2)$$

where $\nabla^s u_t$ is the rate of strains. Formula (4.9) becomes into the following form:

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\xi_t|^2 dx + \nu \int_{\mathbb{R}^3} |\nabla \xi_t|^2 dx = \frac{1}{2\nu} \int_{\mathbb{R}^3} (\xi_t \cdot u_t)^2 dx - \nu \int_{\mathbb{R}^3} (\widehat{\text{Ric}}^t \xi_t, \xi_t) dx. \quad (5.3)$$

This formula says that the variation of vorticity in time and in space can be explicitly measured by using helicity and the associated intrinsic Ricci tensor. Formula (4.11) has the form

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^3} \xi_t \cdot u_t dx &= -\nu \int_{\mathbb{R}^3} (\nabla \times \xi_t) \cdot \xi_t dx - \nu \int_{\mathbb{R}^3} \nabla \xi_t \cdot \nabla u_t dx \\ &\quad - \nu \int_{\mathbb{R}^3} \xi_t \cdot \widehat{\text{Ric}}^t u_t dx + \frac{1}{2\nu} \int_{\mathbb{R}^3} (\xi_t \cdot u_t) |u_t|^2 dx, \end{aligned} \quad (5.4)$$

which shows how the helicity $\int_{\mathbb{R}^3} \xi_t \cdot u_t dx$ varies.

6 Appendix

6.1 Proof of Proposition 3.3

We first give a complete proof of Proposition 3.3 by following the proof of Theorem 3.5.1 in [24], and emphasize the steps we have to modify.

Proof. Let i_x be the injectivity radius at x and suppose that

$$i_M = \inf\{i_x; x \in M\} > 0. \quad (6.1)$$

This means that the ball $B(x, i_M)$ does not meet the cut-locus C_x of x . We prepare what we will need for proving (3.10).

Let $x \in B(x_0, i_M/2)^c$ which maybe is closed to or in C_{x_0} . Let $\gamma_x : [0, \eta(x)] \rightarrow M$ be a distance-minimizing geodesic connecting x_0 and x , parameterized by length. Then $\gamma_x(i_M/4) \notin C_x$ or $x \notin C_{\gamma_x(i_M/4)}$. Put $y = \gamma_x(i_M/4)$. Then $d_M(x_0, x) = d_M(x_0, y) + d_M(y, x)$. Since C_y is closed, there is $\varepsilon_0 > 0$ such that

$$B(x, \varepsilon_0) \cap C_y = \emptyset.$$

We suppose that such ε_0 is valid for all x (in fact, we will restrict ourselves in a compact set). Let $\varepsilon < \varepsilon_0 \wedge \frac{i_M}{8}$, and define

$$D_\varepsilon = \{x \in M; d_M(x, C_{x_M}) < \varepsilon\}.$$

We claim that

$$D_\varepsilon \subset B(x_M, i_M/2)^c. \quad (6.2)$$

In fact, if there exists $x \in D_\varepsilon$ such that $d_M(x, x_M) < i_M/2$; there is $z \in C_{x_M}$ such that $d_M(x, z) < \varepsilon$; then $d_M(x_M, z) \leq d_M(x_M, x) + d_M(x, z) < i_M$ which contradicts the definition of i_M . Let γ_x be the geodesic considered above. Then $x \notin C_y$ with $y = \gamma_x(i_M/4)$.

Now introduce the stopping times σ_q by $\sigma_0 = 0$ and

$$\sigma_q = \inf\{t > \sigma_{q-1}; d_M(\pi(r_t), \pi(r_{\sigma_{q-1}})) = \varepsilon\}.$$

Let $t > 0$ and set $t_q = t \wedge \sigma_q$. Then

$$\rho(r_t) - \rho(r_0) = \sum_{q=1}^{+\infty} \left(\rho(r_{t_q}) - \rho(r_{t_{q-1}}) \right). \quad (6.3)$$

(i) If $\pi(r_{t_{q-1}}) \notin D_\varepsilon$, then for $s \in [t_{q-1}, t_q]$, $\pi(r_s) \notin C_{x_M}$. Applying Itô formula, we have

$$\rho(r_{t_q}) - \rho(r_{t_{q-1}}) = \sum_{k=1}^n \int_{t_{q-1}}^{t_q} (\mathcal{L}_{A_k} \rho)(r_s) dW_s^k + \int_{t_{q-1}}^{t_q} (L_s \rho)(r_s) ds, \quad (6.4)$$

where $L_s = \frac{1}{2} \Delta_{O(M)} + \mathcal{L}_{V_s}$

(ii) Set $x_q = \pi(r_{t_q})$. If $x_{q-1} \in D_\varepsilon$, then by discussion at beginning, there is y_{q-1} on a distance-minimizing geodesic γ connecting x_M and x_{q-1} such that $d_M(x_M, y_{q-1}) = \frac{i_M}{4}$ and $x_{q-1} \notin C_{y_{q-1}}$ and for $s \in [t_{q-1}, t_q]$,

$$d_M(\pi(r_s), x_{q-1}) \leq \varepsilon < \varepsilon_0.$$

Therefore $\pi(r_s) \notin C_{y_{q-1}}$. Let $\rho_q^*(r) = d_M(\pi(r), y_{q-1})$. Applying Itô formula to ρ_q^* , we have

$$\rho_q^*(r_{t_q}) - \rho_q^*(r_{t_{q-1}}) = \sum_{k=1}^n \int_{t_{q-1}}^{t_q} (\mathcal{L}_{A_k} \rho_q^*)(r_s) dW_s^k + \int_{t_{q-1}}^{t_q} (L_s \rho_q^*)(r_s) ds.$$

On one hand

$$d_M(x_M, x_{q-1}) = d_M(x_M, y_{q-1}) + d_M(x_{q-1}, y_{q-1}) \quad \text{or} \quad \rho(r_{t_{q-1}}) = \frac{i_M}{4} + \rho_q^*(r_{t_{q-1}}),$$

and on the other hand

$$d_M(x_M, x_q) \leq d_M(x_M, y_{q-1}) + d_M(x_q, y_{q-1}) \quad \text{or} \quad \rho(r_{t_q}) \leq \frac{i_M}{4} + \rho_q^*(r_{t_q}).$$

It follows that

$$\rho(r_{t_q}) - \rho(r_{t_{q-1}}) \leq \rho_q^*(r_{t_q}) - \rho_q^*(r_{t_{q-1}}).$$

Therefore there exists $\hat{L}_q \geq 0$ such that

$$\rho(r_{t_q}) - \rho(r_{t_{q-1}}) = \rho_q^*(r_{t_q}) - \rho_q^*(r_{t_{q-1}}) - \hat{L}_q.$$

Define

$$\tau_R = \inf\{t > 0, d_M(x_M, \pi(r_t)) > R\}.$$

As did in [24], page 95, we get

$$\rho(r_{t \wedge \tau_R}) - \rho(r_0) = \beta_{t \wedge \tau_R} + \int_0^{t \wedge \tau_R} (L_s \rho)(r_s) ds - \hat{L}_\varepsilon(t \wedge \tau_R) + R_\varepsilon(t \wedge \tau_R),$$

where

$$\hat{L}_\varepsilon(t) = \sum_{q=1}^{+\infty} \hat{L}_q \mathbf{1}_{D_\varepsilon}(\pi(r_{t_{q-1}}))$$

which converges to $\hat{L}(t)$ as $\varepsilon \rightarrow 0$. The term $R_\varepsilon(t) = m_\varepsilon(t) + b_\varepsilon(t)$ with $m_\varepsilon(t)$ the same as in [24], page 95, so that

$$\mathbb{E}(|m_\varepsilon(t)|^2) \leq 4 \int_0^t \mathbb{E}(\mathbf{1}_{D_{2\varepsilon}}(\pi(r_s))) ds.$$

Therefore for any compact subset $K \subset B(x_M, R)$,

$$\begin{aligned} \int_{\pi^{-1}(K)} \mathbb{E}(|m_\varepsilon(t \wedge \tau_R)|^2) dr_0 &\leq 4 \int_0^t \int_{\pi^{-1}(K)} \mathbb{E}(\mathbf{1}_{D_{2\varepsilon}}(\pi(r_s \wedge \tau_R))) dr_0 ds \\ &\rightarrow 4 \int_0^t \int_{\pi^{-1}(K)} E(\mathbf{1}_{C_{x_M}}(\pi(r_s \wedge \tau_R))) dr_0 ds \leq 4 \int_0^t \int_M \mathbf{1}_{C_{x_M}}(x) dx ds = 0. \end{aligned}$$

The term $b_\varepsilon(t)$ has to be modified such that

$$b_\varepsilon(t) = \sum_{q=1}^{+\infty} \left[\int_{t_{q-1}}^{t_q} \left(L_s \rho_q^*(r_s) - L_s \rho(r_s) \right) ds \right] \mathbf{1}_{D_\varepsilon}(\pi(r_{t_{q-1}})).$$

By (3.6) and (3.8), we have to control the term $1/\rho$. For $x_{q-1} \in D_\varepsilon$ and for $s \in [t_{q-1}, t_q]$,

$$d_M(x_M, x_s) \geq d_M(x_M, x_{q-1}) - d_M(x_{q-1}, x_s) \geq \frac{i_M}{2} - \varepsilon \geq \frac{3i_M}{8},$$

and

$$d_M(y_{q-1}, x_s) \geq d_M(x_M, x_s) - d_M(x_M, y_{q-1}) \geq \frac{3i_M}{8} - \frac{i_M}{4} = \frac{i_M}{8}.$$

Therefore, according to (3.7), since $x_s = \pi(r_s) \in D_{2\varepsilon}$, there exists a constant $\alpha > 0$ such that

$$\int_{t_{q-1}}^{t_q} \left| \left(L_s \rho_q^*(r_s) - L_s \rho(r_s) \right) \right| ds \mathbf{1}_{D_\varepsilon}(\pi(r_{t_{q-1}})) \leq \alpha \int_{t_{q-1}}^{t_q} (1 + |V_s(r_s)|) \mathbf{1}_{D_{2\varepsilon}}(\pi(r_s)) ds.$$

It follows that

$$\mathbb{E}(|b_\varepsilon(t)|) \leq \alpha \mathbb{E} \left(\int_0^t (1 + |V_s(r_s)|) \mathbf{1}_{D_{2\varepsilon}}(\pi(r_s)) ds \right). \quad (6.5)$$

Again by hypothesis (2.16), there is a constant $c_0 > 0$ such that $\text{vol}(B(x_0, \delta)) \leq e^{c_0 \delta}$, and therefore for a constant $\lambda_0 > 0$,

$$C_M = \int_{O(M)} \exp(-\lambda_0 d_M^2(\pi(r_0), x_0)) dr_0 < +\infty.$$

Define the probability measure $d\mu$ on $O(M)$ by

$$d\mu(r_0) = \frac{1}{C_M} \exp(-\lambda_0 d_M^2(\pi(r_0), x_0)) dr_0. \quad (6.6)$$

Now integrating with respect to $d\mu(r_0)$, we get

$$\begin{aligned} &\int_0^t \int_{\pi^{-1}(K)} \mathbb{E} \left((1 + |V_s(r_s)|) \mathbf{1}_{D_{2\varepsilon}}(\pi(r_s)) \mathbf{1}_{(s < \tau_R)} \right) d\mu(r_0) ds \\ &\rightarrow \int_0^t \int_{\pi^{-1}(K)} \mathbb{E} \left((1 + |V_s(r_s)|) \mathbf{1}_{C_{x_M}}(\pi(r_s)) \mathbf{1}_{(s < \tau_R)} \right) d\mu(r_0) ds \\ &\leq \sqrt{t} \left(\int_0^t \int_M |v_s(x)|^2 \mathbf{1}_{C_{x_M}}(x) dx ds \right)^{1/2} = 0, \end{aligned}$$

under the hypothesis (3.11). The proof of Proposition 3.3 is complete. \square

6.2 Bismut Formulae and Proof of Proposition 2.9

In this part, we will first present a nice derivative formulae for heat semigroup \mathbf{T}_t on differential p -forms obtained by Elworthy and Li in [18] and by Driver and Thalmaier in [13]. We keep notations introduced in Section 3. Let A_1, \dots, A_n be the canonical horizontal vector fields on $O(M)$. Consider the SDE on $O(M)$

$$dr_t = \sum_{i=1}^n A_i(r_t) \circ dW_t^i, \quad r|_{t=0} = r_0. \quad (6.7)$$

Assume that the Ricci tensor is bounded below $\text{Ric} \geq -\kappa$. Then SDE (6.7) is stochastic complete (see [34]). Set $x_t = \pi(r_t)$ with $x_0 = \pi(r_0)$. Then (x_t) is a semi-martingale on M , with respect to which stochastic integral can be defined (see [7]). Then we can write

$$dx_t = \pi(r_t) \circ dr_t = \sum_{i=1}^n d\pi(r_t) A_i(r_t) \circ dW_t^i = r_t \circ dW_t.$$

Therefore $W_t = \int_0^t r_s^{-1} \circ dx_s$, which is anti-development of $\{x_t; t \geq 0\}$. Set

$$B_t = r_0 W_t = \int_0^t //_s^{-1} \circ dx_s,$$

where $//_s = r_s \circ r_0^{-1}$ is Itô stochastic parallel translation along path $\{x_t; t \geq 0\}$. Recall that Weitzenböck formula for p -differential forms reads as follows [25, 18]:

$$\square = -\Delta + \mathcal{R}_p^\#, \quad (6.8)$$

where $\Delta\phi = \text{Trace}(\nabla\nabla\phi)$ for a p -form ϕ , and $\mathcal{R}_p^\# : \Lambda^p(M) \rightarrow \Lambda^p(M)$ is a tensor, called Weitzenböck curvature. For $p = 1$, $\mathcal{R}_1 = \text{Ric}^\#$ is Ricci tensor. As in [18], $\mathcal{R}_p(x)$ is an endomorphism of p -vectors, that is, $\mathcal{R}_p(x) : \wedge^p T_x M \rightarrow \wedge^p T_x M$. For $r \in O(M)$, define $\hat{\mathcal{R}}_p(r) = r \circ \mathcal{R}_p(\pi(r)) \circ r^{-1}$, more precisely, for $a_i, b_j \in \mathbb{R}^n$,

$$\langle \hat{\mathcal{R}}_p(r)(a_1 \wedge \dots \wedge a_p), b_1 \wedge \dots \wedge b_p \rangle = \langle \hat{\mathcal{R}}_p(\pi(r))(ra_1 \wedge \dots \wedge ra_p), rb_1 \wedge \dots \wedge rb_p \rangle.$$

Consider the heat equation on p -forms:

$$\frac{d\phi_t}{dt} = -\frac{1}{2}\square\phi_t, \quad \phi|_{t=0} = \phi_0. \quad (6.9)$$

By definition $\mathbf{T}_t\phi_0 = \phi_t$. Consider the following resolvent equation on $\wedge^p \mathbb{R}^n$

$$\frac{d\hat{Q}_t^p}{dt} = -\frac{1}{2}\hat{\mathcal{R}}_p(r_t) \cdot \hat{Q}_t^p, \quad \hat{Q}_0^p = \text{Id}. \quad (6.10)$$

Define $Q_t^p : \wedge^p(T_{x_0}M) \rightarrow \wedge^p(T_{x_t}M)$ par $Q_t^p V_0 = r_t \hat{Q}_t^p(r_0^{-1}V_0)$. It is well-known (see [18]) that

$$(\mathbf{T}_t\phi)(V_0) = \mathbb{E}(\langle \phi_{x_t}, V_t \rangle) = \mathbb{E}\left(\langle F_\phi(r_t), \hat{Q}_t^p(r_0^{-1}V_0) \rangle\right), \quad (6.11)$$

where F_ϕ is defined in (3.1) if ϕ is a differential 1-form, and $F_\phi(r) \in \wedge^p(\mathbb{R}^n)$ is such that $\langle F_\phi(r), a_1 \wedge \dots \wedge a_p \rangle = \langle \phi(\pi(r)), ra_1 \wedge \dots \wedge ra_p \rangle$ where $a_1, \dots, a_p \in \mathbb{R}^n$.

Proposition 6.1. *Assume that*

$$\mathcal{R}_p \geq -\kappa_p, \quad \kappa \in \mathbb{R}. \quad (6.12)$$

Then

$$|\mathbf{T}_t \phi| \leq e^{\kappa_p t/2} |\phi|. \quad (6.13)$$

Proof. Using (6.10) and (6.12), we have

$$\frac{d}{dt} |\hat{Q}_t^p(r_0 V_0)|^2 = -\langle \hat{\mathcal{R}}_p(r_t) Q_t^p(r_0 V_0), Q_t^p(r_0 V_0) \rangle \leq -\kappa_p |\hat{Q}_t^p(r_0 V_0)|^2.$$

The Gronwall lemma yields that $|\hat{Q}_t^p(r_0 V_0)| \leq e^{\kappa_p t/2} |V_0|$. Since $|F_\phi| = |\phi|$, (6.11) yields inequality (6.13). \square

For simplicity, for $p = 1$, we still denote κ instead of κ_1 . In the case for 1-forms,

$$|\mathbf{T}_t \phi| \leq e^{\kappa t/2} \mathbf{T}_t^M |\phi|. \quad (6.14)$$

To our purpose, we only state the formula for 1-form established by Elworthy and Li; although it was stated for the case of compact Riemannian manifolds in [18], but it remains valid in non-compact cases as did by Driver and Thalmaier in [13], section 6.

Theorem 6.2. *For 1-form ϕ and a vector field v ,*

$$(\square \mathbf{T}_t \phi, v) = -\frac{4}{t^2} \mathbb{E} \left[(\phi_{x_t}, Q_t^1 \int_{t/2}^t (Q_s^1)^{-1} dM_s(v)) \right] \quad (6.15)$$

where $dM_s(v) = dM_s^1(v) + dM_s^2(v)$ with

$$dM_s^1(v) = \theta_{//_s dB_s} Q_s^2 \left(\int_0^{t/2} (Q_r^2)^{-1} (//_r dB_r \wedge Q_r^1(v)) \right), \quad (6.16)$$

where θ is annihilation operator, and

$$dM_s^2(v) = //_s dB_s \left(\int_0^{t/2} \langle Q_r^1(v), //_r dB_r \rangle \right). \quad (6.17)$$

Let $\{\varepsilon_1, \dots, \varepsilon_n\}$ be the canonical basis of \mathbb{R}^n and set $e_j = r_0 \varepsilon_j$. Then $\{e_1, \dots, e_n\}$ is an orthonormal basis of $T_{x_0} M$. By definition of θ , the term

$$\left\langle \theta_{//_s dB_s} \left(Q_s^2 \int_0^{t/2} (Q_r^2)^{-1} (//_r dB_r \wedge Q_r^1(v)) \right), //_s e_j \right\rangle$$

may be identified with the following

$$\left\langle Q_s^2 \int_0^{t/2} (Q_r^2)^{-1} (//_r dB_r \wedge Q_r^1(v)), //_s dB_s \wedge //_s e_j \right\rangle.$$

Hence

$$dM_s^1(v) = \sum_{k,j=1}^n \left\langle Q_s^2 \int_0^{t/2} (Q_r^2)^{-1} (//_r dB_r \wedge Q_r^1(v)), //_s e_k \wedge //_s e_j \right\rangle //_s e_j dB_s^k,$$

and

$$dM_s^2(v) = \sum_{k=1}^n \left(\int_0^{t/2} \langle Q_r^1(v), //_r dB_r \rangle \right) //_s e_k dB_s^k.$$

Therefore $dM_s(v) = \sum_{k=1}^n (a_k(s) + b_k(s)) dB_s^k$ with

$$a_k(s) = \sum_{j=1}^n \left\langle Q_s^2 \int_0^{t/2} (Q_r^2)^{-1} (//_r dB_r \wedge Q_r^1(v)), //_s e_k \wedge //_s e_j \right\rangle //_s e_j.$$

and $b_k(s) = \left(\int_0^{t/2} \langle Q_r^1(v), //_r dB_r \rangle \right) //_s e_k$. It is obvious that $\langle a_k(s), b_k(s) \rangle = 0$.

Lemma 6.3. *The quadratic variation $dM_s(v) \cdot dM_s(v)$ of $M_s(v)$ admits the following expression*

$$dM_s(v) \cdot dM_s(v) = 2 \left\| Q_s^2 \int_0^{t/2} (Q_r^2)^{-1} (//_r dB_r \wedge Q_r^1(v)) \right\|_{\Lambda^2}^2 + \left(\int_0^{t/2} \langle Q_r^1(v), //_r dB_r \rangle \right)^2.$$

Theorem 6.4. *Assume that (6.12) holds for $p = 1$ and 2. Then for any differential 1-form ϕ ,*

$$\|\mathbf{T}_t \phi\|_2 \leq \frac{2}{t} e^{3\kappa+t/2} \sqrt{2(n-1)e^{3\kappa_2^+ t/2} + 1} \|\phi\|_2, \quad t > 0. \quad (6.18)$$

Proof. By Theorem 6.2,

$$\begin{aligned} |(\square \mathbf{T}_t \phi, v)| &\leq \frac{4}{t^2} \sqrt{\mathbb{E}(|\phi(x_t)|^2)} \cdot \left(\mathbb{E} \left[\left| Q_t^1 \int_{t/2}^t (Q_s^1)^{-1} dM_s(v) \right|^2 \right] \right)^{1/2} \\ &\leq \frac{4e^{\kappa t/2}}{t^2} \sqrt{\mathbb{E}(|\phi(x_t)|^2)} \cdot \left(\mathbb{E} \left[\left| \int_{t/2}^t (Q_s^1)^{-1} dM_s(v) \right|^2 \right] \right)^{1/2}. \end{aligned} \quad (6.19)$$

Note that $(Q_t^p)^{-1}$ enjoys the same kind of equations as (6.10). Thus $\|(Q_t^p)^{-1}\| \leq e^{\kappa_p t/2}$ under (6.12), so that

$$\begin{aligned} \mathbb{E} \left[\left| \int_{t/2}^t (Q_s^1)^{-1} dM_s(v) \right|^2 \right] &\leq \mathbb{E} \left[\int_{t/2}^t \sum_{k=1}^n \left| (Q_s^1)^{-1} (a_k(s) + b_k(s)) \right|^2 \right] \\ &\leq e^{\kappa t} \mathbb{E} \left[\int_{t/2}^t dM_s(v) \cdot dM_s(v) \right] = e^{\kappa t} (I_1(s) + I_2(s)), \end{aligned}$$

where

$$\begin{aligned} I_1(s) &= \mathbb{E} \left[\int_{t/2}^t 2 \left\| Q_s^2 \int_0^{t/2} (Q_r^2)^{-1} (//_r dB_r \wedge Q_r^1(v)) \right\|_{\Lambda^2}^2 ds \right] \\ I_2(s) &= \mathbb{E} \left[\int_{t/2}^t \left(\int_0^{t/2} \langle Q_r^1(v), //_r dB_r \rangle \right)^2 ds \right]. \end{aligned}$$

It is obvious that $I_2(s) \leq \frac{t^2 e^{\kappa_2^+ t/2}}{4} |v|^2$ and

$$I_1(s) \leq 2e^{\kappa_2 s} \int_{t/2}^t \mathbb{E} \left[\left\| \int_0^{t/2} (Q_r^2)^{-1} (//_r dB_r \wedge Q_r^1(v)) \right\|_{\Lambda^2}^2 \right] ds \quad (6.20)$$

Since we have

$$(Q_r^2)^{-1}(\llcorner_r dB_r \wedge Q_r^1(v)) = \sum_{k=1}^n (Q_r^2)^{-1}(\llcorner_r e_k \wedge Q_r^1(v)) dB_r^k,$$

so that

$$\begin{aligned} \mathbb{E} \left[\left\| \int_0^{t/2} (Q_r^2)^{-1}(\llcorner_r dB_r \wedge Q_r^1(v)) \right\|_{\Lambda^2}^2 \right] &= \sum_{k=1}^n \int_0^{t/2} \left\| (Q_r^2)^{-1}(\llcorner_r e_k \wedge Q_r^1(v)) \right\|^2 dr \\ &\leq \sum_{k=1}^n \int_0^{t/2} e^{\kappa_2 r} \|\llcorner_r e_k \wedge Q_r^1(v)\|^2 dr. \end{aligned}$$

But

$$\|\llcorner_r e_k \wedge Q_r^1(v)\|^2 = |Q_r^1(v)|^2 - \langle \llcorner_r e_k, Q_r^1(v) \rangle^2,$$

we therefore have

$$\sum_{k=1}^n \|\llcorner_r e_k \wedge Q_r^1(v)\|^2 = (n-1)|Q_r^1(v)|^2 \leq (n-1)e^{\kappa r} |v|^2.$$

To simplify calculation, we note that $e^{\kappa_p r} \leq e^{\kappa_p^+ t/2}$ since $r \in [0, t/2]$. Substituting these bounds first in (6.20), then together in (6.19), we finally get

$$|\square \mathbf{T}_t \phi| \leq \frac{2}{t} e^{3\kappa^+ t/2} \sqrt{2(n-1)e^{3\kappa_2^+ t/2} + 1} \sqrt{\mathbf{T}_t^M |\phi|^2},$$

and the result (6.18) follows. \square

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