

# Supplementary material for: Projected gradient descent for non-convex sparse spike estimation

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## I. DEFINITIONS AND PROPERTIES

For  $x_0 = \sum_{i=1,k} a_i \delta_{t_i}$ , we have  $y_l = (Ax_0)_l = \sum_{i=1,k} a_i e^{-j\langle \omega_l, t_i \rangle}$ .  
The Fourier transform of a function  $f$  supported on  $\mathbb{R}^d$  is

$$\hat{f}(\omega) = \int_{t \in bR^d} e^{-j\langle \omega, t \rangle} f(t) dt \quad (1)$$

Let  $E = \mathbb{R}^d$ . The Gaussian function  $G$  defined by  $G(u) = \frac{\sigma^d}{(\sqrt{2\pi})^d} e^{-\frac{\sigma^2}{2} \|u\|_2^2}$ . Then  $\hat{G}(v) = e^{-\frac{1}{2\sigma^2} \|v\|_2^2}$ .

## II. PROOFS

*Proof of Lemma III.1..* Using the fact that the  $\omega_l$  sample exactly the support of the Féjer Kernel and that, by definition,

$$K_f(t) = \sum_{l=1,m} \hat{K}(\omega_l) e^{j\langle \omega_l, t \rangle}, \quad (2)$$

we have

$$\begin{aligned} z(t) &= \sum_{l=1,m} d_l \left( \sum_{i=1,k} a_i c_l e^{-j\langle \omega_l, t_i \rangle} \right) e^{j\langle \omega_l, t \rangle} \\ &= \sum_{i=1,k} a_i \sum_{l=1,m} d_l c_l e^{j\langle \omega_l, t-t_i \rangle} \\ &= \sum_{i=1,k} a_i K_f(t-t_i). \end{aligned} \quad (3)$$

□

*Proof of Lemma III.2..* Using the properties of the Fourier transform  $\hat{G}$  of the Gaussian function  $G$  and  $\mathbb{E}$  being the expectation with respect to the distribution of the frequencies  $\omega_l$ , we have

$$\begin{aligned} \mathbb{E}(z(t)) &= \sum_{i=1,k} a_i \sum_{l=1,m} d_l c_l \mathbb{E}(e^{j\langle \omega_l, t-t_i \rangle}) \\ &= \sum_{i=1,k} a_i \sum_{l=1,m} d_l c_l \hat{G}(t_i - t) \\ &= \sum_{i=1,k} a_i K_g(t_i - t). \end{aligned} \quad (4)$$

Using the formula  $\mathbb{E}(|z(t) - E(z(t))|^2) = \mathbb{E}(|z(t)|^2) - |E(z(t))|^2$  and the previous Lemma, we just need to calculate

$$\begin{aligned} \mathbb{E}(|z(t)|^2) &= \mathbb{E} \left( \left| \sum_{i=1,k} a_i \sum_{l=1,m} d_l c_l e^{j\langle \omega_l, t-t_i \rangle} \right|^2 \right) \\ &= \mathbb{E} \left( \sum_{i_1=1,k} a_{i_1} \sum_{l=1,m} d_l c_l e^{j\langle \omega_l, t-t_{i_1} \rangle} \sum_{i_2=1,k} a_{i_2} \sum_{l=1,m} \bar{d}_l \bar{c}_l e^{-j\langle \omega_l, t-t_{i_2} \rangle} \right) \\ &= \sum_{i_1=1,k} \sum_{i_2=1,k} a_{i_1} a_{i_2} \sum_{l=1,m} \sum_{r=1,m} d_l c_l \bar{d}_r \bar{c}_r \mathbb{E} \left( e^{j\langle \omega_l, t-t_{i_1} \rangle} e^{-j\langle \omega_r, t-t_{i_2} \rangle} \right) \end{aligned} \quad (5)$$

For  $r = l$ ,

$$\mathbb{E} \left( e^{j\langle \omega_l, t-t_{i_1} \rangle} e^{-j\langle \omega_r, t-t_{i_2} \rangle} \right) = \hat{G}(t_{i_2} - t_{i_1}) \quad (6)$$

For  $r \neq l$ ,

$$\begin{aligned} \mathbb{E} \left( e^{j\langle \omega_l, t-t_{i_1} \rangle} e^{-j\langle \omega_r, t-t_{i_2} \rangle} \right) &= \mathbb{E} \left( e^{j\langle \omega_l, t-t_{i_1} \rangle} \right) \mathbb{E} \left( e^{-j\langle \omega_r, t-t_{i_2} \rangle} \right) \\ &= \hat{G}(t-t_{i_1}) \hat{G}(t-t_{i_2}) \end{aligned} \quad (7)$$

Hence

$$\begin{aligned}
\mathbb{E}(|z(t)|^2) &= \sum_{i_1=1,k} \sum_{i_2=1,k} a_{i_1} a_{i_2} \left( \sum_{l=1,m} |d_l c_l|^2 \hat{G}(t_{i_2} - t_{i_1}) + \sum_{l \neq r} d_l c_l \bar{d}_r \bar{c}_r \hat{G}(t - t_{i_1}) \hat{G}(t - t_{i_2}) \right) \\
&= \sum_{i_1=1,k} \sum_{i_2=1,k} a_{i_1} a_{i_2} \left( \sum_{l=1,m} \frac{1}{m^2} K_g(t_{i_2} - t_{i_1}) + \sum_{l \neq r} \frac{1}{m^2} K_g(t - t_{i_1}) K_g(t - t_{i_2}) \right) \\
&= \left(1 - \frac{1}{m}\right) \left( \sum_{i=1,k} a_i K_g(t - t_i) \right)^2 + \frac{1}{m} \sum_{i_1=1,k} \sum_{i_2=1,k} a_{i_1} a_{i_2} \frac{1}{m} K_g(t_{i_2} - t_{i_1}) \\
&= \left(1 - \frac{1}{m}\right) |E(z(t))|^2 + \frac{1}{m} \|x_0\|_K^2
\end{aligned} \tag{8}$$

Going back to the variance,

$$\mathbb{E}(|z(t) - E(z(t))|^2) = \frac{1}{m} |E(z(t))|^2 + \frac{1}{m} \|x_0\|_{K_g}^2 \tag{9}$$

□

*Proof of Lemma III.3..*

$$\|\sqrt{\epsilon_g^d} z_d\|_2^2 = \epsilon_g^d \sum_{i \in \Gamma} |z(t_i)|^2 \tag{10}$$

As  $z$  is continuous on  $\mathcal{B}_2(R)$ , it is integrable over  $\mathcal{B}_2(R)$  and this sum is approximating the Riemann integral of  $t \rightarrow |z(t)|^2$  over  $\mathcal{B}_2(R)$  as the grid step size  $\epsilon_g$  of  $\Gamma$  converges to 0. □