

**SUPPLEMENTARY MATERIAL FOR: PROJECTED GRADIENT DESCENT FOR
NON-CONVEX SPARSE SPIKE ESTIMATION**

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1. DEFINITIONS AND PROPERTIES

For $x_0 = \sum_{i=1,k} a_i \delta_{t_i}$, we have $y_l = (Ax_0)_l = \sum_{i=1,k} a_i e^{-j\langle \omega_l, t_i \rangle}$.
The Fourier transform of a function f supported on \mathbb{R}^d is

$$\hat{f}(\omega) = \int_{t \in bR^d} e^{-j\langle \omega, t \rangle} f(t) dt \quad (1)$$

Let $E = \mathbb{R}^d$. The Gaussian function G defined by $G(u) = \frac{\sigma^d}{(\sqrt{2\pi})^d} e^{-\frac{\sigma^2}{2} \|u\|_2^2}$. Then $\hat{G}(v) = e^{-\frac{1}{2\sigma^2} \|v\|_2^2}$.

2. PROOFS

Proof of Lemma 3.1. Using the fact that the ω_l sample exactly the support of the Féjer Kernel and that, by definition,

$$K_f(t) = \sum_{l=1,m} \hat{K}(\omega_l) e^{j\langle \omega_l, t \rangle}, \quad (2)$$

we have

$$\begin{aligned} z(t) &= \sum_{l=1,m} d_l \left(\sum_{i=1,k} a_i c_l e^{-j\langle \omega_l, t_i \rangle} \right) e^{j\langle \omega_l, t \rangle} \\ &= \sum_{i=1,k} a_i \sum_{l=1,m} d_l c_l e^{j\langle \omega_l, t-t_i \rangle} \\ &= \sum_{i=1,k} a_i K_f(t-t_i). \end{aligned} \quad (3)$$

□

Proof of Lemma 3.2. Using the properties of the Fourier transform \hat{G} of the Gaussian function G and \mathbb{E} being the expectation with respect to the distribution of the frequencies ω_l , we have

$$\begin{aligned} \mathbb{E}(z(t)) &= \sum_{i=1,k} a_i \sum_{l=1,m} d_l c_l \mathbb{E}(e^{j\langle \omega_l, t-t_i \rangle}) \\ &= \sum_{i=1,k} a_i \sum_{l=1,m} d_l c_l \hat{G}(t_i-t) \\ &= \sum_{i=1,k} a_i K_g(t_i-t). \end{aligned} \quad (4)$$

Using the formula $\mathbb{E}(|z(t) - E(z(t))|^2) = \mathbb{E}(|z(t)|^2) - |E(z(t))|^2$ and the previous Lemma, we just need to calculate

$$\begin{aligned} \mathbb{E}(|z(t)|^2) &= \mathbb{E} \left(\left| \sum_{i=1,k} a_i \sum_{l=1,m} d_l c_l e^{j\langle \omega_l, t-t_i \rangle} \right|^2 \right) \\ &= \mathbb{E} \left(\sum_{i_1=1,k} a_{i_1} \sum_{l=1,m} d_l c_l e^{j\langle \omega_l, t-t_{i_1} \rangle} \sum_{i_2=1,k} a_{i_2} \sum_{l=1,m} \bar{d}_l \bar{c}_l e^{-j\langle \omega_l, t-t_{i_2} \rangle} \right) \\ &= \sum_{i_1=1,k} \sum_{i_2=1,k} a_{i_1} a_{i_2} \sum_{l=1,m} \sum_{r=1,m} d_l c_l \bar{d}_r \bar{c}_r \mathbb{E} \left(e^{j\langle \omega_l, t-t_{i_1} \rangle} e^{-j\langle \omega_r, t-t_{i_2} \rangle} \right) \end{aligned} \quad (5)$$

For $r = l$,

$$\mathbb{E} \left(e^{j\langle \omega_l, t-t_{i_1} \rangle} e^{-j\langle \omega_r, t-t_{i_2} \rangle} \right) = \hat{G}(t_{i_2} - t_{i_1}) \quad (6)$$

For $r \neq l$,

$$\begin{aligned}\mathbb{E}\left(e^{j\langle\omega_l, t-t_{i_1}\rangle} e^{-j\langle\omega_r, t-t_{i_2}\rangle}\right) &= \mathbb{E}\left(e^{j\langle\omega_l, t-t_{i_1}\rangle}\right) \mathbb{E}\left(e^{-j\langle\omega_r, t-t_{i_2}\rangle}\right) \\ &= \hat{G}(t-t_{i_1}) \hat{G}(t-t_{i_2})\end{aligned}\quad (7)$$

Hence

$$\begin{aligned}\mathbb{E}(|z(t)|^2) &= \sum_{i_1=1,k} \sum_{i_2=1,k} a_{i_1} a_{i_2} \left(\sum_{l=1,m} |d_l c_l|^2 \hat{G}(t_{i_2}-t_{i_1}) + \sum_{l \neq r} d_l c_l \bar{d}_r \bar{c}_r \hat{G}(t-t_{i_1}) \hat{G}(t-t_{i_2}) \right) \\ &= \sum_{i_1=1,k} \sum_{i_2=1,k} a_{i_1} a_{i_2} \left(\sum_{l=1,m} \frac{1}{m^2} K_g(t_{i_2}-t_{i_1}) + \sum_{l \neq r} \frac{1}{m^2} K_g(t-t_{i_1}) K_g(t-t_{i_2}) \right) \\ &= \left(1 - \frac{1}{m}\right) \left(\sum_{i=1,k} a_i K_g(t-t_i) \right)^2 + \frac{1}{m} \sum_{i_1=1,k} \sum_{i_2=1,k} a_{i_1} a_{i_2} \frac{1}{m} K_g(t_{i_2}-t_{i_1}) \\ &= \left(1 - \frac{1}{m}\right) |E(z(t))|^2 + \frac{1}{m} \|x_0\|_K^2\end{aligned}\quad (8)$$

Going back to the variance,

$$\mathbb{E}(|z(t) - E(z(t))|^2) = \frac{1}{m} |E(z(t))|^2 + \frac{1}{m} \|x_0\|_{K_g}^2 \quad (9)$$

□

Proof of Lemma 3.4.

$$\|\sqrt{\epsilon_g^d} z_d\|_2^2 = \epsilon_g^d \sum_{i \in \Gamma} |z(t_i)|^2 \quad (10)$$

As z is continuous on $\mathcal{B}_2(R)$, it is integrable over $\mathcal{B}_2(R)$ and this sum is approximating the Riemann integral of $t \rightarrow |z(t)|^2$ over $\mathcal{B}_2(R)$ as the grid step size ϵ_g of Γ converges to 0. □