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Exponential sums with reducible polynomials

Cécile Dartyge* Greg Martin†

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Abstract: Hooley proved that if $f \in \mathbb{Z}[X]$ is irreducible of degree ≥ 2 , then the fractions $\{r/n\}$, $0 < r < n$ with $f(r) \equiv 0 \pmod{n}$, are uniformly distributed in $]0, 1[$. In this paper we study such problems for reducible polynomials of degree 2 and 3 and for finite products of linear factors. In particular, we establish asymptotic formulas for exponential sums over these normalized roots.

Key words and phrases: exponentials sums, equidistribution of polynomial roots

1 Introduction

Let $f(X)$ be a polynomial of degree at least 2 with integer coefficients, and let h be a nonzero integer. We consider for $x \geq 1$ the exponential sums

$$S(f, x, h) = \sum_{n \leq x} \sum_{\substack{r \pmod{n} \\ f(r) \equiv 0 \pmod{n}}} e\left(\frac{hr}{n}\right) \quad (1)$$

with the standard notation $e(t) = \exp(2i\pi t)$; our interest is in fixed f and h while x tends to infinity. In the case $h = 1$ we will write simply $S(f, x)$. Hooley [11, Theorem 1] proved that if f is irreducible, then $S(f, x, h) = o(x)$ when $x \rightarrow \infty$. By Weyl's criterion, this implies that the fractions r/n , where $0 < r < n$ and $f(r) \equiv 0 \pmod{n}$, are uniformly distributed in $]0, 1[$. The condition that $\deg f \geq 2$ is necessary in this result, as the roots of a linear polynomial are not equidistributed in $]0, 1[$. For example,

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if $f(n) = n + b$, the exponential term in equation (1) is $e(-hb/n) = 1 + O(1/n)$ for fixed h and b , and then $S(f, x, h) = x + O(\log x)$. In general, when $f(n) = an + b$ one can similarly obtain a formula of the form $S(f, x, h) = xC(f, h) + o(x)$ for some constant $C(f, h)$.

When $f(n) = n^2 - D$ with D not a square, Hooley [10] obtained (using a different method) the more precise bound $S(f, x, h) \ll_{f,h} x^{3/4} \log x$. The exponent $3/4$ in this result was improved to $2/3$ via the theory of automorphic forms by Hejhal [8], Bykovski [2], and Zavorotny [21]. A variant of this problem is the distribution of the fractions r/p , where $0 < r < p$ and $f(r) \equiv 0 \pmod{p}$, for primes p . Duke, Friedlander, and Iwaniec [5] proved that for $f(X) = aX^2 + 2bX + c$ with $ac - b^2 > 0$, we have

$$\sum_{p \leq x} \sum_{\substack{0 \leq r < p \\ f(r) \equiv 0 \pmod{p}}} e\left(\frac{hr}{p}\right) = o(\pi(x)).$$

In all of the above results, the polynomial f is assumed to be irreducible; what can we say about the sums $S(f, x, h)$ when f is reducible? In this paper we examine reducible polynomials of degree 2 and 3 (in which case f has at least one linear factor) and on polynomials that factor completely into linear factors. The hope is to obtain, not just an upper bound, but an actual asymptotic formula for $S(f, x, h)$, analogously to the situation described above where f is itself linear.

For example, Sitar and the second author [15] obtained a general bound for reducible quadratic polynomials f with discriminant D :

$$S(f, x, h) \ll \sqrt{D} \prod_{p|h} \left(1 + \frac{7}{\sqrt{p}}\right) x (\log x)^{\sqrt{2}-1} (\log \log x)^{5/2}. \quad (2)$$

Since the number of summands of $S(f, x, h)$ has order $x \log x$ in this case, a consequence is that the fractions r/n , where $0 < r < n$ and $f(r) \equiv 0 \pmod{n}$, are uniformly distributed in $]0, 1[$ for reducible quadratic polynomials. However, this bound is still large enough to disguise a potential asymptotic main term of size x caused by the roots of the linear factors of f .

In our first theorem, which is proved in Section 3, we provide such an asymptotic formula for exponential sums with reducible quadratics:

Theorem 1. *Let a, b, c, d be fixed integers with $ac \neq 0$, $(a, b) = (c, d) = 1$, and $ad \neq bc$, and set $f(n) = (an + b)(cn + d)$. Then for any nonzero integer h , there exists $C(f, h) \in \mathbb{C}$ such that for any $\varepsilon > 0$, we have*

$$S(f, x, h) = xC(f, h) + O(x^{4/5+\varepsilon}), \quad (3)$$

where the implicit constant depends on f , h , and ε . When $(h, ad - bc) = 1$, the error term in equation (3) can be improved to $O(x^{3/4+\varepsilon})$.

The proof of this theorem provides an explicit but complicated formula for $C(f, h)$ (see equation (13) below). In the particular case $h = 1$, we obtain $S(f, x) = xC(f, 1) + O(x^{4/5+\varepsilon})$ with

$$C(f, 1) = \left(\frac{\mu(a)}{a} + \frac{\mu(c)}{c}\right) \frac{6}{\pi^2} \prod_{p|ac} \left(1 - \frac{1}{p^2}\right)^{-1}.$$

(Note in particular that it is possible for $C(f, 1)$ to equal 0, namely if neither a nor c is squarefree; in such a case, Theorem 1 is technically an upper bound rather than an asymptotic formula. The analogous pedantic comment applies to Theorem 4 below.)

Our second result handles the case $f(n) = n(n^2 + 1)$. In Section 4 we will prove:

Theorem 2. *For $f(n) = n(n^2 + 1)$, we have*

$$S(f, x) = \frac{3x}{4} \prod_{p \equiv 1 \pmod{4}} \left(1 - \frac{2}{p^2}\right) + O\left(\frac{x(\log \log x)^{7/2}}{(\log x)^{1-1/\sqrt{2}}}\right).$$

It is also possible to generalize Theorem 2 to $S(f, x, h)$ for a general nonzero integer h , as in the work of Hooley, though we do not do so here. It is likely possible to extend Theorem 2 to general products of two polynomials $f_1 f_2$ with f_1 linear and f_2 irreducible quadratic by adapting some ideas of [18] or [1]. However, such a generalization with a general f_2 in place of $n^2 + 1$ is not straightforward; the case where f_2 has positive discriminant seems more difficult.

The next result, which we prove in Sections 5 and 6.1, handles in detail a special product of three linear polynomials:

Theorem 3. *For $f(n) = n(n+1)(2n+1)$, we have*

$$S(f, x) = x \prod_{p \geq 3} \left(1 - \frac{2}{p^2}\right) + O\left(\frac{x(\log \log x)^6}{(\log x)^2}\right).$$

It was clear that this result could be generalized to arbitrary products of three linear polynomials. However, after reading an earlier version of this manuscript, the anonymous referee pointed out that the method can be extended to give an asymptotic formula for $S(f, x, h)$ when f is the product of four linear polynomials, as well as a nontrivial upper bound for $S(f, x, h)$ when f is a product of any number of linear polynomials. We prove the following result in Sections 6.2 and 7:

Theorem 4. *Let h be a nonzero integer, let $k \geq 3$ be an integer, and let $f(n) = \prod_{i=1}^k (a_i n + b_i)$ where the coefficients $a_1, b_1, \dots, a_k, b_k$ are integers satisfying $(a_j, b_j) = 1$ for all $1 \leq j \leq k$ and $a_i b_j \neq a_j b_i$ for all $1 \leq i < j \leq k$.*

(i) *If $k = 3$ or $k = 4$, then there exists $C(f, h) \in \mathbb{C}$ such that*

$$S(f, x, h) = C(f, h)x + O(x(\log x)^{k-5}(\log \log x)^6).$$

(ii) *If $k \geq 5$, then $S(f, x, h) \ll x(\log x)^{k-5}(\log \log x)^6$.*

In both cases, the implicit constants depend on f and h .

As the number of fractions r/n , where $0 < r < n$ and $f(r) \equiv 0 \pmod{n}$, for all $n \leq x$ has order of magnitude $x(\log x)^{\kappa-1}$, where κ is the number of distinct irreducible factors of f , these theorems all imply that the appropriate sequences of fractions are uniformly distributed in $]0, 1[$.

It is also worth remarking that it is only for convenience that we hypothesize that the linear factors in Theorems 1 and 4 are not proportional to one another. Indeed, more generally consider a polynomial

$f = g^2 h$: the roots of f modulo squarefree integers n are unaffected by the repeated factor g^2 , while if $p^2 \mid n$ then the roots of $g^2 \pmod{n}$ form arithmetic progressions with common difference n/p , and thus their contribution to the exponential sum $S(f, x, h)$ vanishes completely.

Before proceeding to the proofs of our theorems, we establish in the next section a lemma that is used repeatedly throughout the paper. We adopt the following notation and conventions throughout this paper: when a and b are relatively prime integers, \bar{a}_b will denote an integer such that $\bar{a}_b a \equiv 1 \pmod{b}$. Furthermore, when the context will be clear, we will also use the simplified notation \bar{a} for this multiplicative inverse of a to the implied modulus b , which is often the denominator of the fraction in whose numerator \bar{a} appears. The letter p usually denotes a prime number. We adopt the convention throughout that $e(t) = \exp(2i\pi t)$ unless t contains an expression of the form $\frac{\bar{a}}{b}$ where $(a, b) > 1$, in which case $e(t) = 0$. For example, the expression $\sum_{0 \leq n \leq p-1} e(\frac{\bar{n}(n-1)}{p})$ is to be interpreted as $0 + 0 + \sum_{2 \leq n \leq p-1} e(\frac{\bar{n}_p(n-1)_p}{p})$.

2 Exponential sums involving multiplicative inverses

We begin by establishing the following lemma on exponential sums (a slight variation of a result of Hooley [12, Lemma 3]), which is the crucial tool of our paper.

Lemma 1. *For any $y < z$, $q \geq 2$, $t \in \mathbb{Z}$ and $(m, q) = 1$ we have*

$$\sum_{\substack{y < n \leq z \\ (n, q) = 1 \\ n \equiv u \pmod{m}}} e\left(\frac{t\bar{n}_q}{q}\right) = \frac{z-y}{mq} \mu\left(\frac{q}{(q, t)}\right) \frac{\varphi(q)}{\varphi(q/(t, q))} + O(\sqrt{q(t, q)} \cdot \tau(q) \log q).$$

In particular,

$$\sum_{\substack{y < n \leq z \\ (n, q) = 1}} e\left(\frac{t\bar{n}_q}{q}\right) = \frac{z-y}{q} \mu\left(\frac{q}{(q, t)}\right) \frac{\varphi(q)}{\varphi(q/(t, q))} + O(\sqrt{q(t, q)} \cdot \tau(q) \log q). \quad (4)$$

(While we have written the statement of the lemma, for ease of reference, with explicit subscripts on the multiplicative inverses \bar{n}_q and explicit coprimality conditions of summation, we immediately revert to our conventions of suppressing these notational signals.)

Proof. We begin by collecting the summands according to the value of $n \pmod{q}$, which we then detect

using a further additive character:

$$\begin{aligned}
 \sum_{\substack{y < n \leq z \\ n \equiv u \pmod{m}}} e\left(\frac{t\bar{n}}{q}\right) &= \sum_{a \pmod{q}} e\left(\frac{t\bar{a}}{q}\right) \sum_{\substack{y < n \leq z \\ n \equiv a \pmod{q} \\ n \equiv u \pmod{m}}} 1 \\
 &= \sum_{a \pmod{q}} e\left(\frac{t\bar{a}}{q}\right) \sum_{\substack{y < n \leq z \\ n \equiv u \pmod{m}}} \frac{1}{q} \sum_{h=0}^{q-1} e\left(\frac{h(n-a)}{q}\right) \\
 &= \frac{1}{q} \sum_{h=0}^{q-1} \left(\sum_{a \pmod{q}} e\left(\frac{t\bar{a} - ha}{q}\right) \right) \left(\sum_{\substack{y < n \leq z \\ n \equiv u \pmod{m}}} e\left(\frac{hn}{q}\right) \right), \tag{5}
 \end{aligned}$$

with the convention $e(t\bar{a}/q) = 0$ if $(a, q) > 1$ as mentioned in the end of the introduction. The $h = 0$ summand contributes the main term:

$$\begin{aligned}
 \frac{1}{q} \left(\sum_{a \pmod{q}} e\left(\frac{t\bar{a}}{q}\right) \right) \left(\sum_{\substack{y < n \leq z \\ n \equiv u \pmod{m}}} 1 \right) &= \frac{1}{q} \left(\mu\left(\frac{q}{(q, t)}\right) \frac{\varphi(q)}{\varphi(q/(t, q))} \right) \left(\frac{z-y}{m} + O(1) \right) \\
 &= \frac{z-y}{qm} \mu\left(\frac{q}{(q, t)}\right) \frac{\varphi(q)}{\varphi(q/(t, q))} + O(1),
 \end{aligned}$$

since the first sum on the left-hand side is a complete Ramanujan sum which has been evaluated classically (see [16, equation (4.7)]). As for the summands where $h \neq 0$, the first inner sum on the right-hand side of equation (5) is a complete Kloosterman sum, which was shown by Hooley [9, Lemma 2] (using Weil's bounds for exponential sums) to be $\ll \sqrt{q(t, q)} \cdot \tau(q)$; therefore

$$\begin{aligned}
 \frac{1}{q} \sum_{h=1}^{q-1} \left(\sum_{a \pmod{q}} e\left(\frac{t\bar{a} - ha}{q}\right) \right) \left(\sum_{\substack{y < n \leq z \\ n \equiv u \pmod{m}}} e\left(\frac{hn}{q}\right) \right) \\
 \ll \sqrt{q(t, q)} \cdot \tau(q) \cdot \frac{1}{q} \sum_{h=1}^{q-1} \left| \sum_{\substack{y < n \leq z \\ n \equiv u \pmod{m}}} e\left(\frac{hn}{q}\right) \right| \\
 = \sqrt{q(t, q)} \cdot \tau(q) \cdot \frac{1}{q} \sum_{h=1}^{q-1} \left| \sum_{(y-u)/m < \lambda \leq (z-u)/m} e\left(\frac{h(m\lambda + u)}{q}\right) \right| \\
 = \sqrt{q(t, q)} \cdot \tau(q) \cdot \frac{1}{q} \sum_{\eta=1}^{q-1} \left| e\left(\frac{\eta \bar{m} u}{q}\right) \sum_{(y-u)/m < \lambda \leq (z-u)/m} e\left(\frac{\eta \lambda}{q}\right) \right|,
 \end{aligned}$$

since $(m, q) = 1$ and thus the change of variables $\eta = hm$ permutes the nonzero residue classes modulo q . This last sum is a geometric series and is consequently $\ll \|\eta/q\|^{-1}$, where $\|u\|$ is the distance from u to the nearest integer. Therefore we have the following bound for these summands:

$$\ll \sqrt{q(t, q)} \cdot \tau(q) \cdot \frac{2}{q} \sum_{\eta=1}^{q/2} \frac{q}{\eta} \ll \sqrt{q(t, q)} \cdot \tau(q) \log q,$$

as required. □

3 Reducible quadratics (Theorem 1)

Throughout this section, we consider $f(n) = (an + b)(cn + d)$ with a, b, c, d as in Theorem 1. All implicit constants in this section may depend on f, h , and (where appropriate) ε .

Define $\Delta = ad - bc$. The simultaneous congruences $an + b \equiv cn + d \equiv 0 \pmod{p}$ have a solution only when $-b\bar{a} \equiv -d\bar{c} \pmod{p}$, or equivalently when $p \mid \Delta$. We start with the case $(h, ac\Delta) = 1$; in the following section we indicate how to deal with the general case.

3.1 The case $(ac\Delta, h) = 1$

We begin by handling some coprimality problem between the denominators n and Δ .

Lemma 2. *We may write*

$$S(f, x, h) = \sum_{\substack{g \mid \Delta \\ (g, ac) = 1}} \mu^2(g) \sum_{\substack{m \leq x/g \\ (m, \Delta) = 1}} \sum_{\substack{r \pmod{mg} \\ (ar+b)(cr+d) \equiv 0 \pmod{mg}}} e\left(\frac{hr}{mg}\right).$$

Proof. First we sort by $g = (n, \Delta)$:

$$S(f, x, h) = \sum_{g \mid \Delta} \sum_{\substack{m \leq x/g \\ (m, \Delta/g) = 1}} \sum_{\substack{r \pmod{mg} \\ (ar+b)(cr+d) \equiv 0 \pmod{mg}}} e\left(\frac{hr}{mg}\right).$$

If $p \mid g \mid \Delta$ and $p \mid a$ then $p \mid c$ as well, in which case $(ax + b)(cx + d) \equiv bd \not\equiv 0 \pmod{p}$ and there are no roots modulo p . Similarly, if $p \mid (g, c)$, then f has no root modulo p . Thus we can add the condition of summation $(g, ac) = 1$.

We can also assume that $p^2 \nmid mg$ for every $p \mid \Delta$. Indeed, if $mg = qp^2$ for such a prime p , and if r is a root of $(ax + b)(cx + d)$ modulo qp^2 , then both $ar + b \equiv 0 \pmod{p}$ and $cr + d \equiv 0 \pmod{p}$ since $p \mid \Delta$; and then all of $r + qp, r + 2qp, \dots, r + (p - 1)qp$ are roots of $(ax + b)(cx + d)$ modulo qp^2 , since

$$f(r + \lambda qp) \equiv (ar + b)(cr + d) + \lambda qp(a(cr + d) + b(ar + b)) \equiv 0 \pmod{qp^2}.$$

The exponential sum over these roots $r, r + qp, \dots, r + (p - 1)qp$ vanishes (here we use the assumption $(h, \Delta) = 1$, so that $p \nmid h$), which justifies omitting these terms.

In particular, we may assume that g is squarefree (whence the introduction of the factor $\mu^2(g)$) and that $(m, g) = 1$ (which combines with the existing condition $(m, \Delta/g) = 1$ to yield simply $(m, \Delta) = 1$), completing the proof. \square

Next we separate the congruence conditions between the two factors of f .

Lemma 3. *Suppose that $g \mid \Delta$ is squarefree, $(g, c) = 1$, and $(m, \Delta) = 1$. The roots of $(ar + b)(cr + d) \equiv 0 \pmod{mg}$ are in one-to-one correspondence with the factorizations $k\ell = m$ with $(k, \ell) = 1$, $(k, a) = 1$, and $(\ell, c) = 1$. The root r corresponds to the solution of the system of congruences*

$$\begin{aligned} ar + b &\equiv 0 \pmod{k} \\ cr + d &\equiv 0 \pmod{\ell} \\ r &\equiv r_g \pmod{g}, \end{aligned}$$

where r_g is the residue class $r_g \equiv -d\bar{c} \pmod{g}$.

Proof. It is straightforward to check that the factorization corresponding to a root r of $(ar+b)(cr+d) \equiv 0 \pmod{mg}$ is $k = (ar+b, m)$ and $\ell = (cr+d, m)$, and that this correspondence is the inverse function to the correspondence described in the statement of the lemma. \square

By the Chinese remainder theorem, the solution of the system of congruences given in Lemma 3 can be written as

$$r = -b\bar{a}_k\bar{\ell}_k g\bar{g}_k - d\bar{c}_\ell k\bar{k}_\ell g\bar{g}_\ell + r_g k\bar{k}_g \bar{\ell}_g.$$

We thus obtain

$$\begin{aligned} S(f, x, h) &= \sum_{\substack{g|\Delta \\ (g, ac)=1}} \mu^2(g) \sum_{\substack{m \leq x/g \\ (m, \Delta)=1}} \sum_{\substack{k\ell=m \\ (k, \ell)=1 \\ (k, a)=(\ell, c)=1}} e\left(\frac{-hb\bar{a}_k\bar{\ell}_k g\bar{g}_k - hd\bar{c}_\ell k\bar{k}_\ell g\bar{g}_\ell + hr_g k\bar{k}_g \bar{\ell}_g}{mg}\right) \\ &= \sum_{\substack{g|\Delta \\ (g, ac)=1}} \mu^2(g) \sum_{\substack{k\ell \leq x/g \\ (k, \ell)=1 \\ (k, a)=(\ell, c)=1 \\ (k\ell, \Delta)=1}} e\left(\frac{-hb\bar{a}_k\bar{\ell}_k \bar{g}_k}{k}\right) e\left(\frac{-hd\bar{c}_\ell k\bar{g}_\ell}{\ell}\right) e\left(\frac{hr_g k\bar{k}_g \bar{\ell}_g}{g}\right). \end{aligned}$$

We split this into two sums $S(f, x) = S_1(x) + S_2(x)$, where $k \leq \sqrt{x/g}$ in $S_1(x)$ and $k > \sqrt{x/g}$ in $S_2(x)$.

In $S_1(x)$, we use the standard “inversion formula” (obtained with Bézout’s identity) for $(u, v) = 1$,

$$\frac{\bar{u}}{v} + \frac{\bar{v}}{u} \equiv \frac{1}{uv} \pmod{1}. \quad (6)$$

This formula, used on the second exponential factor in the inner sum above, allows us to move ℓ out of the denominators of the exponential terms and write $S_1(x)$ as

$$\sum_{\substack{g|\Delta \\ (g, ac)=1}} \mu^2(g) \sum_{\substack{k \leq \sqrt{x/g} \\ (k, a\Delta)=1}} \sum_{\substack{\ell \leq x/gk \\ (\ell, k)=(\ell, c\Delta)=1}} e\left(\frac{-hb\bar{a}_k\bar{\ell}_k \bar{g}_k}{k}\right) e\left(\frac{hd\bar{\ell}_{cgk}}{cgk}\right) e\left(\frac{-hd}{cgk\ell}\right) e\left(\frac{hr_g k\bar{k}_g \bar{\ell}_g}{g}\right).$$

Next we remark that $e(-d/cgk\ell) = 1 + O(1/gk\ell)$. This effect of this error term is sufficiently small:

$$\begin{aligned} S_1(x) &= \sum_{\substack{g|\Delta \\ (g, ac)=1}} \mu^2(g) \sum_{\substack{k \leq \sqrt{x/g} \\ (k, a\Delta)=1}} \sum_{\substack{\ell \leq x/gk \\ (\ell, k)=(\ell, c\Delta)=1}} e\left(\frac{(-bcg\bar{a}_k\bar{g}_k + d + ckr_g\bar{k}_g)h\bar{\ell}_{cgk}}{cgk}\right) \\ &\quad + O((\log x)^2), \end{aligned}$$

since $\bar{\ell}_k \equiv \bar{\ell}_{cgk} \pmod{k}$ and $\bar{\ell}_g \equiv \bar{\ell}_{cgk} \pmod{g}$. (We remind the reader that the implicit constants in this section may depend on f , h , and ε .)

We rewrite the inner sum over ℓ as

$$\sum_{\substack{\ell \leq x/gk \\ (\ell, ck\Delta)=1}} e\left(\frac{\frac{\Delta}{g}(-bcg\bar{a}_k\bar{g}_k + d + ckr_g\bar{k}_g)h\bar{\ell}_{ck\Delta}}{ck\Delta}\right), \quad (7)$$

and then apply equation (4) with $q = ck\Delta$ and $t = ht'$, where $t' = \frac{\Delta}{g}(-bcg\bar{a}_k\bar{g}_k + d + ckr_g\bar{k}_g)$. Since $d + ckr_g\bar{k}_g \equiv d + cr_g \equiv 0 \pmod{g}$, we see that t' is a multiple of Δ but is relatively prime to both c and k . We thus obtain

$$S_1(x) = x \sum_{\substack{g|\Delta \\ (g,ac)=1}} \mu^2(g) \sum_{\substack{k \leq \sqrt{x/g} \\ (k,ac\Delta)=1}} \frac{\mu\left(\frac{ck}{(ck,h)}\right)}{cg\Delta k^2} \frac{\varphi(ck\Delta)}{\varphi(ck/(ck,h))} \\ + O\left(\sum_{\substack{g|\Delta \\ (g,ac)=1}} \mu^2(g) \sum_{\substack{k \leq \sqrt{x/g} \\ (k,a\Delta)=1}} (k(k,h))^{1/2+\varepsilon} \right).$$

In the previous formula we could insert the condition $(c,k) = 1$ in the main term, because the fact that $(h,c) = 1$ means that $\mu(ck/(ck,h)) = 0$ if $(c,k) > 1$. The error term is $O(x^{3/4+\varepsilon})$. Let $T_1(x)$ denote the main term. By the coprimality conditions among a, c, Δ, k , we get

$$T_1(x) = x \sum_{\substack{g|\Delta \\ (g,ac)=1}} \frac{\mu^2(g)\mu\left(\frac{c}{(c,h)}\right)}{cg\Delta} \frac{\varphi(c\Delta)}{\varphi\left(\frac{c}{(c,h)}\right)} \sum_{\substack{k \leq \sqrt{x/g} \\ (k,ac\Delta)=1}} \frac{\mu\left(\frac{k}{(k,h)}\right)}{k^2} \frac{\varphi(k)}{\varphi\left(\frac{k}{(k,h)}\right)}.$$

Let λ_h denote the function in the inner sum over k :

$$\lambda_h(k) = \frac{\mu\left(\frac{k}{(k,h)}\right)}{k^2} \frac{\varphi(k)}{\varphi\left(\frac{k}{(k,h)}\right)}.$$

Note that $\lambda_h(k)$ is a multiplicative function of k . When $p \nmid h$, we have $\lambda_h(p) = -1/p^2$ and $\lambda_h(p^v) = 0$ for $v \geq 2$. When $p \mid h$, if $v_p(h)$ is the p -adic valuation of h , we have:

$$\lambda_h(p^v) = \begin{cases} \frac{\varphi(p^v)}{p^{2v}} & \text{if } v \leq v_p(h) \\ -\frac{1}{p^{v+1}} & \text{if } v = v_p(h) + 1 \\ 0 & \text{if } v \geq v_p(h) + 2; \end{cases}$$

in particular, $|\lambda_h(p^v)| \leq p^{v_p(h)}/p^{2v}$ always, which implies that $|\lambda_h(n)| \leq h/n^2 \ll 1/n^2$ since h is fixed. We deduce that

$$\sum_{\substack{k \leq \sqrt{x/g} \\ (k,ac\Delta)=1}} \lambda_h(k) = \sum_{(k,ac\Delta)=1} \lambda_h(k) + O(hx^{-1/2+\varepsilon}),$$

from which we obtain $T_1(x) = C_1(h, a, c, \Delta)x$ with

$$C_1(h, a, c, \Delta) = \frac{6}{\pi^2} \frac{\mu\left(\frac{c}{(c,h)}\right)\varphi(c\Delta)}{c\Delta\varphi\left(\frac{c}{(c,h)}\right)} \prod_{\substack{p|\Delta \\ (p,ac)=1}} \left(1 + \frac{1}{p}\right) \\ \prod_{p|ahc\Delta} \left(1 - \frac{1}{p^2}\right)^{-1} \prod_{\substack{p|h \\ (p,ac\Delta)=1}} \left(1 + \frac{1}{p} - \frac{2}{p^{v_p(h)+1}}\right). \quad (8)$$

In the particular case $h = 1$, this gives

$$S_1(x) = \frac{6x}{\pi^2} \frac{\mu(c)}{c} \prod_{p|ac} \left(1 - \frac{1}{p^2}\right)^{-1} + O(x^{3/4+\varepsilon}).$$

In this last computation, we have used the fact that if $p \mid (c, \Delta)$, then $p \mid a$. In the same way, we derive the asymptotic formula

$$S_2(x) = C_1(h, c, a, \Delta)x + O(x^{3/4+\varepsilon}) \quad (9)$$

which, when $h = 1$, simplifies to

$$S_2(x) = \frac{6x}{\pi^2} \frac{\mu(a)}{a} \prod_{p|ac} \left(1 - \frac{1}{p^2}\right)^{-1} + O(x^{3/4+\varepsilon}).$$

This finishes the proof of Theorem 1 in the case $(h, ac\Delta) = 1$.

3.2 The case $(h, ac\Delta) \neq 1$

The main difference when dealing with the case $(h, ac\Delta) \neq 1$ is that we lose some cancellation observed in the proof of Lemma 2. We use the notation $m \mid n^\infty$ to indicate that $p \mid m \Rightarrow p \mid n$, and we sort the summands according to the prime factors they share with $(h, ac\Delta)$:

$$S(f, x, h) = \sum_{\delta \mid (h, ac\Delta)^\infty} \sum_{\substack{n \leq x/\delta \\ (n, (h, ac\Delta))=1}} \sum_{\substack{0 \leq r < \delta n \\ f(r) \equiv 0 \pmod{\delta n}}} e\left(\frac{hr}{\delta n}\right). \quad (10)$$

Since $\deg f = 2$, the number of roots of $f(r) \equiv 0 \pmod{\delta n}$ is less than $2^{\omega(\delta n)}$. Defining $B = x^{1/5-\varepsilon}$ for some sufficiently small $\varepsilon > 0$, we split this sum as $S(f, x, h) = S_>(x, h) + S_<(x, h)$, where $\delta > B$ in $S_>(x, h)$ and $\delta \leq B$ in $S_<(x, h)$. When δ is large, a trivial upper bound is sufficient:

$$\begin{aligned} S_>(x, h) &\ll x \log x \sum_{\substack{\delta \mid (h, ac\Delta)^\infty \\ \delta > B}} \frac{2^{\omega(\delta)}}{\delta} \\ &\ll x \log x \sum_{\delta \mid (h, ac\Delta)^\infty} \frac{\delta^{\varepsilon_1}}{\delta} \left(\frac{\delta}{B}\right)^{1-2\varepsilon_1} = \frac{x \log x}{B^{1-2\varepsilon_1}} \prod_{p \mid (h, ac\Delta)} \left(1 - \frac{1}{p^{\varepsilon_1}}\right)^{-1}. \end{aligned} \quad (11)$$

When δ is not large, we adapt the method of the previous section. Since $(n, \delta) = 1$, we have by the Chinese remainder theorem:

$$S_<(x, h) = \sum_{\substack{\delta \mid (h, ac\Delta)^\infty \\ \delta \leq B}} \sum_{\substack{n \leq x/\delta \\ (n, (h, ac\Delta))=1}} \left(\sum_{\substack{0 \leq r_0 < \delta \\ f(r_0 n) \equiv 0 \pmod{\delta}}} e\left(\frac{hr_0}{\delta}\right) \right) \left(\sum_{\substack{0 \leq r_1 < n \\ f(r_1 \delta) \equiv 0 \pmod{n}}} e\left(\frac{hr_1}{n}\right) \right).$$

In order to suppress the dependence of n in the summation in r_0 , we split this sum according to n modulo δ :

$$S_{\leq}(x, h) = \sum_{\substack{\delta | (h, ac\Delta)^\infty \\ \delta \leq B}} \sum_{\substack{0 \leq \alpha < \delta \\ (\alpha, \delta) = 1}} \left(\sum_{\substack{0 \leq r_0 < \delta \\ f(r_0\alpha) \equiv 0 \pmod{\delta}}} e\left(\frac{hr_0}{\delta}\right) \right) \\ \times \left(\sum_{\substack{n \leq x/\delta \\ (n, (h, ac\Delta)) = 1 \\ n \equiv \alpha \pmod{\delta}}} \sum_{\substack{0 \leq r_1 < n \\ f(r_1\delta) \equiv 0 \pmod{n}}} e\left(\frac{hr_1}{n}\right) \right).$$

Let $G(\alpha, \delta)$ denote the above sum over r_0 and $H(\alpha, \delta)$ the double sum over n and r_1 , so that

$$S_{\leq}(x, h) = \sum_{\substack{\delta | (h, ac\Delta)^\infty \\ \delta \leq B}} \sum_{\substack{0 \leq \alpha < \delta \\ (\alpha, \delta) = 1}} G(\alpha, \delta) H(\alpha, \delta). \quad (12)$$

It is convenient to notice now that the number of summands in the sum defining $G(\alpha, \delta)$ is at most the number of roots of $f(r) \pmod{\delta}$, and hence that sum is bounded by a constant depending on f : indeed, a bound for that number of roots by Sitar and the second author [15, Lemma 3.4] implies that

$$|G(\alpha, \delta)| \leq \Delta^{1/2} 2^{\omega(\delta)} \leq \Delta^{1/2} 2^{\omega(ac\Delta)}.$$

We handle the term $H(\alpha, \delta)$ in the same way as in the previous section. We write $\Delta = \Delta_1 \Delta_2$ with $(\Delta_1, h) = 1$ and $\Delta_2 \mid (h, \Delta)^\infty$. A version of Lemma 2 with Δ_1 in place of Δ gives

$$H(\alpha, \delta) = \sum_{\substack{g | \Delta_1 \\ (g, ac) = 1}} \mu^2(g) \sum_{\substack{m \leq x/g\delta \\ (m, \Delta) = 1 \\ gm \equiv \alpha \pmod{\delta}}} \sum_{\substack{0 \leq r_1 < mg \\ f(r_1\delta) \equiv 0 \pmod{mg}}} e\left(\frac{hr_1}{mg}\right).$$

The corresponding sum $S_1(x)$ is then of type

$$S_1(x) = \sum_{\substack{g | \Delta_1 \\ (g, ac) = 1}} \mu^2(g) \sum_{\substack{k \leq \sqrt{x/\delta g} \\ (k, a\Delta) = 1}} \sum_{\substack{\ell \leq x/gk\delta \\ (\ell, k) = (\ell, c\Delta) = 1 \\ gk\ell \equiv \alpha \pmod{\delta}}} e\left(\frac{(-bcg\bar{a}_k\bar{g}_k + d + ckr_g\bar{k}_g)h\bar{\ell}_{cgk}}{cgk}\right) \\ + O((\log x)^2).$$

Since $(\alpha, \delta) = 1$, the condition $gk\ell \equiv \alpha \pmod{\delta}$ implies that $(g, \delta) = 1$. The analogue of equation (7) is now

$$\Sigma_1 := \sum_{\substack{\ell \leq x/gk \\ (\ell, ck\Delta) = 1 \\ gk\ell \equiv \alpha \pmod{\delta}}} e\left(\frac{\frac{\Delta_1}{g}(-bcg\bar{a}_k\bar{g}_k + d + ckr_g\bar{k}_g)h\bar{\ell}_{ck\Delta}}{ck\Delta_1}\right),$$

in which the only real difference from before is the congruence $gk\ell \equiv \alpha \pmod{\delta}$. Let δ_2 be the least common multiple $\delta_2 = [\delta, \Delta_2]$; we still have $(\delta_2, ck\Delta_1) = 1$. The two conditions $gk\ell \equiv \alpha \pmod{\delta}$ and $(\ell, \Delta_2) = 1$ can be expressed using congruences modulo δ_2 :

$$\Sigma_1 = \sum_{\substack{0 \leq \beta < \delta_2/\delta \\ (\alpha \bar{k}_\delta + \delta \beta, \delta_2) = 1}} \sum_{\substack{\ell \leq x/gk \\ (\ell, ck\Delta_1) = 1 \\ \ell \equiv \alpha \bar{k}_\delta + \beta \delta \pmod{\delta_2}}} e\left(\frac{\frac{\Delta_1}{g}(-bcg\bar{a}_k\bar{g}_k + d + ckr_g\bar{k}_g)h\bar{\ell}_{ck\Delta}}{ck\Delta_1}\right).$$

We apply Lemma 1 in the same way as in the case $(\Delta, h) = 1$ to obtain:

$$S_{\leq}(x, h) = xC(h, a, c, \Delta_1) \sum_{\substack{\delta | (h, ac\Delta)^\infty \\ \delta \leq B}} \sum_{\substack{0 \leq \alpha < \delta \\ (\alpha, \delta) = 1}} \frac{G(\alpha, \delta)}{\delta^2} + O\left(x^{3/4+\varepsilon} \sum_{\substack{\delta | (h, ac\Delta)^\infty \\ \delta \leq B}} \delta^{1/4}\right).$$

If we open the sum $G(\alpha, \delta)$ and exchange the order of summation with α , we find Ramanujan sums:

$$\begin{aligned} \sum_{\substack{0 \leq \alpha < \delta \\ (\alpha, \delta) = 1}} G(\alpha, \delta) &= \sum_{\substack{0 \leq r_0 < \delta \\ f(r_0) \equiv 0 \pmod{\delta}}} \sum_{\substack{0 \leq \alpha < \delta \\ (\alpha, \delta) = 1}} e\left(\frac{hr_0\bar{\alpha}}{\delta}\right) \\ &= \sum_{\substack{0 \leq r_0 < \delta \\ f(r_0) \equiv 0 \pmod{\delta}}} \mu\left(\frac{\delta}{(hr_0, \delta)}\right) \frac{\varphi(\delta)}{\varphi\left(\frac{\delta}{(hr_0, \delta)}\right)}. \end{aligned}$$

Since $G(\alpha, \delta) \ll 1$ as previously remarked, the sum over δ in the main term converges as B tends to ∞ (note that the sum is not over all integers δ but rather only those integers with prime factors in a fixed finite set, which is a very sparse sequence), and the error resulting from replacing the finite sum over δ with the infinite series is $\ll x/B^{1-\varepsilon}$. In the error term, the $\delta^{1/4}$ roughly comes from the summation of the $k^{1/2}$ with $k \leq (x/g\delta)^{1/2}$ and with a trivial summation of the sum over α . This error term is $O(x^{3/4+\varepsilon}B^{1/4})$.

This ends the proof of Theorem 1, with

$$C(f, h) = C(h, a, c, \Delta_1) \sum_{\delta | (h, ac\Delta)^\infty} \frac{1}{\delta^2} \sum_{\substack{0 \leq r_0 < \delta \\ f(r_0) \equiv 0 \pmod{\delta}}} \mu\left(\frac{\delta}{(hr_0, \delta)}\right) \frac{\varphi(\delta)}{\varphi\left(\frac{\delta}{(hr_0, \delta)}\right)}, \quad (13)$$

where $C(h, a, c, \Delta_1)$ is as in equation (8).

4 Linear times an irreducible quadratic (Theorem 2)

In this section we prove Theorem 2 concerning the polynomial $f(t) = t(t^2 + 1)$.

4.1 First step: splitting $S(f, x)$

Since $(n, n^2 + 1) = 1$ we can write $S(f, x)$ as a sort of convolution, as in the previous section. The following lemma is elementary:

Lemma 4. *Let $g(t)$ be a polynomial with integer coefficients with $g(0) = \pm 1$, and let m be a positive integer. The roots of $rg(r) \equiv 0 \pmod{m}$ are in one-to-one correspondence with the factorizations $k\ell = m$ with $(k, \ell) = 1$ and corresponding roots v of $g(v) \equiv 0 \pmod{m}$. The root r corresponds to the solution modulo m of the system of congruences*

$$\begin{aligned} r &\equiv 0 \pmod{k} \\ r &\equiv v \pmod{\ell}. \end{aligned}$$

We have by Lemma 4

$$S(f, x) = \sum_{n \leq x} \sum_{\substack{k\ell=n \\ (k, \ell)=1}} \sum_{\substack{0 \leq v < \ell \\ v^2+1 \equiv 0 \pmod{\ell}}} e\left(\frac{\bar{k}v}{\ell}\right).$$

Let $y_1 = x^{1/3}(\log x)^{-A}$ and $y_2 = x^{1/3}(\log x)^B$ with $A, B > 0$ to be chosen. We split the sum $S(x)$ into three parts according to the size of k : $S(f, x) = S_1(x) + S_2(x) + S_3(x)$ with

$$\begin{aligned} S_1(x) &= \sum_{y_2 < k \leq x} \sum_{\substack{\ell \leq x/k \\ (k, \ell)=1}} \sum_{\substack{0 \leq v < \ell \\ v^2+1 \equiv 0 \pmod{\ell}}} e\left(\frac{\bar{k}v}{\ell}\right) \\ S_2(x) &= \sum_{y_1 < k \leq y_2} \sum_{\substack{\ell \leq x/k \\ (k, \ell)=1}} \sum_{\substack{0 \leq v < \ell \\ v^2+1 \equiv 0 \pmod{\ell}}} e\left(\frac{\bar{k}v}{\ell}\right) \\ S_3(x) &= \sum_{1 \leq k \leq y_1} \sum_{\substack{\ell \leq x/k \\ (k, \ell)=1}} \sum_{\substack{0 \leq v < \ell \\ v^2+1 \equiv 0 \pmod{\ell}}} e\left(\frac{\bar{k}v}{\ell}\right). \end{aligned}$$

We shall see momentarily that in $S_1(x)$, it is possible to use equation (4) in the same way as in the proof of Theorem 1; the main term in our asymptotic formula for $S(f, x)$ arises from this sum. This approach works only when k is sufficiently large (or ℓ sufficiently small), which is to say when k is slightly bigger than $x^{1/3}$. This is our motivation for the choice of y_2 .

The converse is true for $S_3(x)$, in which ℓ is the largest parameter. In this case we use the fact that the second factor is quadratic. We can apply a lemma of Gauss on the correspondence of the roots of $n^2 + 1 \equiv 0 \pmod{\ell}$ and certain representations $\ell = r^2 + s^2$ as the sum of two squares. This approach works when $k = o(x^{1/3})$, which is why we choose y_1 close to $x^{1/3}$.

The remaining range $k \in]y_1, y_2]$ is covered by a direct application of Hooley's result [11]. Since y_1 and y_2 are close together, Hooley's general bound applied to the irreducible polynomial $X^2 + 1$ is sufficient.

4.2 The first two sums

In the sum $S_1(x)$, the variable k is large and thus we arrange for some cancellation in the sum over this variable:

$$S_1(x) = \sum_{\ell < x/y_2} \sum_{\substack{0 \leq v < \ell \\ v^2+1 \equiv 0 \pmod{\ell}}} \sum_{\substack{y_2 < k \leq x/\ell \\ (k, \ell)=1}} e\left(\frac{\bar{k}v}{\ell}\right).$$

Let $\rho(m)$ denote the number of roots modulo m of the polynomial $n^2 + 1$:

$$\rho(m) = \#\{0 \leq v < m : v^2 + 1 \equiv 0 \pmod{m}\}. \quad (14)$$

For any $\ell < x/y_2$ and any $0 \leq v < \ell$ with $v^2 + 1 \equiv 0 \pmod{\ell}$, we apply equation (4) to bound the inner sum in k :

$$\begin{aligned} S_1(x) &= \sum_{\ell \leq x/y_2} \sum_{\substack{0 \leq v < \ell \\ v^2 + 1 \equiv 0 \pmod{\ell}}} \left(\left(\frac{x/\ell - y_2}{\ell} \right) \mu(\ell) + O(\sqrt{\ell} \cdot \tau(\ell) \log \ell) \right) \\ &= x \sum_{\ell \leq x/y_2} \frac{\mu(\ell) \rho(\ell)}{\ell^2} - y_2 \sum_{\ell \leq x/y_2} \frac{\mu(\ell)}{\ell} + O\left(\sum_{\ell \leq x/y_2} \sqrt{\ell} \cdot \tau(\ell) \rho(\ell) \log \ell \right) \\ &= x \prod_p \left(1 - \frac{\rho(p)}{p^2} \right) + O(y_2 x^\varepsilon + (x/y_2)^{3/2} (\log x)^5). \\ &= x \prod_p \left(1 - \frac{\rho(p)}{p^2} \right) + O(x (\log x)^{5-B}). \end{aligned}$$

In the third equality above, we used the calculation

$$\begin{aligned} \sum_{\ell \leq x/y_2} \sqrt{\ell} \cdot \tau(\ell) \rho(\ell) \log \ell &\ll \left(\frac{x}{y_2} \right)^{3/2} \log x \sum_{\ell \leq x/y_2} \frac{\tau(\ell) \rho(\ell)}{\ell} \\ &\ll \left(\frac{x}{y_2} \right)^{3/2} \log x \prod_{p \leq x/y_2} \left(1 + \sum_{k=1}^{\infty} \frac{\tau(p^k) \rho(p^k)}{p^k} \right), \end{aligned}$$

followed by the fact that $\rho(p^k) \leq 2$ for all prime powers p^k . (We could in fact replace the exponent $5 - B$ by $2 - 3B/2$, using the fact that $\rho(p^k) = 0$ if $p \equiv 3 \pmod{4}$, but that improvement is not significant for our purposes.)

In the sum $S_2(x)$, the variable k is in a crucial range (corresponding to when the size of k is close to $\sqrt{\ell}$) where the methods for both $S_1(x)$ and $S_3(x)$ fail. The bound for $S_2(x)$ will be a direct consequence of the work of Hooley:

Lemma 5. *Let $P(X) \in \mathbb{Z}[X]$ be an irreducible polynomial of degree $n \geq 2$. If $hk \neq 0$ then we have*

$$\sum_{\substack{\ell \leq x \\ (\ell, k) = 1}} \sum_{\substack{v \pmod{\ell} \\ P(v) \equiv 0 \pmod{\ell}}} e\left(\frac{h\bar{k}v}{\ell}\right) \ll_{h,P} x^{\frac{(\log \log x)^{(n^2+1)/2}}{(\log x)^{\delta_n}}},$$

where $\delta_n = (n - \sqrt{n})/n!$.

The case $k = 1$ is [11, Theorem 1]; the proof can be adapted with no difficulty for all $k \in \mathbb{N}$ and provides then a result that is uniform in k . The dependence on h in this result of Hooley is due only to the appearance of (h, ℓ) in certain intermediate computations.

We apply this lemma with $P(X) = X^2 + 1$ and replacing x by x/k :

$$S_2(x) \ll \sum_{y_1 \leq k \leq y_2} \frac{x (\log \log x)^{5/2}}{k (\log x)^{1-1/\sqrt{2}}} \ll x \frac{(\log \log x)^{7/2}}{(\log x)^{1-1/\sqrt{2}}},$$

using the fact that $\log(y_2/y_1) \ll \log \log x$.

4.3 The sum $S_3(x)$

In this section we use the special shape of the polynomial $n^2 + 1$ to find an upper bound for $S_3(x)$. Following the ideas of the two articles of Hooley [10, 12] concerning $\tau(n^2 + 1)$ and $P^+(n^2 + 1)$ (the number of divisors of $n^2 + 1$ and the largest prime factor of $n^2 + 1$, respectively), we employ the Gauss correspondence between the roots of $v^2 + 1 \equiv 0 \pmod{\ell}$ and certain representations of $\ell = r^2 + s^2$ as the sum of two squares. Indeed for such integers r, s with $(rs, \ell) = 1$, we have $(\bar{r}s)^2 + 1 \equiv 0 \pmod{\ell}$. The parameter \bar{k} in the exponential gives rise to some coprimality problems. The first author [3] resolved such a difficulty when k is squarefree; in equation (15) we will use an elegant formula of Wu and Xi [20] to handle the general case.

As in the proof of Theorem 1, in the following argument the condition $k_2 = (k, r^\infty)$ means that $p \mid k_2 \Rightarrow p \mid r$ and $(k/k_2, r) = 1$.

Lemma 6. *For $\ell > 1$, there is a one-to-one correspondence between the representations of ℓ by the form $\ell = r^2 + s^2$ with $(r, s) = 1$, $r > 0, s > 0$ and the solutions of the congruence $v^2 + 1 \equiv 0 \pmod{\ell}$. This bijection is given by:*

$$\frac{v}{\ell} \equiv \frac{\bar{s}}{r} - \frac{s}{r(r^2 + s^2)} \pmod{1}.$$

For $k \geq 1$, $k = k_1 k_2$ with $k_2 = (k, r^\infty)$ we have

$$\frac{\bar{k}v}{\ell} \equiv \frac{-\overline{rk_2(r^2 + s^2)}}{k_1 s} + \frac{r}{ks(r^2 + s^2)} - \frac{\overline{rk_1 s(r^2 + s^2)}}{k_2} \pmod{1}. \quad (15)$$

The first part of this lemma is proved in detail in the book of Smith [19, Art. 86], while equation (15) is [20, Lemma 7.4].

By this lemma, still using the notation $k = k_1 k_2$, we have

$$S_3(x) = \sum_{\substack{k_1 k_2 \leq y_1 \\ (k_1, k_2) = 1}} \sum_{\substack{r^2 + s^2 \leq x/(k_1 k_2) \\ (r, s) = 1, r > 0, s > 0 \\ (k_1 k_2, r^\infty) = k_2}} e\left(\frac{-\overline{rk_2(r^2 + s^2)}}{k_1 s} + \frac{r}{ks(r^2 + s^2)} - \frac{\overline{rk_1 s(r^2 + s^2)}}{k_2}\right).$$

First we remove the term $e(r/ks(r^2 + s^2))$: since

$$e\left(\frac{r}{ks(r^2 + s^2)}\right) = 1 + O\left(\frac{1}{ksr}\right),$$

replacing this term by 1 results in a corresponding error in $S_3(x)$ that is $O((\log x)^4)$.

Following the notation of several authors, we denote by $(k_1 s)^\flat$ and $(k_1 s)^\sharp$ the squarefree and squarefull part, respectively, of $k_1 s$. Since $((k_1 s)^\flat, (k_1 s)^\sharp) = 1$, we can use the Chinese remainder theorem as in the proof of Theorem 1:

$$\frac{1}{k_1 s} \equiv \frac{\overline{(k_1 s)^\sharp}}{(k_1 s)^\flat} + \frac{\overline{(k_1 s)^\flat}}{(k_1 s)^\sharp} \pmod{1}.$$

Inserting this in $S_3(x)$, we obtain

$$S_3(x) = \sum_{\substack{k_1 k_2 \leq y_1 \\ (k_1, k_2) = 1}} \sum_{\substack{r^2 + s^2 \leq x/(k_1 k_2) \\ (r, s) = 1, r > 0, s > 0 \\ (k_1 k_2, r^\infty) = k_2}} K(r)W(r) + O((\log x)^4), \quad (16)$$

with

$$K(r) = e\left(\frac{-rk_2(r^2 + s^2)(k_1 s)^\sharp}{(k_1 s)^\flat}\right)$$

$$W(r) = e\left(\frac{-rk_2(r^2 + s^2)(k_1 s)^\flat}{(k_1 s)^\sharp} - \frac{rk_1 s(r^2 + s^2)}{k_2}\right).$$

Let $S_4(x)$ denote the contribution to $S_3(x)$ of the k_1, k_2, r, s such that $(k_1 s)^\sharp > (\log x)^{45}$ or $k_2 > (\log x)^5$, and $S_5(x)$ the remaining contribution, that is, the contribution of the k_1, k_2, r, s such that $k_2 \leq (\log x)^5$ and $(k_1 s)^\sharp \leq (\log x)^{45}$.

First we prove that

$$S_4(x) \ll x(\log x)^{-3}. \quad (17)$$

We remark that if m^2 is the largest square divisor of $(k_1 s)^\sharp$ then $m^2 \geq ((k_1 s)^\sharp)^{2/3}$. We deduce that when $(k_1 s)^\sharp > (\log x)^{45}$, there exists $m > (\log x)^{15}$ such that $m^2 \mid (k_1 s)^\sharp$. We can write this divisor in the following way: $m^2 = u^2 v^2 w^2$ with $u^2 \mid k_1$, $v^2 \mid s$, and $w \mid (k_1, s)$. Thus we have $\max\{u^2, v^2, w^2\} \geq m^{2/3}$ and there exists $d > (\log x)^5$ such that $d^2 \mid k_1$ or $d^2 \mid s$ or $d \mid (k_1, s)$. In the first case (when $d^2 \mid k_1$), the contribution of the k_1, k_2, r, s is less than

$$\sum_{(\log x)^5 < d \leq y_1} \sum_{d^2 k_1 k_2 < y_1} \sum_{\max\{r, s\} \ll x^{1/2}/(d^2 k_1 k_2)^{1/2}} 1 \ll x(\log x)^{-3}.$$

Similarly, in the second case (when $d^2 \mid s$), we have a contribution less than

$$\sum_{k_1 k_2 < y_1} \sum_{r \ll (x/k_1 k_2)^{1/2}} \sum_{(\log x)^5 < d} \sum_{s \ll x^{1/2}/(d^2 k_1^{1/2} k_2^{1/2})} 1 \ll x(\log x)^{-3}.$$

Finally the contribution of the terms with $d \mid (k_1, s)$ is less than

$$\sum_{d > (\log x)^5} \sum_{dk_1 k_2 \leq y_1} \sum_{\max\{r, ds\} \ll (x/dk_1 k_2)^{1/2}} 1 \ll x(\log x)^{-3}.$$

It remains to evaluate the contribution to $S_4(x)$ of the terms where $k_2 > (\log x)^5$. Since $k_2 = (k_1 k_2, r^\infty)$, we have $q(k_2) \mid r$ where $q(k_2) = \prod_{p \mid k_2} p$ is the squarefree kernel of k_2 . Thus, the contribution of r is bounded by $x^{1/2}/(q(k_2)(k_1 k_2)^{1/2})$, and then the corresponding summation of all the k_1, k_2, r, s with $k_2 > (\log x)^5$ is less than

$$\sum_{(\log x)^5 < k_2 < y_1} \sum_{k_2 k_1 < y_1} \frac{x}{k_1 k_2 q(k_2)} \ll x(\log x)^{-3} \sum_{k_2 \geq 1} \frac{1}{q(k_2) k_2^{1/5}} \ll x(\log x)^{-3}.$$

The rest of this section is devoted to the sum $S_5(x)$, which can be written as

$$S_5(x) = \sum_{\substack{k_1 k_2 \leq y_1 \\ (k_1, k_2)=1 \\ k_2 \leq (\log x)^5}} \sum_{\substack{r^2 + s^2 \leq x/(k_1 k_2) \\ r, s > 0, (r, s)=1 \\ q(k_2) | r \\ (k_1, r)=1 \\ (k_1 s)^\sharp \leq (\log x)^{45}}} K(r)W(r).$$

If we replace r by $q(k_2)r'$, the sum over r' has the shape

$$S_R = \sum_{r' < R, (r', s)=1} K(r'q(k_2))W(r'q(k_2))$$

for some quantity $R = R(s, k_1, k_2) \ll x^{1/2}/(q(k_2)\sqrt{k_1 k_2})$. It is then standard to complete the sum:

$$S_R = \frac{1}{k_1 k_2 s} \sum_{h=1}^{k_1 k_2 s} \left(\sum_{\substack{a=1 \\ (a, s)=1}}^{k_1 k_2 s} K(aq(k_2))W(aq(k_2))e\left(\frac{ah}{k_1 k_2 s}\right) \right) \left(\sum_{r' < R} e\left(-\frac{hr'}{k_1 k_2 s}\right) \right).$$

As before, the inner sum over r' is geometric and is $\ll \min\{R, \|h/k_1 k_2 s\|^{-1}\}$. Let S_a denote the inner sum over the variable a , which is a complete sum. Applying the Chinese remainder theorem many times, we have:

$$S_a = \prod_{p|(k_1 s)^\flat} \left(\sum_{a \pmod{p}} K_p(a)e(ha/p) \right) \prod_{p^\nu \| k_2(k_1 s)^\sharp} \left(\sum_{a \pmod{p^\nu}} W_{p^\nu}(a)e(ha/p^\nu) \right),$$

where

$$K_p(a) = e\left(\frac{-aq(k_2)\overline{k_2(a^2 q(k_2)^2 k_2^2 (k_1 s/p)^2 + s^2)}}{p}\right).$$

and W_{p^ν} is an exponential term modulo p^ν whose argument is a similar rational function in a . Since $k_2(k_1 s)^\sharp \leq (\log x)^{50}$, a trivial bound for the sums on the $a \pmod{p^\nu}$ when $p^\nu \mid k_2(k_1 s)^\sharp$ is sufficient, yielding

$$\begin{aligned} S_a &\ll \prod_{p|(k_1 s)^\flat} \left| \sum_{a \pmod{p}} K_p(a)e(ha/p) \right| \prod_{p^\nu \| k_2(k_1 s)^\sharp} p^\nu \\ &\ll (\log x)^{50} \prod_{p|(k_1 s)^\flat} \left| \sum_{a \pmod{p}} K_p(a)e(ha/p) \right|. \end{aligned}$$

(It is in fact possible to find in [20, Appendix B] a useful nontrivial bound for the sums that we have estimated trivially.)

Since $(k_1 s)^\flat$ is squarefree, we can apply Weil's bound for exponential sums of a rational function. The formulation we use is a particular case derived from [4, equation (3.5.2)].

Lemma 7. *Let $\mathbb{P}^1(\mathbb{F}_p)$ be the projective line on \mathbb{F}_p , and let $f: \mathbb{P}^1(\mathbb{F}_p) \rightarrow \mathbb{P}^1(\mathbb{F}_p)$ be a nonconstant rational function. For all $u \in \mathbb{P}^1(\mathbb{F}_p)$, let $v_u(f)$ be the order of the pole of f at u if $f(u) = \infty$ and $v_u(f) = 0$ otherwise. Then we have*

$$\left| \sum_{\substack{u \in \mathbb{P}^1(\mathbb{F}_p) \\ f(u) \neq \infty}} e\left(\frac{f(u)}{p}\right) \right| \leq \sum_{v_u(f) \neq 0} (1 + v_u(f)) p^{1/2} \quad (18)$$

Since $K_p(a)e(ha/p)$ has at most 3 poles (including the pole at ∞), which are simple, we have

$$\left| \sum_{a \pmod{p}} K_p(a)e(ha/p) \right| \leq 6\sqrt{p},$$

from which we deduce that

$$|S_a| \leq 6^{\omega((k_1s)^b)} \sqrt{k_1s} (\log x)^{50}.$$

Returning to S_R , we have obtained

$$S_R \ll \frac{R}{k_1 k_2 s} 6^{\omega((k_1s)^b)} \sqrt{k_1s} (\log x)^{50} + 6^{\omega((k_1s)^b)} \sqrt{k_1s} (\log x)^{51},$$

which gives the following upper bound for $S_5(x)$:

$$\begin{aligned} S_5(x) &\ll (\log x)^{51} \sum_{\substack{k_1 k_2 \leq y_1 \\ k_2 \leq (\log x)^s}} \sum_{s \ll \sqrt{x/(k_1 k_2)}} 6^{\omega((k_1s)^b)} \left(\frac{\sqrt{x}}{q(k_2) \sqrt{k_1 k_2 s}} + \sqrt{k_1 s} \right) \\ &\ll x^{3/4} y_1^{3/4} (\log x)^{63}. \end{aligned}$$

If we take $y_1 = x^{1/3} (\log x)^{-100}$ we obtain $S_5 \ll x (\log x)^{-12}$, which is enough for the proof of Theorem 2.

5 Three linear factors (Theorem 3)

In this section we consider one of the simplest cases of a product of three linear factors, namely the case $f(n) = n(n+1)(2n+1)$. Since $(n, (n+1)(2n+1)) = (n+1, 2n+1) = 1$, our exponential sum is now

$$\begin{aligned} S(f, x) &= \sum_{n \leq x} \sum_{\substack{r \pmod{n} \\ f(r) \equiv 0 \pmod{n}}} e\left(\frac{r}{n}\right) \\ &= \sum_{n_1 n_2 n_3 \leq x} \sum_{\substack{r \pmod{n_1 n_2 n_3} \\ n_1 | r \\ r+1 \equiv 0 \pmod{n_2} \\ 2r+1 \equiv 0 \pmod{n_3}}} e\left(\frac{r}{n_1 n_2 n_3}\right); \end{aligned}$$

note that the inner sum has one term when n_1, n_2 , and n_3 are pairwise coprime and n_3 is odd, and no terms otherwise.

Let $y_2 = x^{1/3} (\log x)^B$ with $B > 0$ to be specified. As in the previous sections we split the sum $S(f, x)$,

writing $S(f, x) = \sum_{i=1}^4 S_i(x)$ where

$$\begin{aligned} S_1(x) &= \sum_{\substack{n_1 n_2 n_3 \leq x \\ n_1 > y_2}} \sum_{\substack{r \pmod{n_1 n_2 n_3} \\ n_1 | r \\ r+1 \equiv 0 \pmod{n_2} \\ 2r+1 \equiv 0 \pmod{n_3}}} e\left(\frac{r}{n_1 n_2 n_3}\right) \\ S_2(x) &= \sum_{\substack{n_1 n_2 n_3 \leq x \\ n_1 \leq y_2 \\ n_2 > y_2}} \sum_{\substack{r \pmod{n_1 n_2 n_3} \\ n_1 | r \\ r+1 \equiv 0 \pmod{n_2} \\ 2r+1 \equiv 0 \pmod{n_3}}} e\left(\frac{r}{n_1 n_2 n_3}\right) \\ S_3(x) &= \sum_{\substack{n_1 n_2 n_3 \leq x \\ n_1, n_2 \leq y_2 \\ n_3 > y_2}} \sum_{\substack{r \pmod{n_1 n_2 n_3} \\ n_1 | r \\ r+1 \equiv 0 \pmod{n_2} \\ 2r+1 \equiv 0 \pmod{n_3}}} e\left(\frac{r}{n_1 n_2 n_3}\right) \\ S_4(x) &= \sum_{\substack{n_1 n_2 n_3 \leq x \\ n_1, n_2, n_3 \leq y_2}} \sum_{\substack{r \pmod{n_1 n_2 n_3} \\ n_1 | r \\ r+1 \equiv 0 \pmod{n_2} \\ 2r+1 \equiv 0 \pmod{n_3}}} e\left(\frac{r}{n_1 n_2 n_3}\right). \end{aligned}$$

5.1 The first three sums

Using a method similar to Section 2 above, the solution r of the congruences in the above sums can be written as

$$r = n_1 \left(-\overline{(n_1 n_3)}_{n_2} n_3 - \overline{(2n_1 n_2)}_{n_3} n_2 \right),$$

and therefore the exponential summand in the $S_i(x)$ becomes

$$e\left(\frac{r}{n_1 n_2 n_3}\right) = e\left(\frac{-\overline{n_1 n_3}}{n_2} - \frac{\overline{2n_1 n_2}}{n_3}\right). \quad (19)$$

For $S_1(x)$ this gives:

$$S_1(x) = \sum_{\substack{n_2 n_3 \leq x/y_2 \\ (2n_2, n_3)=1}} \sum_{y_2 < n_1 \leq x/(n_2 n_3)} e\left(\frac{\bar{n}_1 (n_3 \overline{(n_3)}_{n_2} - n_2 \overline{(2n_2)}_{n_3})}{n_2 n_3}\right).$$

We apply equation (4) with $t = n_3 \overline{(n_3)}_{n_2} - n_2 \overline{(2n_2)}_{n_3}$. In this case $(t, n_2 n_3) = 1$ and we obtain:

$$S_1(x) = \sum_{\substack{n_2 n_3 \leq x/y_2 \\ (2n_2, n_3)=1}} \left(\left(\frac{x/(n_2 n_3) - y_2}{n_2 n_3} \right) \mu(n_2 n_3) + O(\sqrt{n_2 n_3} \cdot \tau(n_2 n_3) \log x) \right).$$

The error term is $O((x/y_2)^{3/2}(\log x)^5)$ which is sufficiently small if B is large enough, and therefore

$$\begin{aligned} S_1(x) &= x \sum_{(2n_2, n_3)=1} \frac{\mu(n_2)\mu(n_3)}{n_2^2 n_3^2} + O(y_2 x^\epsilon) + O((x/y_2)^{3/2}(\log x)^5) \\ &= x \frac{6}{\pi^2} \sum_{n_2} \frac{\mu(n_2)}{n_2^2} \prod_{p|2n_2} \left(1 - \frac{1}{p^2}\right)^{-1} + O((x/y_2)^{3/2}(\log x)^5) \\ &= x \frac{6}{\pi^2} \prod_{p \geq 3} \left(1 - \frac{1}{p^2 - 1}\right) + O((x/y_2)^{3/2}(\log x)^5). \end{aligned}$$

We handle the sum $S_2(x)$ in the same way, but this time summing first over n_2 instead of n_1 . Applying the inversion formula (6), we can rewrite equation (19) in the following way:

$$\begin{aligned} e\left(\frac{r}{n_1 n_2 n_3}\right) &= e\left(\frac{\bar{n}_2}{n_1 n_3} - \frac{1}{n_1 n_2 n_3} - \frac{\overline{2n_1 n_2}}{n_3}\right) \\ &= e\left(\frac{\bar{n}_2(1 - \overline{(2n_1)_{n_3}} n_1)}{n_1 n_3}\right) + O\left(\frac{1}{n_1 n_2 n_3}\right). \end{aligned}$$

The error term $O(1/n_1 n_2 n_3)$ yields a contribution to $S_2(x)$ that is less than $O((\log x)^3)$. Then we apply equation (4):

$$S_2(x) = \sum_{\substack{n_1 n_3 \leq x/y_2 \\ n_1 \leq y_2 \\ (2n_1, n_3)=1}} \left(\left(\frac{x}{n_1 n_3} - y_2 \right) \frac{\mu(n_1 n_3)}{n_1 n_3} + O(\sqrt{n_1 n_3} \cdot \tau(n_1 n_3) \log x) \right).$$

We finish in the same way as for $S_1(x)$, obtaining the same asymptotic formula.

For $S_3(x)$ the corresponding method is to write

$$e\left(\frac{r}{n_1 n_2 n_3}\right) = e\left(\frac{\bar{n}_3(1 - \overline{(n_1)_{n_2}} 2n_1)}{2n_1 n_2}\right) + O\left(\frac{1}{n_1 n_2 n_3}\right),$$

and then after applying equation (4)

$$S_3(x) = \sum_{\substack{n_1 n_2 \leq x/y_2 \\ \max\{n_1, n_2\} \leq y_2 \\ (n_1, n_2)=1}} \left(\left(\frac{x}{n_1 n_2} - y_2 \right) \frac{\mu(2n_1 n_2)}{2n_1 n_2} + O(\sqrt{n_1 n_2} \cdot \tau(n_1 n_2) \log x) \right).$$

The corresponding main term this time is

$$S_3(x) = x \sum_{(n_1, n_2)=1} \frac{\mu(2n_1 n_2)}{2n_1^2 n_2^2} = -\frac{4x}{\pi^2} \prod_{p \geq 3} \left(1 - \frac{1}{p^2 - 1}\right) + O((x/y_2)^{3/2}(\log x)^5).$$

Summing these contributions of $S_1(x)$, $S_2(x)$, and $S_3(x)$ in the decomposition at the start of this section, we deduce that

$$\begin{aligned} S(f, x) &= \frac{8x}{\pi^2} \prod_{p \geq 3} \left(1 - \frac{1}{p^2 - 1}\right) + O(|S_4(x)| + (x/y_2)^{3/2} (\log x)^5) \\ &= x \prod_{p \geq 3} \left(1 - \frac{2}{p^2}\right) + O(|S_4(x)| + x/(\log x)^{3B/2-5}). \end{aligned}$$

5.2 The sum $S_4(x)$

It remains to handle $S_4(x)$. Let $y_1 = x^{1/3}(\log x)^{-A}$ with $A > 0$ to be specified. Let I denote the interval $I = [y_1, y_2]$. First we remark that the number of summands for which $\min\{n_1, n_2, n_3\} < y_1$ is $\ll y_1 y_2^2$, and hence

$$S_4(x) = \sum_{\substack{n_1 n_2 n_3 \leq x \\ n_1, n_2, n_3 \in I}} \sum_{\substack{r \pmod{n_1 n_2 n_3} \\ n_1 | r \\ r+1 \equiv 0 \pmod{n_2} \\ 2r+1 \equiv 0 \pmod{n_3}}} e\left(\frac{r}{n_1 n_2 n_3}\right) + O\left(\frac{x}{(\log x)^{A-2B}}\right).$$

We introduce a new parameter $z = \exp(\log x / (10 \log \log x))$. We now write $n_2 = a_2 b_2$, $n_3 = a_3 b_3$ with $P^+(a_2 a_3) \leq z < P^-(b_2 b_3)$ where $P^+(n)$ and $P^-(n)$ are, respectively, the largest and smallest prime factors of n . Using two more parameters v and w , we split $S_4(x)$ as $S_4(x) = S_5(x) + S_6(x) + S_7(x) + O(x(\log x)^{2B-A})$, where $\max(a_2, a_3) \leq v$ in $S_5(x)$, $\max(a_2, a_3) > w$ in $S_6(x)$, and finally $v < \max\{a_2, a_3\} \leq w$ in $S_7(x)$.

In $S_5(x)$, since a_2, a_3 are small we have:

$$\begin{aligned} S_5(x) &\leq \sum_{a_2, a_3 \leq v} \sum_{n_1, a_2 b_2 \in I} \sum_{b_3 \leq x/(n_1 n_2 a_3)} 1 \\ &\ll \frac{x}{\log z} \sum_{a_2, a_3 \leq v} \sum_{n_1, a_2 b_2 \in I} \frac{1}{n_1 n_2 a_3} \ll \frac{x(\log v)^2 (\log \log x)^2}{(\log z)^2}. \end{aligned} \tag{20}$$

The $(\log z)^2$ above comes from the sieving conditions on b_2 and b_3 . In particular, we have used the following inequality

$$\sum_{y_1/a_2 \leq b_2 \leq y_2/a_2} \frac{1}{b_2} \ll \frac{\log \log x}{\log z}, \tag{21}$$

which can be derived by partial summation from [1, Proposition 1]. The bound (20) is sufficiently small when $(\log v)(\log \log x) = o(\log z)$ (we will eventually choose v to be a power of $\log x$). We remark that this step is the main obstacle to having an upper bound less than $x/(\log x)^2$ in the error term in Theorem 3.

For $S_6(x)$, we estimate each summand trivially by 1; therefore we may assume that $a_3 > w$ is abnormally large (the bound when $a_2 > w$ is exactly the same) and that n_2 is unrestricted. Following some ideas of Hooley [12], we note that if $a_3 > w$, then either $\omega(a_3) \geq \log w / (2 \log z)$, or else there exists $d > w^{1/4}$ such that $d^2 \mid a_3$. We therefore have (ignoring here the condition that b_3 has no small prime

factors)

$$\begin{aligned}
 S_6(x) &\leq \sum_{n_1, n_2 \in I} \sum_{w^{1/4} < d < y_2} \sum_{b_3 \leq x/(n_1 n_2 d^2)} 1 + \sum_{n_1, n_2 \in I} \sum_{\substack{a_3 < y_2 \\ \omega(a_3) \geq (\log w)/(2 \log z)}} \sum_{b_3 \leq x/(n_1 n_2 a_3)} 1 \\
 &\ll \sum_{n_1, n_2 \in I} \sum_{d > w^{1/4}} \frac{x}{n_1 n_2 d^2} + \sum_{n_1, n_2 \in I} \sum_{a_3 \leq y_2} \frac{x 2^{\omega(a_3) - \log w/(2 \log z)}}{n_1 n_2 a_3} \\
 &\ll \frac{x(\log \log x)^2}{w^{1/4}} + x 2^{-\log w/(2 \log z)} (\log \log x)^2 (\log x)^2.
 \end{aligned} \tag{22}$$

It remains to handle the term $S_7(x)$. We begin in the same way as for $S_1(x)$:

$$S_7(x) = \sum_{\substack{a_2 b_2, a_3 b_3 \in I \\ P^+(a_2 a_3) \leq z < P^-(b_2 b_3) \\ v < \max(a_2, a_3) \leq w}} \sum_{n_1 \leq x/(n_2 a_3 b_3)} e\left(\frac{\bar{n}_1(a_3 b_3 \overline{(a_3 b_3)})_{n_2} - n_2 \overline{(2n_2)}_{a_3 b_3}}{n_2 a_3 b_3}\right). \tag{23}$$

Unfortunately, equation (4) is not sufficient here. Since a_3 is not too small and b_3 is not too big, the denominator has three factors not too small and we can apply the recent work of Wu and Xi [20] on the q -analog of the van der Corput method. Such an approach was initiated by Heath–Brown [7] and developed by Graham and Ringrose [6], and more recently by Irving [13, 14] and by Wu and Xi [20], where the arithmetic exponent pairs are obtained when the denominator has good factorization properties. As in the proof of Theorem 2, we denote by n^\sharp and n^\flat the squarefull and squarefree parts of the integer n ; and we write

$$e\left(\frac{\bar{n}_1(a_3 b_3 \overline{(a_3 b_3)})_{n_2} - n_2 \overline{(2n_2)}_{a_3 b_3}}{n_2 a_3 b_3}\right) = K(n_1) W(n_1),$$

with

$$\begin{aligned}
 K(n_1) &= e\left(\frac{\bar{n}_1(n_2 a_3 b_3)^\sharp [a_3 b_3 \overline{(a_3 b_3)})_{n_2} - n_2 \overline{(2n_2)}_{a_3 b_3}]}{(n_2 a_3 b_3)^\flat}\right), \\
 W(n_1) &= e\left(\frac{\bar{n}_1(n_2 a_3 b_3)^\flat [a_3 b_3 \overline{(a_3 b_3)})_{n_2} - n_2 \overline{(2n_2)}_{a_3 b_3}]}{(n_2 a_3 b_3)^\sharp}\right).
 \end{aligned}$$

Lemma 8. *Uniformly for any integers $\alpha, A, N, \delta \in \mathbb{N}$, $q = q_1 q_2 q_3$ squarefree integer such that $(\alpha \delta, q) = 1$ and any rational function R with integer coefficients, we have*

$$\begin{aligned}
 \sum_{\substack{A < n \leq A+N \\ (n, q)=1}} e\left(\frac{\alpha \bar{n}}{q} + \frac{R(n)}{\delta}\right) &\ll N^{1/2} q_3^{1/2} + N^{3/4} q_2^{1/4} \\
 &\quad + N^{3/4} q_1^{1/8} \delta^{1/4} 3^{\omega(q_1)} \left(\frac{N}{q_1} + \log(q_1 \delta)\right)^{1/4}.
 \end{aligned}$$

We emphasize that the implicit constant above is absolute, and in particular does not depend on R : we handle the contribution of this term $R(n)/\delta$ quite trivially. In our application, the denominator δ is small, so the prospects for cancellation are modest in any case.

We prove Lemma 8 in the next section; assuming the lemma for the moment, we can complete the proof of Theorem 3. We apply Lemma 8 to the inner sum in equation (23) with $q = n_2^b b_3^b a_3^b$ and $\delta = n_2^\sharp b_3^\sharp a_3^\sharp$, where $N = x/(n_2 a_3 b_3)$. After doing so, by positivity we may again assume that $v < a_3 \leq w$ (the bound when $v < a_2 \leq w$ is exactly the same) and ignore all restrictions upon n_2 . We obtain

$$S_7(x) \ll \sum_{n_2 \in I} \sum_{\substack{a_3 b_3 \in I \\ v < a_3 \leq w}} \left\{ \sqrt{\frac{x}{n_2 a_3 b_3}} \sqrt{a_3^b} + \left(\frac{x}{n_2 a_3 b_3} \right)^{3/4} (b_3^b)^{1/4} \right. \\ \left. + \left(\frac{x}{n_2 a_3 b_3} \right)^{3/4} (n_2^b)^{1/8} (n_2^\sharp a_3^\sharp b_3^\sharp)^{1/4} 3^{\omega(n_2^b)} \left(\frac{x}{n_2 n_2^b a_3 b_3} + \log x \right)^{1/4} \right\}.$$

We now have to compute all the different sums:

$$S_7(x) \ll \sqrt{xy_2} \sqrt{w} (\log z)^{-1} + (xy_2)^{3/4} v^{-1/4} + x^{23/24+\varepsilon}. \quad (24)$$

Theorem 3 now follows from the estimates (20), (22), and (24) upon taking $w = x^{1/24}$, $v = (\log x)^{4B}$, $B = 10$, and $A = 30$, for example.

6 Short exponential sums

6.1 Proof of Lemma 8

In this section we prove Lemma 8, which will complete the proof of Theorem 3. This lemma is in fact a variant of a particular case of a result of Wu and Xi [20, Theorem 3.1 and Proposition 3.2]. While we do not need to introduce significant new ideas, the results of [20] cannot be applied directly in our context because we need a more precise version of the function N^ε in our error bounds. Careful attention to their paper reveals that it is possible to adapt some arguments to replace this N^ε by a quantity of the type $C^{\omega(q_1)}(\log N)^\alpha$. In many circumstances such a refinement is not necessary, but for us it is important due to the very restricted range of the factor a_3 .

For brevity we will write $J =]A, A + N]$, $W(n) = e(R(n)/\delta)$, and

$$E(J) = \sum_{n \in J} e\left(\frac{\alpha \bar{n}}{q}\right) W(n)$$

for the sum to be estimated. We begin by remarking that we may assume that $q_2 < N$ and $q_3 < N$, for otherwise the lemma is trivial. For any function Ψ and any $h \in \mathbb{Z}$ we define

$$\Delta_h(\Psi)(n) = \Psi(x) \overline{\Psi(x+h)}.$$

Lemma 9. *Let $q = q_1 q_2$ with $(q_1, q_2) = 1$, $J =]A, A + N]$ an interval and $\Psi_i: \mathbb{Z}/q_i \mathbb{Z} \rightarrow \mathbb{C}$. Then for $1 \leq L \leq N/q_2$, we have*

$$\left| \sum_{n \in J} \Psi_1(n) \Psi_2(n) \right|^2 \ll \|\Psi_2\|_\infty \left(L^{-1} N^2 + L^{-1} N \sum_{0 < |\ell| \leq L} \left| \sum_{\substack{n \in J \\ n + \ell q_2 \in J}} \Delta_{\ell q_2}(\Psi_1)(n) \right| \right).$$

This formula, which Wu and Xi call an A -process by analogy with the A -process of the classical van der Corput method, follows from the proof of [20, Lemma 3.1].

We apply this lemma with

$$\psi_1(n) = e\left(\frac{\alpha \bar{q}_3 \bar{n}}{q_1 q_2}\right) W(n), \quad \psi_2(n) = e\left(\frac{\alpha \overline{q_1 q_2} \bar{n}}{q_3}\right).$$

Writing $L_3 = \lceil N/q_3 \rceil$ (which is at least 1), this gives (see also the beginning of the proof of [20, Theorem 3.1])

$$|E(J)|^2 \ll L_3^{-1} N^2 + L_3^{-1} N \sum_{0 < |\ell_3| \leq L_3} |U(\ell_3)|, \quad (25)$$

with

$$U(\ell_3) = \sum_{n \in J(\ell_3)} e\left(\frac{\alpha \bar{q}_3 (\bar{n} - \overline{(n + \ell_3 q_3)})}{q_1 q_2}\right) W(n) \overline{W(n + \ell_3 q_3)}$$

where $J(\ell_3)$ is some interval contained in J . Note that we may write $U(\ell_3) = \sum_{n \in J} \Psi_3(n) \Psi_4(n)$ where

$$\begin{aligned} \psi_3(n) &= e\left(\frac{\alpha \bar{q}_2 \bar{q}_3 (\bar{n} - \overline{(n + \ell_3 q_3)})}{q_1}\right) W(n) \overline{W(n + \ell_3 q_3)} \mathbf{1}_{J(\ell_3)}(n), \\ \psi_4(n) &= e\left(\frac{\alpha \bar{q}_1 \bar{q}_3 (\bar{n} - \overline{(n + \ell_3 q_3)})}{q_2}\right), \end{aligned} \quad (26)$$

where $\mathbf{1}_{J(\ell_3)}$ is the indicator function of $J(\ell_3)$. We again apply Lemma 9 to each $U(\ell_3)$, writing $L_2 = \lceil N/q_2 \rceil$ (which again is at least 1); we have written $\psi_3(n)$ as in equation (26) so as to make this choice of L_2 valid even when the length of $J(\ell_3)$ is much smaller than N . We obtain

$$|U(\ell_3)|^2 \ll L_2^{-1} N^2 + L_2^{-1} N \sum_{0 < |\ell_2| \leq L_2} |U(\ell_2, \ell_3)|, \quad (27)$$

with now

$$U(\ell_2, \ell_3) = \sum_{n \in J(\ell_2, \ell_3)} e\left(\frac{F(n)}{q_1}\right) \tilde{W}(n),$$

where $J(\ell_2, \ell_3)$ is some interval contained in $J(\ell_3)$, and

$$\begin{aligned} F(n) &= \alpha \bar{q}_2 \bar{q}_3 [\bar{n} - \overline{(n + \ell_3 q_3)} - \overline{(n + \ell_2 q_2)} + \overline{(n + \ell_2 q_2 + \ell_3 q_3)}], \\ \tilde{W}(n) &= W(n) \overline{W(n + \ell_3 q_3)} \overline{W(n + \ell_2 q_2)} W(n + \ell_2 q_2 + \ell_3 q_3). \end{aligned} \quad (28)$$

Then we complete the above sum over $n \in J(\ell_2, \ell_3)$:

$$\begin{aligned} U(\ell_2, \ell_3) &= \frac{1}{q_1 \delta} \sum_{a=1}^{q_1 \delta} e\left(\frac{F(a)}{q_1}\right) \tilde{W}(a) \sum_{h=1}^{q_1 \delta} \sum_{n \in J(\ell_2, \ell_3)} e\left(\frac{h(a-n)}{q_1 \delta}\right) \\ &\ll \frac{N}{q_1 \delta} \left| \sum_{a=1}^{q_1 \delta} e\left(\frac{F(a)}{q_1}\right) \tilde{W}(a) \right| + \sum_{1 \leq h < q_1 \delta} \frac{1}{h} \sum_{a=1}^{q_1 \delta} e\left(\frac{F(a)}{q_1}\right) \tilde{W}(a) e\left(\frac{ha}{q_1 \delta}\right). \end{aligned}$$

We denote by $\Sigma_a(h)$ the inner sum on a in the second term and perform the same manipulations as for the sums S_a in the proof of Theorem 2, resulting in

$$\begin{aligned}\Sigma_a(h) &= \sum_{a=1}^{q_1\delta} e\left(\frac{F(a)}{q_1}\right) \widetilde{W}(a) e\left(\frac{ha}{q_1\delta}\right) \\ &= \sum_{u=1}^{\delta} \widetilde{W}(q_1u) e\left(\frac{hu}{\delta}\right) \prod_{p|q_1} \sum_{v=1}^p e\left(\frac{F(v\delta q_1/p) + hv}{p}\right).\end{aligned}\tag{29}$$

The function F in equation (28) can be rewritten in the following way, with $\lambda = \alpha\overline{q_2q_3}$:

$$F(n) = \frac{\lambda G(n)}{n(n + \ell_2q_2)(n + \ell_3q_3)(n + \ell_2q_2 + \ell_3q_3)},$$

where $G(n)$ is a polynomial with constant term $\ell_2q_2\ell_3q_3(\ell_2q_2 + \ell_3q_3)$ (the exception being when $p \mid (\ell_2q_2 + \ell_3q_3)$, in which case we actually have $F(n) = 2\lambda\ell_2q_2\ell_3q_3/(n(n + \ell_2q_2)(n + \ell_3q_3))$). If $p \nmid \ell_2\ell_3$, the function $F(v\delta q_1/p) + hv$ of v has at most 5 poles, each pole being simple (including the pole at ∞); this is most clearly seen from the definition (28) of $F(n)$. Then by Lemma 7 we have

$$\left| \sum_{v=1}^p e\left(\frac{F(v\delta q_1/p) + hv}{p}\right) \right| \leq 10\sqrt{p}$$

when $p \nmid \ell_2\ell_3$. We deduce from equation (29) that

$$|\Sigma_a(h)| \leq \sum_{u=1}^{\delta} \prod_{\substack{p|q_1 \\ p \nmid \ell_2\ell_3}} 10\sqrt{p} \prod_{\substack{p|q_1 \\ p \nmid \ell_2\ell_3}} p \leq 10^{\omega(q_1)} \delta \sqrt{q_1} (q_1, \ell_2\ell_3)^{1/2}.$$

Then

$$\sum_{\ell_2 \leq L_2} |U(\ell_2, \ell_3)| \ll \delta \sqrt{q_1} \left(\log q_1 + \frac{N}{q_1} \right) 10^{\omega(q_1)} \sum_{\ell_2 \leq L_2} (q_1, \ell_2\ell_3)^{1/2}.\tag{30}$$

For the sum on ℓ_2 we have for any $\ell_3 \leq L_3$:

$$\sum_{\ell_2 \leq L_2} (q_1, \ell_2\ell_3)^{1/2} \leq \sum_{d|q_1} \sqrt{d} (\ell_3, q_1/d)^{1/2} \sum_{\ell_2 \leq L_2/d} 1 \ll \tau(q_1) L_2 (q_1, \ell_3)^{1/2}.\tag{31}$$

Inserting this bound into equation (30) and tracing the results back through equations (27) and (25) results in

$$E(J) \ll NL_3^{-1/2} + NL_2^{-1/4} + N^{3/4} \left(\frac{N}{q_1\delta} + \log(q_1\delta) \right)^{1/4} q_1^{1/8} \delta^{1/4} 10^{\omega(q_1)/4} \tau(q_1)^{3/4}.$$

It remains to replace L_2 by $[N/q_2]$ and L_3 by $[N/q_3]$, and to observe that

$$10^{\omega(q_1)/4} \tau(q_1)^{3/4} = 80^{\omega(q_1)/4} \leq 3^{\omega(q_1)},$$

to finish the proof of Lemma 8.

6.2 Generalization of Lemma 8, the A^k -process

In this section, we indicate how to iterate the ideas of the proof of Lemma 8 to obtain bounds for short exponential sums whose denominator can be decomposed into $k + 1$ factors. This generalization relies on techniques from two important papers of Irving [13, 14]; again, our contribution here consists mainly in replacing a factor of N^ε by a more precise error term.

Essentially, we would like to apply Lemma 9 consecutively k times. Irving has given a precise formulation of this iteration; to enounce his result, we need to introduce some notation corresponding to the iterates of the $\Delta_h(\Psi)$ used in the previous section. For any complex-valued function f , define

$$f(n; h_1, \dots, h_k) = \prod_{S \subset \{1, \dots, k\}} f\left(n + \sum_{s \in S} h_s\right)^{\sigma(S)},$$

where $\sigma(S)$ denotes that the complex conjugate is taken when $\#S$ is odd. With this notation, we may quote [14, Lemma 2.2]:

Lemma 10. *Let $k \in \mathbb{N}$ and $q_0, \dots, q_k \in \mathbb{N}$. For each $0 \leq i \leq k$, let $f_i: \mathbb{Z} \rightarrow \mathbb{C}$ be a function with period q_i such that $f_i(n) \ll 1$. Set $q = q_0 \cdots q_k$ and $f(n) = \prod_{i=0}^k f_i(n)$. If I is any interval of length at most N , then*

$$\left| \sum_{n \in I} f(n) \right|^{2^k} \ll_k \sum_{j=1}^k N^{2^k - 2^{k-j}} \frac{2^{k-j}}{q_{k-j+1}} + N^{2^k - k - 1} \frac{q}{q_0} \sum_{0 < |h_1| \leq N/q_1} \cdots \sum_{0 < |h_k| \leq N/q_k} \left| \sum_{n \in I(h_1, \dots, h_k)} f_0(n; q_1 h_1, \dots, q_k h_k) \right|, \quad (32)$$

where $I(h_1, \dots, h_k)$ is a subinterval of I .

We also quote the following combinatorial lemma [13, Lemma 4.5]:

Lemma 11. *Let $p \geq 3$ be prime and $h_1, \dots, h_k \in \mathbb{F}_p$. Suppose that for every $b \in \mathbb{F}_p$, the number of subsets $S \subset \{1, \dots, k\}$ with $b = \sum_{s \in S} h_s$ is even. Then some h_i must equal 0.*

We are now prepared to establish our generalization of Lemma 8.

Lemma 12. *Uniformly for any integers α, A, N , any positive integers δ, k , any positive integers q_0, \dots, q_k such that $q = q_0 \cdots q_k$ is squarefree and coprime to $\alpha\delta$, and any rational function R with integer coefficients,*

$$\sum_{\substack{A < n \leq A+N \\ (n, q) = 1}} e\left(\frac{\alpha \bar{n}}{q} + \frac{R(n)}{\delta}\right) \ll_k \sum_{j=1}^k N^{1-1/2^j} q_j^{1/2^j} + N^{1-1/2^k} q_0^{1/2^{k+1}} \delta^{1/2^k} (2^{k+2} + 4)^{\omega(q_0)/2^k} \left(\left(\frac{N}{q_0}\right)^{1/2^k} + (\log q)^{1/2^k} \right).$$

Proof. We apply Lemma 10 with

$$f(n) = e\left(\frac{\alpha \bar{n}}{q}\right) W(n) = e\left(\frac{\alpha \bar{n}(q/q_0)}{q_0} + \frac{R(n)}{\delta}\right) \prod_{i=1}^k e\left(\frac{\alpha \bar{n}(q/q_i)}{q_i}\right),$$

where as in the previous section $W(n) = e(R(n)/\delta)$. Our desire at this point is to apply Weil's bound to the sums over n that arise from equation (32). After the same manipulations as in the proof of Lemma 8, the analogue of equation (29) is now

$$\begin{aligned} \Sigma_a(m) &= \sum_{u=1}^{\delta} W(aq_0; q_1 h_1, \dots, q_k h_k) e\left(\frac{mu}{\delta}\right) \\ &\times \prod_{p|q_0} \sum_{v=1}^p e\left(\frac{mv + \sum_{S \subset \{1, \dots, k\}} (-1)^{\#S} \overline{\alpha(v\delta q_0/p + \sum_{s \in S} h_s q_s)}}{p}\right). \end{aligned} \quad (33)$$

In order to apply Weil's bound (Lemma 7), we need to confirm that the argument of the exponential is nonconstant modulo p (even if $p \mid m$).

Note that the numerator in equation (33) can be written as

$$\begin{aligned} mv + \sum_{S \subset \{1, \dots, k\}} (-1)^{\#S} \overline{\alpha(v\delta q_0/p + \sum_{s \in S} h_s q_s)} \\ = mv + \alpha \sum_{t \pmod{p}} \overline{(v\delta q_0/p + t)} \sum_{\substack{S \subset \{1, \dots, k\} \\ \sum_{s \in S} h_s q_s \equiv t \pmod{p}}} (-1)^{\#S} \end{aligned} \quad (34)$$

When $p \nmid h_1 \cdots h_k$ (so that indeed $p \nmid h_1 q_1 \cdots h_s q_s$), Lemma 11 implies that at least one of the inner sums on the right-hand side of equation (34) has an odd number of terms, and in particular (by considering its parity) is nonzero. In particular, the numerator in equation (33) is nonconstant, and thus Lemma 7 can be applied.

The number of poles of this numerator is at most $2^k + 1$. When $p \nmid h_1 \cdots h_k$, Lemma 7 implies that the corresponding sum is less than $2(2^k + 1)\sqrt{p}$. We deduce that $\Sigma_a(m) \ll \delta(2^{k+1} + 2)^{\omega(q_0)} \sqrt{q_0} (q_0, h_1 \cdots h_k)^{1/2}$. The proof concludes with quite similar computations to those in the proof of Lemma 8, with the function $\tau(q_1)$ from equation (31) replaced by $\sum_{d_1 \cdots d_k | q_0} 1/\sqrt{d_1 \cdots d_k} \ll 2^{\omega(q_0)}$. \square

7 Product of k linear factors

This section is devoted to the proof of Theorem 4. We will skip some details when the arguments are similar to the previous proofs. All implicit constants in this section may depend on f and h .

Define $A = \prod_{1 \leq i < j \leq k} (a_i b_j - a_j b_i) \prod_{i=1}^k a_i$. The first step follows the beginning of the proof of Theorem 1 in Section 3.2. However, since f has more than two linear factors, the discussions related to the greatest common divisor of A , h , and the denominators n are more delicate. This is why in our splitting analogous of (10), the summation of δ will be for $\delta \mid (hA)^\infty$ instead of $\delta \mid (h, A)^\infty$. We keep the notations $S(f, x, h) = S_>(x, h) + S_\leq(x, h)$, where in $S_>(x, h)$ the parameter $\delta \mid (hA)^\infty$ exceeds B , but now with $B = (\log x)^{10k}$ instead of $x^{1/5-\varepsilon}$. We bound $S_>(x, h)$ by $x(\log x)^{k-1} B^{-1+\varepsilon}$ with $\varepsilon > 0$ arbitrarily small as in equation (11) (indeed $\varepsilon = 1/10$ will suffice for us).

For $S_\leq(x, h)$, however, an analogous version of equation (12) is not sufficient. This is due to the fact that a trivial summation on $\delta < B$ would bring a factor $B > (\log x)^k$ into our error terms, and we can win only a factor $(\log x)^4$ in an error term (denoted by $T_5^{(1)}(r, \delta)$ later in this section) arising from certain denominators n that are the product of several divisors of the same size.

Nevertheless, we still begin by applying the Chinese remainder theorem as in Section 3.2:

$$S_{\leq}(f, x, h) = \sum_{\substack{\delta | (Ah)^{\infty} \\ \delta \leq B}} \sum_{\substack{0 \leq r < \delta \\ f(r) \equiv 0 \pmod{\delta}}} T(r, \delta), \quad (35)$$

with now

$$T(r, \delta) = \sum_{\substack{n \leq x/\delta \\ (n, Ah)=1}} \sum_{\substack{0 \leq r_1 < n \\ f(r_1 \delta) \equiv 0 \pmod{n}}} e\left(\frac{hr_1}{n} + \frac{hr\bar{n}}{\delta}\right).$$

The number of roots r in equation (35) is $O(A^{\omega(\delta)})$ (see Nagell [17, p. 90, Theorem 54] for an even more precise result) and thus $O(1)$ with our conventions for implicit constants. Since our polynomial f is the product of k linear functions, generalizations of Lemma 3 and equation (6) allow us to write

$$T(r, \delta) = \sum_{\substack{m_1 \cdots m_k \leq x/\delta \\ (m_1 \cdots m_k, Ah)=1}} \sum_{\substack{0 \leq r_1 < m_1 \cdots m_k \\ m_i | f_i(r_1 \delta) \ (1 \leq i \leq k)}} e\left(\frac{hr_1}{m_1 \cdots m_k} + \frac{hr\bar{m}_1 \cdots \bar{m}_k}{\delta}\right),$$

where $(m_i, m_j) = 1$ for all $1 \leq i < j \leq k$ and

$$r_1 \equiv -\bar{\delta} \sum_{i=1}^k \bar{a}_i b_i \prod_{\substack{j=1 \\ j \neq i}}^k m_j \left(\prod_{\substack{j=1 \\ j \neq i}}^k \overline{m_j} \right)_{m_i} \pmod{m_1 \cdots m_k},$$

so that

$$e\left(\frac{hr_1}{m_1 \cdots m_k}\right) = \prod_{i=1}^k e\left(\frac{-h\bar{\delta} \bar{a}_i b_i \prod_{\substack{j=1 \\ j \neq i}}^k \overline{m_j}}{m_i}\right). \quad (36)$$

We set $y = x^{1/3}(\log x)^{10k}$ and write $T(r, \delta) = T_1(r, \delta) + T_2(r, \delta)$, where the sum $T_2(\alpha, \delta)$ contains precisely those summands for which $\max\{m_1, \dots, m_k\} > y$.

We decompose $T_2(r, \delta) = S_1 + \dots + S_k$ into k sums, where each S_i is defined by the conditions $m_i > y$ and $m_j \leq y$ for $j < i$. In each S_i , we use the inversion formula (6) and apply Lemma 1 as in the proof of Theorem 3. This yields an asymptotic formula of the shape $S_i = xC_i(f, h, \delta) + O((x/y)^{3/2}\sqrt{\delta}(\log x)^k)$ for some constants $C_i(f, h, \delta)$ similar to the constants found in Section 3.2 in the proof of Theorem 1.

These contributions comprise the main term in Theorem 4 when $k = 3$ and $k = 4$. It remains to find an upper bound for $T_1(r, \delta)$. This upper bound will be $o(x)$ only for $k = 3$ and $k = 4$, which allows for our asymptotic formula in those cases; for $k \geq 5$ it provides only a nontrivial upper bound as stated in Theorem 4.

We split the sum $T_1(r, \delta)$ into $k!$ subsums according to the ordering of the m_i . Let $T_1^{(1)}(r, \delta)$ denote the subsum of $T_1(r, \delta)$ with the additional condition that $m_1 \geq m_2 \geq \dots \geq m_k$; the estimate we find for this subsum will hold for all $k!$ subsums. We would like to deal with $T_1^{(1)}(r, \delta)$ simply by applying Lemma 12; however, this lemma is not efficient enough when m_3 is close to m_1 . Consequently, we make one more splitting $T_1^{(1)}(r, \delta) = T_3^{(1)}(r, \delta) + T_4^{(1)}(r, \delta)$, where $T_3^{(1)}(r, \delta)$ consists of the terms for which $m_3 \leq m_1(\log x)^{-k4^k}$.

Concentrating first on $T_3^{(1)}(r, \delta)$, we do dyadic splittings in all k variables, writing $T_3^{(1)}(r, \delta)$ as $O(\log^k x)$ subsums $S(M_1, \dots, M_k)$ that are restricted to m_i satisfying $M_i < m_i \leq 2M_i$ for all $1 \leq i \leq k$; here the bounds M_i are powers of 2, satisfying $2^k M_1 \cdots M_k \leq x$, such that $M_1 \leq y$ while $2M_i \leq y$ for $2 \leq i \leq k$. The ordering of the m_i also implies that $M_1 \geq M_2 \geq \dots \geq M_k$.

By the inversion formula (6), we can eliminate the variable m_1 in the denominator in equation (36):

$$e\left(-\frac{hb_1 a_1 \overline{\delta m_2 \cdots m_k}}{m_1}\right) = e\left(\frac{hb_1 \overline{m_1}}{a_1 \delta m_2 \cdots m_k}\right) e\left(\frac{hb_1}{a_1 \delta m_1 \cdots m_k}\right),$$

The second exponential term above is $1 + O(1/\delta M_1 \cdots M_k)$ and can be replaced by 1 with an admissible error. Thus we have, for some λ depending on $m_2, \dots, m_k, \delta, r, f$ but not on m_1 ,

$$\begin{aligned} S(M_1, \dots, M_k) &= \sum_{\substack{M_i < m_i \leq 2M_i \ (2 \leq i \leq k) \\ (m_2 \cdots m_k, Ah)=1}} \sum_{\substack{M_1 < m_1 \leq 2M_1 \\ \max\{m_2, \dots, m_k\} \leq m_1 \leq x/(m_1 \cdots m_2) \\ (m_1, Ahm_2 \cdots m_k)=1}} e\left(\frac{\lambda \overline{m_1}}{a_1 \delta m_2 \cdots m_k}\right) + E, \end{aligned}$$

where E is a sufficiently small error term. Next we separate the squarefree and the squarefull parts of this denominator. With the same notation as in the proof of Theorem 3, we obtain a formula of the shape

$$e\left(\frac{\lambda \overline{m_1}}{a_1 m_2 \cdots m_k}\right) = e\left(\frac{\alpha \overline{m_1}}{(a_1 m_2 \cdots m_k)^b} + \frac{R(n)}{\delta (a_1 m_2 \cdots m_k)^\sharp}\right).$$

Writing $q^\sharp = (a_1 m_2 \cdots m_k)^\sharp$, we find that Lemma 12 gives the estimate

$$\begin{aligned} S(M_1, \dots, M_k) &\ll x \sum_{j=3}^k (M_j/M_1)^{1/2^{j-2}} \\ &\quad + x(\delta q^\sharp)^{1/2^k} (\log x)^{2^k+3} (M_2^{-1/2^k} + (\log x)^{1/2^k} M_2^{1/2^{k+1}} M_1^{-1/2^k}). \end{aligned}$$

Since $M_k \leq \dots \leq M_3 \leq M_1 (\log x)^{-k4^k}$, this estimate saves enough powers of $\log x$ to compensate for the number of subsums and the summation over δ ; we obtain a bound for $T_3^{(1)}(r, \delta)$ that is admissible for Theorem 4.

It remains to handle $T_4^{(1)}(r, \delta)$, where a direct application of Lemma 12 is not sufficient. We will proceed as in the sum $S_4(x)$ in Section 5.2. Recalling that $m_1 \geq m_2 \geq \dots \geq m_k$, we let $\ell \in \{2, \dots, k\}$ be the largest index such that $m_\ell > m_1 (\log x)^{-k4^k}$. For $2 \leq i \leq \ell$, we factor $m_i = c_i d_i$ with $P^+(c_i) \leq z < P^-(d_i)$ where $z = \exp(\log x / (800k \log \log x))$. We also introduce the parameters $w = x^{1/(50k)}$ and $v = (\log x)^{10k4^k}$.

Let $T_6^{(1)}(r, \delta)$ be the analogue of the sum $S_6(x)$ in Section 5.2, which is the contribution of the (m_1, \dots, m_k) such that $c_i > w$ for some $i \in \{2, \dots, \ell\}$. With a computation similar to the one in equation (22), we find:

$$T_6^{(1)}(r, \delta) \ll \frac{x(\log x)^k}{w^{1/4}} + x 2^{-\log w / (2 \log z)} (\log x)^k \ll x (\log x)^{k-8 \log 2},$$

where the final estimate is sufficient since $8 \log 2 > 5$.

We next examine $T_7^{(1)}(r, \delta)$, the analogue of $S_7(x)$ in Section 5.2, corresponding to the case where there is at most one $i \in \{2, \dots, \ell\}$ with $c_i \leq v$. Without loss of generality we can suppose that $c_2 \leq v$ and $c_i \in [v, w]$ for all $3 \leq i \leq \ell$. We introduce some dyadic splittings, similar to those in our treatment of $T_3^{(1)}(r, \delta)$, of $T_7^{(1)}(r, \delta)$ into $O((\log x)^k)$ sums $S_7(M_1, \dots, M_k)$ with $M_i < c_i d_i \leq 2M_i$ for $2 \leq i \leq \ell$ and $M_i < m_i \leq 2M_i$ for $i \in \{1\} \cup \{\ell + 1, \dots, k\}$. We apply Lemma 12 with k replaced by $k + \ell - 4$ and with the parameters

$$\begin{aligned} N &= 2M_1, \\ q_0 &= c_2 d_2, \\ q_1 &= d_3, q_2 = d_4, \dots, q_{\ell-2} = d_\ell, \\ q_{\ell-1} &= m_{\ell+1}, q_\ell = m_{\ell+2}, \dots, q_{k-2} = m_k, \\ q_{k-1} &= c_3, q_k = c_4, \dots, q_{k+\ell-4} = c_\ell. \end{aligned}$$

With the notations $q^\sharp = (a_1 m_2 \cdots m_k)^\sharp$ and $k_1 = k + \ell - 4$, Lemma 12 gives

$$\begin{aligned} S_7(M_1, \dots, M_k) &= \sum_{m_2, \dots, m_k}^* \left[\sum_{j=1}^{\ell-2} M_1^{1-1/2^j} d_{j+2}^{1/2^j} + \sum_{j=\ell-1}^{k-2} M_1^{1-1/2^j} m_{j+2}^{1/2^j} + \sum_{j=k-1}^{k+\ell-4} M_1^{1-1/2^j} c_{j-k+4}^{1/2^j} \right. \\ &\quad \left. + M_1^{1-1/2^{k_1}} M_2^{1/2^{k_1+1}} (\delta q^\sharp)^{1/2^{k_1}} (2^{k_1} + 4)^{\omega(m_2)/2^{k_1}} \left(\left(\frac{M_1}{M_2} \right)^{1/2^{k_1}} + (\log x)^{1/2^{k_1}} \right) \right], \end{aligned}$$

where the asterisk on the sum indicates that the m_j and the c_j, d_j satisfy the previously indicated conditions. At this point, computations analogous to those at the end of Section 5.2 give a bound for $T_7^{(1)}(r, \delta)$ that is sufficiently small.

Finally, we examine $T_5^{(1)}(r, \delta)$, the analogue of $S_5(x)$ in Section 5.2, corresponding to the case where $c_i \leq v$ for at least two indices $i \in \{2, \dots, \ell\}$. The estimation of this part of the sum is where we fail to obtain an asymptotic formula for $S(f, x, h)$ when $k \geq 5$.

Without loss of generality we can suppose that $c_2 \leq v$ and $c_3 \leq v$. If we define $I(m_1) = [m_1/(\log x)^{k_1}, m_1]$, then

$$T_5^{(1)}(r, \delta) \ll \sum_{m_1 \leq y} \sum_{m_5, \dots, m_k \leq m_1} \sum_{\substack{c_2, c_3 \leq v \\ c_2 d_2 \in I(m_1) \\ c_3 d_3 \in I(m_1)}} \sum_{\substack{m_4 \leq x/(\delta m_1 c_2 d_2 c_3 d_3 m_5 \cdots m_k) \\ m_4 \leq m_1}} 1.$$

(In this line, the sum over m_5, \dots, m_k does not occur if $k = 3$ or $k = 4$; in fact, the case $k = 3$ is handled exactly in the same way as in the estimation of $S_5(x)$ in Section 5.2.) We use equation (21) for the sum over the sifted variables d_2, d_3 , gaining an additional factor of $(\log z)^2$ in the estimate:

$$T_5^{(1)}(r, \delta) \ll \frac{x(\log \log x)^2 (\log x)^{k-3}}{\delta (\log z)^2} \ll x \delta^{-1} (\log \log x)^4 (\log x)^{k-5}. \quad (37)$$

Since the number of roots r is bounded uniformly in δ and $\sum_{\delta | (Ah)^\infty} \delta^{-1} \ll 1$, we see that the bound (37) is sufficiently small. This completes the proof of Theorem 4.

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