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# ASYMPTOTIC BEHAVIOR OF NERNST-PLANCK EQUATION

XINGYU LI

## Abstract

This paper is devoted to the Nernst-Planck system of equations with an external potential of confinement. The main result is concerned with the asymptotic behaviour of the solution of the Cauchy problem. We will prove that the optimal exponential rate of convergence of the solution to the unique stationary solution is determined by the spectral gap of the linearized problem around the minimizer of the free energy. The key issue is to consider an adapted notion of scalar product.

**Keywords:** Nernst-Planck equation; large time asymptotic; free energy; Fisher information; spectral gap.

**AMS subject classifications:** 35B40; 35P15; 35Q70.

## 1. INTRODUCTION

At the end of nineteenth century, Nernst and Planck introduced a system of equations for representing the evolution of charged particles subject to electrostatic forces. The original model is exposed in [22, 24]: electrically charged particles diffuse under the action of a drift caused by an electrostatic potential. Nowadays we use this system in various frameworks like, for instance, phenomenological models for electrolytic behaviour in membranes. The original model is the non-confined Nernst-Planck system. If we take into account a mean-field Poisson coupling, in dimension  $d = 2$ , the system takes the form

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + \nabla \cdot (u \nabla v) \\ v = G_2 * u \\ u(0, x) = n_0 \geq 0 \end{cases} \quad x \in \mathbb{R}^2, t > 0, \quad (1)$$

where  $G_2(x) = -\frac{1}{2\pi} \log|x|$  denotes the Green function of the Laplacian in  $\mathbb{R}^2$ . We shall call this model the *Poisson-Nernst-Planck system*, which was also considered by Debye and Hückel in [12] and is sometimes called the *Debye-Hückel system* in the literature. Up to a sign change in the mean-field term, the model is similar to the Keller-Segel model, which is going to be a source of inspiration (see [9, 10, 11] for more details) for the study of the large time behaviour and this is a reason why we consider the two-dimensional case of the model.

Now let us introduce the notion of *confinement*. In the whole space, particles repel themselves and a well-known *runaway* phenomenon occurs: solutions locally vanish while the mass escapes at infinity. This can be prevented using a container (a bounded domain, with convenient boundary conditions) with walls, or a confinement potential. Actually, it is possible to obtain the bounded domain case as a limit of a whole space case with an external potential of *confinement* taking larger and larger values outside of the domain. Here we shall consider the Poisson-Nernst-Planck system with confinement in  $\mathbb{R}^d$ , where the dimension is  $d = 2$  or  $d = 3$ . The density function  $n$  solves

$$\begin{cases} \frac{\partial n}{\partial t} = \Delta n + \nabla \cdot (n \nabla c) + \nabla \cdot (n \nabla \phi) \\ c = G_d * n \\ n(0, x) = n_0 \geq 0, \quad \int_{\mathbb{R}^d} n(0, x) dx = M > 0 \end{cases} \quad x \in \mathbb{R}^d, t > 0. \quad (2)$$

The convolution kernel  $G_d$  is the Green function of the Laplacian in  $\mathbb{R}^d$ , namely

$$G_2(x) = -\frac{1}{2\pi} \log|x| \quad \text{for any } x \in \mathbb{R}^2 \quad \text{and} \quad G_3(x) = \frac{1}{4\pi|x|} \quad \text{for any } x \in \mathbb{R}^3.$$

In other words, we ask that  $c$  solves the Poisson equation

$$-\Delta c = n \quad x \in \mathbb{R}^d,$$

while  $\phi$  is a given external potential. In the special case of  $d = 2$  and  $\phi(x) = \frac{\mu}{2}|x|^2$  for some  $\mu > 0$ , if we use the change of variables

$$\begin{aligned} u(t, x) &= R^{-d} n(\tau, \xi), \quad v(t, x) = c(\tau, \xi), \\ \xi &= \frac{x}{R}, \quad \tau = \log R, \quad R = R(t) := \sqrt{1 + 2\mu t}, \end{aligned} \quad (3)$$

then we observe that  $(n, c)$  solves (2) if and only if  $(u, v)$  solves (1). Studying the convergence rates of the solutions of (2) amounts to study the intermediate asymptotics of the solutions of (1) when runaway occurs. Obviously, the mass of a solution of (2) is conserved, and we shall write that  $\int_{\mathbb{R}^d} n(t, x) dx = M$  for any  $t \geq 0$ . The mass of a solution of (1) is also conserved, but one can prove that, for a solution of (1), the mass contained in any given compact set in  $\mathbb{R}^2$  decays to zero.

From here on, we shall assume that  $M > 0$  is fixed. Now let us turn our attention to the conditions on the confinement potential. From now on, we shall assume that  $\phi \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^d)$  is such that  $\nabla \phi \in W^{1,\infty}(\mathbb{R}^d)$  and

$$\liminf_{|x| \rightarrow +\infty} \frac{\phi(x)}{\log|x|} > d, \quad (C1)$$

and also that the bounded measure  $e^{-\phi} dx$  admits a spectral gap (or Poincaré inequality, *i.e.*, that there exists a positive constant  $\Lambda_\phi$  such that

$$\int_{\mathbb{R}^d} |\nabla u|^2 e^{-\phi} dx \geq \Lambda_\phi \int_{\mathbb{R}^d} |u|^2 e^{-\phi} dx$$

$$\forall u \in H^1(\mathbb{R}^d; e^{-\phi} dx) \quad \text{such that} \quad \int_{\mathbb{R}^d} u e^{-\phi} dx = 0. \quad (\text{C2})$$

Based on Persson's lemma, a sufficient condition is obtained by requesting that

$$\sigma_\phi := \lim_{r \rightarrow +\infty} \inf_{x \in B_r^c} \left( \frac{1}{4} |\nabla \phi|^2 - \frac{1}{2} \Delta_x \phi \right) > 0 \quad \text{and} \quad \lim_{r \rightarrow +\infty} \inf_{x \in B_r^c} |\nabla \phi| > 0. \quad (\text{C3})$$

Let us refer to [1] for details and further references. We learn from [3, 7] that the stationary solutions  $(n_\infty, c_\infty)$  of (2) are obtained as solutions of the *Poisson-Boltzmann equation*

$$-\Delta c_\infty = n_\infty = M \frac{e^{-c_\infty - \phi}}{\int_{\mathbb{R}^d} e^{-c_\infty - \phi} dx}. \quad (4)$$

Under Assumption (C1) and the additional condition

$$\liminf_{|x| \rightarrow +\infty} \frac{\phi(x)}{\log|x|} > 4 + \frac{M}{2\pi} \quad \text{if} \quad d = 2, \quad (\text{C4})$$

we know (see [1, Lemma 5] and earlier references therein) that the unique solution of (4) is obtained as a minimizer of the *free energy*  $\mathcal{F}$  defined by

$$\mathcal{F}[n] := \int_{\mathbb{R}^d} n \log n dx + \int_{\mathbb{R}^d} n \phi dx + \frac{1}{2} \int_{\mathbb{R}^d} n (-\Delta)^{-1} n dx. \quad (5)$$

Further details are given in Section 2. A simple consequence of the minimization procedure is that

$$\mathcal{F}[n] - \mathcal{F}[n_\infty] \geq 0 \quad \forall n \in L_+^1(\mathbb{R}^d)$$

with the convention that  $\mathcal{F}[n]$  can take the value  $+\infty$  if, for instance  $n \log n$  is not integrable. For sake of brevity, we shall say that  $\phi$  is a *confinement potential satisfying Assumption (C)* if (C1), (C3) and (C4) hold.

Our goal is to study the asymptotic behaviour of a solution of (2) with initial datum  $n_0$  at  $t = 0$  such that  $\mathcal{F}[n_0]$  is finite. It is a standard observation that the free energy  $\mathcal{F}[n(t, \cdot)]$  of a solution of (2) is monotone non-increasing along the flows and obeys to

$$\frac{d}{dt} \mathcal{F}[n(t, \cdot)] = -\mathcal{I}[n(t, \cdot)] \quad (6)$$

where the *Fisher information*  $\mathcal{I}$  is defined by

$$\mathcal{I}[n] := \int_{\mathbb{R}^d} n |\nabla(\log n + c + \phi)|^2 dx.$$

Our main result is that, as  $t \rightarrow +\infty$ ,  $\mathcal{F}[n(t, \cdot)]$  is bounded by  $\mathcal{I}[n(t, \cdot)]$  up to a multiplicative constant which shows that  $n(t, \cdot)$  converges to  $n_\infty$  at an exponential rate. The precise result is not written in terms of the free energy but in terms of a weighted  $L^2$  norm and goes as follows.

**Theorem 1.1.** *Let  $d = 2$  or  $3$  and consider a potential  $\phi$  satisfying (C). Assume that  $n$  solves (2) with initial datum  $n(0, \cdot) = n_0 \in L^2_+(n_\infty^{-1} dx)$ ,  $\int_{\mathbb{R}^d} n_0 dx = M$ , and  $\mathcal{F}[n_0] < \infty$ . Then there exist two positive constants  $C$  and  $\Lambda$  such that*

$$\int_{\mathbb{R}^d} |n(t, \cdot) - n_\infty|^2 n_\infty^{-1} dx \leq C e^{-\Lambda t} \quad \forall t \geq 0.$$

In section 4, we will characterize  $\Lambda$  as the spectral gap of the linearized operator associated with (2) and observe, as a special case, that  $\Lambda = 2\mu$  if  $d = 2$  and  $\phi = \frac{\mu}{2}|x|^2$ , for some  $\mu > 0$ .

Beyond free energy and entropy methods, the study of the large time asymptotics of the Poisson-Nernst-Planck system involves various tools of nonlinear analysis. Proving an exponential rate of convergence is interesting for studies of Poisson-Nernst-Planck systems by methods of scientific computing. Specific methods are needed for the numerical computation of the solutions, see [4, 23]. In [21], Liu and Wang implement at the level of the free energy a finite difference method to compute the numerical solution in a bounded domain. Concerning rates of convergence from a more theoretical point of view, let us mention that the existence of special solutions and self-similar solutions is considered in [5, 6, 17]. We refer to [26] for a discussion of the evolution problem from the point of view of physics.

Variants of the Poisson-Nernst-Planck system with nonlinear diffusions have been considered, for which the sharp rate of convergence is still unknown. Some papers rely on the use of distances related to the  $L^2$ -Wasserstein distance, see [14, 18, 28]. Exponential decay rates should be natural in view of the expected gradient flow structure of the system in this framework. The simpler case of linear diffusions on a bounded domain of  $\mathbb{R}^d$  with  $d \geq 3$  was studied in [7]: the convergence to the stationary solution occurs at an exponential rate. As already mentioned, another related model is the Keller-Segel system in dimension 2. Regularity and asymptotic estimates for this system were discussed in [9, 11] and are a source of inspiration for the present study, in particular concerning the scalar product and the coercivity estimates. For completeness, let us mention that similar ideas have been recently developed in [19] for the

study of a McKean-Vlasov model model of flocking, which also involves a non-local coupling.

This paper is organized as follows. In Section 2 we prove that the minimizer of the free energy  $\mathcal{F}$  is the stationary solution  $(n_\infty, c_\infty)$  and it attracts any solution of (2) as  $t \rightarrow +\infty$ . In Section 3, we show that the relative entropy and the relative Fisher information provide us with two quadratic forms which are related by the linearized evolution operator and prove the spectral gap property of this operator. And in Section 4, we give the proof of Theorem 1.1 and give some additional results.

## 2. MINIMIZERS OF THE FREE ENERGY AND CONVERGENCE TO THE STATIONARY SOLUTION

The main goal of this section is to prove that the minimizer of the free energy  $\mathcal{F}$  is the stationary solution  $(n_\infty, c_\infty)$  considered in the introduction and that it attracts any solution of (2) as  $t \rightarrow +\infty$ .

### 2.1. Minimizers of the free energy and stationary solutions.

**Lemma 2.1.** *Let  $d = 2$  or  $d = 3$  and assume that the potential  $\phi$  satisfies (C). On the set*

$$\mathcal{X} := \left\{ f \in L^1_+(\mathbb{R}^d) : \int_{\mathbb{R}^d} f(x) dx = M, f \log f \in L^1(\mathbb{R}^d), f \phi \in L^1(\mathbb{R}^d) \right\},$$

*the free energy  $\mathcal{F}$  is semi-bounded from below.*

*Proof.* According to Assumption (C1), we know that  $e^{-\phi} \in L^1(\mathbb{R}^d)$ . Set  $\rho(x) := \lambda e^{-\phi}$ , such that  $\int_{\mathbb{R}^d} \rho(x) dx = M$ . Since the function  $x \log x$  is convex, we obtain that  $\int_{\mathbb{R}^d} f \log f dx \geq \int_{\mathbb{R}^d} f \log \rho dx$  by Jensen's inequality. So

$$\begin{aligned} \mathcal{F}[f] &\geq \int_{\mathbb{R}^d} f \log \rho dx + \int_{\mathbb{R}^d} f \phi dx + \frac{1}{2} \int_{\mathbb{R}^d} f (-\Delta)^{-1} f dx \\ &= M \log \lambda + \frac{1}{2} \int_{\mathbb{R}^d} f (-\Delta)^{-1} f dx. \end{aligned}$$

If  $d = 3$ ,  $\int_{\mathbb{R}^d} f (-\Delta)^{-1} f dx \geq 0$  because the Green function  $G_3(x)$  is non-negative. If  $d = 2$ , the result has been established in [?, Corollary 1.2] as a consequence of Assumption (C4).  $\square$

**Lemma 2.2.** *Let  $d = 2$  or  $d = 3$  and assume that the potential  $\phi$  satisfies (C). There exists a unique minimizer  $n_\infty$  of  $\mathcal{F}$  in  $\mathcal{X}$ .*

*Proof.* Standard minimization methods show that a minimizing sequence admits, up to the extraction of a subsequence, a limit which is a minimizer. From the proof above,  $\mathcal{F}$  is lower bounded and satisfies the coercivity inequality. For a fixed minimizer  $n_\infty$ , it should satisfy the Euler-Lagrange equation

$$\log n_\infty + \phi + c_\infty = \lambda, \quad c_\infty = (-\Delta)^{-1} n_\infty,$$

for some Lagrange multiplier  $\lambda$  associated with the mass constraint, which means that  $(n_\infty, c_\infty)$  solve (4). By direct computation, with  $c = (-\Delta)^{-1} n$ , we observe that

$$\mathcal{F}[n] - \mathcal{F}[n_\infty] = \int_{\mathbb{R}^d} n \log\left(\frac{n}{n_\infty}\right) dx + \frac{1}{2} \int_{\mathbb{R}^d} (n - n_\infty)(c - c_\infty) dx.$$

Since  $\int_{\mathbb{R}^d} n dx = \int_{\mathbb{R}^d} n_\infty dx = M$ , we obtain from Jensen's inequality that

$$\int_{\mathbb{R}^d} n \log\left(\frac{n}{n_\infty}\right) dx \geq 0$$

and, according to [9],

$$\int_{\mathbb{R}^d} (n - n_\infty)(c - c_\infty) dx = \int_{\mathbb{R}^d} |\nabla(c - c_\infty)|^2 dx \geq 0.$$

Hence  $\mathcal{F}[n] - \mathcal{F}[n_\infty] \geq 0$  for any  $n \in \mathcal{X}$ , with equality if and only if  $n = n_\infty$ . This means that the minimizer of  $\mathcal{F}$  is unique.  $\square$

We may notice that  $n_\infty$  is radially symmetric if  $\phi$  is radially symmetric, as a consequence of the uniqueness result of Lemma 2.2.

We learn from the proof of [9, Lemma 23] that

$$\max_{|x| \rightarrow \infty} \left| c_\infty + \frac{M}{2\pi} \log|x| \right| < \infty \quad \text{if } d = 2, \quad \max_{|x| \rightarrow \infty} \left| c_\infty - \frac{M}{4\pi|x|} \right| < \infty \quad \text{if } d = 3,$$

and deduce from (4) that, as  $|x| \rightarrow \infty$ ,

$$n_\infty \sim |x|^{\frac{M}{2\pi}} e^{-\phi} \quad \text{if } d = 2, \quad n_\infty \sim e^{-\frac{M}{4\pi|x|} - \phi} \quad \text{if } d = 3. \quad (7)$$

**Proposition 2.1.** *Let  $d = 2$  or  $d = 3$  and assume that the potential  $\phi$  satisfies (C). Then the solutions  $(n_\infty, c_\infty)$  of (4) are such that  $c_\infty$  is bounded if  $d = 3$  and  $\|\nabla c_\infty\|_{L^q(\mathbb{R}^2)}$  is bounded for any  $q \in (2, +\infty]$  if  $d = 2$ .*

*Proof.* From (7) and (C4), we know that  $n_\infty$  is bounded outside of a large centered ball of radius  $R > 0$ . Let us assume that  $|x| \leq R$  and recall that

$$c_\infty(x) = \kappa_3 \int_{\mathbb{R}^3} \frac{e^{-c_\infty(y) - \phi(y)}}{|x - y|} dy \quad \text{if } d = 3, \quad \text{with } \kappa_3 = \frac{M}{4\pi \int_{\mathbb{R}^3} e^{-c_\infty - \phi} dx},$$

$$|\nabla c_\infty(x)| \leq \kappa_2 \int_{\mathbb{R}^2} \frac{e^{-c_\infty(y) - \phi(y)}}{|x - y|} dy \quad \text{if } d = 2, \quad \text{with } \kappa_2 = \frac{M}{2\pi \int_{\mathbb{R}^2} e^{-c_\infty - \phi} dx}.$$

In dimension  $d = 3$ , it is enough to observe that  $c_\infty = (-\Delta)^{-1}n_\infty \leq 0$  and deduce the bound

$$0 \leq c_\infty(x) \leq \kappa_3 \int_{\mathbb{R}^3} \frac{e^{-\phi(y)}}{|x-y|} dy.$$

In dimension  $d = 2$ , we deduce from (7) and (C) that for  $R > 0$  large enough, there exists a constant  $\kappa > 0$  such that

$$n_\infty(x) \leq n_\infty(x) \mathbb{1}_{|x| < R} + \kappa \mathbb{1}_{|x| \geq R} |x|^{-4},$$

which allows us to write

$$c_\infty(x) \geq -\frac{M}{2\pi} \log(2R) - \frac{\kappa}{2\pi} \int_{|y| \geq R} \frac{\log|x-y|}{|y|^4} dy$$

for any  $x \in \mathbb{R}^2$  such that  $|x| \leq R$ . Reinjecting this estimate in the expression of  $|\nabla c_\infty(x)|$  completes the proof. The bound on  $\|\nabla c_\infty\|_{L^q(\mathbb{R}^2)}$  follows by observing that  $|\nabla c_\infty(x)| \sim |x|^{-1}$  as  $|x| \rightarrow +\infty$ .  $\square$

**2.2. Uniform bounds on the solution of (2).** We establish bounds on the solution  $n(t, \cdot)$  of (2) which are independent of  $t$ .

**Lemma 2.3.** *Let  $d = 2$  or  $d = 3$  and assume that the potential  $\phi$  satisfies (C). For any solution  $n$  of (2), there exists a constant  $\mathcal{C} > 0$  and a time  $T > 0$  such that*

$$\|n(t, \cdot)\|_{L^p} \leq \mathcal{C} \quad \forall t \geq T, \quad \forall p \in (1, +\infty].$$

*Proof.* For any integer  $k$ , set  $n_{0,k} = \min(n_0, k)$ , then  $n_{0,k} \in L^p(\mathbb{R}^d)$  for any  $p \geq 1$ . The solution  $n(t, \cdot)$  of the equation (2) with initial data  $n_{0,k}$  is in  $L^p(\mathbb{R}^d)$  for any  $t > 0$  by the Maximum Principle. Since, by assumption,  $|\nabla\phi|$  satisfies a Lipschitz condition, there exists a constant  $C > 0$  such that  $\Delta\phi \leq C$ , and we have the estimate

$$\begin{aligned} \frac{1}{p-1} \frac{d}{dt} \int_{\mathbb{R}^d} n(t, x)^p dx &= -p \int_{\mathbb{R}^d} |\nabla n|^2 n^{p-2} dx - \int_{\mathbb{R}^d} n^{p+1} dx + \int_{\mathbb{R}^d} n^p \Delta\phi dx \\ &\leq - \int_{\mathbb{R}^d} n^{p+1} dx + C \int_{\mathbb{R}^d} n^p dx. \end{aligned}$$

Using Hölder's inequality  $(\int_{\mathbb{R}^d} n dx)^{\frac{1}{p}} (\int_{\mathbb{R}^d} n^{p+1} dx)^{\frac{p-1}{p}} \geq \int_{\mathbb{R}^d} n^p dx$ , we obtain that

$$\int_{\mathbb{R}^d} n^{p+1} dx \geq M^{-\frac{1}{p-1}} \left( \int_{\mathbb{R}^d} n^p dx \right)^{\frac{p}{p-1}}$$

With  $z(t, \cdot) = \int_{\mathbb{R}^d} n(t, \cdot)^p dx$ , the problem reduces to the differential inequality

$$\frac{1}{p-1} z' \leq -M^{-\frac{1}{p-1}} z^{\frac{p}{p-1}} + C z$$



using  $\int_{\mathbb{R}^d} n_{0,k} dx \leq M$ . It is elementary to prove that

$$z(t) \leq (2C)^{p-1} M \quad \forall t \geq 4C$$

and conclude that the bound

$$\|n(t, \cdot)\|_{L^p(\mathbb{R}^d)} \leq (2C)^{\frac{p-1}{p}} M^{\frac{1}{p}}$$

has a uniform upper bound in the limit as  $p \rightarrow +\infty$ . See [10] for further details on a similar estimate.  $\square$

**Corollary 2.1.** *Let  $d = 2$  or  $d = 3$  and assume that the potential  $\phi$  satisfies (C). For any solution  $n$  of (2) with initial datum  $n_0 \in L^1_+(\mathbb{R}^d)$  such that  $\mathcal{F}[n_0] < +\infty$ , there exists a constant  $\mathcal{C} > 0$  and a time  $T > 0$  such that*

$$\|\nabla c(t, \cdot)\|_{L^q(\mathbb{R}^d)} \leq \mathcal{C} \quad \forall t \geq T, \quad \forall q \in (2, +\infty].$$

*Proof.* The method is inspired from [11, Section 3]. If  $h = (-\Delta)^{-1}\rho$ , then

$$|\nabla h(x)| \leq \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{R}^d} \frac{\rho(y)}{|x-y|^{d-1}} dx$$

can be estimated by splitting the integral into two parts corresponding to  $|x-y| \leq 1$  and  $|x-y| > 1$ . By applying twice Hölder's inequality, we deduce from

$$\begin{aligned} \frac{1}{|\mathbb{S}^{d-1}|} \int_{\|x-y\| < 1} \frac{\rho(y)}{|x-y|^{d-1}} dy &\leq d^{\frac{d}{d+1}} |\mathbb{S}^{d-1}|^{-\frac{1}{d+1}} \|\rho\|_{L^{d+1}(\mathbb{R}^d)} \\ \frac{1}{|\mathbb{S}^{d-1}|} \int_{\|x-y\| \geq 1} \frac{\rho(y)}{|x-y|^{d-1}} dy &\leq \frac{1}{|\mathbb{S}^{d-1}|} \|\rho\|_{L^1(\mathbb{R}^d)} \end{aligned}$$

that

$$\|\nabla((-\Delta)^{-1}\rho)\|_{L^\infty(\mathbb{R}^d)} \leq \|\rho\|_{L^1(\mathbb{R}^d)} + d^{\frac{d}{d+1}} |\mathbb{S}^{d-1}|^{-\frac{1}{d+1}} \|\rho\|_{L^{d+1}(\mathbb{R}^d)} \quad (8)$$

for any  $\rho \in L^1 \cap L^{d+1}(\mathbb{R}^d)$ . Applying it with  $\rho = n(t, \cdot)$  and  $c = (-\Delta)^{-1}n$  and using Minkowski's inequality  $\|\nabla c(t, \cdot)\|_{L^q(\mathbb{R}^d)} \leq \|\nabla c(t, \cdot) - \nabla c_\infty\|_{L^q(\mathbb{R}^d)} + \|\nabla c_\infty\|_{L^q(\mathbb{R}^d)}$ , the result follows from the estimate  $\|\nabla c(t, \cdot) - \nabla c_\infty\|_{L^2(\mathbb{R}^d)}^2 \leq 2\mathcal{F}[n_0]$  together with Proposition 2.1 and Lemma 2.3.  $\square$

**2.3. Convergence to stationary solutions.** The next step is to establish the convergence *without rate* of the solution of (2) to the stationary solution. For later purpose, let us recall the Aubin-Lions compactness lemma. A simple statement goes as follows (see [16] for more details).

**Lemma 2.4. (Aubin-Lions Lemma)** *Take  $T > 0$ ,  $p \in (1, \infty)$ , and let  $(f_k)_{k \in \mathbb{N}}$  be a bounded sequence of functions in  $L^p(0, T; H)$ , where  $H$  is a Banach space. If  $(f_k)_{k \in \mathbb{N}}$  is bounded in  $L^p(0, T; V)$ , where  $V$  is compactly imbedded in  $H$  and if  $(\partial f_k / \partial t)_{k \in \mathbb{N}}$  is bounded in  $L^p(0, T; V')$  uniformly with respect to  $k \in \mathbb{N}$ , where  $V'$  is the dual space of  $V$ , then  $(f_k)_{k \in \mathbb{N}}$  is relatively compact in  $L^p(0, T; H)$ .*

With this result in hand, we are in a position to prove the following result.

**Proposition 2.2.** *Suppose that  $d = 2$  or  $3$ . Let  $n$  be the solution of (2) and assume that the potential  $\phi$  satisfies (C). Then for any  $p \in [1, \infty)$  and any  $q \in [2, \infty)$ , we have*

$$\lim_{t \rightarrow \infty} \|n(t, \cdot) - n_\infty\|_{L^p(\mathbb{R}^d)} = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \|\nabla c(t, \cdot) - \nabla c_\infty\|_{L^q(\mathbb{R}^d)} = 0.$$

*Proof.* Since  $\mathcal{F}[n(t, \cdot)]$  is nonnegative and decreasing, by (6) we know that

$$\lim_{t \rightarrow \infty} \int_t^\infty \mathcal{F}[n(s, \cdot)] ds = 0. \quad (9)$$

This means that the sequence  $(n_k, c_k)_{k \in \mathbb{N}}$ , defined by  $n_k(t, \cdot) = n(t + k, \cdot)$ ,  $c_k = (-\Delta)^{-1} n_k$ , is such that  $\nabla n_k + n_k \nabla c_k + n_k \nabla \phi$  strongly converges to 0 in  $L^2(\mathbb{R}^+ \times \mathbb{R}^d)$ . By lemma 2.4, this shows that  $(n_k)_{k \in \mathbb{N}}$  is relatively compact and converges, up to the extraction of a subsequence, to a limit  $\bar{n}$ . Up to the extraction of an additional subsequence,  $(c_k)_{k \in \mathbb{N}}$  converges to  $\bar{c} = (-\Delta)^{-1} \bar{n}$  so that we may pass to the limit in the quadratic term and know that

$$\nabla \bar{n} + \bar{n} \nabla \bar{c} + \bar{n} \nabla \phi = 0, \quad -\Delta \bar{c} = \bar{n}.$$

Since mass is conserved by passing to the limit, we conclude that  $\bar{n} = n_\infty$  and  $\bar{c} = c_\infty$ . The limit is uniquely defined, so it is actually the whole family  $(n(t, \cdot))_{t > 0}$  which converges as  $t \rightarrow +\infty$  to  $n_\infty$  and  $\lim_{t \rightarrow +\infty} \mathcal{F}[n(t, \cdot)] = \mathcal{F}[n_\infty]$ , then proving by the Csiszár-Kullback inequality that  $\lim_{t \rightarrow \infty} \|n(t, \cdot) - n_\infty\|_{L^1(\mathbb{R}^d)} = 0$  (see [? ]) and  $\lim_{t \rightarrow \infty} \|\nabla c(t, \cdot) - \nabla c_\infty\|_{L^2(\mathbb{R}^d)} = 0$ . The result for any  $p \in [1, \infty)$  and any  $q \in [2, \infty)$  follows by Hölder interpolation.  $\square$

#### 2.4. Uniform convergence in $L^\infty$ norm in the harmonic potential case.

The issue of the convergence of  $n(t, \cdot)$  to  $n_\infty$  and of  $\nabla c(t, \cdot)$  to  $\nabla c_\infty$  in  $L^\infty(\mathbb{R}^d)$  was left open in Section 2.3. As in the case of the Keller-Segel model, see [10], better results can be achieved in the case of the harmonic potential.

**Proposition 2.3.** *Set  $d = 2$ ,  $\phi = \frac{\mu}{2} |x|^2$ , for some  $\mu > 0$ . Then for any solution  $n$  of (2) is such that*

$$\lim_{t \rightarrow +\infty} \|n(t, \cdot) - n_\infty\|_{L^\infty(\mathbb{R}^d)} = 0.$$

*Proof.* The main tool is the Duhamel formula: see [11] for more details. We have

$$n(t, x) = \int_{\mathbb{R}^2} K(t, x, y) n_0(y) dy - \int_0^t \int_{\mathbb{R}^2} \nabla K(t-s, x, y) \cdot n(s, y) \nabla c(s, y) dy ds$$

where  $K(t, x, y)$  is the Green function of the Fokker-Planck equation

$$\frac{\partial n}{\partial t} = \Delta n + \mu \nabla(nx)$$

which is

$$K(t, x, y) := \frac{\mu}{2\pi(1-e^{-2t})} e^{-\frac{\mu|x-e^{-t}y|^2}{2(1-e^{-2t})}}$$

and from the semi-group property we get that

$$\begin{aligned} n(t+1, x) &= \int_{\mathbb{R}^2} K(t, x, y) n(t, y) dy \\ &\quad - \int_t^{t+1} \int_{\mathbb{R}^2} \nabla K(t+1-s, x, y) \cdot n(s, y) \nabla c(s, y) dy ds. \end{aligned} \quad (10)$$

Notice that the stationary solution  $n_\infty$  is a fixed-point of the evolution map, that is,

$$\begin{aligned} n_\infty(x) &= \int_{\mathbb{R}^2} K(t, x, y) n_\infty(y) dy \\ &\quad - \int_t^{t+1} \int_{\mathbb{R}^2} \nabla K(t+1-s, x, y) \cdot n_\infty(y) \nabla c_\infty(y) dy ds. \end{aligned} \quad (11)$$

Buy doing the difference between (10) and (11), we have

$$\begin{aligned} &n(t+1, x) - n_\infty(x) \\ &= \int_{\mathbb{R}^2} K(t, x, y) (n(t, y) - n_\infty(y)) dy \\ &\quad - \int_t^{t+1} \int_{\mathbb{R}^2} \nabla K(t+1-s, x, y) (n(s, y) \nabla c(s, y) dy - n_\infty(y) \nabla c_\infty(y)) ds. \end{aligned}$$

Hence

$$\begin{aligned} \|n(t+1, x) - n_\infty(x)\|_{L^\infty(\mathbb{R}^2)} &\leq \|K(t, x, y)\|_{L^\infty(\mathbb{R}_x^2; L^r(\mathbb{R}_y^2))} \|n(t, x) - n_\infty\|_{L^1(\mathbb{R}^2)} \\ &\quad + \int_0^1 \|\nabla K(s, x, y)\|_{L^\infty(\mathbb{R}_x^2; L^r(\mathbb{R}_y^2))} ds \mathcal{R}(t) \end{aligned}$$

where  $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$  with  $p \in (2, \infty)$ ,  $q \in [2, \infty)$ ,  $r \in (1, 2)$ , and

$$\begin{aligned} \mathcal{R}(t) &:= \sup_{s \in (t, t+1)} \left( \|n(s, \cdot)\|_{L^p(\mathbb{R}^2)} \|\nabla c(s, \cdot) - \nabla c_\infty\|_{L^q(\mathbb{R}^2)} \right. \\ &\quad \left. + \|\nabla c_\infty\|_{L^q(\mathbb{R}^2)} \|n(s, \cdot) - n_\infty\|_{L^p(\mathbb{R}^2)} \right). \end{aligned} \quad (12)$$

Notice that

$$\nabla K = \frac{\mu^2(e^{-t}y - x)}{2\pi(1-e^{-2t})} e^{-\frac{\mu|x-e^{-t}y|^2}{2(1-e^{-2t})}}$$

allows us to compute

$$\|\nabla K\|_{L^r(\mathbb{R}_y^2)} = \frac{\mu^2}{2\pi(1-e^{-2t})} \left( \int_{\mathbb{R}^2} |x|^r e^{-\frac{\mu r |x|^2}{2(e^{2t}-1)}} dx \right)^{\frac{1}{r}} = \kappa(r) e^{3t} \left( \frac{e^{2t}-1}{\mu} \right)^{-\frac{3}{2} + \frac{1}{r}}$$

where  $\kappa(r) = \left( \int_0^\infty x^r e^{-\frac{1}{2}x^2} dx \right)^{\frac{1}{r}}$ . So  $\|\nabla K\|_{L^r(\mathbb{R}_y^2)}$  is integrable in  $t \in (0, 1)$  if and only if  $1 \leq r < 2$ . From Proposition 2.2,  $\mathcal{R}(t)$  converges to 0, which completes the proof.  $\square$

### 3. COERCIVITY RESULT OF QUADRATIC FORMS

In this section, we study the quadratic forms associated with the free energy  $\mathcal{F}$  and the Fisher information  $\mathcal{I}$  when we Taylor expand these functionals around the stationary solution  $(n_\infty, c_\infty)$  defined by (4). Let us consider a smooth perturbation  $n = f n_\infty$  of  $n_\infty$  such that  $\int_{\mathbb{R}^d} f n_\infty dx = 0$  and suppose that  $g c_\infty := (-\Delta)^{-1}(f n_\infty)$ . We define

$$Q_1[f] := \lim_{\varepsilon \rightarrow 0} \frac{2}{\varepsilon^2} \mathcal{F}[n_\infty(1 + \varepsilon f)] = \int_{\mathbb{R}^d} f^2 n_\infty dx + \int_{\mathbb{R}^d} |\nabla(g c_\infty)|^2 dx,$$

$$Q_2[f] := \lim_{\varepsilon \rightarrow 0} \frac{2}{\varepsilon^2} \mathcal{I}[n_\infty(1 + \varepsilon f)] = \int_{\mathbb{R}^d} |\nabla(f + g c_\infty)|^2 n_\infty dx.$$

**3.1. A spectral gap inequality.** According to [1, Section 3.2], if the potential  $\phi$  satisfies (C1), (C2) and (C3), then there exists a positive constant  $\mathcal{C}_\star$ , such that

$$\int_{\mathbb{R}^d} |\nabla h|^2 n_\infty dx \geq \mathcal{C}_\star \int_{\mathbb{R}^d} h^2 n_\infty dx$$

$$\forall f \in H^1(\mathbb{R}^d, n_\infty dx) \quad \text{such that} \quad \int_{\mathbb{R}^d} h n_\infty dx = 0. \quad (13)$$

Here  $n_\infty$  is the stationary solution given by (4).

**Proposition 3.1.** *Let  $d = 2$  or  $d = 3$  and assume that the potential  $\phi$  satisfies (C). Then for any  $f \in H^1(\mathbb{R}^d, n_\infty dx)$  such that  $\int_{\mathbb{R}^d} f n_\infty dx = 0$ , we have*

$$Q_2[f] \geq \mathcal{C}_\star Q_1[f].$$

*Proof.* We apply (13) to  $h(x) = f(x) + g c_\infty(x) - \frac{1}{M} \int_{\mathbb{R}^d} g c_\infty n_\infty dx$ . Notice that  $\int_{\mathbb{R}^d} h(x) n_\infty dx = 0$  from  $\int_{\mathbb{R}^d} f n_\infty dx = 0$  and  $\int_{\mathbb{R}^d} n_\infty(x) dx = M$ . So

we obtain that

$$\begin{aligned}
Q_2[f] &= \int_{\mathbb{R}^d} |\nabla(f + g c_\infty)|^2 n_\infty dx \\
&\geq \mathcal{C}_\star \int_{\mathbb{R}^d} (f + g c_\infty)^2 n_\infty dx - \frac{\mathcal{C}_\star}{M} \left( \int_{\mathbb{R}^d} g c_\infty n_\infty dx \right)^2 \\
&= \mathcal{C}_\star \int_{\mathbb{R}^d} f(f + g c_\infty) n_\infty dx + \mathcal{C}_\star \int_{\mathbb{R}^d} g c_\infty (f + g c_\infty) n_\infty dx \\
&\quad - \frac{\mathcal{C}_\star}{M} \left( \int_{\mathbb{R}^d} g c_\infty n_\infty dx \right)^2 \\
&= \mathcal{C}_\star Q_1[f] + \mathcal{C}_\star \int_{\mathbb{R}^d} f n_\infty g c_\infty dx + \mathcal{C}_\star \int_{\mathbb{R}^d} (g c_\infty)^2 n_\infty dx \\
&\quad - \frac{\mathcal{C}_\star}{M} \left( \int_{\mathbb{R}^d} g c_\infty n_\infty dx \right)^2.
\end{aligned}$$

Let us study the term  $\int_{\mathbb{R}^d} f n_\infty g c_\infty dx$ . Obviously  $f n_\infty$  is in  $L^2(\mathbb{R}^d)$  because  $n_\infty$  is bounded. Moreover, for any  $p \in (1, 2)$ , from Hölder's inequality, we infer that

$$\int_{\mathbb{R}^d} |f|^p n_\infty^p dx \leq \left( \int_{\mathbb{R}^d} f^2 dx \right)^{\frac{p}{2}} \left( \int_{\mathbb{R}^d} n_\infty^{\frac{2p}{2-p}} dx \right)^{\frac{2-p}{2}} < \infty$$

because  $n_\infty \in L^1 \cap L^\infty(\mathbb{R}^d)$ . When  $d = 3$ , we directly obtain from the Hardy-Littlewood-Sobolev inequality that  $\int_{\mathbb{R}^d} f n_\infty g c_\infty dx$  is well defined and equal to  $\int_{\mathbb{R}^d} |\nabla g c_\infty|^2 dx$ . When  $d = 2$ , by log-Hölder interpolation,  $|f n_\infty| \log|f n_\infty|$  is integrable. From the logarithmic Hardy-Littlewood-Sobolev inequality (see [? ]), we also know that  $\int_{\mathbb{R}^d} f n_\infty g c_\infty dx$  is well defined and learn from [9] that the function  $\nabla(g c_\infty)$  is bounded in  $L^2(\mathbb{R}^2)$  using the fact that  $\int_{\mathbb{R}^d} f n_\infty dx = 0$ . In a word, this means that

$$\int_{\mathbb{R}^d} f n_\infty g c_\infty dx = \int_{\mathbb{R}^d} |\nabla g c_\infty|^2 dx$$

for  $d = 2$  or  $3$ . Next, let us notice that

$$\begin{aligned}
&\mathcal{C}_\star \int_{\mathbb{R}^d} (g c_\infty)^2 n_\infty dx - \frac{\mathcal{C}_\star}{M} \left( \int_{\mathbb{R}^d} g c_\infty n_\infty dx \right)^2 \\
&= \frac{\mathcal{C}_\star}{M} \int_{\mathbb{R}^d} (g c_\infty)^2 n_\infty dx \int_{\mathbb{R}^d} n_\infty dx - \frac{\mathcal{C}_\star}{M} \left( \int_{\mathbb{R}^d} g c_\infty n_\infty dx \right)^2
\end{aligned}$$

is nonnegative by Hölder's inequality. Altogether, we conclude that

$$Q_2[f] \geq \mathcal{C}_\star Q_1[f] + \mathcal{C}_\star \int_{\mathbb{R}^d} |\nabla(g c_\infty)|^2 dx \geq \mathcal{C}_\star Q_1[f].$$

□

**3.2. Optimal spectral gap in a special case.** As a conclusion, let us give the optimal coercivity constant in the special case that the dimension  $d = 2$  and the harmonic function  $\phi = \frac{\mu}{2}|x|^2, \mu > 0$ ,

**Lemma 3.1.** *Suppose that  $d = 2$ ,  $\phi = \frac{\mu}{2}|x|^2$ , where  $\mu > 0$ . Then for any  $f \in H^1(\mathbb{R}^2, n_\infty dx)$  such that  $\int_{\mathbb{R}^2} f n_\infty dx = 0$ , we have*

$$Q_2[f] \geq \mu Q_1[f].$$

*Proof.* We establish the proof into three steps.

**Step 1. Radially symmetric functions and cumulated densities.** We first consider the case of a spherically symmetric function  $f$ . The problem is reduced to solving an ordinary differential equation, for which we use a reformulation in terms of *cumulated densities*. Let

$$\Phi(s) := \frac{1}{2\pi} \int_{B(0, \sqrt{s})} n_\infty(x) dx, \quad \phi(s) := \frac{1}{2\pi} \int_{B(0, \sqrt{s})} (f n_\infty)(x) dx$$

and

$$\Psi(s) := \frac{1}{2\pi} \int_{B(0, \sqrt{s})} c_\infty(x) dx, \quad \psi(s) := \frac{1}{2\pi} \int_{B(0, \sqrt{s})} (g c_\infty)(x) dx.$$

Notice that  $n_\infty$  and  $c_\infty$  are both radial, so they can be regarded as functions of  $r = |x|$ . We can easily infer that

$$n_\infty(\sqrt{s}) = 2\Phi'(s), \quad n'_\infty(\sqrt{s}) = 4\sqrt{s}\Phi''(s)$$

and

$$c_\infty(\sqrt{s}) = 2\Psi'(s), \quad c'_\infty(\sqrt{s}) = 4\sqrt{s}\Psi''(s).$$

The Poisson equation  $-\sqrt{s}c'_\infty(\sqrt{s}) = \Phi(s)$  can henceforth be rephrased as

$$-4s\Psi'' = \Phi \tag{14}$$

while the equation for the density,

$$n'_\infty(\sqrt{s}) + \mu\sqrt{s}n_\infty(\sqrt{s}) + n_\infty(\sqrt{s})c'_\infty(\sqrt{s}) = 0,$$

is now equivalent to

$$\Phi'' + \frac{\mu}{2}\Phi' + 2\Phi'\Psi'' = 0. \tag{15}$$

After eliminating  $\Psi''$  from (14) and (15), we can get that  $\Phi$  satisfies the ordinary differential equation

$$\Phi'' + \frac{\mu}{2}\Phi' - \frac{1}{2s}\Phi\Phi' = 0 \tag{16}$$

with initial data  $\Phi(0) = 0$  and  $\Phi'(0) = a$ . The solutions of the ODE are parameterized in terms of  $a > 0$ .

Let us consider the linearized operator

$$\mathcal{L}f := \frac{1}{n_\infty} \nabla \cdot [f n_\infty \nabla (g c_\infty)].$$

If  $f$  solves  $-\mathcal{L}f = \lambda f$ , computations similar to the above ones show that

$$(n_\infty f)(\sqrt{s}) = 2\phi'(s), \quad (n_\infty f')(\sqrt{s}) = 4\sqrt{s}\phi''(s) - 2\frac{n'_\infty}{n_\infty}\phi'(s)$$

which is equivalent to

$$(g c_\infty)(\sqrt{s}) = 2\psi'(s), \quad (g c_\infty)'(\sqrt{s}) = 4\sqrt{s}\psi''(s). \quad (17)$$

Using (17), we find that

$$-\sqrt{s}(g c_\infty)'(\sqrt{s}) = \phi(s), \quad \sqrt{s}((n_\infty f')(\sqrt{s}) + n_\infty(g c_\infty)'(\sqrt{s})) + \lambda\phi(s) = 0.$$

After eliminating  $\Psi$  and  $\psi$ , we get that  $\Phi$  and  $\phi$  satisfy the equation

$$\phi'' + \frac{\mu s - \Phi}{2s}\phi' + \frac{\lambda - 2\Phi'}{4s}\phi = 0. \quad (18)$$

Next we check that  $\phi = s\Phi'(s)$  is a nonnegative solution of (18) with  $\lambda = 2\mu$ . In fact, (18) is equivalent to

$$2s\phi'' + (\mu s - \Phi)\phi' + (\mu - \Phi')\phi = 0$$

which is

$$(2(s\phi' - \phi) + (\mu s - \Phi)\phi)' = 0.$$

notice that when  $\phi = s\phi'$ ,

$$2(s\phi' - \phi) + (\mu s - \Phi)\phi = s(2s\Phi'' + (\mu s - \Phi)\phi') = 0.$$

Hence  $\lambda = 2\mu$  is an eigenvalue of the linearized operator  $\mathcal{L}f$ .

**Step 2.** *Characterization of the radial ground state.* Let us prove that  $2\mu$  is the lowest positive eigenvalue corresponding to a radial eigenfunction. Assume by contradiction that  $\mathcal{L}$  admits an eigenvalue  $\lambda \in (0, 2\mu)$  with eigenfunction  $f_1$  and define the corresponding function  $\phi_1$  that satisfy (18). Let us consider various cases depending on the zeros of  $\phi$ .

- Assume that  $\phi_1$  is always strictly positive or strictly negative in  $(0, \infty)$ . Suppose without losing generality that  $\phi_1(s) > 0$  in  $(0, \infty)$ . On the one hand, if we multiply (18) written for the eigenvalue  $2\mu$  and for the eigenvalue  $\lambda$  respectively by  $\phi_1$  and  $\phi$ , we obtain that

$$\phi_1\phi'' - \frac{\Phi''}{\Phi'}\phi_1\phi' + \frac{2\mu - 2\Phi'}{4s}\phi_1\phi = 0,$$

$$\phi\phi_1'' - \frac{\Phi''}{\Phi'}\phi\phi_1' + \frac{\lambda - 2\Phi'}{4s}\phi\phi_1 = 0.$$

By subtracting the second identity from the first one, we have

$$\frac{\phi_1'\phi(s) - \phi_1\phi'(s)}{\Phi'(s)} \Big|_0^\infty = \int_{(0, \infty)} \frac{2\mu - \lambda}{4s}\phi\phi_1 ds > 0. \quad (19)$$

On the other hand, define

$$h(s) := \frac{1}{2\pi} \int_{B(0, \sqrt{s})} f_1^2 n_\infty(r) dr.$$

From the cumulated mass formulation of Step 1, we find that

$$h'(s) = \frac{1}{2} f_1^2 n_\infty(\sqrt{s}) = \frac{2\Phi'(s)^2}{n_\infty(\sqrt{s})}$$

is in  $L^1(0, \infty)$ . So, for some constant  $\kappa > 0$ , we have

$$\begin{aligned} \phi_1(s)^2 &= \left( \int_{(s, \infty)} \phi_1'(s) ds \right)^2 \leq \left( \int_{(s, \infty)} \frac{\phi_1'(s)^2}{n_\infty(\sqrt{s})} ds \right) \left( \int_{(s, \infty)} n_\infty(\sqrt{s}) ds \right) \\ &\leq \kappa \int_{(s, \infty)} s^{-\frac{\alpha}{2}} e^{-\frac{s}{2}} ds \leq \kappa e^{-\frac{\mu s}{4}} \end{aligned}$$

when  $s$  is large enough. As a consequence, we know that

$$\lim_{s \rightarrow \infty} \phi_1(s) = \lim_{s \rightarrow \infty} \phi(s) = \lim_{s \rightarrow \infty} s\phi(s) = 0.$$

We also claim that

$$\lim_{s \rightarrow \infty} s\phi_1'(s) = 0. \quad (20)$$

In fact, for any large enough  $x_1, x_2$ , by integrating on  $(x_1, x_2)$ , we have

$$\begin{aligned} \phi_1'(x_2) - \phi_1'(x_1) + \frac{\mu}{2} (\phi_1(x_2) - \phi_1(x_1)) - \frac{\Phi}{2s} (\phi_1(x_2) - \phi_1(x_1)) \\ - \int_{x_1}^{x_2} \phi_1 \frac{s\phi' - \Phi}{2s^2} ds + \int_{(x_1, x_2)} \frac{\lambda - \Phi'}{4s} \phi_1 ds = 0. \end{aligned}$$

Using again that  $\phi_1(s) \leq \kappa e^{-\frac{\mu s}{4}}$ , we get that there exists a constant  $c_2$  which is independent of  $x_1$  and  $x_2$ , such that  $|\phi_1'(x_2) - \phi_1'(x_1)| \leq c_2$ . So  $\phi_1'(s)$  is bounded. As a result,  $\phi_1''(s)$  is also bounded, with a bound  $c_3$ . If (20) is not true, then there exists a constant  $c_1$  and a strictly increasing, diverging sequence  $(s_k)_{k \in \mathbb{N}}$  such that  $s_k \phi_1'(s_k) \geq c_1$ . For any interval  $(s_k, \infty)$ , we have that

$$\frac{c_1}{s_k} \leq C \sqrt{e^{-\frac{\mu s_k}{4}}}$$

which is impossible as  $k \rightarrow \infty$ . So from (20), we obtain that

$$\lim_{s \rightarrow \infty} \frac{\phi_1' \phi(s) - \phi_1 \phi'(s)}{\Phi'(s)} = \lim_{s \rightarrow \infty} s\phi_1' - \phi_1 \left( 1 + \frac{s\phi''}{\Phi'} \right) = \lim_{s \rightarrow \infty} s\phi_1' - \phi_1 \left( 1 - \frac{\mu s - \Phi}{2} \right) = 0. \quad (21)$$

From (19), (21), we have

$$0 = \frac{\phi_1' \phi(s) - \phi_1 \phi'(s)}{\Phi'(s)} \Big|_0^\infty = \int_{(0, \infty)} \frac{2\mu - \lambda}{4s} \phi \phi_1 ds > 0$$

a contradiction.



• Assume that  $\phi_1$  has a zero in  $(0, \infty)$ . By Sturm comparison theorem (see [13]), we get that

$$\phi(s) = s\Phi'(s)$$

has a zero in  $(0, \infty)$ . It means that

$$n_\infty(\sqrt{s}) = 2\Phi'(s)$$

has a zero between  $(0, \infty)$ . But according to the definition of  $n_\infty$ , it is impossible. Hence we have shown that  $2\mu$  is the best constant.

**Step 3.** *Spherical harmonics decomposition.*

We now deal with the non-radial modes of  $\mathcal{L}$ . Notice that  $n_\infty$  and  $c_\infty$  are radial functions: we can use a spherical harmonics decomposition as in [11]. In dimension  $d = 2$ , we use radial coordinates and a Fourier decomposition for the angular variables. On the  $k^{\text{th}}$  mode we can write the operator  $\mathcal{L}$  corresponding to the radial functions  $f$  and  $g$  as

$$\begin{aligned} -f'' - \frac{f'}{r} + \frac{k^2 f}{r^2} + (\mu r + c'_\infty)(f' + (g c_\infty)') - n_\infty f &= \lambda f, \\ -(g c_\infty)'' - \frac{(g c_\infty)'}{r} + \frac{k^2 g c_\infty}{r^2} &= n_\infty f, \end{aligned}$$

for any integer  $k \geq 1$ . It is obvious that in non-radial functions,  $k = 1$  realizes the infimum of the spectrum of  $\mathcal{L}$ . We now check that when  $k = 1$ ,  $\lambda = \mu$  and  $f = -n'_\infty/n_\infty$  is an eigenstate. In fact, we can choose  $g c_\infty = -c'_\infty$ , so that  $f = \mu r + c'_\infty$ , and notice that

$$-c''_\infty - \frac{c'_\infty}{r} = n_\infty, \quad f' + \frac{f}{r} = 2\mu + c''_\infty + \frac{c'_\infty}{r} = 2\mu - n_\infty,$$

for the first equation, and

$$\begin{aligned} -f'' - \frac{f'}{r} + \frac{k^2 f}{r^2} + (\mu r + c'_\infty)(f' + (g c_\infty)') + n_\infty f \\ = -\left(f' + \frac{f}{r}\right)' - n'_\infty + \mu(\mu r + c'_\infty) = \mu f \end{aligned}$$

for the second equation, while

$$-(g c_\infty)'' - \frac{(g c_\infty)'}{r} + \frac{k^2 g c_\infty}{r^2} = -\left(c''_\infty + \frac{c'_\infty}{r}\right) = -n'_\infty = n_\infty f.$$

It is easy to prove that  $f$  is nonnegative and that  $f_1(r) := r f(r)$  solves  $-\mathcal{L}f_1 = (\lambda + \mu)f_1$  among the radial functions: we are back to the Step 2 and find that  $\lambda = \mu$ .

Let us summarize: the spectral gap  $\lambda$  associated with the operator  $\mathcal{L}$  is achieved either among radial functions and  $\lambda = 2\mu$  in this sense, or it is achieved among the functions in one of the non-radial components (in the sense of harmonics decomposition), which has to be the  $k = 1$

component, and in that case we have found that  $\lambda + \mu = 2\mu$ , that is  $\lambda = \mu$ . Obviously  $\lambda = \mu$  is optimal, which completes the proof of Lemma 3.1.  $\square$

#### 4. LINEARIZED EQUATION AND THE LARGE TIME BEHAVIOUR

This section is primarily devoted to the proof Theorem 1.1 but also collects some additional results.

**4.1. The scalar product and the linearized operator.** We adapt the strategy of [11]. Notice that

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^d} f_1 f_2 n_\infty dx + \int_{\mathbb{R}^d} n_\infty f_1 (G_d * (f_2 n_\infty)) dx \quad (22)$$

is a scalar product on the admissible set

$$\mathcal{A} := \left\{ f \in L^2(\mathbb{R}^d, n_\infty dx) : \int_{\mathbb{R}^d} f n_\infty dx = 0 \right\}$$

because  $Q_1[f] = \langle f, f \rangle$ . Now come back to the Poisson-Nernst-Planck system with confinement (2). For any  $x \in \mathbb{R}^d$  and  $t \geq 0$ , let us set

$$n(t, x) = n_\infty(x) (1 + f(t, x)), \quad c(t, x) = c_\infty (1 + g(t, x))$$

and rewrite the evolution problem in terms of  $f$  and  $g$  as

$$n_\infty \frac{\partial f}{\partial t} = \Delta(n_\infty f) + \nabla \cdot (n_\infty f \nabla \phi) + \nabla \cdot (n_\infty \nabla(c_\infty g) + n_\infty f \nabla c_\infty + n_\infty f \nabla(c_\infty g)).$$

After observing that

$$\Delta(n_\infty f) + \nabla \cdot (n_\infty f \nabla \phi) + \nabla \cdot (n_\infty f \nabla c_\infty) = \nabla \cdot (n_\infty \nabla f),$$

it turns out that

$$n_\infty \frac{\partial f}{\partial t} = \nabla \cdot (n_\infty \nabla f) + \nabla \cdot (n_\infty \nabla(c_\infty g)) + \nabla \cdot (n_\infty f \nabla(c_\infty g)).$$

Hence  $(f, g)$  solves

$$\begin{cases} \frac{\partial f}{\partial t} - \mathcal{L}f = \frac{1}{n_\infty} \nabla \cdot [f n_\infty \nabla(g c_\infty)] & x \in \mathbb{R}^d, t > 0 \\ -\Delta(g c_\infty) = f n_\infty \end{cases} \quad (23)$$

for any  $x \in \mathbb{R}^d$ ,  $t \geq 0$ , where the linear operator  $\mathcal{L}$  is defined by

$$\mathcal{L}f := \frac{1}{n_\infty} \nabla [n_\infty \nabla(f + g c_\infty)].$$

**Lemma 4.1.** *The linearized operator  $\mathcal{L}$  is self-adjoint on  $\mathcal{A}$  with the scalar product defined in (22), which means that  $\langle f_1, \mathcal{L}f_2 \rangle = \langle \mathcal{L}f_1, f_2 \rangle$  for any  $f_1, f_2 \in \mathcal{A}$ , and moreover,*

$$-\langle f, \mathcal{L}f \rangle = Q_2[f]$$

for any  $f \in \mathcal{A}$ .

*Proof.* Set  $g_1 c_\infty = (-\Delta)^{-1}(f_1 n_\infty)$ ,  $g_2 c_\infty = (-\Delta)^{-1}(f_2 n_\infty)$ . By direct computation, we obtain that

$$\begin{aligned} \langle \mathcal{L} f_1, f_2 \rangle &= \int_{\mathbb{R}^d} f_2 \nabla \cdot (n_\infty \nabla (f_1 + g_1 c_\infty)) dx + \int_{\mathbb{R}^d} g_2 c_\infty \nabla \cdot (n_\infty \nabla (f_1 + g_1 c_\infty)) dx \\ &= - \int_{\mathbb{R}^d} n_\infty \nabla (f_1 + g_1 c_\infty) \cdot \nabla (f_2 + g_2 c_\infty) dx, \end{aligned}$$

which proves the lemma.  $\square$

#### 4.2. Proof of Theorem 1.1.

*Proof.* For the equations (23), we find that

$$\frac{d}{dt} Q_1[f] = -2 Q_2[f] - 2 \lambda(t) \quad \text{with} \quad \lambda(t) := \int_{\mathbb{R}^d} \nabla (f + g c_\infty) \cdot f n_\infty \nabla (g c_\infty) dx.$$

According to the Cauchy-Schwarz inequality, we have that

$$(\lambda(t))^2 \leq Q_2[f] \int_{\mathbb{R}^d} f^2 n_\infty dx \|\nabla (g c_\infty)\|_{L^\infty(\mathbb{R}^d)}^2 \leq Q_2[f] Q_1[f] \|\nabla (g c_\infty)\|_{L^\infty(\mathbb{R}^d)}^2.$$

So we obtain

$$\frac{d}{dt} Q_1[f] \leq -2 \left( 1 - \frac{\|\nabla (g c_\infty)\|_{L^\infty(\mathbb{R}^d)}}{\sqrt{\mathcal{C}_*}} \right) Q_2[f] \leq -2 \mathcal{C}_* \left( 1 - \frac{\|\nabla (g c_\infty)\|_{L^\infty(\mathbb{R}^d)}}{\sqrt{\mathcal{C}_*}} \right) Q_1[f].$$

We know from Proposition 2.2 that  $\lim_{t \rightarrow +\infty} \|\nabla (g c_\infty)\|_{L^\infty(\mathbb{R}^d)} = 0$ , which proves that

$$\limsup_{t \rightarrow \infty} e^{2(\mathcal{C}_* - \varepsilon)t} Q_1[f(t, \cdot)] < \infty$$

for any  $\varepsilon \in (0, \mathcal{C}_*)$ . It remains to prove that we can also obtain this estimate with  $\varepsilon = 0$ .

Suppose that  $\mathcal{C}_*$  is the optimal constant without losing generality. Let us give a more accurate estimate of  $\lambda(t)$ . If  $d = 2$ , according to (8) applied to  $\rho = f n_\infty$ , we have

$$\|\nabla (g c_\infty)\|_{L^\infty} \leq C(\|f n_\infty\|_{L^1} + \|f n_\infty\|_{L^3})$$

where

$$\|f n_\infty\|_{L^1} \leq \sqrt{M} \|f \sqrt{n_\infty}\|_{L^2}, \quad \|f n_\infty\|_{L^3} \leq \|f \sqrt{n_\infty}\|_{L^2}^{\frac{2}{3}} \|f n_\infty\|_{L^\infty}^{\frac{1}{3}} \|n_\infty\|_{L^\infty}^{\frac{1}{3}}.$$

Notice that from  $\|f \sqrt{n_\infty}\|_{L^2}^2 \leq Q_1[f]$ , we deduce that

$$\|\nabla (g c_\infty)\|_{L^\infty} = O(Q_1[f(t, \cdot)]^{\frac{1}{3}}) \tag{24}$$

which leads to

$$\lambda(t) \leq O(Q_1[f(t, \cdot)]^{\frac{4}{3}}) \quad \text{as} \quad t \rightarrow +\infty.$$

As a result, we read from

$$\frac{d}{dt} Q_1[f] \leq -2\mathcal{E}_* Q_1[f] + O\left((Q_1[f])^{\frac{4}{3}}\right)$$

that

$$\limsup_{t \rightarrow \infty} e^{2\mathcal{E}_* t} Q_1[f(t, \cdot)] < \infty.$$

When  $d = 3$ , we have the estimate

$$\|\nabla(g c_\infty)\|_{L^\infty} \leq C (\|f n_\infty\|_{L^1} + \|f n_\infty\|_{L^4})$$

and similarly obtain that

$$\|f n_\infty\|_{L^1} \leq \sqrt{M} \|f \sqrt{n_\infty}\|_{L^2}, \quad \|f n_\infty\|_{L^4} \leq \|f \sqrt{n_\infty}\|_{L^2}^{\frac{1}{2}} \|f n_\infty\|_{L^\infty}^{\frac{1}{2}} \|n_\infty\|_{L^\infty}^{\frac{1}{4}}.$$

Using again  $\|f \sqrt{n_\infty}\|_{L^2}^2 \leq Q_1[f]$ , we have

$$\|\nabla(g c_\infty)\|_{L^\infty} = O\left(Q_1[f(t, \cdot)]^{\frac{1}{4}}\right) \quad (25)$$

which allows us to write that

$$\lambda(t) \leq O\left(Q_1[f(t, \cdot)]^{\frac{5}{4}}\right) \quad \text{as } t \rightarrow +\infty.$$

We conclude as above, which completes the proof of Theorem 1.1.  $\square$

**4.3. Uniform rate of convergence.** Let us give additional results on the convergence in various norms of the solution of (2) to the stationary solution.

**Corollary 4.1.** *Under the assumptions of Theorem 1.1, if  $\phi(x) = \frac{1}{2}|x|^2$ , the solution  $n$  of (2) is such that*

$$\|n(t, \cdot) - n_\infty\|_{L^p} = O\left(e^{-\frac{t}{p}}\right) \quad \text{and} \quad \|\nabla c(t, \cdot) - \nabla c_\infty\|_{L^q} = O\left(e^{-\frac{t(q+2d)}{(d+1)q}}\right)$$

as  $t \rightarrow +\infty$ , for any  $p \in (1, \infty)$  and any  $q \in (2, \infty)$ . Additionally, if  $d = 2$ , then

$$\|n(t, \cdot) - n_\infty\|_{L^\infty} = O\left(e^{-\lambda t}\right)$$

as  $t \rightarrow +\infty$ , for any  $\lambda < 1$ .

*Proof.* From the Cauchy-Schwarz inequality, we read that

$$\|n(t, \cdot) - n_\infty\|_{L^1(\mathbb{R}^d)} \leq \left( \|n_\infty\|_{L^1(\mathbb{R}^d)} \int_{\mathbb{R}^d} \frac{|n(t, \cdot) - n_\infty|^2}{n_\infty} dx \right)^{\frac{1}{2}} \leq \sqrt{CM} e^{-t}$$

for some  $C > 0$  if  $t$  is taken large enough, and we also know also that

$$\|n(t, \cdot) - n_\infty\|_{L^p(\mathbb{R}^d)} = O\left(e^{-\frac{t}{p}}\right) \quad (26)$$

for any  $p \in [1, \infty)$ . By definition of  $Q_1[f]$ , we have

$$\|\nabla c(t, \cdot) - \nabla c_\infty\|_{L^2(\mathbb{R}^d)} \leq \sqrt{C} e^{-t}$$

for some  $C > 0$  if  $t$  is taken large enough, according to Lemma 3.1. Moreover, according to (24), (25) and Theorem 1.1, we obtain that

$$\|\nabla c(t, \cdot) - \nabla c_\infty\|_{L^\infty(\mathbb{R}^d)} = O\left(e^{-\frac{t}{d+1}}\right). \quad (27)$$

This proves that

$$\|\nabla c(t, \cdot) - \nabla c_\infty\|_{L^q(\mathbb{R}^d)} = O\left(e^{-\frac{t(q+2d)}{(d+1)q}}\right)$$

for any  $q \in [2, \infty)$  by interpolating between (26) and (27).

The proof of the case  $d = 2$  is inspired by [11, Remark 5]. We reconsider  $\mathcal{R}(t)$  defined in (12) in Section 2 with  $p = \frac{7r}{5r-4}$ ,  $q = \frac{7r}{2r-3}$ . We obtain from Corollary 4.1 that

$$\|n(t, \cdot) - n_\infty\|_{L^\infty(\mathbb{R}^2)} = O\left(e^{-\frac{5r-4}{7r}t}\right).$$

This is the first step of a proof by induction. If

$$\|n(t, \cdot) - n_\infty\|_{L^\infty(\mathbb{R}^2)} = O\left(e^{-at}\right),$$

then one has

$$\|n(t, \cdot) - n_\infty\|_{L^\infty(\mathbb{R}^2)} = O\left(e^{-\frac{5r-4+(2r+4)a}{7r}t}\right).$$

By iterating this estimate infinitely many times, we finally have

$$\|n(t, \cdot) - n_\infty\|_{L^\infty(\mathbb{R}^2)} = O\left(e^{-\lambda t}\right)$$

for any  $\lambda < 1$ . The proof of the corollary is complete.  $\square$

**4.4. Intermediate asymptotics of the Nernst-Planck equation with Poisson term.** Let us come back to the equation (1). The self-similar solution of (1) has the expression

$$u_\infty(x, t) = \frac{1}{1+2t} n_\infty \left( \frac{1}{2} \log(1+2t), \frac{x}{\sqrt{1+2t}} \right), \quad (28)$$

$$v_\infty(x, t) = c_\infty \left( \frac{1}{2} \log(1+2t), \frac{x}{\sqrt{1+2t}} \right), \quad (29)$$

where  $(n_\infty, c_\infty)$  are the stationary solutions of (2) given by (4) with the harmonic potential  $\phi(x) = \frac{1}{2}|x|^2$ . Using Theorem 1.1 and Corollary 4.1, we achieve a result on the *intermediate asymptotics* for the solutions of the Nernst-Planck equation with Poisson term in absence of any external potential of confinement.

**Theorem 4.1.** *Assume that  $u$  solves (1) with initial datum  $u(0, \cdot) = n_0 \in L^2_+(n_\infty^{-1} dx)$ ,  $\int_{\mathbb{R}^d} n_0 dx = M$ , and  $\mathcal{F}[n_0] < \infty$ . Let us consider the self-similar solution defined by (28) and (29) of mass  $M$ . Then, as  $t \rightarrow +\infty$ , we have*

(i) for any  $p \in (1, \infty)$  and any  $\lambda < 1$ ,

$$\|u(t, \cdot) - u_\infty\|_{L^1(\mathbb{R}^2)} = O\left((1+2t)^{-\frac{1}{2}}\right),$$

$$\|u(t, \cdot) - u_\infty\|_{L^p(\mathbb{R}^2)} = O\left((1+2t)^{-\frac{\lambda}{2} - \frac{d(p-1)}{2p}}\right),$$

(ii) for any  $q \in (2, \infty)$  and any  $\lambda < 1$ ,

$$\|\nabla v(t, \cdot) - \nabla v_\infty\|_{L^2(\mathbb{R}^2)} = O\left((1+2t)^{-1 + \frac{d}{4}}\right),$$

$$\|\nabla v(t, \cdot) - \nabla v_\infty\|_{L^q(\mathbb{R}^2)} = O\left((1+2t)^{-\frac{\lambda+1}{2} + \frac{d}{2q}}\right).$$

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