A Taylor series based continuation method for the periodic solutions of a wide range of dynamical systems

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Abstract. This extended abstract is a companion of the one called *A Taylor series based continuation method for solutions of Ordinary Differential Equations*. It is shown how a quadratic framework allows to solve a large class of dynamical systems. A brief introduction on the continuation method is followed by an explanation on the quadratic rewriting of systems of equations. A few types of dynamical systems that can be treated using this framework are finally given.

Introduction

This paper emphasizes how a quadratic framework can handle a wide range of applications for the continuation of dynamical systems. The continuation method used, the so-called Asymptotic Numerical Method (ANM) that has first been described in [2], relies on a high-order Taylor series representation of the solution-branch. This technique has already proven its efficiency for a lot of applications in enginering, mechanics or acoustics for example. While some implementations relying on automatic differentiation do exist, see [1] for example, the choice is made here to work with a quadratic framework. A generic implementation of this latest approach which minimizes problem-dependent implementation has been developed [3]. A simplified scheme of it is represented in figure 1. It is based on the numerical continuation of algebraic systems of the form

$$R(V) = 0 \tag{1}$$

where $V \in \mathbb{R}^{n+1}$ and $R(V) \in \mathbb{R}^n$ is an analytic function of its argument. This system is always written in a quadratic format as a prerequisite of the method. Again, this formalism is not a constraint that we suffer but a choice that allows to treat a very wide range of problems as shown in [5]. The system (1) can then be written

$$\tilde{R}(\tilde{V}) = C + L(\tilde{V}) + Q(\tilde{V}, \tilde{V}) = 0 \tag{2}$$

where $\tilde{V} \in \mathbb{R}^{N+1}, N \geq n$ and C, L and Q are respectively a constant, a linear and a quadratic operator. This step can be tough and some hints are given in the second section. Again, the key idea of the ANM is to compute a high-order Taylor series development of the solution-branch. Let consider that $\tilde{V}_0$ is a regular solution of the system (2). Let $\tilde{V}_1$ be a tangent vector at $\tilde{V}_0$. The classical arc-length parameter $a = \tilde{V}_1^t(\tilde{V}_0 - \tilde{V})$ is introduced and the solution-branch around $\tilde{V} = \tilde{V}_0$ is written as a power series with respect to a

$$\tilde{V}(a) = \tilde{V}_0 + a\tilde{V}_1 + a^2\tilde{V}_2 + a^3\tilde{V}_3 + \cdots \tag{3}$$

The quadratic framework

The quadratic rewriting of the system allows to compute the terms in the development of the series explicitly with discrete convolution for the non-linear terms (see [3] for some details about this section). If the solutions seeked are periodic functions of time expanded in Fourier series as it is the case in this paper, the product of two variables in the frequency domain is similarly represented by an explicit discrete convolution. Another huge advantage of the quadratic rewriting is that the exact Jacobian matrix of the system (2) is known explicitly if the constant, linear and quadratic operators are known, and can be derived automatically from these operators.

Rational variables

The equations with inversions $y = \frac{1}{x}$ or monomials of degree more than two $y = x^3$ can be rewritten either implicitly $yx = 1$ or with the use of auxiliary variables $\begin{cases} y - xr = 0 \\ r - x^2 = 0 \end{cases}$ respectively as detailed in [3].

Transcendental functions

Transcendental non-linearities like $y = \exp(x)$ require a specific treatment. The idea is to write the differential form of this equation in a quadratic format. For this simple example, the differential form is $\mathrm{d}y = \exp(x)\mathrm{d}x$. It is easy to recast it using $y = \exp(x)$ to obtain $\mathrm{d}y = y\mathrm{d}x$. Apart for the term of the series (3) at order 0 that shall still be computed using $y = \exp(x)$, all the terms of higher order can be computed with the differentiated form which is quadratic. More details are given about these type of computation in the book [4].

Classes of systems that can be treated

The quadratic rewriting described in the previous section allows to treat a very wide range of dynamical systems with the same continuation tool. For clarity this section is reduced to periodic solutions of dynamical systems, but in the companion extended abstract, the case of static solutions, quasi-periodic solutions and transients will be treated. For now, there are 5 different types of differential equations that have been treated.
The Ordinary Differential Equations (ODE), Differential-Algebraic Equations (DAE) and Implicit Differential Equations (IDE) are illustrated through the same example of the pendulum, using three different formulations of the problem.

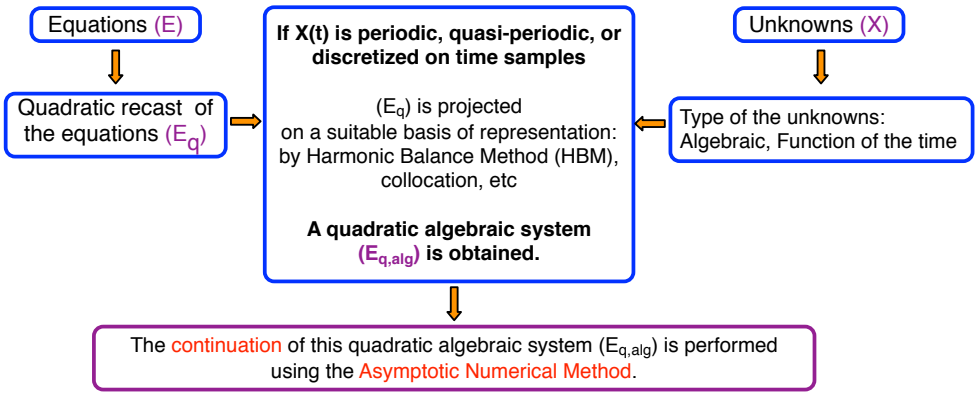

Figure 1: This scheme describes from a general perspective the implementation of the Asymptotic Numerical Method used in this work.

ODE : second order polar coordinates

This is the very classical equation of motion of the pendulum.

$$\ddot{\theta} + \sin(\theta) = 0 \tag{4}$$

DAE : second order Cartesian coordinates

This projection on fixed axis of the equations of the motion leads to an algebraic equation to constrain the length of the pendulum. In addition, the tension of the string T appears in the equations.

$$\begin{cases} \ddot{x} + Tx & = 0 \\ \ddot{y} + Ty - 1 & = 0 \\ x^2 + y^2 - 1 & = 0 \end{cases} \tag{5}$$

IDE : conservation of the energy

The pendulum is a conservative mechanical system meaning that its energy h is a constant of the motion. It leads to this implicit differential equation:

$$\frac{1}{2}\dot{\theta}^2 + 1 - \cos(\theta) = h \tag{6}$$

Other types of systems

In addition to these three widespread types of systems, some very recent works have shown that equations with fractional derivatives [6] or time-delayed variables could also be treated using the quadratic rewriting presented in this paper.

Conclusion

The full paper corresponding to this extended abstract and to the companion extended abstract will show how to take advantage from a quadratic rewriting to carry out the continuation of solutions of a large class of dynamical systems. The method requires very few effort to switch between the different types of systems. Its implementation is freely available online on a dedicated website http://manlab.lma.cnrs-mrs.fr/.

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