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Abstract

We investigate the Vlasov-Poisson equations perturbed by a strong external uniform magnetic field. We study the asymptotic behavior of the solutions, based on averaging techniques. We analyze the case of general initial conditions. By filtering out the oscillations, we are led to a profile. We prove strong convergence results and establish second order estimates.

Keywords: Vlasov-Poisson system, two-scale analysis, averaging, homogenization.

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1 Introduction

We consider a population of charged particles of mass m, charge q, whose density is denoted by $f = f(t, x, v)$, depending on time t, position x and velocity v. In order to study the magnetic confinement, we focus on the asymptotic behavior of the Vlasov-Poisson equations, with strong external magnetic field [16, 17, 18, 24, 25, 11, 13, 22, 3, 4, 5, 6, 14, 19, 15, 12].

Neglecting the curvature of the magnetic lines, that is $\mathbf{B}^\varepsilon = (0, 0, B^\varepsilon) = (0, 0, B/\varepsilon)$, we are led to

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + \frac{q}{m} \left\{ E[f^\varepsilon(t)](x) + B^\varepsilon \, {}^\perp v \right\} \cdot \nabla_v f^\varepsilon = 0, \quad (t, x, v) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2. \quad (1)$$

The notation ${}^\perp(\cdot)$ stands for the rotation of angle $-\pi/2$ of the velocity $v = (v_1, v_2)$, *i.e.*, ${}^\perp v = \mathcal{R}(-\pi/2)v = (v_2, -v_1)$, $v = (v_1, v_2) \in \mathbb{R}^2$ and $\varepsilon > 0$ is a small parameter related to the ratio between the cyclotronic period and the advection time scale. The

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electric field $E[f^\varepsilon(t)] = -\nabla_x \Phi[f^\varepsilon(t)]$ derives from a potential, satisfying the Poisson equation

$$-\varepsilon_0 \Delta_x \Phi[f^\varepsilon(t)] = q \int_{\mathbb{R}^2} f^\varepsilon(t,x,v)\, \mathrm{d}v, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2$$

where ε_0 represents the electric permittivity. Appealing to the fundamental solution $z \to -\frac{1}{2\pi} \ln |z|, z \in \mathbb{R}^2 \setminus \{0\}$, we have

$$\Phi[f^\varepsilon(t)](x) = -\frac{q}{2\pi\varepsilon_0} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|x-x'| f^\varepsilon(t,x',v')\, \mathrm{d}v'\mathrm{d}x' \tag{2}$$

and

$$E[f^\varepsilon](x) = \frac{q}{2\pi\varepsilon_0} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f^\varepsilon(x',v') \frac{x-x'}{|x-x'|^2}\, \mathrm{d}v'\mathrm{d}x'. \tag{3}$$

For any particle density $f = f(x,v)$, we denote by $E[f]$ the Poisson electric field associated to the charge density $\rho[f] = q \int_{\mathbb{R}^2} f(\cdot,v)\, \mathrm{d}v$. We also use the notation $j[f] = q \int_{\mathbb{R}^2} f(\cdot,v) v\, \mathrm{d}v$ for the current density of f. We complete the above system by the initial condition

$$f^\varepsilon(0,x,v) = f_{\mathrm{in}}(x,v), \quad (x,v) \in \mathbb{R}^2 \times \mathbb{R}^2. \tag{4}$$

Very recently, the Vlasov-Poisson equations with strong external non homogeneous magnetic field have been studied, when considering well prepared initial densities [9, 14]. As usual, we are looking for quantities with small variations over a cyclotronic period (the guiding center), and we average with respect to the fast cyclotronic motion in order to obtain the effective Vlasov equation. Following this strategy, most of the previous studies provided formal or rigorous (based on compactness arguments) gyrokinetic approximations for the transport of charged particles under the action of strong external magnetic fields. The subject matter in [9] was to derive second order regular reformulations of the Vlasov-Poisson equations with strong magnetic field and to perform the error analysis. One of the key point was to split the advection field of the Vlasov equation into a fast and slow dynamics, such that the guiding center is left (exactly) invariant by the fast dynamics. It was shown that in this case the fast dynamics becomes periodic (even for a non homogeneous magnetic field), and therefore the homogenization procedure simplifies a lot : instead of taking ergodic means, it is enough to average over one period.

When the magnetic field is uniform, it is possible to go further in our analysis, by considering smooth initial particle densities, not necessarily well prepared. We mention that most of the studies concentrate only on models with well prepared initial particle densities. We intend to extend the analysis in [9] for general initial conditions. The asymptotic behavior is more complicated because the particle densities $(f^\varepsilon)_{\varepsilon>0}$ present fast oscillations in time. We appeal to a two scale approach by working in a extended phase space supplemented by a fast time variable $s = t/\varepsilon$. Up to a second order term, the oscillations of the family $(f^\varepsilon)_{\varepsilon>0}$ can be described in terms of a profile solving a regular reformulation of the Vlasov-Poisson equations, see also [25].

Theorem 1.1

Let $f_{\mathrm{in}} \in C_c^2(\mathbb{R}^2 \times \mathbb{R}^2)$ be a non negative, smooth, compactly supported particle density. We denote by $(f^\varepsilon)_{\varepsilon>0}$ the solutions of the Vlasov-Poisson equations (1), (2) *with*

uniform external magnetic field $\frac{B}{\varepsilon} \neq 0$*, corresponding to the initial condition* $f_{\rm in}$*. Then for any* $T \in \mathbb{R}_+$*, there is a constant* $C_T > 0$ *such that for any* $\varepsilon > 0$

$$\sup_{t\in[0,T]} \left\{ \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left[f^\varepsilon(t,x,v) - \tilde{F}\left(t, x + \varepsilon \frac{{}^\perp v}{\omega_c}, \mathcal{R}(\omega_c \frac{t}{\varepsilon})\left(v - \varepsilon \frac{{}^\perp E[\tilde{F}(t)]}{B}\right)\right)\right]^2 \mathrm{d}v\mathrm{d}x \right\}^{1/2} \leq C_T \varepsilon^2$$

where $\tilde{F}$ *is the solution of*

$$\partial_t \tilde{F} + \varepsilon \frac{{}^\perp E[\tilde{F}(t)]}{B} \cdot \nabla_X \tilde{F} + \varepsilon \frac{{}^\perp (j[\tilde{F}(t)] - \rho[\tilde{F}(t)]\tilde{V})}{2\varepsilon_0 B} \cdot \nabla_{\tilde{V}} \tilde{F} = 0, \quad (t, X, \tilde{V}) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2$$

corresponding to the initial condition

$$\tilde{F}(0, X, \tilde{V}) = f_{\rm in}\left(X - \varepsilon \frac{{}^\perp \tilde{V}}{\omega_c}, \tilde{V} + \varepsilon \frac{{}^\perp E[f_{\rm in}](X)}{B}\right), \quad (X, \tilde{V}) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

Our paper is organized as follows. In Section 2 we derive formally the effective Vlasov-Poisson equations. We introduce the average operators along the fast dynamics and discuss their main properties. The effective model is studied in Section 3, and we summarize its properties. The well posedness of this model is a direct consequence of the well posedness of the vorticity formulation for the 2D incompressible Euler equations. Section 4 is devoted to the error analysis. We establish second order estimates, by constructing a suitable corrector on the extended phase space, supplemented by the fast time variable. Some generalizations are indicated in the last section.

2 Asymptotic analysis by formal arguments

The well posedness of the Vlasov-Poisson problem is well known, see [1] for weak solutions, and [26, 21, 23, 2] for strong solutions. Essentially the same arguments provide the global existence and uniqueness for the strong solution of the Vlasov-Poisson problem with external magnetic field cf. Theorem 2.1 [9].

Theorem 2.1
Consider a non negative, smooth, compactly supported initial particle density $f_{\rm in} \in C^1_c(\mathbb{R}^2 \times \mathbb{R}^2)$ *and a smooth magnetic field* $B \in C^1_b(\mathbb{R}^2)$*. There is a unique particle density* $f \in C^1(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2)$*, whose restriction on* $[0,T] \times \mathbb{R}^2 \times \mathbb{R}^2$ *is compactly supported for any* $T \in \mathbb{R}_+$*, whose Poisson electric field is smooth* $E[f] \in C^1(\mathbb{R}_+ \times \mathbb{R}^2)$*, satisfying*

$$\partial_t f + v \cdot \nabla_x f + \frac{q}{m}\left(E[f(t)] + B\,{}^\perp v\right) \cdot \nabla_v f = 0, \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2$$

$$E[f(t)](x) = \frac{q}{2\pi\varepsilon_0} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(t,x',v') \frac{x - x'}{|x - x'|^2}\, \mathrm{d}v'\mathrm{d}x', \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2$$

$$f(0,x,v) = f_{\rm in}(x,v), \quad (x,v) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

Moreover, if for some integer $k \geq 2$ *we have* $f_{\rm in} \in C^k_c(\mathbb{R}^2 \times \mathbb{R}^2), B \in C^k_b(\mathbb{R}^2)$*, then* $f \in C^k(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2)$ *and* $E[f] \in C^k(\mathbb{R}_+ \times \mathbb{R}^2)$*.*

We assume that the initial particle density is smooth

$$f_{\rm in} \geq 0, \quad f_{\rm in} \in C^2_c(\mathbb{R}^2 \times \mathbb{R}^2)$$

and that the magnetic field is uniform $B^\varepsilon = B/\varepsilon \neq 0$. We know by Theorem 2.1 that for every $\varepsilon > 0$, there is a unique strong solution $f^\varepsilon \in C^2(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2)$ (whose restriction on $[0,T]\times\mathbb{R}^2\times\mathbb{R}^2$ is compactly supported for any $T \in \mathbb{R}_+$), $E^\varepsilon := E[f^\varepsilon] \in C^2(\mathbb{R}_+ \times \mathbb{R}^2)$ for the Vlasov-Poisson problem with external magnetic field $B^\varepsilon = B/\varepsilon$. By standard arguments, we also have uniform estimates with respect to $\varepsilon > 0$ for the L^∞ norm of the electric field $E^\varepsilon|_{[0,T]\times\mathbb{R}^2}$ and the size of the support of the particle density $f^\varepsilon|_{[0,T]\times\mathbb{R}^2\times\mathbb{R}^2}$. Let us denote by $(X^\varepsilon, V^\varepsilon)(t; t_0, x, v)$ the characteristics associated to (1)

$$\frac{\mathrm{d}X^\varepsilon}{\mathrm{d}t} = V^\varepsilon(t;t_0,x,v), \ \frac{\mathrm{d}V^\varepsilon}{\mathrm{d}t} = \frac{q}{m}\left[E^\varepsilon(t,X^\varepsilon(t;t_0,x,v)) + \frac{B}{\varepsilon}\,{}^\perp V^\varepsilon(t;t_0,x,v)\right]$$

$$X^\varepsilon(t_0;t_0,x,v) = x, \quad V^\varepsilon(t_0;t_0,x,v) = v.$$

The strong external magnetic field induces a large cyclotronic frequency with respect to the reciprocal advection time scale, and therefore a fast dynamics. Indeed, by introducing the characteristic scales $(\bar{t}, \bar{x}, \bar{v})$ for time, length, velocity, we have $\bar{t} = \bar{x}/\bar{v}$ and $\omega_c^\varepsilon \bar{t} \sim 1/\varepsilon$. We use the notations $\omega_c^\varepsilon = qB^\varepsilon/m = \omega_c/\varepsilon, \omega_c = qB/m \sim 1/\bar{t}$. It is well known that the guiding center, $X^\varepsilon(t) + \varepsilon\ {}^\perp V^\varepsilon(t)/\omega_c$ has small variations over one cyclotronic period [14, 22, 25]. Another quantity having small variations is $\mathcal{R}\left(\omega_c t/\varepsilon\right)\left[V^\varepsilon(t) - \varepsilon\frac{{}^\perp E^\varepsilon(t,X^\varepsilon(t))}{B}\right]$, see [20].

Lemma 2.1
The following quantities

$$X^\varepsilon(t) + \varepsilon\ {}^\perp V^\varepsilon(t)/\omega_c, \quad \mathcal{R}\left(\omega_c t/\varepsilon\right)\left[V^\varepsilon(t) - \varepsilon\frac{{}^\perp E^\varepsilon(t,X^\varepsilon(t))}{B}\right]$$

have small variations in time.

Proof.
By direct computations we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[X^\varepsilon(t) + \varepsilon\frac{{}^\perp V^\varepsilon(t)}{\omega_c}\right] = \varepsilon\frac{{}^\perp E^\varepsilon(t,X^\varepsilon(t))}{B}.$$

Notice also that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[V^\varepsilon(t) - \varepsilon\frac{{}^\perp E^\varepsilon(t,X^\varepsilon(t))}{B}\right] = \frac{\omega_c}{\varepsilon}\,{}^\perp\left[V^\varepsilon(t) - \varepsilon\frac{{}^\perp E^\varepsilon(t,X^\varepsilon(t))}{B}\right] - \varepsilon\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{{}^\perp E^\varepsilon(t,X^\varepsilon(t))}{B}\right]$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\mathcal{R}\left(\omega_c t/\varepsilon\right)\left[V^\varepsilon(t) - \varepsilon\frac{{}^\perp E^\varepsilon(t,X^\varepsilon(t))}{B}\right]\right\} = -\varepsilon\mathcal{R}\left(\omega_c t/\varepsilon\right)\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{{}^\perp E^\varepsilon(t,X^\varepsilon(t))}{B}\right].$$

□

Motivated by the above calculation, we introduce the relative velocity with respect to the electric cross field drift

$$\tilde{v} = v - \varepsilon \frac{{}^{\perp}E^{\varepsilon}(t,x)}{B} \tag{5}$$

and the new particle density

$$\tilde{f}^{\varepsilon}(t,x,\tilde{v}) = f^{\varepsilon}\left(t,x,\tilde{v}+\varepsilon\frac{{}^{\perp}E[f^{\varepsilon}(t)](x)}{B}\right), \quad (x,\tilde{v}) \in \mathbb{R}^2\times\mathbb{R}^2. \tag{6}$$

It is easily seen that

$$\rho[\tilde{f}^{\varepsilon}(t)] = q\int_{\mathbb{R}^2}\tilde{f}^{\varepsilon}(t,\cdot,\tilde{v})\,\mathrm{d}\tilde{v} = q\int_{\mathbb{R}^2} f^{\varepsilon}(t,\cdot,v)\,\mathrm{d}v = \rho[f^{\varepsilon}(t)], \quad t\in\mathbb{R}_+$$

and

$$\begin{aligned} j[f^{\varepsilon}(t)] &= q\textstyle\int_{\mathbb{R}^2} f^{\varepsilon}(t,\cdot,v)v\,\mathrm{d}v = q\int_{\mathbb{R}^2}\tilde{f}^{\varepsilon}(t,\cdot,\tilde{v})\left(\tilde{v}+\varepsilon\frac{{}^{\perp}E^{\varepsilon}(t)}{B}\right)\mathrm{d}\tilde{v}\\ &= j[\tilde{f}^{\varepsilon}(t)] + \varepsilon\frac{{}^{\perp}E^{\varepsilon}(t)}{B}\rho[\tilde{f}^{\varepsilon}(t)], \quad t\in\mathbb{R}_+. \end{aligned}$$

Therefore the Poisson electric fields corresponding to the particle densities $f^{\varepsilon}, \tilde{f}^{\varepsilon}$ coincide

$$E[f^{\varepsilon}(t)] = E[\tilde{f}^{\varepsilon}(t)], \quad t\in\mathbb{R}_+$$

and we can use the same notation $E^{\varepsilon}(t)$ for denoting them. We introduce the notations

$$B_0 := |B| > 0, \quad \omega_0 := |\omega_c| > 0.$$

Observe that the new particle densities $(\tilde{f}^{\varepsilon})_{\varepsilon>0}$ are smooth, $\tilde{f}^{\varepsilon}\in C^2(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{R}^2)$ and that the restrictions to $[0,T]\times\mathbb{R}^2\times\mathbb{R}^2$ are compactly supported, uniformly with respect to $\varepsilon\in]0,1]$, for any $T\in\mathbb{R}_+$. We obtain the following problem in the new coordinates $(x,\tilde{v})$

$$\begin{aligned} \partial_t\tilde{f}^{\varepsilon} + \left(\tilde{v}+\varepsilon\frac{{}^{\perp}E^{\varepsilon}}{B}\right)\cdot\nabla_x\tilde{f}^{\varepsilon} - \varepsilon\left[\partial_t\left(\frac{{}^{\perp}E^{\varepsilon}}{B}\right) + \partial_x\left(\frac{{}^{\perp}E^{\varepsilon}}{B}\right)\left(\tilde{v}+\varepsilon\frac{{}^{\perp}E^{\varepsilon}}{B}\right)\right]\cdot\nabla_{\tilde{v}}\tilde{f}^{\varepsilon}\\ + \frac{\omega_c}{\varepsilon}\,{}^{\perp}\tilde{v}\cdot\nabla_{\tilde{v}}\tilde{f}^{\varepsilon} = 0, \quad (t,x,\tilde{v})\in\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{R}^2 \end{aligned}$$

$$\tilde{f}^{\varepsilon}(0,x,\tilde{v}) = f_{\mathrm{in}}\left(x,\tilde{v}+\varepsilon\frac{{}^{\perp}E[f_{\mathrm{in}}](x)}{B}\right), \quad (x,\tilde{v})\in\mathbb{R}^2\times\mathbb{R}^2.$$

Thanks to the continuity equation

$$\partial_t\rho[f^{\varepsilon}] + \mathrm{div}_x j[f^{\varepsilon}] = 0$$

the time derivative of the electric field E^{ε} can be written in terms of the particle density f^{ε} (or $\tilde{f}^{\varepsilon}$)

$$\begin{aligned} \partial_t E[f^{\varepsilon}] &= \frac{1}{2\pi\varepsilon_0}\int_{\mathbb{R}^2}\partial_t\rho[f^{\varepsilon}(t)](x-x')\frac{x'}{|x'|^2}\,\mathrm{d}x'\\ &= -\frac{1}{2\pi\varepsilon_0}\int_{\mathbb{R}^2}\mathrm{div}_x j[f^{\varepsilon}](x-x')\frac{x'}{|x'|^2}\,\mathrm{d}x'\\ &= -\frac{1}{2\pi\varepsilon_0}\mathrm{div}_x\int_{\mathbb{R}^2}\frac{x'}{|x'|^2}\otimes j[f^{\varepsilon}(t)](x-x')\,\mathrm{d}x'\\ &= -\frac{1}{2\pi\varepsilon_0}\mathrm{div}_x\int_{\mathbb{R}^2}\frac{x-x'}{|x-x'|^2}\otimes\left(j[\tilde{f}^{\varepsilon}(t)](x') + \varepsilon\frac{{}^{\perp}E^{\varepsilon}(t,x')}{B}\rho[\tilde{f}^{\varepsilon}(t)](x')\right)\mathrm{d}x'. \end{aligned}$$

The change of coordinates (5), (6) leads to the problem

$$\partial_t \tilde{f}^\varepsilon + \varepsilon a^\varepsilon[\tilde{f}^\varepsilon(t)] \cdot \nabla_{x,\tilde{v}} \tilde{f}^\varepsilon + \frac{b^\varepsilon(x,\tilde{v})}{\varepsilon} \cdot \nabla_{x,\tilde{v}} \tilde{f}^\varepsilon = 0, \quad (t,x,\tilde{v}) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2 \tag{7}$$

$$\tilde{f}^\varepsilon(0,x,\tilde{v}) = f_{\text{in}}\left(x, \tilde{v} + \varepsilon \frac{{}^\perp E[f_{\text{in}}](x)}{B}\right), \quad (x,\tilde{v}) \in \mathbb{R}^2 \times \mathbb{R}^2$$

where $b^\varepsilon \cdot \nabla_{x,\tilde{v}} = \varepsilon \tilde{v} \cdot \nabla_x + \omega_c \, {}^\perp\tilde{v} \cdot \nabla_{\tilde{v}}$ and for any particle density $\tilde{f}$, $a^\varepsilon[\tilde{f}] \cdot \nabla_{x,\tilde{v}}$ stands for the vector field

$$a^\varepsilon[\tilde{f}] \cdot \nabla_{x,\tilde{v}} = \frac{{}^\perp E[\tilde{f}]}{B} \cdot \nabla_x + \frac{1}{2\pi\varepsilon_0 B} \mathrm{div}_x \int_{\mathbb{R}^2} \frac{{}^\perp(x-x')}{|x-x'|^2} \otimes \left(j[\tilde{f}] + \varepsilon \frac{{}^\perp E[\tilde{f}]}{B} \rho[\tilde{f}] \right)(x') \, \mathrm{d}x' \cdot \nabla_{\tilde{v}}$$
$$- \partial_x \left(\frac{{}^\perp E[\tilde{f}]}{B} \right) \left(\tilde{v} + \varepsilon \frac{{}^\perp E[\tilde{f}]}{B} \right) \cdot \nabla_{\tilde{v}}.$$

Notice that the vector fields $a^\varepsilon[\tilde{f}] \cdot \nabla_{x,\tilde{v}}$ and $b^\varepsilon \cdot \nabla_{x,\tilde{v}}$ are divergence free. The characteristic flow of the vector field $b^\varepsilon \cdot \nabla_{x,\tilde{v}} = \varepsilon \tilde{v} \cdot \nabla_x + \omega_c \, {}^\perp\tilde{v} \cdot \nabla_{\tilde{v}}$

$$\frac{\mathrm{d}\mathcal{X}^\varepsilon}{\mathrm{d}s} = \varepsilon \tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v}), \quad \frac{\mathrm{d}\tilde{\mathcal{V}}^\varepsilon}{\mathrm{d}s} = \omega_c \, {}^\perp\tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v}), \quad \mathcal{X}^\varepsilon(0;x,\tilde{v}) = x, \quad \tilde{\mathcal{V}}^\varepsilon(0;x,\tilde{v}) = \tilde{v}$$

is given by

$$\mathcal{X}^\varepsilon(s;x,\tilde{v}) = x + \varepsilon[I_2 - \mathcal{R}(-\omega_c s)] \frac{{}^\perp\tilde{v}}{\omega_c}, \quad \tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v}) = \mathcal{R}(-\omega_c s)\tilde{v}, \quad \omega_c = \frac{qB}{m}. \tag{8}$$

It is periodic, and has the same period as the characteristic flow ($\mathcal{X}(s;x,\tilde{v}) = x, \tilde{\mathcal{V}}(s;x,\tilde{v}) = \mathcal{R}(-s\omega_c)\tilde{v}$) of the vector field $b(x,\tilde{v}) \cdot \nabla_{x,\tilde{v}} = \omega_c \, {}^\perp\tilde{v} \cdot \nabla_{\tilde{v}}$

$$S^\varepsilon(x,\tilde{v}) = S(x,\tilde{v}) = S, \quad S := \frac{2\pi}{|\omega_c|}, \quad (x,\tilde{v}) \in \mathbb{R}^2 \times \mathbb{R}^2, \quad \varepsilon > 0.$$

The properties of these flows are summarized below. The proof details are left to the reader (see also Proposition 3.1 [9]).

Proposition 2.1
We denote by $(\mathcal{X}^\varepsilon(s;x,\tilde{v}), \tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v}))$ the characteristic flow of the autonomous vector field $b^\varepsilon(x,\tilde{v}) \cdot \nabla_{x,\tilde{v}}$

$$\frac{\mathrm{d}\mathcal{X}^\varepsilon}{\mathrm{d}s} = \varepsilon \tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v}), \quad \frac{\mathrm{d}\tilde{\mathcal{V}}^\varepsilon}{\mathrm{d}s} = \omega_c \, {}^\perp\tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v}), \quad \mathcal{X}^\varepsilon(0;x,\tilde{v}) = x, \quad \tilde{\mathcal{V}}^\varepsilon(0;x,\tilde{v}) = \tilde{v}$$

and by $(\mathcal{X}(s;x,\tilde{v}), \tilde{\mathcal{V}}(s;x,\tilde{v}))$ the characteristic flow of the autonomous vector field $b(x,\tilde{v}) \cdot \nabla_{x,\tilde{v}} = \omega_c \, {}^\perp\tilde{v} \cdot \nabla_{\tilde{v}}$

$$\frac{\mathrm{d}\mathcal{X}}{\mathrm{d}s} = 0, \quad \frac{\mathrm{d}\tilde{\mathcal{V}}}{\mathrm{d}s} = \omega_c \, {}^\perp\tilde{\mathcal{V}}(s;x,\tilde{v}), \quad \mathcal{X}(0;x,\tilde{v}) = x, \quad \tilde{\mathcal{V}}(0;x,\tilde{v}) = \tilde{v}.$$

1. *For any $(x,\tilde{v}) \in \mathbb{R}^2 \times \mathbb{R}^2$ and $\varepsilon > 0$, the characteristic $s \to (\mathcal{X}^\varepsilon, \tilde{\mathcal{V}}^\varepsilon)(s;x,\tilde{v})$ is S-periodic, with $S = 2\pi/\omega_0$.*

2. *For any* $(x,\tilde{v})\in\mathbb{R}^2\times\mathbb{R}^2$ *and* $\varepsilon>0$ *we have*

$$|\mathcal{X}^\varepsilon(s;x,\tilde{v})-\mathcal{X}(s;x,\tilde{v})|=|\mathcal{X}^\varepsilon(s;x,\tilde{v})-x|\leq\varepsilon\frac{2}{\omega_0}|\tilde{v}|,\ \ s\in\mathbb{R}$$

and

$$\tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v})=\tilde{\mathcal{V}}(s;x,\tilde{v})=\mathcal{R}(-s\omega_c)\tilde{v},\ \ s\in\mathbb{R}.$$

3. *For any continuous function* $u\in C(\mathbb{R}^2\times\mathbb{R}^2)$ *we define the averages along the flows of* $b\cdot\nabla_{x,\tilde{v}}$, $b^\varepsilon\cdot\nabla_{x,\tilde{v}}$

$$\langle u\rangle(x,\tilde{v})=\frac{1}{S}\int_0^S u(\mathcal{X}(s;x,\tilde{v}),\tilde{\mathcal{V}}(s;x,\tilde{v}))\,\mathrm{d}s,\ \ (x,\tilde{v})\in\mathbb{R}^2\times\mathbb{R}^2$$

$$\langle u\rangle_\varepsilon(x,\tilde{v})=\frac{1}{S}\int_0^S u(\mathcal{X}^\varepsilon(s;x,\tilde{v}),\tilde{\mathcal{V}}^\varepsilon(s;x,\tilde{v}))\,\mathrm{d}s,\ \ (x,\tilde{v})\in\mathbb{R}^2\times\mathbb{R}^2.$$

For any $R_x,R_{\tilde{v}}\in\mathbb{R}_+$ *we have*

$$\|\langle u\rangle\|_{L^\infty(B(R_x)\times B(R_{\tilde{v}}))}\leq\|u\|_{L^\infty(B(R_x)\times B(R_{\tilde{v}}))}$$

$$\|\langle u\rangle_\varepsilon\|_{L^\infty(B(R_x)\times B(R_{\tilde{v}}))}\leq\|u\|_{L^\infty(B(R_x^\varepsilon)\times B(R_{\tilde{v}}))},\ \ R_x^\varepsilon=R_x+2\varepsilon R_{\tilde{v}}/\omega_0$$

where $B(R)$ *stands for the closed ball of radius* R *in* $\mathbb{R}^2$.

4. *If* u *is Lipschitz continuous, then for any* $(x,\tilde{v})\in\mathbb{R}^2\times\mathbb{R}^2$ *and* $\varepsilon>0$ *we have*

$$\frac{|\langle u\rangle_\varepsilon(x,\tilde{v})-\langle u\rangle(x,\tilde{v})|}{\varepsilon}\leq\mathrm{Lip}(u)\frac{2}{\omega_0}|\tilde{v}|.$$

5. *For any function* $u\in C_c^1(\mathbb{R}^2\times\mathbb{R}^2)$ *we have the inequality*

$$\|u-\langle u\rangle\|_{L^2(\mathbb{R}^2\times\mathbb{R}^2)}\leq\frac{2\pi}{\omega_0}\|b\cdot\nabla_{x,\tilde{v}}u\|_{L^2(\mathbb{R}^2\times\mathbb{R}^2)}.$$

6. *For any function* $u\in C^1(\mathbb{R}^2\times\mathbb{R}^2)$, *we have* $\langle u\rangle\in C^1(\mathbb{R}^2\times\mathbb{R}^2)$ *and*

$$\langle\nabla_x u\rangle=\nabla_x\langle u\rangle,\ \ \tilde{v}\cdot\nabla_{\tilde{v}}\langle u\rangle=\langle\tilde{v}\cdot\nabla_{\tilde{v}}u\rangle,\ \ {}^\perp\tilde{v}\cdot\nabla_{\tilde{v}}\langle u\rangle=0.$$

In order to filter out the fast oscillations corresponding to the vector field $\frac{b^\varepsilon(x,\tilde{v})}{\varepsilon}\cdot\nabla_{x,\tilde{v}}$, we perform one more change of coordinates

$$\tilde{f}^\varepsilon(t,x,\tilde{v})=\tilde{F}^\varepsilon(t,X,\tilde{V}),\ \ (X,\tilde{V})=(\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(-t/\varepsilon;x,\tilde{v}).\tag{9}$$

By applying the chain rule, we obtain

$$\partial_t\tilde{F}^\varepsilon+\varepsilon\partial_{x,\tilde{v}}(\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(-t/\varepsilon;(\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(t/\varepsilon;X,\tilde{V}))a^\varepsilon[\tilde{F}^\varepsilon(t)_{-t/\varepsilon}]_{t/\varepsilon}\cdot\nabla_{X,\tilde{V}}\tilde{F}^\varepsilon=0\tag{10}$$

where we have used the notations

$$\tilde{F}^\varepsilon(t)_{-t/\varepsilon}=\tilde{F}^\varepsilon(t)\circ(\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(-\frac{t}{\varepsilon}),\ \ a^\varepsilon[\tilde{F}^\varepsilon(t)_{-t/\varepsilon}]_{t/\varepsilon}=a^\varepsilon[\tilde{F}^\varepsilon(t)\circ(\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(-\frac{t}{\varepsilon})]\circ(\mathcal{X}^\varepsilon,\tilde{\mathcal{V}}^\varepsilon)(\frac{t}{\varepsilon}).$$

As the characteristic flow $(\mathcal{X}^\varepsilon, \tilde{\mathcal{V}}^\varepsilon)$ in (8) is linear, the jacobian matrix simply writes for any $(x, \tilde{v}) \in \mathbb{R}^2 \times \mathbb{R}^2$

$$\partial_{x,\tilde{v}}(\mathcal{X}^\varepsilon, \tilde{\mathcal{V}}^\varepsilon)(-t/\varepsilon; x, \tilde{v}) = \begin{pmatrix} I_2 & \frac{\varepsilon}{\omega_c}\mathcal{R}(-\pi/2)[I_2 - \mathcal{R}(\omega_c t/\varepsilon)] \\ O_2 & \mathcal{R}(\omega_c t/\varepsilon) \end{pmatrix}$$

and therefore (10) becomes

$$\begin{aligned}\partial_t \tilde{F}^\varepsilon + \varepsilon \left\{ a_x^\varepsilon[\tilde{F}^\varepsilon(t)_{-t/\varepsilon}]_{t/\varepsilon} + \frac{\varepsilon}{\omega_c}\mathcal{R}(-\pi/2)[I_2 - \mathcal{R}(\omega_c t/\varepsilon)] a_{\tilde{v}}^\varepsilon[\tilde{F}^\varepsilon(t)_{-t/\varepsilon}]_{t/\varepsilon} \right\} \cdot \nabla_X \tilde{F}^\varepsilon \\ + \varepsilon \mathcal{R}(\omega_c t/\varepsilon) a_{\tilde{v}}^\varepsilon[\tilde{F}^\varepsilon(t)_{-t/\varepsilon}]_{t/\varepsilon} \cdot \nabla_{\tilde{V}} \tilde{F}^\varepsilon = 0, \quad (t, X, \tilde{V}) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2.\end{aligned}$$

We have obtained a two scale problem and we expect that the asymptotic behavior when ε becomes small will follow by averaging with respect to the fast time variable $s = t/\varepsilon$. As we are looking for second order approximations, we only need to average, with respect to s, when ε is small

$$a_x^\varepsilon[\tilde{F}^\varepsilon(t)_{-s}]_s \cdot \nabla_X \tilde{F}^\varepsilon \quad \text{and} \quad \mathcal{R}(\omega_c s) a_{\tilde{v}}^\varepsilon[\tilde{F}^\varepsilon(t)_{-s}]_s \cdot \nabla_{\tilde{V}} \tilde{F}^\varepsilon$$

up to terms of order ε. By the second statement in Proposition 2.1, observe that

$$\begin{aligned} E[\tilde{F}^\varepsilon(t)_{-s}]_s(X, \tilde{V}) &= \frac{q}{2\pi\varepsilon_0} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \tilde{F}^\varepsilon(t, (\mathcal{X}^\varepsilon, \tilde{\mathcal{V}}^\varepsilon)(-s; x', \tilde{v}')) \frac{\mathcal{X}^\varepsilon(s; X, \tilde{V}) - x'}{|\mathcal{X}^\varepsilon(s; X, \tilde{V}) - x'|^2} \, \mathrm{d}\tilde{v}' \mathrm{d}x' \\ &= \frac{q}{2\pi\varepsilon_0} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \tilde{F}^\varepsilon(t, X', \tilde{V}') \frac{\mathcal{X}^\varepsilon(s; X, \tilde{V}) - \mathcal{X}^\varepsilon(s; X', \tilde{V}')}{|\mathcal{X}^\varepsilon(s; X, \tilde{V}) - \mathcal{X}^\varepsilon(s; X', \tilde{V}')|^2} \, \mathrm{d}\tilde{V}' \mathrm{d}X' \\ &= E[\tilde{F}^\varepsilon(t)](X) + \mathcal{O}(\varepsilon) \end{aligned} \tag{11}$$

and therefore we deduce that

$$\frac{\omega_c}{2\pi} \int_0^{2\pi/\omega_c} a_x^\varepsilon[\tilde{F}^\varepsilon(t)_{-s}]_s \cdot \nabla_X \tilde{F}^\varepsilon(t) \, \mathrm{d}s = \frac{{}^\perp E[\tilde{F}^\varepsilon(t)](X)}{B} \cdot \nabla_X \tilde{F}^\varepsilon + \mathcal{O}(\varepsilon). \tag{12}$$

We concentrate now on the average of $\mathcal{R}(\omega_c s) a_{\tilde{v}}^\varepsilon[\tilde{F}^\varepsilon(t)_{-s}]_s \cdot \nabla_{\tilde{V}} \tilde{F}^\varepsilon$. We only need to consider the contributions of order 1 in $a_{\tilde{v}}^\varepsilon[\tilde{F}^\varepsilon(t)_{-s}]_s$, that is, those of

$$\begin{aligned} &\frac{q}{2\pi\varepsilon_0 B} \left(\mathrm{div}_x \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{{}^\perp(\cdot - x')}{|\cdot - x'|^2} \otimes \tilde{v}' \tilde{F}^\varepsilon(t, (\mathcal{X}^\varepsilon, \tilde{\mathcal{V}}^\varepsilon)(-s; x', \tilde{v}')) \, \mathrm{d}\tilde{v}' \mathrm{d}x' \right) (\mathcal{X}^\varepsilon(s; X, \tilde{V})) \\ &= \frac{q}{2\pi\varepsilon_0 B} \mathrm{div}_X \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{{}^\perp(\mathcal{X}^\varepsilon(s; X, \tilde{V}) - \mathcal{X}^\varepsilon(s; X', \tilde{V}'))}{|\mathcal{X}^\varepsilon(s; X, \tilde{V}) - \mathcal{X}^\varepsilon(s; X', \tilde{V}')|^2} \otimes \mathcal{R}(-\omega_c s) \tilde{V}' \tilde{F}^\varepsilon(t, X', \tilde{V}') \, \mathrm{d}\tilde{V}' \mathrm{d}X' \end{aligned}$$

and

$$-\left(\partial_x \left(\frac{{}^\perp E[\tilde{F}^\varepsilon(t)_{-s}]}{B} \right) \tilde{v} \right)_s (X, \tilde{V}) = \frac{\mathcal{R}(\pi/2)}{B} \partial_X E[\tilde{F}^\varepsilon(t)_{-s}]_s \mathcal{R}(-\omega_c s) \tilde{V}.$$

Notice that, up to terms of order ε, the average of the first contribution writes

$$\begin{aligned}
&\frac{\omega_c}{2\pi}\int_0^{\frac{2\pi}{\omega_c}} \mathcal{R}(\omega_c s)\frac{q}{2\pi\varepsilon_0 B}\mathrm{div}_X \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{{}^\perp(X-X')}{|X-X'|^2}\otimes \mathcal{R}(-\omega_c s)\tilde V' \tilde F^\varepsilon(t,X',\tilde V')\,\mathrm{d}\tilde V'\mathrm{d}X'\mathrm{d}s\\
&= \frac{q\mathcal{R}(-\pi/2)}{2\pi\varepsilon_0 B}\mathrm{div}_X \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{\omega_c}{2\pi}\int_0^{\frac{2\pi}{\omega_c}} \frac{\mathcal{R}(\omega_c s)(X-X')}{|X-X'|^2}\otimes \mathcal{R}(-\omega_c s)\tilde V'\mathrm{d}s \tilde F^\varepsilon(t,X',\tilde V')\,\mathrm{d}\tilde V'\mathrm{d}X'\\
&= \frac{q\mathcal{R}(-\pi/2)}{2\pi\varepsilon_0 B}\mathrm{div}_X \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{(X-X')\otimes\tilde V' - {}^\perp(X-X')\otimes {}^\perp\tilde V'}{2|X-X'|^2}\tilde F^\varepsilon(t,X',\tilde V')\,\mathrm{d}\tilde V'\mathrm{d}X'\\
&= \frac{q\mathcal{R}(-\pi/2)}{2\pi\varepsilon_0 B}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{(X-X')\otimes\tilde V' - {}^\perp(X-X')\otimes {}^\perp\tilde V'}{2|X-X'|^2}\nabla_{X'}\tilde F^\varepsilon(t,X',\tilde V')\,\mathrm{d}\tilde V'\mathrm{d}X'.
\end{aligned}$$

By using the properties of the fundamental solution $z\to -\frac{1}{2\pi}\ln|z|$, it is easily seen that

$$\mathrm{div}_{X'}\frac{(X-X')\otimes\tilde V' - {}^\perp(X-X')\otimes {}^\perp\tilde V'}{2|X-X'|^2} = 0,\quad X\neq X'$$

and thus the above computations lead to

$$\begin{aligned}
&\frac{q\mathcal{R}(-\pi/2)}{2\pi\varepsilon_0 B}\lim_{r\searrow 0}\int_{\mathbb{R}^2}\int_{|X-X'|>r}\frac{(X-X')\otimes\tilde V' - {}^\perp(X-X')\otimes {}^\perp\tilde V'}{2|X-X'|^2}\nabla_{X'}\tilde F^\varepsilon(t,X',\tilde V')\,\mathrm{d}X'\,\mathrm{d}\tilde V'\\
&= \frac{q\mathcal{R}(-\pi/2)}{2\pi\varepsilon_0 B}\lim_{r\searrow 0}\int_{\mathbb{R}^2}\int_{|X-X'|=r}\tilde F^\varepsilon\frac{(X-X')\otimes\tilde V' - {}^\perp(X-X')\otimes {}^\perp\tilde V'}{2|X-X'|^2}\frac{X-X'}{|X-X'|}\,\mathrm{d}\sigma(X')\,\mathrm{d}\tilde V'\\
&= \frac{q\mathcal{R}(-\pi/2)}{2\pi\varepsilon_0 B}\lim_{r\searrow 0}\frac{1}{2r}\int_{|X-X'|=r}\int_{\mathbb{R}^2}\tilde F^\varepsilon(t,X',\tilde V')\tilde V'\,\mathrm{d}\tilde V'\mathrm{d}\sigma(X')\\
&= \frac{{}^\perp j[\tilde F^\varepsilon(t)](X)}{2\varepsilon_0 B}.
\end{aligned}\tag{13}$$

Similarly, up to terms of order ε, the second contribution is, thanks to (11)

$$\begin{aligned}
\frac{\omega_c}{2\pi B}\int_0^{2\pi/\omega_c}\mathcal{R}(\pi/2)\mathcal{R}(\omega_c s)\partial_X E[\tilde F^\varepsilon(t)]\mathcal{R}(-\omega_c s)\tilde V\mathrm{d}s &= -\frac{\mathrm{trace}(\partial_X E[\tilde F^\varepsilon(t)])}{2B}{}^\perp\tilde V\\
&= -\frac{\mathrm{div}_X E[\tilde F^\varepsilon(t)]}{2B}{}^\perp\tilde V = -\frac{\rho[\tilde F^\varepsilon(t)]^\perp\tilde V}{2\varepsilon_0 B}.
\end{aligned}\tag{14}$$

Combining (13), (14) we obtain

$$\frac{\omega_c}{2\pi}\int_0^{2\pi/\omega_c}\mathcal{R}(\omega_c s)a^\varepsilon_{\tilde v}[\tilde F^\varepsilon(t)_{-s}]_s\cdot\nabla_{\tilde V}\tilde F^\varepsilon\mathrm{d}s = \frac{{}^\perp\{j[\tilde F^\varepsilon(t)]-\rho[\tilde F^\varepsilon(t)]\tilde V\}}{2\varepsilon_0 B}\cdot\nabla_{\tilde V}\tilde F^\varepsilon + \mathcal{O}(\varepsilon).\tag{15}$$

Thanks to (12), (15) we are led to the model

$$\partial_t\tilde F + \varepsilon\frac{{}^\perp E[\tilde F(t)]}{B}\cdot\nabla_X\tilde F + \varepsilon\frac{{}^\perp\{j[\tilde F(t)]-\rho[\tilde F(t)]\tilde V\}}{2\varepsilon_0 B}\cdot\nabla_{\tilde V}\tilde F = 0,\ (t,X,\tilde V)\in\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{R}^2\tag{16}$$

which is supplemented by the initial condition

$$\tilde F(0,X,\tilde V) = f_{\mathrm{in}}\left(X-\varepsilon\frac{{}^\perp\tilde V}{\omega_c},\tilde V+\varepsilon\frac{{}^\perp E[f_{\mathrm{in}}](X)}{B}\right),\quad (X,\tilde V)\in\mathbb{R}^2\times\mathbb{R}^2.\tag{17}$$

3 The effective Vlasov-Poisson equations

We expect that solving (16) together with the initial condition (17), will provide a second order approximation for (1), (2). Although the above solution depends on ε, we use the notation $\tilde{F}$, saying that it is an approximation, when ε becomes small. The well posedness of the limit model (16), (17) is a direct consequence of the well posedness of the vorticity formulation for the $2D$ incompressible Euler equations, see also Lemma 3.3 [25]. Indeed, integrating (16) with respect to the velocity leads to the Euler equations

$$\partial_t \rho[\tilde{F}] + \frac{^\perp E}{B^\varepsilon} \cdot \nabla_X \rho[\tilde{F}] = 0, E(t,X) = \frac{1}{2\pi\varepsilon_0} \int_{\mathbb{R}^2} \rho[\tilde{F}(t)](X') \frac{X - X'}{|X - X'|^2} \, \mathrm{d}X', (t,X) \in \mathbb{R}_+ \times \mathbb{R}^2$$

which allows us to determine ρ and E. Multiplying (16) by $\tilde{V}$ and integrating with respect to the velocity give a transport equation for the current density as well

$$\partial_t j[\tilde{F}] + \left(\frac{^\perp E}{B^\varepsilon} \cdot \nabla_X \right) j[\tilde{F}] = 0$$

and finally the particle density $\tilde{F}$ comes by solving the linear transport equation (16), with smooth advection field

$$\frac{^\perp E}{B^\varepsilon} \cdot \nabla_X + \frac{1}{2\varepsilon_0 B^\varepsilon} \, ^\perp (j - \rho \tilde{V}) \cdot \nabla_{\tilde{V}}.$$

The proof details are left to the reader.

Theorem 3.1 *Consider a non negative, smooth, compactly supported initial particle density $\tilde{F}_{\mathrm{in}} \in C^1_c(\mathbb{R}^2 \times \mathbb{R}^2)$ and a uniform magnetic field $B^\varepsilon = \frac{B}{\varepsilon} \neq 0$. There is a unique particle density $\tilde{F} \in C^1(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2)$ whose restriction on $[0,T] \times \mathbb{R}^2 \times \mathbb{R}^2$ is compactly supported for any $T \in \mathbb{R}_+$, whose Poisson electric field is smooth $E[\tilde{F}] \in C^1(\mathbb{R}_+ \times \mathbb{R}^2)$, satisfying* (16), (17). *Moreover, if for some integer $k \geq 2$ we have $\tilde{F}_{\mathrm{in}} \in C^k_c(\mathbb{R}^2 \times \mathbb{R}^2)$, then $\tilde{F} \in C^k(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2)$ and $E[\tilde{F}] \in C^k(\mathbb{R}_+ \times \mathbb{R}^2)$.*

The properties of the above limit model are summarized below : conservation of the total kinetic energy, conservation of the total electric energy, invariance of $\frac{1}{2}|j[\tilde{F}] - \rho[\tilde{F}]\tilde{V}|^2$, invariance under the rotations in the velocity space.

Remark 3.1

1. *The total kinetic energy is conserved. Indeed, multiplying* (16) *by $m|\tilde{V}|^2/2$ and integrating with respect to velocity yields*

$$\partial_t \int_{\mathbb{R}^2} m \frac{|\tilde{V}|^2}{2} \tilde{F} \, \mathrm{d}\tilde{V} + \frac{^\perp E[\tilde{F}(t)]}{B^\varepsilon} \cdot \nabla_X \int_{\mathbb{R}^2} m \frac{|\tilde{V}|^2}{2} \tilde{F} \, \mathrm{d}\tilde{V} = \frac{m}{2\varepsilon_0 B^\varepsilon} \int_{\mathbb{R}^2} \, ^\perp (j[\tilde{F}(t)] - \rho[\tilde{F}(t)]\tilde{V}) \cdot \tilde{V} \tilde{F} \, \mathrm{d}\tilde{V} = 0$$

   and our conclusion follows by integrating also with respect to X.

2. *The total electric energy is conserved. Using the fundamental solution of the Poisson equation in* $\mathbb{R}^2$*, we have*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2\varepsilon_0}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e(X-X')\rho[\tilde{F}(t)](X)\rho[\tilde{F}(t)](X')\,\mathrm{d}X'\mathrm{d}X\\
&=\frac{1}{\varepsilon_0}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e(X-X')\rho[\tilde{F}(t)](X')\partial_t\rho[\tilde{F}(t)](X)\,\mathrm{d}X'\mathrm{d}X\\
&=\int_{\mathbb{R}^2}\Phi[\tilde{F}(t)](X)\partial_t\rho[\tilde{F}(t)](X)\,\mathrm{d}X-\int_{\mathbb{R}^2}\Phi[\tilde{F}(t)](X)\mathrm{div}_X\left(\rho[\tilde{F}(t)]\frac{{}^\perp E[\tilde{F}(t)]}{B^\varepsilon}\right)\,\mathrm{d}X\\
&=\int_{\mathbb{R}^2}\nabla_X\Phi[\tilde{F}(t)]\cdot\frac{{}^\perp E[\tilde{F}(t)]}{B^\varepsilon}\rho[\tilde{F}(t)](X)\,\mathrm{d}X=0.
\end{aligned}$$

3. *The function* $(X,\tilde{V})\to\frac{1}{2}|j[\tilde{F}(t)](X)-\rho[\tilde{F}(t)](X)\tilde{V}|^2$ *is an invariant of the transport operator in* (16)*. Indeed, thanks to the mass and momentum balances, we deduce that*

$$\begin{aligned}
&\left(\partial_t+\frac{{}^\perp E[\tilde{F}(t)]}{B^\varepsilon}\cdot\nabla_X+\frac{{}^\perp(j[\tilde{F}(t)]-\rho[\tilde{F}(t)]\tilde{V})}{2\varepsilon_0 B^\varepsilon}\cdot\nabla_{\tilde{V}}\right)\frac{|j[\tilde{F}(t)]-\rho[\tilde{F}(t)]\tilde{V}|^2}{2}\\
&=-\rho[\tilde{F}(t)]\frac{{}^\perp(j[\tilde{F}(t)]-\rho[\tilde{F}(t)]\tilde{V})}{2\varepsilon_0 B^\varepsilon}\cdot(j[\tilde{F}(t)]-\rho[\tilde{F}(t)]\tilde{V})=0.
\end{aligned}$$

4. *The model* (16), (17) *is invariant under rotation in the velocity space. More exactly, if* $\tilde{F}$ *solves* (16), (17)*, then* $\tilde{F}_\theta(t,X,\tilde{V})=\tilde{F}(t,X,\mathcal{R}(\theta)\tilde{V})$ *solves* (16) *together with the initial condition*

$$\tilde{F}_\theta(0,X,\tilde{V})=\tilde{F}_{\mathrm{in}}(X,\mathcal{R}(\theta)\tilde{V}),\quad (X,\tilde{V})\in\mathbb{R}^2\times\mathbb{R}^2.$$

In particular, if the initial particle density satisfies $b\cdot\nabla_{X,\tilde{V}}\tilde{F}_{\mathrm{in}}=0$*, then, thanks to the uniqueness, we have* $b\cdot\nabla_{X,\tilde{V}}\tilde{F}(t)=0$ *at any time* $t\in\mathbb{R}_+$*.*

4 The error analysis

The solution of (16), (17) will allow us to describe the asymptotic behavior of the family $(f^\varepsilon)_{\varepsilon>0}$ corresponding to the initial condition f_{in}, when ε becomes small. Let us introduce the particle density

$$\tilde{f}(t,s,x,\tilde{v})=\tilde{F}(t,(\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v})),\quad (t,s,x,\tilde{v})\in\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2$$

where $\tilde{F}$ solves (16), (17) and $(\mathcal{X},\tilde{\mathcal{V}})$ is the characteristic flow associated to the vector field $b(x,\tilde{v})\cdot\nabla_{x,\tilde{v}}=\omega_c{}^\perp\tilde{v}\cdot\nabla_{\tilde{v}}$, see Proposition 2.1. The idea is to compare $\tilde{f}^\varepsilon(t,x,\tilde{v})$ with respect to $\overline{f}(t,x+\varepsilon^\perp\tilde{v}/\omega_c,\tilde{v})$, where $\overline{f}(t,x,\tilde{v})=\tilde{f}(t,t/\varepsilon,x,\tilde{v})$. By direct computation we check that

$$\partial_t\overline{f}+\varepsilon\left(\frac{{}^\perp E[\overline{f}(t)]}{B}\cdot\nabla_x\overline{f}+\frac{{}^\perp(j[\overline{f}(t)]-\rho[\overline{f}(t)]\tilde{v})}{2\varepsilon_0 B}\cdot\nabla_{\tilde{v}}\overline{f}\right)+\frac{b(x,\tilde{v})}{\varepsilon}\cdot\nabla_{x,\tilde{v}}\overline{f}=0.\quad(18)$$

We introduce the application $T^\varepsilon(x,\tilde v)=(x+\varepsilon\,{}^\perp\tilde v/\omega_c,\tilde v)$. Notice that $\overline{f}(0)\circ T^\varepsilon$ is a second order approximation of $\tilde f^\varepsilon(0)$ in $L^2(\mathbb{R}^2\times\mathbb{R}^2)$

$$\left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[\tilde f^\varepsilon(0,x,\tilde v)-\overline{f}\left(0,x+\varepsilon\frac{{}^\perp\tilde v}{\omega_c},\tilde v\right)\right]^2\,\mathrm{d}\tilde v\mathrm{d}x\right\}^{1/2} \tag{19}$$
$$=\left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[f_{\rm in}\left(x,\tilde v+\varepsilon\frac{{}^\perp E[f_{\rm in}]}{B}(x)\right)-\tilde F\left(0,x+\varepsilon\frac{{}^\perp\tilde v}{\omega_c},\tilde v\right)\right]^2\,\mathrm{d}\tilde v\mathrm{d}x\right\}^{1/2}$$
$$=\left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[f_{\rm in}\left(x,\tilde v+\varepsilon\frac{{}^\perp E[f_{\rm in}]}{B}(x)\right)-f_{\rm in}\left(x,\tilde v+\varepsilon\frac{{}^\perp E[f_{\rm in}]}{B}(x+\varepsilon\,{}^\perp\tilde v/\omega_c)\right)\right]^2\,\mathrm{d}\tilde v\mathrm{d}x\right\}^{1/2}$$
$$\leq C\varepsilon^2.$$

By introducing a corrector term, we will prove that $\overline{f}\circ T^\varepsilon$ is a second order approximation of $\tilde f^\varepsilon$ in $L^\infty_{\rm loc}(\mathbb{R}_+;L^2(\mathbb{R}^2\times\mathbb{R}^2))$. We mention that the asymptotic behavior for a very similar problem has been investigated in Theorem 1.2 [25], but without indicating the convergence rate. Our goal is to complete the asymptotic analysis by justifying the second order approximation. The arguments developed for well prepared initial particle densities in [9] apply for general initial particle densities as well, justifying the robustness of our method. For any smooth particle density $\tilde f\in C^1_c(\mathbb{R}^2\times\mathbb{R}^2)$ we use the notations

$$a[\tilde f]\cdot\nabla_{x,\tilde v}=\frac{{}^\perp E[\tilde f]}{B}\cdot\nabla_x+\left[\frac{\mathrm{div}_x}{2\pi\varepsilon_0 B}\int_{\mathbb{R}^2}\frac{{}^\perp(x-x')}{|x-x'|^2}\otimes j[\tilde f](x')\,\mathrm{d}x'-\partial_x\left(\frac{{}^\perp E[\tilde f]}{B}\right)\tilde v\right]\cdot\nabla_{\tilde v}$$

and

$$\langle a\rangle\,[\tilde f]\cdot\nabla_{x,\tilde v}=\frac{{}^\perp E[\tilde f]}{B}\cdot\nabla_x+\frac{{}^\perp(j[\tilde f]-\rho[\tilde f]\tilde v)}{2\varepsilon_0 B}\cdot\nabla_{\tilde v}.$$

For constructing the corrector, we need essentially to invert the transport operator $\partial_s+b\cdot\nabla_{x,\tilde v}$ on the subspace of functions with zero average with respect to the characteristic flow of the vector field $\partial_s+b\cdot\nabla_{x,\tilde v}$, see [7, 8]. The expression of the corrector is explicit and follows by direct computations. Its smoothness and uniform boundedness with respect to the fast time variable will be crucial when establishing the error estimate. We consider particle densities depending also on the fast time variable and therefore we work in the phase space $(s,x,\tilde v)\in\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2$.

Proposition 4.1 *Assume that $\tilde f=\tilde f(s,x,\tilde v)\in C^1(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2)$ is $S=\frac{2\pi}{|\omega_c|}$ periodic and uniformly compactly supported in $(x,\tilde v)$ with respect to $s\in\mathbb{R}$, such that*

$$\partial_s\tilde f+b(x,\tilde v)\cdot\nabla_{x,\tilde v}\tilde f=0,\quad (s,x,\tilde v)\in\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2.$$

Then we have the equality

$$a[\tilde f(s)]\cdot\nabla_{x,\tilde v}\tilde f-\langle a\rangle\,[\tilde f(s)]\cdot\nabla_{x,\tilde v}\tilde f+(\partial_s+b(x,\tilde v)\cdot\nabla_{x,\tilde v})\tilde f^2=0$$

where

$$\tilde{f}^2 = \frac{\cos(2\omega_c s)}{8\omega_c\pi\varepsilon_0 B}\mathrm{div}_x \int_{\mathbb{R}^2}\left[\frac{{}^\perp(x-x')}{|x-x'|^2}\otimes {}^\perp j[\tilde{F}] + \frac{x-x'}{|x-x'|^2}\otimes j[\tilde{F}]\right]\,\mathrm{d}x'\cdot\nabla_{\tilde{V}}\tilde{F}(x,\mathcal{R}(\omega_c s)\tilde{v})$$
$$+\frac{\sin(2\omega_c s)}{8\omega_c\pi\varepsilon_0 B}\mathrm{div}_x \int_{\mathbb{R}^2}\left[\frac{(x-x')}{|x-x'|^2}\otimes {}^\perp j[\tilde{F}] - \frac{{}^\perp(x-x')}{|x-x'|^2}\otimes j[\tilde{F}]\right]\,\mathrm{d}x'\cdot\nabla_{\tilde{V}}\tilde{F}(x,\mathcal{R}(\omega_c s)\tilde{v})$$
$$+\frac{\cos(2\omega_c s)}{4\omega_c}\left[\mathcal{R}(\pi/2)\partial_x\left(\frac{E[\tilde{F}]}{B}\right)+\partial_x\left(\frac{E[\tilde{F}]}{B}\right)\mathcal{R}(-\pi/2)\right] : {}^\perp\nabla_{\tilde{V}}\tilde{F}\otimes\mathcal{R}(\omega_c s)\tilde{v}$$
$$-\frac{\sin(2\omega_c s)}{4\omega_c}\left[\partial_x\left(\frac{E[\tilde{F}]}{B}\right)-\mathcal{R}(\pi/2)\partial_x\left(\frac{E[\tilde{F}]}{B}\right)\mathcal{R}(-\pi/2)\right] : {}^\perp\nabla_{\tilde{V}}\tilde{F}\otimes\mathcal{R}(\omega_c s)\tilde{v}.$$

Proof. The particle density $\tilde{f}$ satisfies the constraint $(\partial_s + b\cdot\nabla_{x,\tilde{v}})\tilde{f}=0$ and therefore we have

$$\tilde{f}(s,x,\tilde{v}) = \tilde{f}(0,(\mathcal{X},\tilde{\mathcal{V}})(-s;x,\tilde{v})),\quad (s,x,\tilde{v})\in\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2.$$

Therefore there is a function $\tilde{F}\in C^1_c(\mathbb{R}^2\times\mathbb{R}^2)$ such that

$$\tilde{f}(s,x,\tilde{v}) = \tilde{F}(x,\mathcal{R}(\omega_c s)\tilde{v}),\quad (s,x,\tilde{v})\in\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2.$$

Observe that $\rho[\tilde{f}(s)]=\rho[\tilde{F}]$, $j[\tilde{f}(s)]=\mathcal{R}(-\omega_c s)j[\tilde{F}]$ and

$$\nabla_{\tilde{v}}\tilde{f}(s,x,\tilde{v}) = \mathcal{R}(-\omega_c s)\nabla_{\tilde{V}}\tilde{F}(x,\mathcal{R}(\omega_c s)\tilde{v}),\quad (s,x,\tilde{v})\in\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2.$$

Notice that

$$\frac{1}{2\pi\varepsilon_0 B}\mathrm{div}_x\int_{\mathbb{R}^2}\frac{{}^\perp(x-x')}{|x-x'|^2}\otimes j[\tilde{f}(s)](x')\,\mathrm{d}x'\cdot\nabla_{\tilde{v}}\tilde{f}(s) - \frac{{}^\perp j[\tilde{f}(s)]}{2\varepsilon_0 B}\cdot\nabla_{\tilde{v}}\tilde{f}(s) \tag{20}$$
$$= \frac{\mathrm{div}_x}{2\pi\varepsilon_0 B}\int_{\mathbb{R}^2}\mathcal{R}(\omega_c s)\frac{{}^\perp(x-x')}{|x-x'|^2}\otimes\mathcal{R}(-\omega_c s)j[\tilde{F}](x')\,\mathrm{d}x'\cdot(\nabla_{\tilde{V}}\tilde{F})(x,\mathcal{R}(\omega_c s)\tilde{v})$$
$$-\frac{{}^\perp j[\tilde{F}](x)}{2\varepsilon_0 B}\cdot(\nabla_{\tilde{V}}\tilde{F})(x,\mathcal{R}(\omega_c s)\tilde{v}).$$

As already observed in (13), the average with respect to s of

$$\frac{1}{2\pi\varepsilon_0 B}\mathrm{div}_x\int_{\mathbb{R}^2}\mathcal{R}(\omega_c s)\frac{{}^\perp(x-x')}{|x-x'|^2}\otimes\mathcal{R}(-\omega_c s)j[\tilde{F}](x')\,\mathrm{d}x'$$

coincides with ${}^\perp j[\tilde{F}]/(2\varepsilon_0 B)$, and therefore we have

$$\frac{\mathrm{div}_x}{2\pi\varepsilon_0 B}\int_{\mathbb{R}^2}\mathcal{R}(\omega_c s)\frac{{}^\perp(x-x')}{|x-x'|^2}\otimes\mathcal{R}(-\omega_c s)j[\tilde{F}](x')\,\mathrm{d}x' - \frac{{}^\perp j[\tilde{F}](x)}{2\varepsilon_0 B}$$
$$=\frac{\mathrm{div}_x}{2\pi\varepsilon_0 B}\int_{\mathbb{R}^2}\left[\cos^2(\omega_c s)-\frac{1}{2}\right]\frac{{}^\perp(x-x')}{|x-x'|^2}\otimes j[\tilde{F}] + \left[\sin^2(\omega_c s)-\frac{1}{2}\right]\frac{x-x'}{|x-x'|^2}\otimes{}^\perp j[\tilde{F}]\,\mathrm{d}x'$$
$$+\frac{\mathrm{div}_x}{2\pi\varepsilon_0 B}\int_{\mathbb{R}^2}\cos(\omega_c s)\sin(\omega_c s)\left[\frac{{}^\perp(x-x')}{|x-x'|^2}\otimes{}^\perp j[\tilde{F}]+\frac{(x-x')}{|x-x'|^2}\otimes j[\tilde{F}]\right]\,\mathrm{d}x'$$

whose zero average primitive with respect to s is

$$\frac{\sin(2\omega_c s)}{8\omega_c\pi\varepsilon_0 B}\mathrm{div}_x\int_{\mathbb{R}^2}\left[\frac{{}^\perp(x-x')}{|x-x'|^2}\otimes j[\tilde F]-\frac{x-x'}{|x-x'|^2}\otimes {}^\perp j[\tilde F]\right]\,\mathrm{d}x'$$
$$-\frac{\cos(2\omega_c s)}{8\omega_c\pi\varepsilon_0 B}\mathrm{div}_x\int_{\mathbb{R}^2}\left[\frac{{}^\perp(x-x')}{|x-x'|^2}\otimes {}^\perp j[\tilde F]+\frac{x-x'}{|x-x'|^2}\otimes j[\tilde F]\right]\,\mathrm{d}x'.$$

Coming back to (20), and taking into account that $\nabla_{\tilde V}\tilde F(x,\mathcal{R}(\omega_c s)\tilde v)$ belongs to the kernel of $\partial_s+b\cdot\nabla_{x,\tilde v}$, since it depends only on the invariants $x,\mathcal{R}(\omega_c s)\tilde v$ of $\partial_s+b\cdot\nabla_{x,\tilde v}$, it is easily seen that

$$\frac{\mathrm{div}_x}{2\pi\varepsilon_0 B}\int_{\mathbb{R}^2}\frac{{}^\perp(x-x')}{|x-x'|^2}\otimes j[\tilde f(s)]\,\mathrm{d}x'\cdot\nabla_{\tilde v}\tilde f(s)-\frac{{}^\perp j[\tilde f(s)]}{2\varepsilon_0 B}\cdot\nabla_{\tilde v}\tilde f(s)+(\partial_s+b\cdot\nabla_{x,\tilde v})\tilde f_I^2=0$$

where

$$\tilde f_I^2=\frac{\cos(2\omega_c s)}{8\omega_c\pi\varepsilon_0 B}\mathrm{div}_x\int_{\mathbb{R}^2}\left[\frac{{}^\perp(x-x')}{|x-x'|^2}\otimes {}^\perp j[\tilde F]+\frac{x-x'}{|x-x'|^2}\otimes j[\tilde F]\right]\,\mathrm{d}x'\cdot\nabla_{\tilde V}\tilde F(x,\mathcal{R}(\omega_c s)\tilde v)$$
$$+\frac{\sin(2\omega_c s)}{8\omega_c\pi\varepsilon_0 B}\mathrm{div}_x\int_{\mathbb{R}^2}\left[\frac{(x-x')}{|x-x'|^2}\otimes {}^\perp j[\tilde F]-\frac{{}^\perp(x-x')}{|x-x'|^2}\otimes j[\tilde F]\right]\,\mathrm{d}x'\cdot\nabla_{\tilde V}\tilde F(x,\mathcal{R}(\omega_c s)\tilde v). \tag{21}$$

Similarly we obtain

$$-\partial_x\left(\frac{{}^\perp E[\tilde f(s)]}{B}\right)\tilde v\cdot\nabla_{\tilde v}\tilde f+\frac{\rho[\tilde f(s)]}{2\varepsilon_0 B}\,{}^\perp\tilde v\cdot\nabla_{\tilde v}\tilde f+(\partial_s+b\cdot\nabla_{x,\tilde v})\tilde f_{II}^2=0$$

where

$$\tilde f_{II}^2=\frac{\cos(2\omega_c s)}{4\omega_c}\left[\mathcal{R}(\pi/2)\partial_x\left(\frac{E[\tilde F]}{B}\right)+\partial_x\left(\frac{E[\tilde F]}{B}\right)\mathcal{R}(-\pi/2)\right]:{}^\perp\nabla_{\tilde V}\tilde F\otimes\mathcal{R}(\omega_c s)\tilde v$$
$$-\frac{\sin(2\omega_c s)}{4\omega_c}\left[\partial_x\left(\frac{E[\tilde F]}{B}\right)-\mathcal{R}(\pi/2)\partial_x\left(\frac{E[\tilde F]}{B}\right)\mathcal{R}(-\pi/2)\right]:{}^\perp\nabla_{\tilde V}\tilde F\otimes\mathcal{R}(\omega_c s)\tilde v. \tag{22}$$

Our conclusion follows by combining (21), (22). □

Remark 4.1 *Notice that $\tilde f^2=\tilde f^2(s,x,\tilde v)$ is continuous, $S=\frac{2\pi}{|\omega_c|}$ periodic and uniformly compactly supported in $(x,\tilde v)$ with respect to s. Moreover, if $\tilde f\in C^k(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2)$ for some integer $k\geq 2$, then $\tilde f^2\in C^{k-1}(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2)$.*

We appeal to the application $T^\varepsilon(x,\tilde v)=(x+\varepsilon\,{}^\perp\tilde v/\omega_c,\tilde v)$. We have

$$\partial T^\varepsilon b^\varepsilon=b\circ T^\varepsilon,\;\; b\cdot\nabla_{x,\tilde v}=\omega_c\,{}^\perp\tilde v\cdot\nabla_{\tilde v},\;\; b^\varepsilon\cdot\nabla_{x,\tilde v}=\varepsilon\tilde v\cdot\nabla_x+\omega_c\,{}^\perp\tilde v\cdot\nabla_{\tilde v}. \tag{23}$$

Finally we are ready to prove that $\overline{f}(t)\circ T^\varepsilon$, where $\overline{f}(t,x,\tilde v)=\tilde F(t,x,\mathcal{R}(\omega_c t/\varepsilon)\tilde v)$, is a second order approximation of $\tilde f^\varepsilon(t,x,\tilde v)$.

Proof. (of Theorem 1.1)
Estimating the error between $\tilde{f}^\varepsilon(t), \overline{f}(t)\circ T^\varepsilon$ is not of all obvious. A direct comparison between the above densities does not lead to the desired error estimate. We have to use a corrector term. More exactly, for any $t\in\mathbb{R}_+$, the particle density $(s,x,\tilde{v})\to \tilde{f}(t,s,x,\tilde{v})=\tilde{F}(t,(\mathfrak{X},\tilde{\mathcal{V}})(-s;x,\tilde{v}))$ belongs to $C^2(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2)$, is $S=\frac{2\pi}{|\omega_c|}$ periodic with respect to s and uniformly compactly supported in $(x,\tilde{v})$, with respect to $s\in\mathbb{R}$ and $t\in[0,T], T\in\mathbb{R}_+$. By definition, for any $t\in\mathbb{R}_+$, the particle density $\tilde{f}(t)$ satisfies the constraint $(\partial_s+b\cdot\nabla_{x,\tilde{v}})\tilde{f}(t)=0$. Thanks to Proposition 4.1, there is $\tilde{f}^2(t,s,x,\tilde{v})$ such that

$$a[\tilde{f}(t,s)]\cdot\nabla_{x,\tilde{v}}\tilde{f}-\langle a\rangle\,[\tilde{f}(t,s)]\cdot\nabla_{x,\tilde{v}}\tilde{f}+(\partial_s+b\cdot\nabla_{x,\tilde{v}})\tilde{f}^2=0. \tag{24}$$

It is easily seen, by the explicit formula of $\tilde{f}^2$, that the corrector $\tilde{f}^2$ belongs to $C^1(\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}^2\times\mathbb{R}^2)$, is $S=\frac{2\pi}{|\omega_c|}$ periodic with respect to s, and uniformly compactly supported in $(x,\tilde{v})$ with respect to $s\in\mathbb{R}$ and $t\in[0,T], T\in\mathbb{R}_+$. Taking $s=t/\varepsilon$ in (24), multiplying by ε and combining with (18) yield

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\overline{f}(t)+\varepsilon^2\tilde{f}^2(t,t/\varepsilon)\right\}+\varepsilon a[\overline{f}(t)]\cdot\nabla_{x,\tilde{v}}\overline{f}(t)+\frac{b}{\varepsilon}\cdot\nabla_{x,\tilde{v}}\left\{\overline{f}(t)+\varepsilon^2\tilde{f}^2(t,t/\varepsilon)\right\}\\ =\varepsilon^2\partial_t\tilde{f}^2(t,t/\varepsilon,x,\tilde{v}).\end{aligned}$$

Using the corrector $\tilde{f}^2$ led to a model similar to (7), nevertheless the transport operators appearing in the terms $\frac{b}{\varepsilon}\cdot\nabla_{x,\tilde{v}}\left\{\overline{f}(t)+\varepsilon^2\tilde{f}^2(t,t/\varepsilon)\right\}$, $\frac{b^\varepsilon}{\varepsilon}\cdot\nabla_{x,\tilde{v}}\tilde{f}^\varepsilon$ are different, and the key point is to take advantage of the map T^ε. After composition with T^ε, the above equation becomes, thanks to (23)

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\overline{f}(t)\circ T^\varepsilon+\varepsilon^2\tilde{f}^2(t,t/\varepsilon)\circ T^\varepsilon\right\}+\varepsilon a[\overline{f}(t)]\circ T^\varepsilon\cdot(\nabla\overline{f}(t))\circ T^\varepsilon\\ &\quad+\frac{b^\varepsilon}{\varepsilon}\cdot\nabla\left\{\overline{f}(t)\circ T^\varepsilon+\varepsilon^2\tilde{f}^2(t,t/\varepsilon)\circ T^\varepsilon\right\}=\varepsilon^2\partial_t\{\tilde{f}^2(t,t/\varepsilon)\circ T^\varepsilon\}.\end{aligned}$$

Combining with (7) and using the notation $r^\varepsilon(t)=\tilde{f}^\varepsilon(t)-\overline{f}(t)\circ T^\varepsilon-\varepsilon^2\tilde{f}^2(t,t/\varepsilon)\circ T^\varepsilon$, lead to

$$\partial_t r^\varepsilon+\varepsilon a^\varepsilon[\tilde{f}^\varepsilon(t)]\cdot\nabla_{x,\tilde{v}}\tilde{f}^\varepsilon-\varepsilon(a[\overline{f}(t)]\cdot\nabla\overline{f}(t))\circ T^\varepsilon+\frac{b^\varepsilon}{\varepsilon}\cdot\nabla_{x,\tilde{v}}r^\varepsilon=-\varepsilon^2\partial_t\tilde{f}^2(t,t/\varepsilon)\circ T^\varepsilon. \tag{25}$$

As the magnetic field is uniform, the vector fields $a^\varepsilon[\tilde{f}^\varepsilon], b^\varepsilon$ are divergence free and therefore

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\varepsilon a^\varepsilon[\tilde{f}^\varepsilon(t)]\cdot\nabla_{x,\tilde{v}}\tilde{f}^\varepsilon r^\varepsilon\,\mathrm{d}\tilde{v}\mathrm{d}x=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\varepsilon a^\varepsilon[\tilde{f}^\varepsilon(t)]\cdot\nabla(\overline{f}(t)\circ T^\varepsilon+\varepsilon^2\tilde{f}^2(t,t/\varepsilon)\circ T^\varepsilon)r^\varepsilon\,\mathrm{d}\tilde{v}\mathrm{d}x$$

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{b^\varepsilon}{\varepsilon}\cdot\nabla_{x,\tilde{v}}r^\varepsilon\,r^\varepsilon\,\mathrm{d}\tilde{v}\mathrm{d}x=0.$$

Multiplying (25) by $r^\varepsilon(t,x,\tilde v)$ and integrating with respect to $(x,\tilde v)$ imply

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|r^\varepsilon(t)\|_{L^2}^2+\varepsilon\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}a^\varepsilon[\tilde f^\varepsilon(t)]\cdot\nabla(\overline{f}(t)\circ T^\varepsilon)r^\varepsilon\,\mathrm{d}\tilde v\mathrm{d}x\\
&\quad-\varepsilon\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}(a[\overline{f}(t)]\cdot\nabla\overline{f}(t))\circ T^\varepsilon r^\varepsilon\,\mathrm{d}\tilde v\mathrm{d}x\\
&\quad=-\varepsilon^2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\partial_t\tilde f^2(t,t/\varepsilon)\circ T^\varepsilon r^\varepsilon\,\mathrm{d}\tilde v\mathrm{d}x-\varepsilon^3\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}a^\varepsilon[\tilde f^\varepsilon(t)]\cdot\nabla(\tilde f^2(t,t/\varepsilon)\circ T^\varepsilon)r^\varepsilon\,\mathrm{d}\tilde v\mathrm{d}x
\end{aligned}$$

and by Bellman lemma one gets

$$\begin{aligned}
\|r^\varepsilon(t)\|_{L^2}\le\|r^\varepsilon(0)\|_{L^2}&+\varepsilon\int_0^t\|a^\varepsilon[\tilde f^\varepsilon(t')]\cdot\nabla(\overline{f}(t')\circ T^\varepsilon)-(a[\overline{f}(t')]\cdot\nabla\overline{f}(t'))\circ T^\varepsilon\|_{L^2}\,\mathrm{d}t'\\
&+\varepsilon^2\int_0^t\|\partial_t\tilde f^2(t',t'/\varepsilon)\circ T^\varepsilon\|_{L^2}\,\mathrm{d}t'+\varepsilon^3\int_0^t\|a^\varepsilon[\tilde f^\varepsilon(t')]\cdot\nabla(\tilde f^2(t',t'/\varepsilon)\circ T^\varepsilon)\|_{L^2}\,\mathrm{d}t'.
\end{aligned}\tag{26}$$

We are working for $t\in[0,T],T\in\mathbb{R}_+$, and we denote by C any constant depending on m,ε_0,q,T,B and the initial particle density f_{in}, but not on ε. Thanks to (19) we have

$$\|r^\varepsilon(0)\|_{L^2}\le\|\tilde f^\varepsilon(0)-\overline{f}(0)\circ T^\varepsilon\|_{L^2}+\varepsilon^2\|\tilde f^2(0,0)\circ T^\varepsilon\|_{L^2}$$

and thus clearly $\|r^\varepsilon(0)\|_{L^2}\le C\varepsilon^2$. Using the C^1_c regularity of $\tilde f^2$ which comes from the C^2_c regularity of $\tilde F$, it is straightforward that

$$\varepsilon^2\int_0^T\|\partial_t\tilde f^2(t,t/\varepsilon)\circ T^\varepsilon\|_{L^2}\,\mathrm{d}t'+\varepsilon^3\int_0^T\|a^\varepsilon[\tilde f^\varepsilon(t)]\cdot\nabla(\tilde f^2(t,t/\varepsilon)\circ T^\varepsilon)\|_{L^2}\,\mathrm{d}t\le C\varepsilon^2.$$

We claim that

$$\begin{aligned}
\varepsilon\int_0^t\|a^\varepsilon[\tilde f^\varepsilon(t')]\cdot\nabla(\overline{f}(t')\circ T^\varepsilon)-(a[\overline{f}(t')]\cdot\nabla\overline{f}(t'))\circ T^\varepsilon\|_{L^2}\,\mathrm{d}t'\le C\varepsilon^2\\
+C\varepsilon\int_0^t\|\tilde f^\varepsilon(t')-\overline{f}(t')\circ T^\varepsilon\|_{L^2},\quad t\in[0,T].
\end{aligned}\tag{27}$$

This part of the proof relies on the smoothness and the uniform estimates of the densities $(\tilde f^\varepsilon)_{\varepsilon>0}$, together with the elliptic regularity results. Thanks to the uniform bounds

$$\sup_{\varepsilon>0,t\in[0,T]}\{\|\tilde f^\varepsilon(t)\|_{C^1(\mathbb{R}^2\times R^2)}+\|E[\tilde f^\varepsilon(t)]\|_{C^1(\mathbb{R}^2)}\}<+\infty$$

it is easily seen that

$$\|(a^\varepsilon[\tilde f^\varepsilon(t)]-a[\tilde f^\varepsilon(t)])\cdot\nabla(\overline{f}(t)\circ T^\varepsilon)\|_{L^2}\le C\varepsilon,\quad t\in[0,T],\quad\varepsilon>0.$$

Thanks to elliptic regularity results, the quantity

$$\|(\,a[\tilde f^\varepsilon(t)]-a[\overline{f}(t)\circ T^\varepsilon]\,)\cdot\nabla(\overline{f}(t)\circ T^\varepsilon)\|_{L^2}$$

is bounded by the L^2 norms of the charge and current densities

$$\|\rho[\tilde f^\varepsilon(t)]-\rho[\overline{f}(t)\circ T^\varepsilon]\|_{L^2(\mathbb{R}^2)}+\|j[\tilde f^\varepsilon(t)]-j[\overline{f}(t)\circ T^\varepsilon]\|_{L^2(\mathbb{R}^2)}$$

and thus by the L^2 norm of the particle densities $\|\tilde{f}^\varepsilon(t)-\overline{f}(t)\circ T^\varepsilon\|_{L^2(\mathbb{R}^2\times\mathbb{R}^2)}$. We obtain the inequality

$$\|(a[\tilde{f}^\varepsilon(t)]-a[\overline{f}(t)\circ T^\varepsilon])\cdot\nabla(\overline{f}(t)\circ T^\varepsilon)\|_{L^2}\leq C\|\tilde{f}^\varepsilon(t)-\overline{f}(t)\circ T^\varepsilon\|_{L^2(\mathbb{R}^2\times\mathbb{R}^2)},\quad t\in[0,T],\quad \varepsilon>0.$$

The inequality (27) follows immediately, noticing that

$$\|a[\overline{f}(t)\circ T^\varepsilon]\cdot\nabla(\overline{f}(t)\circ T^\varepsilon)-(a[\overline{f}(t)]\cdot\nabla\overline{f}(t))\circ T^\varepsilon\|_{L^2(\mathbb{R}^2\times\mathbb{R}^2)}\leq C\varepsilon,\quad t\in[0,T],\quad \varepsilon>0.$$

Finally (26) writes

$$\|r^\varepsilon(t)\|_{L^2}\leq C\varepsilon^2+C\varepsilon\int_0^t\|\tilde{f}^\varepsilon(t')-\overline{f}(t')\circ T^\varepsilon\|_{L^2}\,\mathrm{d}t'$$

implying that

$$\begin{aligned}\|\tilde{f}^\varepsilon(t)-\overline{f}(t)\circ T^\varepsilon\|_{L^2}&\leq\|r^\varepsilon(t)\|_{L^2}+\varepsilon^2\|\tilde{f}^2(t,t/\varepsilon)\circ T^\varepsilon\|_{L^2}\\&\leq C\varepsilon^2+C\varepsilon\int_0^t\|\tilde{f}^\varepsilon(t')-\overline{f}(t')\circ T^\varepsilon\|_{L^2}\,\mathrm{d}t',\quad t\in[0,T],\quad\varepsilon>0.\end{aligned}$$

By Gronwall lemma we deduce

$$\|\tilde{f}^\varepsilon(t)-\overline{f}(t)\circ T^\varepsilon\|_{L^2}\leq C\varepsilon^2\exp(C\varepsilon t),\quad t\in[0,T],\quad\varepsilon>0.$$

Clearly we have

$$\begin{aligned}\|E[f^\varepsilon(t)]-E[\tilde{F}(t)]\|_{L^2}=\|E[\tilde{f}^\varepsilon(t)]-E[\overline{f}(t)]\|_{L^2}&\leq\|E[\tilde{f}^\varepsilon(t)]-E[\overline{f}(t)\circ T^\varepsilon]\|_{L^2}\\&+\|E[\overline{f}(t)\circ T^\varepsilon]-E[\overline{f}(t)]\|_{L^2}\leq C\varepsilon\end{aligned}$$

and therefore

$$\begin{aligned}&\left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[\tilde{F}\left(t,x+\varepsilon\frac{{}^\perp v}{\omega_c},\mathcal{R}(\omega_c t/\varepsilon)\left(v-\varepsilon\frac{{}^\perp E[\tilde{F}(t)]}{B}\right)\right)-f^\varepsilon(t,x,v)\right]^2\mathrm{d}v\mathrm{d}x\right\}^{1/2}\\&=\left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[\tilde{F}\left(t,x+\frac{\varepsilon}{\omega_c}\left({}^\perp\tilde{v}-\varepsilon\frac{E[f^\varepsilon(t)]}{B}\right),\mathcal{R}(\omega_c t/\varepsilon)\left(\tilde{v}+\varepsilon\frac{{}^\perp E[f^\varepsilon(t)]-{}^\perp E[\tilde{F}(t)]}{B}\right)\right)\right.\right.\\&\left.\left.-\tilde{f}^\varepsilon(t,x,\tilde{v})\right]^2\mathrm{d}\tilde{v}\mathrm{d}x\right\}^{1/2}\\&\leq\|\tilde{f}^\varepsilon(t)-\overline{f}(t)\circ T^\varepsilon\|_{L^2}+\left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[\tilde{F}\left(t,x+\varepsilon\frac{{}^\perp\tilde{v}}{\omega_c},\mathcal{R}(\omega_c t/\varepsilon)\tilde{v}\right)-\right.\right.\\&\left.\left.\tilde{F}\left(t,x+\varepsilon\frac{{}^\perp\tilde{v}}{\omega_c}-\frac{\varepsilon^2}{\omega_c}\frac{E[f^\varepsilon(t)]}{B},\mathcal{R}(\omega_c t/\varepsilon)\tilde{v}+\varepsilon\mathcal{R}(\omega_c t/\varepsilon)\frac{{}^\perp E[f^\varepsilon(t)]-{}^\perp E[\tilde{F}(t)]}{B}\right)\right]^2\mathrm{d}\tilde{v}\mathrm{d}x\right\}^{1/2}\\&\leq C\varepsilon^2+C\varepsilon\|E[f^\varepsilon(t)]-E[\tilde{F}(t)]\|_{L^2}\leq C\varepsilon^2,\quad t\in[0,T],\quad\varepsilon>0.\end{aligned}$$

□

5 Conclusions

We presented regular reformulations for the Vlasov-Poisson equations with uniform magnetic fields and general initial conditions. The effective model comes by averaging over one cyclotronic period, once we have determined a periodic fast dynamics. Certainly, in the framework of the magnetic confinement, a much more interesting case is that of curved magnetic fields.

The above results extend to the three dimensional Vlasov-Poisson system, with strong external curved magnetic field. The same arguments lead to regular reformulations and second order estimates for both well prepared and general initial particle densities. In the three dimensional setting, we emphasize a fast periodic dynamics leaving invariant not only the guiding center and the modulus of the perpendicular velocity, but also the parallel velocity. Averaging over one period allows us to obtain completely explicit effective models. Nevertheless the analysis is much more elaborated due to the combination between the parallel and perpendicular dynamics and to the curvature effects. These studies will be the topic of future works [10].

Another interesting issue will be to handle models with self-consistent magnetic field, that is the Vlasov-Maxwell equations, perturbed by a strong external magnetic field.

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