Mathematical comparison of three alternative laws for linear viscosity

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SEMINAIRE D'ANALYSE CONVEXE

Montpellier 1979, Exposé n° 8

MATHEMATICAL COMPARISON OF THREE ALTERNATIVE LAWS FOR LINEAR VISCOSITY

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1. INTRODUCTION

This paper is restricted to incompressible fluids. Let u_i, $i = 1,2,3$, denote the components of the fluid velocity at the point $\vec{x}$, relative to some inertial, orthonormal frame and let $u_{i,j}$ denote the partial derivative of u_i with respect to the coordinate x_j.

As far as viscosity may be described, at each point, by a stress tensor V_{ij} depending linearly on the local value of the tensor $u_{i,j}$, the following relation, for a materially isotropic fluid, is universally accepted by mechanists

(1.1) $$V_{ij} = \mu(u_{i,j} + u_{j,i})$$

with $\mu \geq 0$, the viscosity coefficient. This tensor, to be added to the "hydrostatic" stress $-p\,\delta_{ij}$ in order to obtain the total stress of the fluid, meets two essential requirements :

(i) it is symmetrical

(ii) it vanishes everywhere if and only if the velocity field corresponds to a rigid motion of the fluid.

Reçu le 3 mai 1979

Let a portion of the fluid be limited, at the considered instant, by a smooth surface S and let $\vec{\nu}$ denote the normal unit vector at the generic point of S, directed toward the considered portion ; the distribution of forces exerted along this surface by the fluid upon the material beyond admits the surface density $\vec{T}(\vec{\nu})$, related in the conventional way to the total stress tensor. When (1.1) is taken into account this yields

$$(1.2) \qquad \vec{T}(\vec{\nu}) = - p\,\vec{\nu} + \mu \frac{d\vec{u}}{d\vec{\nu}} + \mu\, \overrightarrow{\mathrm{grad}}\; \vec{u}.\vec{\nu}$$

where the last term is understood as the gradient of the scalar function $\vec{x} \to \vec{u}(\vec{x}).\vec{\nu}$, with $\vec{\nu}$ treated as a constant.

Disregarding what precedes, some textbooks of engineering or elementary physics start with the simple example of a parallel shear flow ; from this example, they induce as general, instead of (1.2), the following expression for the tension on S

$$(1.3) \qquad \vec{T}'(\vec{\nu}) = - p\,\vec{\nu} + \mu \frac{d\vec{u}}{d\vec{\nu}} \quad .$$

This would involve, for the viscosity stress tensor, the expression

$$(1.4) \qquad V'_{ij} = \mu\, u_{i,j} \quad ,$$

inacceptable by mechanists as it does not meet the requirements (i) and (ii) above. However, this incorrect law of viscosity is immediately found to generate the same Navier - Stokes equations as the correct one. And it has the mathematical advantage of leading to simpler calculation ; on the other hand, its formal analogy with the law of heat conduction may be suggestive.

The purpose of this paper is to discuss some topics regarding to which the replacement of (1.2) by (1.3) turns out to be immaterial.

Meanwhile, another incorrect law of viscosity will be considered ; in fact the identity

(1.5) $$(\text{curl } \vec{u}) \times \vec{\nu} = \frac{d\vec{u}}{d\vec{\nu}} - \overrightarrow{\text{grad}}\ \vec{u}.\vec{\nu}$$

shows that the expression

(1.6) $$\vec{T}''(\vec{\nu}) = - p\,\vec{\nu} + \mu(\overrightarrow{\text{curl}}\ \vec{u}) \times \vec{\nu}$$

for the tension on the surface S is equivalent to (1.2) and (1.3) in the aforesaid case of a parallel shear flow orthogonal to $\vec{\nu}$. If this expression was accepted as general, the corresponding viscosity stress tensor would be

(1.7) $$V''_{ij} = \mu(u_{i,j} - u_{j,i})$$

intrinsically inacceptable by mechanists but leading also to the correct Navier - Stokes equations.

Since the three alternative expressions (1.1), (1.4) and (1.7) for the viscosity stress yield the same Navier - Stokes equations, every boundary problem concerning the fluid, as far as the data of this problem do not involve explicitely the tension on any surface, will admit the same solutions $(\vec{u},p)$ whichever of the three expressions is adopted. Such is the problem of determining the flow in the presence of boundaries with given motion ; this is no more the case on the contrary if some free surface is present.

In view of (1.5), the comparison between $\vec{T}$, $\vec{T}'$ and $\vec{T}''$ sums up into :

(1.8) $$\vec{T}(\nu) = \vec{T}'(\nu) + \mu\ \overrightarrow{\text{grad}}\ (\vec{u}.\vec{\nu})$$

(1.9) $$\vec{T}''(\nu) = \vec{T}'(\nu) - \mu\ \overrightarrow{\text{grad}}\ (\vec{u}.\vec{\nu}) \quad .$$

This yields in particular a known expression of the correct tension

$$\vec{T}(\nu) = - p\, \vec{\nu} + 2\, \mu\, \frac{d\vec{u}}{d\vec{\nu}} + \mu\, \vec{\nu} \times \overrightarrow{\mathrm{curl}}\, \vec{u} \tag{1.10}$$

(cf. R. Berker [1], [2]).

2. RESULTS OF THE PAPER

The following is established in Sect. 3 :

PROPOSITION 1. Let S denote a closed bounded and orientable surface with piecewise continuous normal. Let the velocity field $\vec{u}$ and its first order derivatives be defined and continuous at least in a one-side neighborhood of S, with limits at every regular point of this surface. Then the three expressions $\vec{T}(\vec{\nu})$, $\vec{T}'(\vec{\nu})$ and $\vec{T}''(\vec{\nu})$ of the tension yield the same total on S ; concerning the respective values of the total moment of the tension on S, one has

$$\begin{aligned} \iint_S \vec{x} \times \vec{T}(\vec{\nu})\, d\sigma - \iint_S \vec{x} \times \vec{T}'(\vec{\nu})\, d\sigma \\ = \iint_S \vec{x} \times \vec{T}''(\vec{\nu})\, d\sigma - \iint_S \vec{x} \times T'(\vec{\nu})\, d\sigma \\ = \mu \iint_S \vec{u} \times \vec{\nu}\, d\sigma \quad . \end{aligned} \tag{2.1}$$

Observe that S is not supposed to constitute the boundary of a domain in which $\vec{u}$ would have to be everywhere defined.

Sect. 4 is devoted to the proof of :

PROPOSITION 2. Let S be a smooth surface portion, with the same regularity assumptions for $\vec{u}$ as in the above proposition. If the limit of $\vec{u}$ is the same at every point of S (for instance, S is the boundary of a translating immersed body) , then $\vec{T}(\vec{\nu}) = \vec{T}'(\nu) = \vec{T}''(\vec{\nu})$

<u>at every regular point of</u> S.

The equality $\vec{T} = \vec{T}''$ under such circumstances has already been pointed out by R. Berker [1], [2].

Sect. 5 derives the expressions of the <u>power dissipated by viscosity</u> in a fluid portion D when the three alternative expressions V, V', V" of the viscous stress are considered ; one respectively obtains

(2.2) $$\mathcal{D}(u) = 2\mu \iiint_D e_{ij}\, e_{ij}\, d\tau \quad ,$$

which is the classical result, with $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, and

(2.3) $$\mathcal{D}'(u) = \mu \iiint_D u_{i,j}\, u_{j,i}\, d\tau$$

(2.4) $$\mathcal{D}''(u) = \mu \iiint_D (\overrightarrow{\mathrm{curl}}\ \vec{u})^2\, d\tau \quad .$$

In the special case where $\vec{u}$ vanishes on the boundary of D these three expressions are equal, a fact already observed by J. Serrin [8] ; the proof of this is similar to the derivation of our equ. (6.5).

In Sect. 6 it is established that these three expressions may indifferently be used in the variational characterization of the solution of the classical boundary value problem for Stokes flows, possibly in unbounded domains.

As an example, Sect. 7 applies what precedes to an optimal design problem recently studied [3], [7] : one considers the Stokes flow around a fixed rigid body B, with velocity $\vec{V}$ at infinity ; the volume of B being prescribed, the problem is that of determining the shape of this body in order to minimize the drag or, equivalently, to minimize the power dissipated in the fluid by viscosity. O. Pironneau [7] has established the following characterization of such a shape :

in the corresponding flow, the scalar $|\frac{d\vec{u}}{d\vec{\nu}}|$ has a constant value over the boundary surface. We show that similar conditions can be derived in the general framework of boundary problems whose solutions extremize some functionals. In view of the results of Sect. 6 it is found that Pironneau's condition may equivalently be replaced by one of the following : i) The norm of the vector $\mathrm{rot}\,\vec{u}$ is constant over the boundary ; ii) The norm of the tensor e_{ij} is constant over the boundary.

3. RESULTANT OF THE TENSION ON A CLOSED SURFACE

Let S denote a bounded closed surface as in the statement of Proposition 1. Let f be a real function, continuous as well as its partial derivatives $f_{,i}$ at least in a one-side neighborhood of S, with limits at every point of this surface. Then, denoting by $d\sigma$ the area measure on S, one has

(3.1) $$\iint_S (f_{,i}\,\nu_j - f_{,j}\,\nu_i)\, d\sigma = 0.$$

In fact, when $i \neq j$ the integrand equals one of the components of the vector $\vec{\nu} \times \overrightarrow{\mathrm{grad}}\, f$; by a well known variant of the Kelvin - Stokes theorem, the integral of this vector over a surface portion equals the line integral $\int f\, d\vec{x}$ relative to the boundary curve of the surface portion. Now, as S is closed, its boundary curve vanishes.

By applying (3.1) with $f = u_j$, $j = 1,2,3$, one obtains

$$\iint_S u_{j,i}\,\nu_j\, d\sigma = \iint_S u_{j,j}\,\nu_i\, d\sigma$$

which is zero by the incompressibility assumption. In the notations of Sect. 1, this means the nullity of $\mu \iint_S \overrightarrow{\text{grad}}\, \vec{u}.\vec{\nu}\, d\sigma$; then the first part of Proposition 1 results from (1.8) and (1.9).

Concerning the moments, the components of

$$\iint_S \vec{x} \times \overrightarrow{\text{grad}}\, \vec{u}.\vec{\nu}\, d\sigma \tag{3.2}$$

may be expressed by introducing the fundamental alternate tensor ϵ_{ijk} of the oriented Euclidean three-dimensional space. The i^{th} component of the integrand in (3.2) is $\epsilon_{ijk}\, x_j\, u_{\ell,k}\nu_\ell$; if one observes that $\epsilon_{ijk}\, x_{j,k} = \epsilon_{ijk}\, \delta_{jk} = 0$, this component becomes $\epsilon_{ijk}(x_j u_\ell)_{,k}\, \nu_\ell$, which, in view of (3.1), yields the same integral over the closed surface S as

$$\epsilon_{ijk}(x_j u_\ell)_{,\ell}\, \nu_k = \epsilon_{ijk}\, \delta_{j\ell}\, u_\ell\, \nu_k + \epsilon_{ijk}\, x_j\, u_{\ell,\ell}\, \nu_k = \epsilon_{ijk}\, u_j\, \nu_k \quad ;$$

this is the i^{th} component of $\vec{u} \times \vec{\nu}$, hence (2.1) ; this completes the proof of Proposition 1.

REMARK. Let S be the boundary of a rigid body B immersed in the fluid ; by expressing that $\vec{u}$ equals, at each point of this boundary, the velocity $\vec{v}$ of the body, one obtains

$$\iint_S \vec{u} \times \vec{\nu}\, d\sigma = \iint_S \vec{v} \times \vec{\nu}\, d\sigma = \iiint_B - \overrightarrow{\text{curl}}\, \vec{v}\, d\tau \quad . \tag{3.3}$$

Now $\overrightarrow{\text{curl}}\, \vec{v}$ is twice the angular velocity vector $\vec{\omega}$ of the rigid body ; therefore, if $\mathcal{V}$ denotes the volume of the body, the expression (3.3) equals $-2\, \mathcal{V}\, \vec{\omega}$.

4. TENSION ON A TRANSLATING BOUNDARY

In this section, let S denote a smooth surface portion at every point of which the fluid velocity $\vec{u}$ has the same limit. As before it is supposed that $\vec{u}$ is defined and continuously differentiable at least over some unilateral neighborhood of S and that its partial derivatives have limits at every point of S.

These assumptions entail that, considering a point of S, the differential

$$du_i = u_{i,j}\, dx_j$$

is zero for every $\vec{dx}$ tangent to the surface, i.e. for every (dx_1, dx_2, dx_3) satisfying $\nu_j\, dx_j = 0$. By Lagrange's multiplier theorem, this is equivalent to the existence, for the considered point of S, of $(\lambda_1, \lambda_2, \lambda_3)$ such that

$$u_{i,j} = \lambda_i\, \nu_j \quad .$$

Then the incompressibility condition yields

$$\lambda_j\, \nu_j = 0 \quad .$$

This gives to the components of $\overrightarrow{\text{grad}}\ \vec{u}.\vec{\nu}$ the form

$$u_{j,i}\, \nu_j = \lambda_j\, \nu_i\, \nu_j = 0 \quad .$$

In view of (1.8) and (1.9) this establishes Proposition 2.

5. DISSIPATED POWER

As the nonsymmetric forms of the stress tensors in (1.4) and (1.7) is not consistent with the usual framework of continuum mechanics, there is a priori some risk of contradiction in rapproaching them from the classical formulas concerning the power of such stresses. For this reason, let us first retrace the elementary rationale of the subject.

Let Θ_{ij} denote the components of the (possibly nonsymmetric) stress tensor field of a continuous medium and f_i the components of the volume density of external force. By expressing the balance of momentum for every portion of the medium and applying the divergence theorem, one obtains, if Θ_{ij} is continuously differentiable, the local equations of dynamics

(5.1) $$\rho \gamma_i = \Theta_{ij,j} + f_i \quad ,$$

where ρ denotes the local density and γ_i the components of the acceleration.

Let v_i be the components of a continuously differentiable velocity field (actual or virtual) of the medium. For every bounded portion D of the medium, with piecewise smooth boundary ∂D and $\vec{\nu}$ as normal outward unit vector, the corresponding power of the external forces acting on this portion is

(5.2) $$\begin{aligned} \mathcal{P}_{ext} &= \iiint_D f_i \, v_i \, d\tau + \iint_{\partial D} \Theta_{ij} \, \nu_j \, v_i \, d\sigma \\ &= \iiint_D [(f_i + \Theta_{ij,j}) \, v_i + \Theta_{ij} \, v_{i,j}] \, d\tau \\ &= \iiint_D (\rho \, \gamma_i \, v_i + \Theta_{ij} \, v_{i,j}) \, d\tau \quad . \end{aligned}$$

Therefore, in order to safeguard the d'Alembert "principle", or "principle of virtual power" one is compelled to accept as the definition of the power of "internal forces", or <u>power of the stress field</u> Θ_{ij} , the following

$$(5.3) \qquad \mathcal{P}_{int} = - \iiint_D \Theta_{ij} \, v_{i,j} \, d\tau \ .$$

In particular, by taking as v_i the components u_i of the actual velocity, one obtains that the total power $\mathcal{P}_{int} + \mathcal{P}_{ext}$, corresponding to the actual motion, equals the time-rate of change of kinetic energy.

Coming back to the viscous incompressible fluid, let us consider as Θ_{ij} the three alternative expressions proposed in Sect. 1. Due to the nullity of $u_{i,i}$, the "hydrostatic" part $- p \, \delta_{ij}$ of the stress has zero power in the actual motion ; then $\mathcal{P}_{int}$, for a portion D of the fluid, reduces to the contribution of the viscous stress, with the three respective expressions V_{ij} , V'_{ij} or V''_{ij} . Let us rather write down the negative of this power, also called the <u>power dissipated by viscosity</u>, respectively

$$(5.4) \qquad \mathcal{D}(\vec{u}) = \mu \iiint_D (u_{i,j} + u_{j,i}) \, u_{i,j} \, d\tau$$

$$(5.5) \qquad \mathcal{D}'(\vec{u}) = \mu \iiint_D u_{i,j} \, u_{i,j} \, d\tau$$

$$(5.6) \qquad \mathcal{D}''(\vec{u}) = \mu \iiint_D (u_{i,j} - u_{j,i}) \, u_{i,j} \, d\tau \ .$$

These are three <u>quadratic functionals</u> of the vector field $\vec{u}$ on the domain D. Clearly $\mathcal{D}'(\vec{u})$ is nonnegative ; in order to display the same property for $\mathcal{D}$ or $\mathcal{D}''$, one proceeds to some easy manipulations of indices. Classically, the components

$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ of the strain rate tensor are introduced ; this yields (2.2). Similarly (5.6) entails (2.4). Observe incidentally that $\mathcal{D} + \mathcal{D}'' = 2\,\mathcal{D}'$.

As the $u_{i,j}$ are supposed continuous, the functional $\mathcal{D}(\vec{u})$ vanishes if and only if $e = 0$ throughout D, i.e. the motion of the fluid is rigid ; this is satisfactory from the physical standpoint. On the other hand, $\mathcal{D}'(u)$ vanishes if and only if $u_{i,j} = 0$, i.e. the motion of the fluid is a translation ; and $\mathcal{D}''(\vec{u})$ vanishes if and only if the motion is irrotational.

6. THE EXTREMAL PROPERTY OF STOKES FLOWS

In all the sequel, the volume density of external force is supposed zero. Stokes flows are the motions of an incompressible viscous fluid under such circumstances that the left member in the Navier - Stokes equations

$$\rho\,\gamma_i = -p_{,i} + \mu\,u_{i,jj}$$

turns out to be negligible before the right member. The equations thus obtained

(6.1) $$\mu\,u_{i,jj} = p_{,i}$$

(6.2) $$u_{i,i} = 0$$

express that, at every instant in every part of the fluid, the viscosity stress balances the hydrostatic stress ; hydrostatic stress may be regarded as the reaction associated with the incompressibility constraint (6.2).

Let us consider the simplest boundary value problem concerning these flows : given a bounded domain D, with sufficiently smooth boundary ∂D, to find $(\vec{u},p)$ satisfying (6.1), (6.2) in D and such that $\vec{u}$ agrees on ∂D with a given velocity field $\vec{v}$. The existence of a solution and its uniqueness (up to an additive constant in what regards p) are classical [4] [9]. Every solution of (6.1), (6.2) is C^∞ in the open set D ; we restrict ourselves here to the case where ∂D and $\vec{v}$ are regular enough for the partial derivatives $u_{i,j}$ to have limits at every boundary point. Then one elementarily proves that the solution of the boundary value problem minimizes the functional $\mathfrak{D}(\vec{u})$ in the set of the vector fields possessing the above regularity and agreeing with the two constraints

(6.3) $$u_{i,i} = 0 \quad \text{in } D \;,$$

(6.4) $$\vec{u} = \vec{v} \quad \text{on } \partial D \;.$$

As our two incorrect laws of viscosity generate the same Navier-Stokes equations as the correct one, it may be expected that the solution of this boundary value problem exhibits a similar extremal property with regard to $\mathfrak{D}'(\vec{u})$ and $\mathfrak{D}''(\vec{u})$.

In fact, (5.4), (5.5) and (5.6) show that, up to the constant factor μ, $\mathfrak{D}$ and $\mathfrak{D}''$ differ from $\mathfrak{D}'$ by the term $\iiint_D u_{j,i}\, u_{i,j}\, d\tau$. When the elementary methods of the calculus of variations are applied, this term contributes to the variations of the considered functionals by

(6.5) $$\delta \iiint_D u_{j,i}\, u_{i,j}\, d\tau = 2 \iiint_D u_{i,j}\, (\delta\, u_j)_{,i}\, d\tau$$

$$= 2 \iint_{\partial D} u_{i,j}\, \delta\, u_j\, \nu_i\, d\sigma - 2 \iiint_D u_{i,ji}\, \delta\, u_j\, d\tau$$

which is zero if the variation $\delta\,\vec{u}$ preserves (6.4) and if $\vec{u}$ satisfies (6.3).

Let us turn now to the similar exterior boundary value problem. Henceforth the fluid domain D shall consist at a given instant in all the space outside a bounded body B. The flow is supposed uniform at infinity, i.e. there exists a constant vector $\vec{V}$ such that $|\vec{u}(\vec{x}) - \vec{V}|$ tends to zero when $|\vec{x}|$ tends to infinity. Precise results are available concerning the behavior of the solutions $(\vec{u},p)$ of Stokes equations under such assumptions [6] ; it is proved that, for $|\vec{x}|$ tending to infinity,

$$(6.6) \qquad |\vec{u}(\vec{x}) - \vec{V}| = 0(|\vec{x}|^{-1})$$

$$(6.7) \qquad u_{i,j} = 0(|\vec{x}|^{-2})$$

with the existence of some constant p_∞ such that

$$(6.8) \qquad p - p_\infty = 0(|\vec{x}|^{-2})$$

(for other ways of specifying the conditions at infinity, see [4]).

If in addition the motion of the immersed body is prescribed, one obtains a boundary condition of the form (6.4). Here again the existence and the uniqueness of the solution (u,p) is well known, under sufficient smoothness assumptions regarding the boundary ∂D and the velocity field $\vec{v}$ of this boundary. We restrict ourselves as before to the case where the partial derivatives $u_{i,j}$ possess limits at every boundary point ; then (6.7) guarantees that, in spite of D being unbounded, the functionals $\mathcal{D}(\vec{u})$, $\mathcal{D}'(\vec{u})$, $\mathcal{D}''(\vec{u})$ are well defined. Similarly to the case of a bounded domain, the solution $\vec{u}$ of the problem minimizes the functional $\mathcal{D}(\vec{u})$ over the set of the vector fields agreeing with (6.3), (6.4), (6.6), (6.7). On the other hand, the calculation made in (6.5) may be adapted to prove that <u>this solution minimizes</u> $\mathcal{D}'(\vec{u})$ <u>and</u> $\mathcal{D}''(\vec{u})$ <u>as well</u> : the integration by part involved in this calculation shall first be performed on some bounded domain D_R, namely the part of D enclosed in a large sphere Σ_R with radius R. As a consequence of (6.6), the variation $\delta \vec{u}$ satisfies $|\delta \vec{u}| = O(|x|^{-1})$; hence the contribution of Σ_R vanishes when R tends to infinity.

7. A PROBLEM OF OPTIMAL DESIGN CONCERNING STOKES FLOWS

Let us come back to the exterior boundary value problem formulated in last Section, but restricting ourselves to the special case of a fixed immersed body, i.e. $\vec{v} = 0$. According to Sect. 3 the resultant force $\vec{F}$ experienced by the body may be calculated from the expressions $\vec{T}$, $\vec{T}'$ or $\vec{T}''$ of the tension indifferently. The component of $\vec{F}$ along the direction of the current at infinity constitutes the drag ; up to the factor $|\vec{V}|$, this equals the scalar product $\vec{F}.\vec{V}$.

Several papers have recently been devoted to the following optimal design problem : if the velocity $\vec{V}$ and the volume $\mathcal{V}$ of the body B are prescribed, to determine the shape of D in order to minimize the drag (cf. [3], [5], [7] where references to earlier papers may be found).

One first observes that $\vec{F}.\vec{V}$ equals the power dissipated in the fluid. In [5] and [7] this fact is only inferred from the comparison with the case where $\vec{V}$, instead of beeing the limit of $\vec{u}$ at infinity, constitutes the velocity of a large vessel containing the whole system. Such an inference would call for careful investigation if the flow was regulated by the complete Navier - Stokes equations, i.e. if inertia was taken into account. But in the case of a Stokes flow, we establish easily that

$$(7.1) \qquad \vec{F}.\vec{V} = \mathcal{D}(\vec{u}) = \mathcal{D}'(\vec{u}) = \mathcal{D}''(\vec{u}) .$$

In fact, the energy balance, investigated in Sect. 5 above, may here be calculated for the domain D_R comprised between the body B and a large sphere Σ_R, with radius R. As $\vec{u} = 0$ on the boundary of B, this yields, when the correct law of viscosity is used,

$$2\mu \iiint_{D_R} e_{ij}\, e_{ij}\, d\tau = \iint_{\Sigma_R} \vec{T}(\nu) . \vec{u}\, d\sigma$$

$$= \vec{V} . \iint_{\Sigma_R} \vec{T}(\nu)\, d\sigma + \iint_{\Sigma_R} \vec{T}(\nu) . (\vec{u} - \vec{V})\, d\sigma$$

(here $\vec{\nu}$ denotes the outward normal unit vector at the generic point of Σ_R). In view of (6.6), (6.7) and (6.8), the last integral tends to zero when R tends to infinity. On the other hand, using equation (6.1) (which merely expresses that the total stress is self-equilibrated in every part of the fluid), one obtains

$$\iint_{\Sigma_R} \vec{T}(\vec{\nu})\, d\sigma = \vec{F} \quad .$$

The same reasoning holds for the two incorrect laws of viscosity, hence (7.1).

Let us finish this paper by connecting our results with the optimality conditions derived by O. Pironneau [7] for the considered optimal design problem.

More generally, we shall consider, instead of the boundary value problem for Stokes flows, the following extremal problem :

Denoting again by D the domain outside the bounded body B, let us call P(B) the problem of finding a vector field $\vec{u}$ on D, with partial derivatives $u_{i,j}$ continuous on $D \cup \partial D$, extremizing a given functional

$$I(\vec{u}) = \iiint_D L(x, u_i, u_{i,j})\, d\tau$$

in the set of the vector fields satisfying (6.6) and (6.7) at infinity as well as the two "constraints" $\vec{u} = 0$ over ∂D and $u_{i,j}$ in D. Let us denote by $\vec{u}(B)$ a solution of this problem ; the formulation of the optimal design problem is now : to determine B in order to extremize $I(\vec{u}(B))$.

Let the expected solution B_o of this optimal design problem correspond to the value $\lambda = 0$ in a family B_λ of bodies depending on some real parameter λ. Assume the surface ∂B_o of B_o smooth enough for the surface ∂B_λ of B_λ to be defined as follows, at least if λ is sufficiently close to zero : some scalar functions $\vec{x} \to h(\vec{x})$ is defined on ∂B_o ; let $\vec{\nu}(\vec{x})$ denote the inward normal unit vector at the point $\vec{x}$ of ∂B_o ; then ∂B_λ is represented as the set of the points $\vec{y}(\lambda,\vec{x})$

$$\vec{y}(\lambda,\vec{x}) = \vec{x} + \lambda\, h(\vec{x})\, \vec{\nu}(\vec{x}) \quad ,$$

for $\vec{x}$ ranging over ∂B_o.

Let D_λ denote the space outside B_λ ; let $\vec{u}^\lambda$ be a vector field defined on an open set containing the closure of D_λ. The consideration of the volume swept by the boundary $\partial D_\lambda = \partial B_\lambda$ when λ varies, in the spirit of the classical kinematics of continua, permits the following calculation

$$\frac{d}{d\lambda}\,(\text{volume } B_\lambda) = - \iint_{\partial B_o} h(\vec{x})\, d\sigma(\vec{x}) \quad , \tag{7.2}$$

$$\frac{d}{d\lambda}\, I(\vec{u}^\lambda) = \iiint_{D_\lambda} \frac{d}{d\lambda}\, L(\vec{x}\,,\, u_i^\lambda\,,\, u_{i,j}^\lambda)\, d\tau(\vec{x}) + \iint_{\partial B_\lambda} L(\vec{x}\,,\, u_i^\lambda\,,\, u_{i,j}^\lambda)\, h(\vec{x})\, d\sigma(\vec{x}) \quad . \tag{7.3}$$

After an integration by part, this yields

$$(7.4)\qquad \frac{d}{d\lambda} I(\vec{u}^{\lambda}) = \iiint_{D_\lambda} \left[\frac{\partial L}{\partial u_i} - \left(\frac{\partial L}{\partial u_{i,j}}\right)_{,j}\right] \frac{\partial u_i}{\partial \lambda}\, d\tau$$

$$+ \iint_{\partial B_\lambda} \left[L(\vec{x}, u_i^\lambda, u_{i,j}^\lambda)\, h(\vec{x}) + \frac{\partial L}{\partial u_{i,j}} \nu_j \frac{\partial u_i}{\partial \lambda}\right] d\sigma \quad .$$

If $\vec{u}^{\lambda}$ is the solution of $P(B_\lambda)$, it satisfies

$$(7.5)\qquad u_i^\lambda(\vec{x} + \lambda\, h(\vec{x})\, \vec{\nu}(\vec{x})) = 0$$

for every $\vec{x}$ in ∂B_o and every λ in the considered neighborhood of zero ; by derivation, this entails

$$(7.6)\qquad \frac{\partial u_i^\lambda}{\partial \lambda} + u_{i,k}^\lambda\, h\, \nu_k = 0 \quad .$$

On the other hand, the Euler condition for the problem $P(B_\lambda)$ makes the triple integral in (7.4) vanish. Then (7.4) becomes

$$\frac{d}{d\lambda} I(\vec{u}^{\lambda}) = \iint_{\partial B_\lambda} \left[L(\vec{x}, u_i^\lambda, u_{i,j}^\lambda) - \frac{\partial L}{\partial u_{i,j}} u_{i,k}^\lambda \nu_j \nu_k\right] h\, d\sigma \quad .$$

For B_o to be of optimal shape it is necessary that this derivative vanishes at $\lambda = 0$, for every continuous function $\vec{x} \mapsto h(\vec{x})$ defined on ∂B_o and such that the expression (7.2) is zero. Classically this is equivalent to the existence of a constant C such that

$$(7.7)\qquad L(\vec{x}, u_i^o, u_{i,j}^o) - \frac{\partial L}{\partial u_{i,j}} u_{i,k}^o \nu_j \nu_k = C$$

holds on the surface ∂B_o. Henceforth we shall omit the superscript 0. As $\vec{u} = 0$ on ∂B_o , the reasoning used in Sect. 4 proves the existence of $(\lambda_1, \lambda_2, \lambda_3)$ such that

$$(7.8)\qquad u_{i,k} = \lambda_i \nu_k \quad ;$$

then, in view of $\vec{\nu}$ being a unit vector,

$$u_{i,k} \nu_j \nu_k = \lambda_i \nu_k \nu_j \nu_k = \lambda_i \nu_j = u_{i,j} \quad .$$

Therefore (7.7) takes the form

(7.9) $$L(\vec{x}, u_i, u_{i,j}) - \frac{\partial L}{\partial u_{i,j}} u_{i,j} = C \quad .$$

Let us observe in addition that in the problem of Stokes flows, whichever of the three functionals $\mathcal{D}$, $\mathcal{D}'$ or $\mathcal{D}''$ is considered, the integrand L is a quadratic form relatively to the nine variables $u_{i,j}$, hence a homogeneous function with degree 2. By applying Euler's identity one transforms the condition (7.9) into

(7.10) $$L(\vec{x}, u_i, u_{i,j}) = \text{const} \quad .$$

Here L may indifferently denote any of the integrands in (2.2), (2.3), (2.4), (5.4), (5.6). It may be directly checked that the five conditions obtained in that way are all equivalent, since at every point of ∂B_o, one has, in view of (7.8),

$$u_{i,j}\, u_{j,i} = \lambda_i\, \nu_j\, \lambda_j\, \nu_i$$

which is zero, as the incompressiblity condition yields $\lambda_i \nu_i = \lambda_j \nu_j = u_{i,i} = 0$.

To be more precise, let us recall that the boundary of the optimal body in Stokes flows has been found by Pironneau to admit two singular points. The above reasoning is then to be applied with h vanishing on some arbitrarily small neighborhoods of these two points ; it yields that the necessary condition (7.10) holds at every regular point of S.

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