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A product of invariant random permutations has the same small cycle structure as uniform

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Abstract

We use moment method to understand the cycle structure of the composition of independent invariant permutations. We prove that under a good control on fixed points and cycles of length 2, the limiting joint distribution of the number of small cycles is the same as in the uniform case i.e. for any positive integer k , the number of cycles of length k converges to the Poisson distribution with parameter $\frac{1}{k}$ and is asymptotically independent of the number of cycles of length $k' \neq k$.

1 Introduction and main results

We denote by \mathfrak{S}_n the group of permutations of $\{1, \dots, n\}$, by $\#_k \sigma$ the number of cycles of σ of length k , by $\# \sigma$ the total number of cycles of σ and by $\text{tr}(\sigma) := \#_1 \sigma$.

The cycle structure of a permutation chosen uniformly among the symmetric group \mathfrak{S}_n is well understood (see e.g. [Arratia, Tavaré, and Barbour, 2003] for detailed results). In particular, the following classical result holds:

Theorem 1. [Arratia, Barbour, and Tavaré, 2000, Theorem 3.1] *If σ_n follows the uniform distribution on \mathfrak{S}_n then for any $k \geq 1$,*

$$(1) \quad (\#_1 \sigma_n, \dots, \#_k \sigma_n) \xrightarrow[n \rightarrow \infty]{d} \eta_k := (\xi_1, \xi_2, \dots, \xi_k),$$

where $\xrightarrow[n \rightarrow \infty]{d}$ denotes the convergence in distribution, $\xi_1, \xi_2, \dots, \xi_k$ are independent and the distribution of ξ_d is Poisson of parameter $\frac{1}{d}$.

In this work, we question the universality class of this convergence. We show that a product of conjugation invariant permutations that do not have too many fixed points and cycles of size 2 lies within this class. More precisely, we have the following.

Theorem 2. *Let $m \geq 2$. For $1 \leq \ell \leq m$, let $(\sigma_n^\ell)_{n \geq 1}$ be a sequence of random permutations such that for any $n \geq 1$, $\sigma_n^\ell \in \mathfrak{S}_n$. For any $k \geq 1$, let $t_k^n := \#_k(\prod_{\ell=1}^m \sigma_n^\ell)$. Assume that*

- (H_1) *For any $n \geq 1$, $(\sigma_n^1, \dots, \sigma_n^\ell)$ are independent.*

- *For any $n \geq 1$ and $1 \leq \ell \leq m$, for any $\sigma \in \mathfrak{S}_n$,*

$$(H_2) \quad \sigma^{-1} \sigma_n^\ell \sigma \stackrel{d}{=} \sigma_n^\ell,$$

except maybe for one $\ell \in \{1, \dots, m\}$.

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- There exists $1 \leq i < j \leq m$ such that for any $k \geq 1$,

$$(H_3) \quad \lim_{n \rightarrow \infty} \mathbb{E} \left(\left(\frac{\#_1 \sigma_n^i}{\sqrt{n}} \right)^k \right) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathbb{E} \left(\left(\frac{\#_1 \sigma_n^j}{\sqrt{n}} \right)^k \right) = 0,$$

$$(H_4) \quad \lim_{n \rightarrow \infty} \frac{\mathbb{E}(\#_2 \sigma_n^i)}{n} = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\mathbb{E}(\#_2 \sigma_n^j)}{n} = 0.$$

Then for any $k \geq 1$,

$$(t_1^n, t_2^n, \dots, t_k^n) \xrightarrow[n \rightarrow \infty]{d} \eta_k.$$

This convergence has also been obtained by Mukherjee [2016] for a quite different class of permutations, namely the permutations that are equicontinuous in both coordinates and converging as a permutation (see Definitions in [Mukherjee, 2016]). Here, it is easy to check that for any $\theta \in [0, 1]$, the Ewens distribution with parameter θ satisfies the convergences required in H_3 and H_4 . Our result tells that the product of (at least two) Ewens distributions behaves like a uniform permutation, as far as small cycles are concerned.

In our framework, in the case of two permutations, a weaker result can be obtained without any hypothesis on the cycles of size 2.

Proposition 3. *When $m = 2$, under H_1, H_2 and H_3 , we have convergence of the first moment i.e for any $v \geq 1$,*

$$\lim_{n \rightarrow \infty} \mathbb{E}(t_v^n) = \frac{1}{v}.$$

Note that when one of the permutations σ_n^ℓ follows the uniform distribution, under H_1 , the product also follows the uniform distribution and Theorem 2 is a direct consequence of Theorem 1.

Our motivation to understand the cycle structure of random permutations is the relation, in the case of conjugation invariant permutations, to the longest common subsequence (LCS) of two permutations. For example, for $m = 2$, if $\sigma_n^{-1} \rho_n$ is conjugation invariant and

$$\frac{\#(\sigma_n^{-1} \rho_n)}{\sqrt[6]{n}} \xrightarrow[n \rightarrow \infty]{d} 0.$$

Then for any $s \in \mathbb{R}$,

$$\mathbb{P} \left(\frac{LCS(\sigma_n, \rho_n) - 2\sqrt{n}}{\sqrt[6]{n}} \leq s \right) \xrightarrow[n \rightarrow \infty]{} F_2(s),$$

where F_2 is the cumulative distribution function of the GUE Tracy-Widom distribution.

Another motivation comes from traffic distributions, a non-commutative probability theory introduced by Male [2011] to understand the moments of permutation invariant random matrices. As shown in [Male, 2011], the limit in traffic distribution of uniform permutation matrices is trivial but Theorem 1 can be seen as a second-order result in this framework. It is therefore natural to ask about limiting joint fluctuations for the product of several permutation matrices, which is a really non-commutative case. To emphasize this relation, we rewrite Theorem 2 as follows.

Corollary 4. *Under H_1, H_2, H_3 and H_4 , for any $k \geq 1$, $(\text{tr}(\prod_{i=1}^m \sigma_n^\ell), \text{tr}((\prod_{i=1}^m \sigma_n^\ell)^2), \dots, \text{tr}((\prod_{i=1}^m \sigma_n^\ell)^k))$ converges in distribution to $(\xi_1, \xi_1 + 2\xi_2, \dots, \sum_{d|k} d\xi_d)$, where ξ_1, ξ_2, \dots are independent and the distribution of ξ_d is Poisson of parameter $\frac{1}{d}$.*

The optimality of conditions H_3 and H_4 will be discussed at the end of the paper.

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2 Proof of results

We begin with a few preliminary remarks and simplifications.

First of all, the equivalence between Theorem 2 and Corollary 4 is due to the following classical argument. For any $\sigma \in \mathfrak{S}_n$, if $c_i(\sigma)$ denotes the length of the cycle of σ containing i ,

$$(2) \quad \text{tr}(\sigma^k) = \sum_{i=1}^n \mathbb{1}_{\sigma^k(i)=i} = \sum_{i=1}^n \mathbb{1}_{c_i(\sigma)|k} = \sum_{j|k} \sum_{i=1}^n \mathbb{1}_{c_i(\sigma)=j} = \sum_{j|k} j \#_j \sigma.$$

In the hypothesis H_2 , we assume that one of the permutations, say σ_n^1 , may not have a conjugation invariant distribution. In fact, it is enough to prove of Theorem 2 in the case where all permutations are conjugation invariant. Indeed, if we choose τ_n uniform and independent of the σ -algebra generated by $(\sigma_n^\ell)_{1 \leq \ell \leq m}$, the cycle structure of $\prod_{\ell=1}^m \sigma_n^\ell$ is the same as

$$\tau_n^{-1} \left(\prod_{\ell=1}^m \sigma_n^\ell \right) \tau_n = (\tau_n^{-1} \sigma_n^1 \tau_n) \prod_{\ell=2}^m (\tau_n^{-1} \sigma_n^\ell \tau_n) \stackrel{d}{=} (\tau_n^{-1} \sigma_n^1 \tau_n) \prod_{\ell=2}^m \sigma_n^\ell$$

and $(\tau_n^{-1} \sigma_n^1 \tau_n)$ is also conjugation invariant.

We will prove in full details the case $m = 2$ and indicate briefly at the end of the paper how to extend the proof to a larger number of permutations. In the sequel, σ_n^1 and σ_n^2 will be denoted respectively by σ_n and ρ_n .

2.1 Preliminary results

To prove Theorem 2, we will use the same objects introduced in [Kammoun, 2019, pages 12-13] where one can get further details and examples. To a couple of permutations and a subset of p indices, we will associate a set of $2p$ graphs. For technical reasons, we prefer working with $\sigma_n^{-1} \rho_n$ rather than $\sigma_n \rho_n$: for any $k \geq 1$, we define $\tilde{t}_k^n := \#_k(\sigma_n^{-1} \rho_n)$. Under H_2 , $\sigma_n \stackrel{d}{=} \sigma_n^{-1}$ and consequently under H_1 and H_2 , $\forall k \geq 1$ $(t_1^n, t_2^n, \dots, t_k^n)$ and $(\tilde{t}_1^n, \tilde{t}_2^n, \dots, \tilde{t}_k^n)$ have the same distribution.

Let us now recall the combinatorial objects we will use.

- We denote by \mathbb{G}_k^n the set of oriented simple graphs with vertices $\{1, 2, \dots, n\}$ and having exactly k edges. Given $g \in \mathbb{G}_k^n$, we denote by E_g the set of its edges and by $A_g := [\mathbb{1}_{(i,j) \in E_g}]_{1 \leq i, j \leq n}$ its adjacency matrix.
- A connected component of g is called *trivial* if it does not have any edge and a vertex i of g is called *isolated* if E_g does not contain any edge of the form (i, j) or (j, i) nor a loop (i, i) . Let $g \in \mathbb{G}_k^n$, we denote by \tilde{g} the graph obtained from g after removing isolated vertices.
- We say that two oriented simple graphs g_1 and g_2 are *isomorphic* if one can obtain g_2 by changing the labels of the vertices of g_1 . In particular, if $g_1, g_2 \in \mathbb{G}_k^n$ then g_1, g_2 are isomorphic if and only if there exists a permutation matrix σ such that $A_{g_1} \sigma = \sigma A_{g_2}$.
- Let \mathcal{R} be the equivalence relation such that $g_1 \mathcal{R} g_2$ if \tilde{g}_1 and \tilde{g}_2 are isomorphic. We denote by $\hat{\mathbb{G}}_k := \cup_{n \geq 1} \mathbb{G}_k^n / \mathcal{R}$ the set of equivalence classes of $\cup_{n \geq 1} \mathbb{G}_k^n$ for the relation \mathcal{R} .

Let $n \in \mathbb{N}^*$ and $\sigma, \rho \in \mathfrak{S}_n$. Let $m \in \{1, \dots, n\}$ be fixed.

- We denote by $(i_1^m(\sigma, \rho) = m, i_2^m(\sigma, \rho), \dots, i_{k_m(\sigma, \rho)}^m(\sigma, \rho))$ the cycle of $\sigma^{-1} \circ \rho$ containing m , so that $k_m(\sigma, \rho) := c_m(\sigma^{-1} \circ \rho)$ is the length of this cycle. For $i \leq k_m(\sigma, \rho)$, we define $j_l^m(\sigma, \rho) := \rho(i_l^m(\sigma, \rho))$. In particular, $i_1^m(\sigma, \rho), i_2^m(\sigma, \rho), \dots, i_{k_m(\sigma, \rho)}^m(\sigma, \rho)$ are pairwise distinct and $j_1^m(\sigma, \rho), j_2^m(\sigma, \rho), \dots, j_{k_m(\sigma, \rho)}^m(\sigma, \rho)$ are pairwise distinct. For sake of simplicity, when it is clear, we will use the notations k_m, i_l^m and j_l^m instead of $k_m(\sigma, \rho), i_l^m(\sigma, \rho)$ and $j_l^m(\sigma, \rho)$.

- We denote by $\mathcal{G}_1^m(\sigma, \rho) \in \mathbb{G}_{k_m}^n$ and $\mathcal{G}_2^m(\sigma, \rho) \in \mathbb{G}_{k_m}^n$ the graphs with vertices $\{1, \dots, n\}$ such that

$$E_{\mathcal{G}_1^m(\sigma, \rho)} = \{(i_1^m, j_{k_m}^m)\} \cup \left(\bigcup_{l=1}^{k_m-1} \{(i_{l+1}^m, j_l^m)\} \right) \quad \text{and} \quad E_{\mathcal{G}_2^m(\sigma, \rho)} = \bigcup_{l=1}^{k_m} \{(i_l^m, j_l^m)\}$$

and by g_σ the graph such that $A_{g_\sigma} = \sigma$. By construction, for any positive integer $m \leq n$, $\mathcal{G}_1^m(\sigma, \rho)$ (resp. $\mathcal{G}_2^m(\sigma, \rho)$) is a sub-graph of g_σ (resp. g_ρ). Moreover, we want to emphasize that $\mathcal{G}_1^m(\sigma, \rho)$ and $\mathcal{G}_2^m(\sigma, \rho)$ have the same set of non-isolated vertices.

For $i \in \{1, 2\}$, let $\hat{\mathcal{G}}_i^m(\sigma, \rho)$ be the equivalence class of $\mathcal{G}_i^m(\sigma, \rho)$.

- Let $I = (s_1, s_2, \dots, s_l)$ a set of distinct indices of $\{1, \dots, n\}$. We denote by

$$\mathcal{G}^I(\sigma, \rho) = (\mathcal{G}_1^{s_1}(\sigma, \rho), \mathcal{G}_2^{s_1}(\sigma, \rho), \mathcal{G}_1^{s_2}(\sigma, \rho), \dots, \mathcal{G}_1^{s_l}(\sigma, \rho), \mathcal{G}_2^{s_l}(\sigma, \rho))$$

and

$$\hat{\mathcal{G}}^I(\sigma, \rho) = (\hat{\mathcal{G}}_1^{s_1}(\sigma, \rho), \hat{\mathcal{G}}_2^{s_1}(\sigma, \rho), \hat{\mathcal{G}}_1^{s_2}(\sigma, \rho), \dots, \hat{\mathcal{G}}_1^{s_l}(\sigma, \rho), \hat{\mathcal{G}}_2^{s_l}(\sigma, \rho)).$$

- For $i \in \{1, 2\}$, let $\mathcal{G}_i^{\{1, 2, \dots, k\}}(\sigma, \rho)$ be the graph such that $E_{\mathcal{G}_i^{\{1, 2, \dots, k\}}(\sigma, \rho)} = \bigcup_{l=1}^k E_{\mathcal{G}_i^l(\sigma, \rho)}$ and $\hat{\mathcal{G}}_i^{\{1, 2, \dots, k\}}(\sigma, \rho)$ be the equivalence class of $\mathcal{G}_i^{\{1, 2, \dots, k\}}(\sigma, \rho)$.

Using the conjugation invariance and the relation (2), Theorem 2 is equivalent to the following: under the same hypotheses, for any $v_1, v_2, v_3, \dots, v_k \geq 1$,

$$(*) \quad \lim_{n \rightarrow \infty} \sum_{\hat{g}_i, \hat{g}'_i \in \hat{\mathbb{G}}_{v_i}, 1 \leq i \leq k} n^k \mathbb{P} \left(\hat{\mathcal{G}}^{\{1, 2, \dots, k\}}(\sigma_n, \rho_n) = (\hat{g}_1, \hat{g}'_1, \hat{g}_2, \dots, \hat{g}_k) \right) = C_{v_1, v_2, \dots, v_k},$$

where C_{v_1, v_2, \dots, v_k} is a constant independent of the laws of the permutations. Note that, for any $v_i \geq 1$, $\hat{\mathbb{G}}_{v_i}$ and therefore the number of terms of the sum is finite.

For example, if we take $P(x) = x^2$, we have

$$\begin{aligned} \mathbb{E} \left(P(\hat{t}_1^n) \right) &= \mathbb{E} \left(\left(\sum_{i=1}^n \mathbb{1}_{c_i(\sigma^{-1} \circ \rho) = 1} \right)^2 \right) = \sum_{i=1}^n \mathbb{E} \left(\mathbb{1}_{c_i(\sigma^{-1} \circ \rho) = 1} \right) + \sum_{i \neq j}^n \mathbb{E} \left(\mathbb{1}_{c_i(\sigma^{-1} \circ \rho) = 1} \mathbb{1}_{c_j(\sigma^{-1} \circ \rho) = 1} \right) \\ &= n \mathbb{E} \left(\mathbb{1}_{c_1(\sigma^{-1} \circ \rho) = 1} \right) + (n^2 - n) \mathbb{E} \left(\mathbb{1}_{c_1(\sigma^{-1} \circ \rho) = 1} \mathbb{1}_{c_2(\sigma^{-1} \circ \rho) = 1} \right) \\ &\xrightarrow{n \rightarrow \infty} C_1 + C_{1,1} = 1 + 1 = 2 \end{aligned}$$

Similarly, if we take $P(x, y) = xy$, we obtain $\mathbb{E}(P(\hat{t}_1^n, \hat{t}_2^n)) \xrightarrow{n \rightarrow \infty} C_{1,2} = C_{2,1} = 1$.

Before getting into the proof of (*), let us gather some useful combinatorial and then probabilistic results.

Lemma 5. [Kammoun, 2019, Lemma 15] *If $m_1 \in \{i_l^{m_2}, 1 \leq l \leq k_{m_2}\}$, then $\mathcal{G}_1^{m_1}(\sigma, \rho) = \mathcal{G}_1^{m_2}(\sigma, \rho)$ and $\mathcal{G}_2^{m_1}(\sigma, \rho) = \mathcal{G}_2^{m_2}(\sigma, \rho)$.*

Lemma 6. *For any $m \leq n$, for any permutation $\sigma, \rho \in \mathfrak{S}_n$,*

$$\begin{aligned} k_m(\rho, \sigma) &= k_m(\sigma, \rho), \\ j_\ell^m(\rho, \sigma) &= j_{k_m(\sigma, \rho) - \ell + 1}^m(\sigma, \rho), \quad \forall 1 \leq \ell \leq k_m(\sigma, \rho), \\ i_\ell^m(\rho, \sigma) &= i_{k_m(\sigma, \rho) - \ell + 2}^m(\sigma, \rho), \quad \forall 2 \leq \ell \leq k_m(\sigma, \rho), \\ i_1^m(\rho, \sigma) &= i_1^m(\sigma, \rho) = m, \\ A_{\mathcal{G}_1^m(\sigma, \rho)} &= A_{\mathcal{G}_2^{\rho(m)}(\rho^{-1}, \sigma^{-1})}^T. \end{aligned}$$

Lemma 7. *If all non trivial connected components of $\mathcal{G}_1^{m_1}(\sigma, \rho)$ and $\mathcal{G}_2^{m_1}(\sigma, \rho)$ have 2 vertices then both $\mathcal{G}_1^{m_1}(\sigma, \rho)$ and $\mathcal{G}_2^{m_1}(\sigma, \rho)$ have no 2-cycles .*

Proof. Using the symmetries of the problem (Lemmas 5 and 6), it suffices to prove that if all non trivial connected components of $\mathcal{G}_1^1(\sigma, \rho)$ and $\mathcal{G}_2^1(\sigma, \rho)$ have 2 vertices then it is impossible to have at the same time $(1, 2) \in \mathcal{G}_2^1(\sigma, \rho)$ and $(2, 1) \in \mathcal{G}_2^1(\sigma, \rho)$. To simplify notations, let $k_1 := k_1(\sigma, \rho) = c_1(\sigma^{-1} \circ \rho)$, $i_o^1 := i_o^1(\sigma, \rho)$ and $j_o^1 := j_o^1(\sigma, \rho)$.

Let $A = \{\eta > 1; j_\eta^1 \in \{i_1^1, i_2^1, \dots, i_{\eta-1}^1\} \text{ or } i_\eta^1 \in \{j_1^1, j_2^1, \dots, j_{\eta-1}^1\}\}$. Suppose that $(1, 2) \in \mathcal{G}_2^1(\sigma, \rho)$ and $(2, 1) \in \mathcal{G}_2^1(\sigma, \rho)$ then $k_1 \geq 2$ and there exists a unique $1 < l \leq k_1$ such that $i_l^1 = 2$ and $j_l^1 = 1$ so that A is non-empty. Let $\ell' := \inf(A) \geq 2$. Assume that $\ell' > 2$. If $j_{\ell'}^1 \in \{i_1^1, i_2^1, \dots, i_{\ell'-1}^1\}$, then there exists $\ell'' < \ell'$ such that $j_{\ell'}^1 = i_{\ell''}^1$ and since the component of $\mathcal{G}_2^1(\sigma, \rho)$ containing $i_{\ell'}^1$ has two vertices and by definition $(i_{\ell'}^1, j_{\ell'}^1)$ and $(i_{\ell''}^1, j_{\ell''}^1)$ are two edges of $\mathcal{G}_2^1(\sigma, \rho)$, then $j_{\ell''}^1 = i_{\ell'}^1$. Since $(i_{\ell'}^1, j_{\ell'-1}^1) = (j_{\ell''}^1, j_{\ell'-1}^1)$ and $(i_{\ell''+1}^1, j_{\ell''}^1)$ are edges of $\mathcal{G}_1^1(\sigma, \rho)$ and since $\mathcal{G}_1^1(\sigma, \rho)$ has only connected components of size 2, we have necessarily $i_{\ell''+1}^1 = j_{\ell'-1}^1$. One can check easily that $\ell'' < \ell' - 2$ otherwise either $\mathcal{G}_1^1(\sigma, \rho)$ or $\mathcal{G}_2^1(\sigma, \rho)$ has a loop. Indeed, if $\ell'' = \ell' - 2$, then $(i_{\ell''+1}^1, j_{\ell''+1}^1) = (j_{\ell'-1}^1, j_{\ell''+1}^1) = (j_{\ell'-1}^1, j_{\ell'-1}^1)$ is an edge of $\mathcal{G}_2^1(\sigma, \rho)$ and if $\ell'' = \ell' - 1$, then $(i_{\ell''+1}^1, j_{\ell''}^1) = (j_{\ell'-1}^1, j_{\ell''}^1) = (j_{\ell'-1}^1, j_{\ell'-1}^1)$ is an edge of $\mathcal{G}_1^1(\sigma, \rho)$. This implies that $\ell' - 1 \in A$, which is absurd. $i_{\ell'}^1 \in \{j_1^1, j_2^1, \dots, j_{\ell'-1}^1\}$ can be treated using the same techniques and one can extend easily to $\ell' = 2$. \square

We now introduce the following notation : given $g \in \mathbb{G}_k^n$, we denote by

$$\mathfrak{S}_{n,g} := \{\sigma \in \mathfrak{S}_n; \forall (i, j) \in E_g, \sigma(i) = j\}.$$

In other words, $\mathfrak{S}_{n,g}$ is the set of permutations σ such that g is a sub-graph of g_σ . It is not difficult to prove the two following lemmas.

Lemma 8. *Let $g_1, g'_1, g_2, \dots, g'_k \in \cup_{\ell} \mathbb{G}_\ell^n$ and let g, g' be such that $E_g = \cup_{\ell=1}^k E_{g_\ell}$ and $E_{g'} = \cup_{\ell=1}^k E_{g'_\ell}$. Assume that there exists ρ, σ such that*

$$\mathcal{G}^{\{1,2,\dots,k\}}(\sigma, \rho) = (g_1, g'_1, g_2, \dots, g'_k).$$

Then for any random permutation ρ_n, σ_n ,

$$\begin{aligned} \mathbb{P} \left(\bigcap_{i=1}^k \{\sigma_n \in \mathfrak{S}_{n,g_i}, \rho_n \in \mathfrak{S}_{n,g'_i}\} \right) &= \mathbb{P} \left(\mathcal{G}^{\{1,2,\dots,k\}}(\sigma_n, \rho_n) = (g_1, g'_1, g_2, \dots, g'_k) \right) \\ &= \mathbb{P} \left(\mathcal{G}_1^{\{1,2,\dots,k\}}(\sigma_n, \rho_n) = g, \mathcal{G}_2^{\{1,2,\dots,k\}}(\sigma_n, \rho_n) = g' \right). \end{aligned}$$

Proof. We will only prove the first equality. The second one can be obtained using the same argument.

Let σ', ρ' be two permutations. We have seen that $\mathcal{G}_2^m(\sigma', \rho')$ is a subset of $g_{\rho'}$, so that

$$\mathcal{G}_2^m(\sigma', \rho') = g'_m \Rightarrow \rho' \in \mathfrak{S}_{n,g'_m},$$

and that $\mathcal{G}_1^m(\sigma', \rho')$ is a subset of $g_{\sigma'}$, so that

$$\mathcal{G}_1^m(\sigma', \rho') = g_m \Rightarrow \sigma' \in \mathfrak{S}_{n,g_m}.$$

Consequently,

$$\mathbb{P} \left(\mathcal{G}^{\{1,2,\dots,k\}}(\sigma_n, \rho_n) = (g_1, g'_1, g_2, \dots, g'_k) \right) \leq \mathbb{P} \left(\bigcap_{i=1}^k \{\sigma_n \in \mathfrak{S}_{n,g_i}, \rho_n \in \mathfrak{S}_{n,g'_i}\} \right).$$

Now suppose that there exists ρ', σ' such that

$$\mathcal{G}^{\{1,2,\dots,k\}}(\sigma', \rho') = (g_1, g'_1, g_2, \dots, g'_k).$$

Let σ, ρ such that $\sigma \in \cap_{i=1}^k \mathfrak{S}_{n, g_i}$ and $\rho \in \cap_{i=1}^k \mathfrak{S}_{n, g'_i}$. By definition and by iteration on ℓ , one can check that for any $\ell' \leq k$, $i_{\ell'}^\ell(\sigma', \rho') = i_{\ell'}^\ell(\sigma, \rho)$ and $j_{\ell'}^\ell(\sigma', \rho') = j_{\ell'}^\ell(\sigma, \rho)$. Consequently,

$$\mathcal{G}^{\{1, 2, \dots, k\}}(\sigma, \rho) = (g_1, g'_1, g_2, \dots, g'_k).$$

Finally we obtain

$$\mathbb{P}\left(\mathcal{G}^{\{1, 2, \dots, k\}}(\sigma_n, \rho_n) = (g_1, g'_1, g_2, \dots, g'_k)\right) \geq \mathbb{P}\left(\bigcap_{i=1}^k \{\sigma_n \in \mathfrak{S}_{n, g_i}, \rho_n \in \mathfrak{S}_{n, g'_i}\}\right).$$

□

Lemma 9. [*Kammoun, 2019, Lemma 16*] Let $g_1, g_2 \in \mathbb{G}_k^n$. Assume that there exists $\rho \in \mathfrak{S}_n$ such that $A_{g_2}\rho = \rho A_{g_1}$. If ρ has a fixed point on any non-trivial connected component of g_1 , then $\mathfrak{S}_{n, g_1} \cap \mathfrak{S}_{n, g_2} = \emptyset$ or $A_{g_1} = A_{g_2}$.

Lemma 10. For any graph $g \in \mathbb{G}_k^n$ having f loops, p non-trivial connected components and v non-isolated vertices, for any random permutation σ_n with conjugation invariant distribution on \mathfrak{S}_n ,

$$\mathbb{P}(\sigma_n \in \mathfrak{S}_{n, g}) \leq \frac{\mathbb{P}(\sigma_n(1) = 1, \dots, \sigma_n(f) = f)}{\binom{n-p}{v-p}(v-p)!} \leq \frac{1}{\binom{n-p}{v-p}(v-p)!}.$$

Proof. It is an adaptation of the proof of [*Kammoun, 2019, Corollary 17*]. By conjugation invariance, one can suppose without loss of generality that the loops of g are $(1, 1), (2, 2), \dots, (f, f)$ and the set of non isolated vertices of g are $\{1, 2, \dots, v\}$.

If there exist i, j, l , with $j \neq l$ such that $\{(i, j) \cup (i, l)\} \subset E_g$ or $\{(j, i) \cup (l, i)\} \subset E_g$ then $\mathfrak{S}_{n, g} = \emptyset$. Therefore, if $\mathfrak{S}_{n, g} \neq \emptyset$, then non-trivial connected components of g having w vertices are either cycles of length w or isomorphic to \bar{g}_w , where $A_{\bar{g}_w} = [\mathbb{1}_{j=i+1}]_{1 \leq i, j \leq w}$.

Let $g \in \mathbb{G}_k^n$ such that $\mathfrak{S}_{n, g} \neq \emptyset$. Fix p vertices $x_1 = 1, x_2 = 2, \dots, x_f = f, x_{f+1}, \dots, x_p$ each belonging to a different non-trivial connected components of g . Let $x_{p+1} < x_{p+2} < \dots < x_v$ be such that $\{x_{p+1}, \dots, x_v\} = \{1, 2, \dots, v\} \setminus \{x_1, \dots, x_p\}$ be the other non-isolated vertices. Let

$$F = \{(y_i)_{p+1 \leq i \leq v}; y_i \in \{1, 2, \dots, n\} \setminus \{x_1, \dots, x_p\} \text{ pairwise distinct}\}.$$

Given $y = (y_i)_{p+1 \leq i \leq v} \in F$, we denote by $g_y \in \mathbb{G}_k^n$ the graph isomorphic to g obtained by fixing the labels of x_1, x_2, \dots, x_p and by changing the labels of x_i by y_i for $p+1 \leq i \leq v$. Since non trivial connected components of g of length w are either cycles or isomorphic to \bar{g}_w , if $y \neq y' \in F$, then $g_y \neq g_{y'}$ and by Lemma 9, $\mathfrak{S}_{n, g_y} \cap \mathfrak{S}_{n, g_{y'}} = \emptyset$. Since σ_n is conjugation invariant, we have $\mathbb{P}(\sigma_n \in \mathfrak{S}_{n, g_y}) = \mathbb{P}(\sigma_n \in \mathfrak{S}_{n, g_{y'}}) = \mathbb{P}(\sigma_n \in \mathfrak{S}_{n, g})$. Remark also that for any $y \in F$ and any $i \leq f$, (i, i) is a loop of g_y . Thus, $\mathfrak{S}_{n, g_y} \subset \{\sigma \in \mathfrak{S}_n; \forall i \leq f, \sigma_n(i) = i\}$ and thus

$$\begin{aligned} \mathbb{P}(\sigma_n \in \mathfrak{S}_{n, g}) &= \frac{\sum_{y \in F} \mathbb{P}(\sigma_n \in \mathfrak{S}_{n, g_y})}{\text{card}(F)} = \frac{\mathbb{P}(\sigma_n \in \cup_{y \in F} \mathfrak{S}_{n, g_y})}{\text{card}(F)} \leq \frac{\mathbb{P}(\sigma_n(1) = 1, \dots, \sigma_n(f) = f)}{\binom{n-p}{v-p}(v-p)!} \\ &\leq \frac{1}{\binom{n-p}{v-p}(v-p)!}. \end{aligned}$$

□

Lemma 11. Let σ_n be a random permutation with conjugation invariant distribution on \mathfrak{S}_n such that, for any $k \geq 1$, $\lim_{n \rightarrow \infty} \mathbb{E}\left(\left(\frac{\#\mathbb{1}_{\sigma_n}}{\sqrt{n}}\right)^k\right) = 0$. Then, for any $f \geq 1$,

$$\mathbb{P}(\sigma_n^1(1) = 1, \dots, \sigma_n^1(f) = f) = o(n^{-\frac{f}{2}}).$$

Lemma 12. For any $p \geq 1$, let g be a graph with p non trivial components each having 2 vertices. Assume that at least one of these components is a cycle. Then for any random permutation σ_n with conjugation invariant distribution on \mathfrak{S}_n ,

$$\mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g}) \leq \frac{\mathbb{P}(c_1(\sigma_n) = 2)}{\binom{n-p}{p} p!}.$$

Proof. Remark that by conjugation invariance, one can suppose without loss of generality that the set of non isolated vertices of g are $\{1, 2, \dots, 2p\}$ and that $(1, 2), (2, 1) \in E_g$. Using the same definitions as the previous proof with $f = 0$ and $v = 2p$ and by choosing $x_1 = 1$, we have $\mathfrak{S}_{n,g} \subset \{\sigma \in \mathfrak{S}_n; c_1(\sigma) = 2\}$. Thus,

$$\mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g}) = \frac{\sum_{y \in F} \mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g_y})}{\text{card}(F)} = \frac{\mathbb{P}(\sigma_n \in \cup_{y \in F} \mathfrak{S}_{n,g_y})}{\text{card}(F)} \leq \frac{\mathbb{P}(c_1(\sigma_n) = 2)}{\text{card}(F)} = \frac{\mathbb{P}(c_1(\sigma_n) = 2)}{\binom{n-p}{p} p!}.$$

□

By the previous combinatorial lemmas, we get that the main contribution will come from the following subset of graphs. Let $\mathcal{T}_k^n \subset \mathbb{G}_k^n$ be the set of graphs g having exactly k non trivial component each having one edge and two vertices.

For example, $\mathcal{T}_1^3 = \left\{ \begin{array}{c} \textcircled{1} \rightarrow \textcircled{2}, \textcircled{2} \rightarrow \textcircled{1}, \textcircled{1} \rightarrow \textcircled{3}, \textcircled{3} \rightarrow \textcircled{1}, \textcircled{2} \rightarrow \textcircled{3}, \textcircled{3} \rightarrow \textcircled{2} \end{array} \right\}$. Let $\widehat{\mathcal{T}}_k$ be the equivalence class of the graphs of $\cup_n \mathcal{T}_k^n$.

Their contribution is as follows.

Lemma 13. For any $p \geq 1$, $n \geq 2p$ and any graph $g \in \mathcal{T}_p^n$, for any random permutation σ_n with conjugation invariant distribution on \mathfrak{S}_n ,

$$\frac{1}{\binom{n-p}{p} p!} \left(1 - \frac{p^2 - p}{n - 1} - p \mathbb{P}(\sigma_n(1) = 1) \right) \leq \mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g}) \leq \frac{1}{\binom{n-p}{p} p!}.$$

Proof. The upper bound is due to Lemma 10 with $v = 2p$. Using the conjugation invariance, one can suppose without loss of generality that $E_g = \{(1, i_1), (2, i_2), \dots, (p, i_p)\}$ where $i_j > p$ are all distinct. Let

$$\mathfrak{S}_n^p = \{\sigma \in \mathfrak{S}_n, \forall i \leq p, \sigma(i) > p\}.$$

Remark that $\mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g} | \sigma_n \in \mathfrak{S}_n \setminus \mathfrak{S}_n^p) = 0$. If $\mathbb{P}(\sigma_n \in \mathfrak{S}_n^p) = 0$, then necessarily by conjugation invariance, $1 - \frac{p^2 - p}{n - 1} - p \mathbb{P}(\sigma_n(1) = 1) \leq 0$.

Suppose now that $\mathbb{P}(\sigma_n \in \mathfrak{S}_n^p) \neq 0$. We obtain $\mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g}) = \mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g} | \sigma_n \in \mathfrak{S}_n^p) \mathbb{P}(\sigma_n \in \mathfrak{S}_n^p)$. Using again the conjugation invariance, we obtain

$$\mathbb{P}(\sigma_n \in \mathfrak{S}_{n,g} | \sigma_n \in \mathfrak{S}_n^p) = \frac{1}{\binom{n-p}{p} p!}$$

and

$$\begin{aligned} \mathbb{P}(\sigma_n \in \mathfrak{S}_n^p) &= 1 - \mathbb{P}(\sigma_n \in \mathfrak{S}_n \setminus \mathfrak{S}_n^p) \\ &\geq 1 - \sum_{i=1}^p \mathbb{P}(\sigma_n(i) \leq p) \\ &= 1 - p \left(\mathbb{P}(\sigma_n(1) = 1) + \frac{(1 - \mathbb{P}(\sigma_n(1) = 1))(p - 1)}{n} \right) \\ &\geq 1 - \frac{p^2 - p}{n - 1} - p \mathbb{P}(\sigma_n(1) = 1). \end{aligned}$$

□

2.2 Proof of Proposition 3

Proof. We will adapt the proof of [Kammoun, 2019, Lemma 14]. Let $v_1 \geq 1$ be fixed. In the case $k = 1$, since $C_1 = 1$, (*) holds if we have:

$$\forall \hat{g}_1, \hat{g}_2 \in \hat{\mathbb{G}}_{v_1}, \mathbb{P}((\hat{\mathcal{G}}_1^1(\sigma_n, \rho_n), \hat{\mathcal{G}}_2^1(\sigma_n, \rho_n)) = (\hat{g}_1, \hat{g}_2)) = \frac{C_{\hat{g}_1, \hat{g}_2}}{n} + o\left(\frac{1}{n}\right) \quad \text{and} \quad \sum_{\hat{g}_1, \hat{g}_2 \in \hat{\mathbb{G}}_{v_1}} C_{\hat{g}_1, \hat{g}_2} = C_1 = 1.$$

Let $\hat{g}_1, \hat{g}_2 \in \hat{\mathbb{G}}_{v_1}$ be two unlabeled graphs having respectively p_1 and p_2 connected components and $v \leq 2v_1$ vertices. We denote by

$$p_n(\hat{g}_1, \hat{g}_2) := \mathbb{P}((\hat{\mathcal{G}}_1^1(\sigma_n, \rho_n), \hat{\mathcal{G}}_2^1(\sigma_n, \rho_n)) = (\hat{g}_1, \hat{g}_2)).$$

Let $B_{\hat{g}_1, \hat{g}_2}^n$ be the set of couples $(g_1, g_2) \in (\mathbb{G}_{v_1}^n)^2$ having the same non-isolated vertices such that 1 is a non-isolated vertex of both graphs and, for $i \in \{1, 2\}$, the equivalence class of g_i is \hat{g}_i and there exists σ, ρ such that $\mathcal{G}_1^1(\sigma, \rho) = g_1$ and $\mathcal{G}_2^1(\sigma, \rho) = g_2$. By Lemma 8 and H_1 , we have

$$\begin{aligned} p_n(\hat{g}_1, \hat{g}_2) &= \sum_{(g_1, g_2) \in B_{\hat{g}_1, \hat{g}_2}^n} \mathbb{P}((\mathcal{G}_1^1(\sigma_n, \rho_n), \mathcal{G}_2^1(\sigma_n, \rho_n)) = (g_1, g_2)) \\ (3) \quad &= \sum_{(g_1, g_2) \in B_{\hat{g}_1, \hat{g}_2}^n} \mathbb{P}(\sigma_n \in \mathfrak{S}_{n, g_1}, \rho_n \in \mathfrak{S}_{n, g_2}) = \sum_{(g_1, g_2) \in B_{\hat{g}_1, \hat{g}_2}^n} \mathbb{P}(\sigma_n \in \mathfrak{S}_{n, g_1}) \mathbb{P}(\rho_n \in \mathfrak{S}_{n, g_2}) \end{aligned}$$

Starting from (3), we now distinguish different cases, depending on the structure of \hat{g}_1 and \hat{g}_2 .

- Case 1: \hat{g}_1 and \hat{g}_2 have respectively f_1 and f_2 loops i.e edges of type (i, i) with $f_1 + f_2 > 0$. Then $2p_1 - f_1 \leq v$ and $2p_2 - f_2 \leq v$. Consequently, by Lemmas 10 and 11,

$$\begin{aligned} p_n(\hat{g}_1, \hat{g}_2) &= o\left(n^{\frac{-f_1-f_2}{2}}\right) \sum_{(g_1, g_2) \in B_{\hat{g}_1, \hat{g}_2}^n} \frac{1}{\binom{n-p_1}{v-p_1}(v-p_1)!} \frac{1}{\binom{n-p_2}{v-p_2}(v-p_2)!} \\ &= \frac{\text{card}(B_{\hat{g}_1, \hat{g}_2}^n)}{\binom{n-p_1}{v-p_1}(v-p_1)! \binom{n-p_2}{v-p_2}(v-p_2)!} o\left(n^{\frac{-f_1-f_2}{2}}\right) \\ &\leq \frac{\binom{n-1}{v-1} v!^2 o\left(n^{\frac{-f_1-f_2}{2}}\right)}{\binom{n-p_1}{v-p_1}(v-p_1)! \binom{n-p_2}{v-p_2}(v-p_2)!} = n^{v-1-(v-p_1+v-p_2)} o\left(n^{\frac{-f_1-f_2}{2}}\right) = o(n^{-1}). \end{aligned}$$

- Case 2: \hat{g}_1 and \hat{g}_2 do not contain any loop, so that $p_1 \leq \frac{v}{2}$ and $p_2 \leq \frac{v}{2}$. Then, again by Lemma 10,

$$\begin{aligned} p_n(\hat{g}_1, \hat{g}_2) &\leq \sum_{(g_1, g_2) \in B_{\hat{g}_1, \hat{g}_2}^n} \frac{1}{\binom{n-p_1}{v-p_1}(v-p_1)!} \frac{1}{\binom{n-p_2}{v-p_2}(v-p_2)!} \\ &= \frac{\text{card}(B_{\hat{g}_1, \hat{g}_2}^n)}{\binom{n-p_1}{v-p_1}(v-p_1)! \binom{n-p_2}{v-p_2}(v-p_2)!} \\ &\leq \frac{\binom{n-1}{v-1} v!^2}{\binom{n-p_1}{v-p_1}(v-p_1)! \binom{n-p_2}{v-p_2}(v-p_2)!} = O\left(n^{v-1-(v-p_1+v-p_2)}\right). \end{aligned}$$

Therefore, if $p_1 < \frac{v}{2}$, as $p_1 \leq \frac{v-1}{2}$ we have

$$p_n(\hat{g}_1, \hat{g}_2) = O(n^{-\frac{3}{2}}).$$

The same holds if $p_2 < \frac{v}{2}$ and the only remaining terms are the cases when $p_1 = \frac{v}{2} = v_1$ and $p_2 = \frac{v}{2} = v_1$. In this case, both graphs have necessarily connected components having two vertices. By Lemma 7, we obtain that the only non trivial contribution comes from $\hat{g}_1 = \hat{g}_2 = \widehat{\mathcal{T}}_{v_1}$. By Lemma 13, we obtain

$$\frac{\text{card}(B_{\widehat{\mathcal{T}}_{v_1}, \widehat{\mathcal{T}}_{v_1}}^n)}{\binom{n-p_1}{v-p_1} (v-p_1)! \binom{n-p_2}{v-p_2} (v-p_2)!} \left(1 - O\left(\frac{1}{n}\right)\right) \leq p_n(\widehat{\mathcal{T}}_{v_1}, \widehat{\mathcal{T}}_{v_1}) \leq \frac{\text{card}(B_{\widehat{\mathcal{T}}_{v_1}, \widehat{\mathcal{T}}_{v_1}}^n)}{\binom{n-p_1}{v-p_1} (v-p_1)! \binom{n-p_2}{v-p_2} (v-p_2)!}.$$

Moreover, each element of $B_{\widehat{\mathcal{T}}_{v_1}, \widehat{\mathcal{T}}_{v_1}}^n$ can be characterized by a choice of $i_2^1, i_3^1, \dots, i_{v_1}^1, j_1^1, \dots, j_{v_1}^1$ pairwise distincts in $\{2, 3, \dots, n\}$, so that

$$\text{card}(B_{\widehat{\mathcal{T}}_{v_1}, \widehat{\mathcal{T}}_{v_1}}^n) = \binom{n-1}{2v_1-1} (2v_1-1)!.$$

Since $v = 2p_1 = 2p_2 = 2v_1$, we get that

$$p_n(\widehat{\mathcal{T}}_{v_1}, \widehat{\mathcal{T}}_{v_1}) = \frac{1 + o(1)}{n}.$$

Summarizing all cases, we get that $C_{\hat{g}_1, \hat{g}_2} = 0$ unless $\hat{g}_1 = \hat{g}_2 = \widehat{\mathcal{T}}_{v_1}$, in which case $C_{\widehat{\mathcal{T}}_{v_1}, \widehat{\mathcal{T}}_{v_1}} = 1$.

□

2.3 Proof of Theorem 2

The proof of Theorem 2 is similar to that of Proposition 3. Instead of studying \mathcal{G}_i^1 , we study $\mathcal{G}_i^{\{1,2,\dots,k\}}$. We will prove using the same argument that only the event $\{\sigma, \rho; \forall i \in \{1, 2\}, \mathcal{G}_i^{\{1,2,\dots,k\}}(\sigma, \rho) \in \cup_{p \geq 1} \mathcal{T}_p^n\}$ will contribute to the limit.

Proof of Theorem 2 in the case $m = 2$. Let $\mathbf{v} = (v_1, v_2, \dots, v_k)$ be fixed. If $\forall i \leq k, c_i(\sigma^{-1}\rho) = v_i$, then

$$\mathcal{G}_1^{\{1,2,\dots,k\}}(\sigma, \rho), \mathcal{G}_2^{\{1,2,\dots,k\}}(\sigma, \rho) \in \bigcup_{p \leq \sum_{i=1}^k v_k} \widehat{\mathbb{G}}_p.$$

Since $\bigcup_{p \leq \sum_{i=1}^k v_k} \widehat{\mathbb{G}}_p$ is finite, it is sufficient to prove that for any pair $\hat{g}_1, \hat{g}_2 \in \bigcup_{p \leq \sum_{i=1}^k v_k} \widehat{\mathbb{G}}_p$ having the same number of non-isolated vertices, there exists a constant $C_{\hat{g}_1, \hat{g}_2, \mathbf{v}}$ such that under the assumptions of Theorem 2,

$$\mathbb{P}\left(\left(\widehat{\mathcal{G}}_1^{\{1,2,\dots,k\}}(\sigma_n, \rho_n), \widehat{\mathcal{G}}_2^{\{1,2,\dots,k\}}(\sigma_n, \rho_n)\right) \in (\hat{g}_1, \hat{g}_2) \cap A_{\mathbf{v}}\right) = \frac{C_{\hat{g}_1, \hat{g}_2, \mathbf{v}}}{n^k} + o\left(\frac{1}{n^k}\right),$$

where $A_{\mathbf{v}} := \{\forall i \leq k, c_i(\sigma_n^{-1}\rho_n) = v_i\}$.

Let $\hat{g}_1, \hat{g}_2 \in \bigcup_{p \leq \sum_{i=1}^k v_k} \widehat{\mathbb{G}}_p$ be two unlabeled graphs having respectively p_1 and p_2 connected components and v vertices. Let $B_{\hat{g}_1, \hat{g}_2}^{n, \mathbf{v}}$ be the set of couples (g_1, g_2) with n vertices, having the same non-isolated vertices such that

- $1, 2, \dots, k$ are non-isolated vertices of both graphs,
- for $i \in \{1, 2\}$, the equivalence class of g_i is \hat{g}_i ,
- there exists σ, ρ such that for $i \in \{1, 2\}$, $\mathcal{G}_i^{\{1,2,\dots,k\}}(\sigma, \rho) = g_i$ and $c_i(\sigma^{-1}\rho) = v_i$.

As before, we denote by

$$p_{n,\mathbf{v}}(\hat{g}_1, \hat{g}_2) := \mathbb{P}\left(\left(\hat{\mathcal{G}}_1^{\{1,2,\dots,k\}}(\sigma_n, \rho_n), \hat{\mathcal{G}}_2^{\{1,2,\dots,k\}}(\sigma_n, \rho_n)\right) = (\hat{g}_1, \hat{g}_2) \cap A_{\mathbf{v}}\right)$$

and we have

$$\begin{aligned} p_{n,\mathbf{v}}(\hat{g}_1, \hat{g}_2) &= \sum_{(g_1, g_2) \in B_{\hat{g}_1, \hat{g}_2}^{n,\mathbf{v}}} \mathbb{P}\left(\left(\mathcal{G}_1^{\{1,2,\dots,k\}}(\sigma_n, \rho_n), \mathcal{G}_2^{\{1,2,\dots,k\}}(\sigma_n, \rho_n)\right) = (g_1, g_2)\right) \\ &= \sum_{(g_1, g_2) \in B_{\hat{g}_1, \hat{g}_2}^{n,\mathbf{v}}} \mathbb{P}(\sigma_n \in \mathfrak{S}_{n, g_1}, \rho_n \in \mathfrak{S}_{n, g_2}) = \sum_{(g_1, g_2) \in B_{\hat{g}_1, \hat{g}_2}^{n,\mathbf{v}}} \mathbb{P}(\sigma_n \in \mathfrak{S}_{n, g_1}) \mathbb{P}(\rho_n \in \mathfrak{S}_{n, g_2}). \end{aligned}$$

Starting from there, we distinguish different cases:

- Case 1: \hat{g}_1 and \hat{g}_2 have respectively f_1 and f_2 loops i.e edges of type (i, i) with $f_1 + f_2 > 0$. Then $2p_1 - f_1 \leq v$ and $2p_2 - f_2 \leq v$. Consequently, by Lemmas 10 and 11,

$$\begin{aligned} p_{n,\mathbf{v}}(\hat{g}_1, \hat{g}_2) &= \frac{\text{card}(B_{\hat{g}_1, \hat{g}_2}^{n,\mathbf{v}})}{\binom{n-p_1}{v-p_1}(v-p_1)! \binom{n-p_2}{v-p_2}(v-p_2)!} o\left(n^{\frac{-f_1-f_2}{2}}\right) \\ &\leq \frac{\binom{n-k}{v-k} v!^2 o\left(n^{\frac{-f_1-f_2}{2}}\right)}{\binom{n-p_1}{v-p_1}(v-p_1)! \binom{n-p_2}{v-p_2}(v-p_2)!} = n^{v-k-(v-p_1+v-p_2)} o\left(n^{\frac{-f_1-f_2}{2}}\right) = o(n^{-k}). \end{aligned}$$

- Case 2: \hat{g}_1 and \hat{g}_2 do not contain any loop. Then $p_1 \leq \frac{v}{2}$ and $p_2 \leq \frac{v}{2}$. Consequently,

$$\begin{aligned} p_{n,\mathbf{v}}(\hat{g}_1, \hat{g}_2) &\leq \frac{\text{card}(B_{\hat{g}_1, \hat{g}_2}^{n,\mathbf{v}})}{\binom{n-p_1}{v-p_1}(v-p_1)! \binom{n-p_2}{v-p_2}(v-p_2)!} \\ &\leq \frac{\binom{n-k}{v-k} v!^2}{\binom{n-p_1}{v-p_1}(v-p_1)! \binom{n-p_2}{v-p_2}(v-p_2)!} \\ &\leq C n^{v-k-(v-p_1+v-p_2)}. \end{aligned}$$

Therefore, if $p_1 < \frac{v}{2}$ or $p_2 < \frac{v}{2}$ then $p_{n,\mathbf{v}}(\hat{g}_1, \hat{g}_2) = o(n^{-k})$. The only remaining terms are the cases when $p_1 = \frac{v}{2}$ and $p_2 = \frac{v}{2}$. In this case, both graphs have necessarily only connected components having two vertices. Assume that one of the two graphs has a cycle. Then, by Lemma 12, we have

$$\begin{aligned} p_{n,\mathbf{v}}(\hat{g}_1, \hat{g}_2) &\leq \sum_{(g_1, g_2) \in B_{\hat{g}_1, \hat{g}_2}^{n,\mathbf{v}}} \frac{(\mathbb{P}(c_1(\sigma_n) = 2) + \mathbb{P}(c_1(\rho_n) = 2))}{\binom{n-p_1}{v-p_1}(v-p_1)! \binom{n-p_2}{v-p_2}(v-p_2)!} \\ &\leq C(\mathbb{P}(c_1(\sigma_n) = 2) + \mathbb{P}(c_1(\rho_n) = 2)) n^{-k}. \end{aligned}$$

Under H_4 , we have $\mathbb{P}(c_1(\sigma_n) = 2) + \mathbb{P}(c_1(\rho_n) = 2) = o(1)$ so that $p_{n,\mathbf{v}}(\hat{g}_1, \hat{g}_2) = o(n^{-k})$ as soon as one of the graph has a cycle.

As before, the only non-trivial contributions come from the cases when $\hat{g}_1 = \hat{g}_2 = \widehat{\mathcal{T}}_p$ for some $p \leq \sum_{i=1}^k v_i$ and by Lemma 13, we obtain

$$\frac{\text{card}(B_{\widehat{\mathcal{T}}_p, \widehat{\mathcal{T}}_p}^{n,\mathbf{v}})}{\binom{n-p_1}{v-p_1}(v-p_1)! \binom{n-p_2}{v-p_2}(v-p_2)!} \left(1 - O\left(\frac{1}{n}\right)\right) \leq p_{n,\mathbf{v}}(\widehat{\mathcal{T}}_p, \widehat{\mathcal{T}}_p) \leq \frac{\text{card}(B_{\widehat{\mathcal{T}}_p, \widehat{\mathcal{T}}_p}^{n,\mathbf{v}})}{\binom{n-p_1}{v-p_1}(v-p_1)! \binom{n-p_2}{v-p_2}(v-p_2)!}.$$

One can conclude since, for any $n \geq 2p$,

$$\text{card}(B_{\widehat{\mathcal{T}}_p, \widehat{\mathcal{T}}_p}^{n,\mathbf{v}}) = \text{card}(B_{\widehat{\mathcal{T}}_p, \widehat{\mathcal{T}}_p}^{2p,\mathbf{v}}) \binom{n-k}{2p-k}$$

and consequently, for any $p \leq \sum_{i=1}^k v_i$,

$$C_{\widehat{\mathcal{T}}_p, \widehat{\mathcal{T}}_p, \mathbf{v}} = \frac{\text{card}\left(B_{\widehat{\mathcal{T}}_p, \widehat{\mathcal{T}}_p}^{2p, \mathbf{v}}\right)}{(2p - k)!},$$

and $C_{\widehat{g}_1, \widehat{g}_2, \mathbf{v}} = 0$, as soon as $(\widehat{g}_1, \widehat{g}_2) \notin \{(\widehat{\mathcal{T}}_p, \widehat{\mathcal{T}}_p), p \leq \sum_{i=1}^k v_i\}$. As the constants $C_{\widehat{g}_1, \widehat{g}_2, \mathbf{v}}$ do not depend on the distributions of σ_n and ρ_n , this concludes the proof of Theorem 2 in the case of two permutations. \square

To extend to $m > 2$, we will proceed by induction on the number m of permutations. Our main argument is the following lemma.

Lemma 14. *Let $(\sigma_n^1)_{n \geq 1}, (\sigma_n^2)_{n \geq 1}$ be two sequences of random permutations such that for any $n \geq 1$, $\sigma_n^1, \sigma_n^2 \in \mathfrak{S}_n$. Assume that*

- For any $n \geq 1$, σ_n^1 and σ_n^2 are independent.
- For any $n \geq 1$ and $\ell \in \{1, 2\}$, for any $\sigma \in \mathfrak{S}_n$,

$$\sigma^{-1} \sigma_n^\ell \sigma \stackrel{d}{=} \sigma_n^\ell.$$

- For any $k \geq 1$,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left(\left(\frac{\#_1 \sigma_n^1}{\sqrt{n}} \right)^k \right) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\mathbb{E}(\#_2 \sigma_n^1)}{n} = 0.$$

Then,

$$(4) \quad \lim_{n \rightarrow \infty} \mathbb{E} \left(\left(\frac{\#_1(\sigma_n^1 \sigma_n^2)}{\sqrt{n}} \right)^k \right) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\mathbb{E}(\#_2(\sigma_n^1 \sigma_n^2))}{n} = 0. \quad .$$

Proof. We will only give a sketch of the proof. The idea is to repeat the same study as in the case $m = 2$ in the two particular quantities.

- Take $k \geq 1$ and $v_1 = v_2 = \dots = v_k = 1$. One can show that, under the hypotheses of Lemma 14,

$$\lim_{n \rightarrow \infty} \sum_{\widehat{g}_i, \widehat{g}'_i \in \widehat{\mathbb{G}}_1, 1 \leq i \leq k} n^{\frac{k}{2}} \mathbb{P}(\widehat{\mathcal{G}}^{\{1, 2, \dots, k\}}(\sigma_n^1, \sigma_n^2) = (\widehat{g}_1, \widehat{g}'_1, \widehat{g}_2, \dots, \widehat{g}'_k)) = 0.$$

This leads to the first limit in (4).

- Take $k = 1$ and $v_1 = 2$. One can show that, under the hypotheses of Lemma 14,

$$\forall \widehat{g}_1, \widehat{g}_2 \in \widehat{\mathbb{G}}_2, \lim_{n \rightarrow \infty} \mathbb{P}((\widehat{\mathcal{G}}_1^1(\sigma_n^1, \sigma_n^2), \widehat{\mathcal{G}}_2^1(\sigma_n^1, \sigma_n^2)) = (\widehat{g}_1, \widehat{g}_2)) = 0.$$

This leads to the second limit in (4). \square

3 Further discussion

In this last section, we make a few remarks on the optimality of the assumptions H_3 and H_4 in Theorem 2. We assume hereafter that H_1 and H_2 hold true and consider for the sake of clarity the case $m = 2$.

- The assumption H_3 is optimal in the sense that if

$$\liminf_{n \rightarrow \infty} n^{-\frac{k}{2}} \min(\mathbb{E}((\#_1 \sigma_n)^k), \mathbb{E}((\#_1 \rho_n)^k)) = \varepsilon_k > 0,$$

then

$$\liminf_{n \rightarrow \infty} \mathbb{E}((\#_1(\sigma_n \rho_n))^k) \geq \mathbb{E}(\xi_1^k) + \varepsilon_k^2.$$

Indeed, going back to the equation (*), one can see that in the case $v_1 = v_2 = \dots = v_k = 1$, if \hat{g} is the class of the graph with adjacency matrix Id_k the event $\{(\hat{\mathcal{G}}_1^{1,2,\dots,k}(\sigma_n, \rho_n), \hat{\mathcal{G}}_2^{1,2,\dots,k}(\sigma_n, \rho_n)) = (\hat{g}, \hat{g})\}$ will contribute to the limit, leading to the term ε_k^2 .

- Similarly H_4 is optimal in the sense that if

$$\liminf_{n \rightarrow \infty} \left(\frac{\min(\mathbb{E}(\#_2 \sigma_n), \mathbb{E}(\#_2 \rho_n))}{n} \right) = \varepsilon' > 0,$$

then,

$$\liminf_{n \rightarrow \infty} \mathbb{E}((\#_1(\sigma_n \rho_n))^2) \geq 2 + \varepsilon'^2.$$

Indeed, as above, in the case $v_1 = v_2 = 1$, if \hat{g}' is the class of the graph with adjacency matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the event $\{(\hat{\mathcal{G}}_1^{1,2,\dots,k}(\sigma_n, \rho_n), \hat{\mathcal{G}}_2^{1,2,\dots,k}(\sigma_n, \rho_n)) = (\hat{g}', \hat{g}')\}$ will contribute to the limit.

- Assume now that one of the bounds in H_3 is not satisfied. More precisely, assume that there exists $k \geq 1$ such that

$$\liminf_{n \rightarrow \infty} n^{-\frac{k}{2}} \mathbb{E}((\#_1 \sigma_n)^k) = \varepsilon_k > 0,$$

or

$$\liminf_{n \rightarrow \infty} \frac{\mathbb{E}(\#_2 \sigma_n)}{n} = \varepsilon' > 0.$$

Then, by similar arguments, one can check that the convergences

$$\forall k \geq 1, \lim_{n \rightarrow \infty} n^{-\frac{k}{2}} \mathbb{E}((\#_1 \rho_n)^k) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\mathbb{E}(\#_2 \rho_n)}{n} = 0$$

are a necessary condition to obtain (1) and that the convergences

$$\forall k \geq 1, \lim_{n \rightarrow \infty} n^{-\frac{k}{2}} \mathbb{E}((\#_1 \rho_n)^k) = 0, \quad \limsup_{n \rightarrow \infty} n^{-\frac{k}{2}} \mathbb{E}((\#_1 \sigma_n)^k) < \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\mathbb{E}(\#_2 \rho_n)}{n} = 0$$

are a sufficient condition to obtain (1).

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