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Nicolas Burq, Chenmin Sun

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# TIME OPTIMAL OBSERVABILITY FOR GRUSHIN SCHRÖDINGER EQUATION

NICOLAS BURQ AND CHENMIN SUN

ABSTRACT. We consider two dimensional Grushin Schrödinger equation posed on a finite cylinder  $\Omega = (-1, 1)_x \times \mathbb{T}_y$  with Dirichlet boundary condition. We obtain the sharp observability by any horizontal strip, with the optimal time  $T_* > 0$  depending on the size of the strip. Consequently, we prove the exact controllability for Grushin Schrödinger equation. By exploiting the concentration of eigenfunctions of harmonic oscillator at  $x = 0$ , we also show that the observability fails for any  $T \leq T_*$ .

## 1. INTRODUCTION

1.1. **Motivation.** In this article we are interested in the observation and control for a sub-elliptic Schrödinger equation on a finite cylinder. Observability for linear evolution PDEs has been extensively studied in the past decades. These studies aim to observe solutions of PDEs

$$\partial_t U - \mathcal{L}U = 0, \quad U|_{t=0} = U_0 \quad (1.1)$$

in a small region  $\omega$  (control region) by spending some time. More precisely, it concerns an inequality of the type

$$\|U(0)\|_X \leq C_T \int_0^T \|\mathbf{1}_\omega U(t)\|_X^2 dt. \quad (1.2)$$

If (1.2) holds, then we say that the observability of the equation (1.1) from  $(0, T) \times \omega$  is true. There are two aspects. Firstly, observability is a quantitative version of the unique continuation property. The question of observability from  $(0, T) \times \omega$  can be reformulated as: can sequences of solutions of (1.1) concentrate on sets which do not intersect with  $(0, T) \times \omega$ ? Secondly, it is equivalent to the null controllability (and hence for reversible systems to the exact controllability) of the dual equation of (1.1) (see [Li88]). For hyperbolic PDEs, the observability can be obtained effectively by the theorem of propagation of singularities. In particular, for the wave equation, it is well-known that (see [BLR92],[BG96]) the observability is equivalent to the so called geometric control condition, which says that all the trajectories of the generalized geodesic flow will enter the control region before some prescribed time  $T > 0$ . Moreover, one cannot observe the waves before this geometric time  $T$ .

For Schrödinger equation

$$i\partial_t u + \Delta u = 0$$

on manifolds (with or without boundaries), due to the dispersion, or infinite propagation speed, it was shown by Lebeau ([Le92]) that the observability holds true for arbitrary short time  $T$ , provided that the control region  $\omega$  satisfies the geometric control condition. This geometric restriction on  $\omega$  is not necessary in general. It turns out that the stability(instability) of the geodesic flow of the underlying manifold plays an important role for the concentration of solutions of Schrödinger equation. It was shown in [Ja90],[BZ12],[BBZ13],[AM14],[ALM16],[Ji17] that for any  $T > 0$  and any non-empty open set  $\omega$ , the observability of Schrödinger equation from  $(0, T) \times \omega$  is true, if the underlying manifolds are torus, disk or hyperbolic surfaces. In this article, in some sense, we change dramatically the geometry of the underlying manifold, and are interested in the observability problem for the Schrödinger equation in a hypoelliptic geometry. For our model example, a mixed feature of propagation and dispersion will be revealed, which will crucially influence the validity of the observability, leading to a unique example for which observability and control hold for the Schrödinger equation while it is violated for the heat equation [Ko17].

**1.2. Grushin Schrödinger equation.** We consider the following Grushin Schrödinger equation on a finite cylinder  $\Omega = (-1, 1)_x \times \mathbb{T}_y$  with Dirichlet boundary condition:

$$\begin{cases} i\partial_t u + \Delta_G u = 0, (t, x, y) \in \mathbb{R} \times \Omega \\ u|_{t=0} = u_0(x, y) \in L^2(\Omega), \\ u|_{x=\pm 1} = 0 \end{cases} \quad (1.3)$$

where the Grushin Laplace operator is defined as  $\Delta_G = \partial_x^2 + x^2 \partial_y^2$ , together with domain

$$D(\Delta_G) := \{u \in \mathcal{D}'(\Omega) : \partial_x^2 u, x^2 \partial_y^2 u \in L^2(\Omega) \text{ and } u|_{\partial\Omega} = 0\}.$$

The mass

$$\|u(t)\|_{L^2(\Omega)}^2 = \int_{\Omega} |u(t, x, y)|^2 dx dy$$

and the energy

$$\|u\|_{\dot{H}_G^1(\Omega)} := \int_{\Omega} (|\partial_x u(t, x, y)|^2 + |x \partial_y u(t, x, y)|^2) dx dy$$

are conserved along the flow.

The equation (1.3) can be solved by first taking the Fourier transformation in  $y$  variable, and then solving the spectral problem of 1D harmonic oscillator. More precisely, for fixed  $n \in \mathbb{Z}$ , we denote by  $\varphi_{m,n}(x)$  be the  $m$ -th eigenfunction of the operator  $L_n = -\partial_x^2 + n^2 x^2$  with the domain  $D(L_n) = H^2((-1, 1)) \cap H_0^1((-1, 1))$ , associated with eigenvalues  $\lambda_{m,n}^2$ . We expand the initial data  $u_0(x, y)$  as

$$u_0(x, y) = \sum_{m=0}^{\infty} \sum_{n \in \mathbb{Z}} a_{m,n} \varphi_{m,n}(x) e^{iny},$$

hence, the solution of (1.3) is given by

$$(e^{it\Delta_G} u_0)(t, x, y) = \sum_{m=0}^{\infty} \sum_{n \in \mathbb{Z}} a_{m,n} e^{-it\lambda_{m,n}^2} \varphi_{m,n}(x) e^{iny}.$$

In particular,

- $\text{Spec}(-\Delta_G) = \{\lambda_{m,n}^2 : m \in \mathbb{N}, n \in \mathbb{Z}\}$ .
- Eigenfunctions of  $-\Delta_G$  are

$$\Phi_{m,n}(x, y) = \varphi_{m,n}(x) e^{iny}.$$

A classical question in control theory is whether, given a non-empty open set  $\omega$  and time  $T_0 > 0$ , for  $u_0 \in L^2(\Omega)$ , there exists a control  $f \in L^2((0, T) \times \omega)$  such that the solution of (1.1) satisfies  $u|_{t \geq T_0} = 0$ .

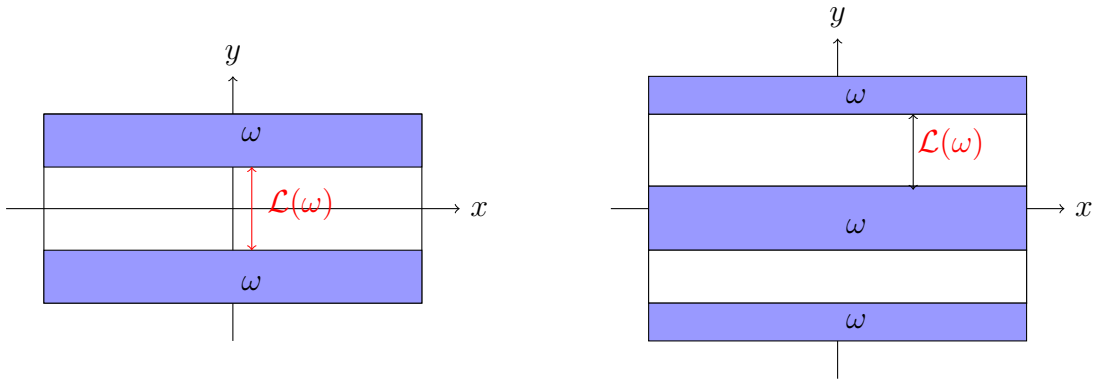
Control and observation problem for the degenerate parabolic equation has attracted lots of attentions in these years. Many results have been obtained, yet, the picture is far from been completed. For the parabolic Grushin equation

$$\partial_t u - \Delta_G u = 0,$$

it turns out that there exists a minimal time  $T^* > 0$  for the observability, if  $\omega$  does not touch the degenerate line (see [BCG14]). It was also shown in [Ko17] that the observability is untrue, if  $\omega$  does not intersect a horizontal strip. Though governed by different mechanism, we present a time-optimal control and observation result for the Grushin Schrödinger equation, as a counterpart of the mentioned degenerate parabolic equations.

**1.3. Main results.** In this article, we consider the horizontal control domain  $\omega$  of the form  $(-1, 1)_x \times I$ , where  $I \subset \mathbb{T}$  is a finite union of intervals. For such  $\omega$ , we define the quantity  $\mathcal{L}(\omega)$  to be the length of the largest interval in  $\Omega \setminus \omega \cap \{x = 0\}$ :

$$\mathcal{L}(\omega) := \sup\{s : \exists y_1, y_2 \in \mathbb{T} \text{ such that } \text{dist}_{\mathbb{T}}(y_1, y_2) = s, \text{ and } [(0, y_1), (0, y_2)] \cap \omega = \emptyset\}.$$



We show that if  $T_0 > \mathcal{L}(\omega)$ , the observability is true:

**Theorem 1.** *Let  $\omega$  be a horizontal strip of  $\Omega$  and assume that  $T_0 > \mathcal{L}(\omega)$ . Then there exists  $C_{T_0} > 0$ , such that for any solution  $u$  to (1.3) with initial data  $u_0 \in L^2(\Omega)$ , the observability*

$$\|u_0\|_{L^2(\Omega)}^2 \leq C_{T_0} \int_0^{T_0} \|u(t)\|_{L^2(\omega)}^2 dt \quad (1.4)$$

*holds true.*

Consequently, by the standard HUM method ([Li88], see for example [BZ12] in the context of Schrödinger equation), we obtain the following exact controllability:

**Theorem 2.** *Assume that  $T_0 > \mathcal{L}(\omega)$ , then for any  $u_0 \in L^2(\Omega)$ , there exists  $g \in L^2((0, T_0) \times \omega)$ , such that the solution of the equation*

$$i\partial_t u + \Delta_G u = \mathbf{1}_\omega g, \quad u|_{t=0} = u_0$$

*satisfies  $u(T_0, \cdot) = 0$ .*

Our next result shows that  $T_0 > \mathcal{L}(\omega)$  is also necessary for the observability. This will be done by constructing examples which contradicts the observability, if  $T_0 \leq \mathcal{L}(\omega)$ . The precise statement is as follows:

**Theorem 3.** *Assume that  $T_0 \leq \mathcal{L}(\omega)$ , then there exists a sequence  $(u_{k,0}) \subset L^2(\Omega)$ , such that*

$$\|u_{k,0}\|_{L^2(\Omega)} = 1, \text{ and } \lim_{k \rightarrow \infty} \int_0^{T_0} \|e^{it\Delta_G} u_{k,0}\|_{L^2(\omega)}^2 dt = 0.$$

The observability for Schrödinger equation is related to the long-time behaviour of the semi-classical Schrödinger equation. Starting from initial data oscillating at space frequency of order  $h^{-1}$ , a simple time rescaling gives

$$i\partial_t u + \Delta u = 0 \Leftrightarrow ih\partial_s v + h^2\Delta v = 0, v(s, x) = u(hs, x).$$

If the geometric control condition is fulfilled, the concentration outside  $\omega$  cannot occur beyond the semi-classical time scale  $s \sim 1$ . This in turn tells us  $t \sim h$ , which is responsible for the short time observability result of Lebeau [Le92]. In the instable cases (torus, disk), we can see dispersion by understanding propagation beyond the semi-classical time scale  $s \sim \epsilon h^{-1}$ ,  $\epsilon \ll 1$ , and this is responsible for the observability results in [BZ12] [AM14],[ALM16] at the classical time scale  $\epsilon$ . To the best of our knowledge, Theorem 1 is the first result for the Schrödinger type equations where we can track dispersion all the way up to the exact semi-classical time scale  $s = \mathcal{L}(\omega)h^{-1}$ .

**Remark 1.1.** The method we used in the proof of Theorem 2, Theorem 1.4 and Theorem 3 would allow one to obtain similar results for more general hypoelliptic operator of the form

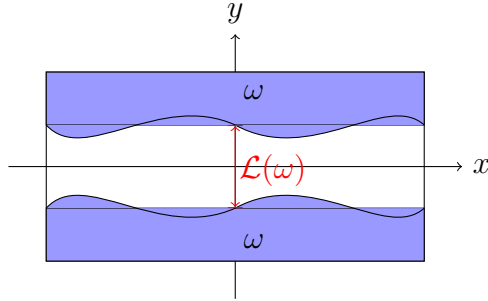
$$P = \partial_x^2 + V(x)\partial_y^2,$$

where  $V(x) \in C^\infty([-1, 1])$ , satisfying

$$V(x) > 0, \text{ if } x \neq 0, \quad V(x) \sim x^2, V'(x) \sim x, V''(x) \sim 1, x \rightarrow 0.$$

A typical example is  $V(x) = \sin^2(\pi x)$ .

**Remark 1.2.** Though the control domain  $\omega$  we considered here is horizontal strips, with a little effort, one can prove the exact controllability with the same time threshold  $T_0 > \mathcal{L}(\omega)$ , when  $\omega$  contains a horizontal strip. Moreover, from the concentration of the first eigenfunction of the semi-classical harmonic oscillator  $-\partial_x^2 + n^2 x^2$ , the exact controllability is untrue, for any  $T > 0$ , if  $\Omega \setminus \omega$  contains a neighborhood of the degenerate line  $\{x = 0\}$ . It would be interesting to know whether the exact controllability is true, if  $\omega$  is only a neighborhood of the verticle interval  $[(0, y_1), (0, y_2)]$ .



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## 2. REFORMULATION OF THE PROBLEM AND OUTLINE OF THE PROOFS

Assume that  $\omega = (-1, 1)_x \times \cup_{k=1}^N (c_k, d_k)$ , where  $(c_k, d_k)$  are disjoint intervals of  $\mathbb{T}$ . For  $\sigma_0 > 0$ , define the smooth function  $\phi \in C^\infty(\mathbb{T})$  by

$$\phi(y) = \sum_{j=1}^N \phi_j(y), \quad \text{where } \phi_j(y) = \begin{cases} 1 & \text{if } y \in (c_j, d_j) \text{ and } \text{dist}(y, \frac{c_j+d_j}{2}) \leq \frac{|d_j-c_j|}{2} - \alpha_0 \\ 0 & \text{otherwise} . \end{cases} \quad (2.1)$$

Since  $T_0 > \mathcal{L}(\omega)$ , for

$$\alpha_0 < \min \left\{ |d_k - c_k|/2, (T_0 - \mathcal{L}(\omega))/2; k = 1, 2, \dots, N \right\},$$

we denote by  $a_0 = \frac{\mathcal{L}(\omega)}{2} + \alpha_0$ . Hence we reformulate Theorem 1 as follows:

**Theorem 4.** *Assume that  $T > a_0$ , then for any  $u_0 \in L^2(\Omega)$ , we have*

$$\|u_0\|_{L^2(\Omega)}^2 \leq C_{T,a_0} \int_{-T}^T \|\phi(y)e^{it\Delta_G}u_0\|_{L^2(\Omega)}^2 dt. \quad (2.2)$$

Throughout this article, we will use the standard notation  $D_y = \frac{1}{i}\partial_y$ ,  $D_x = \frac{1}{i}\partial_x$ .

**2.1. Outline of the non-observability for  $T_0 \leq \mathcal{L}(\omega)$ .** Unlike the classical Schrödinger equation, the Grushin-Schrödinger equation exhibits the features of both wave equation and dispersive equations. To understand the heuristics, consider Grushin-Schrödinger equation posed on  $(x, y) \in \mathbb{R}^2$ :

$$i\partial_t u + \Delta_G u = 0.$$

The solution formula can be written explicitly

$$u(t, x, y) = \frac{1}{\sqrt{2\pi}} \sum_{m=0}^{\infty} \int_{\mathbb{R}} e^{-it(2m+1)|\eta| + iy\eta} \widehat{f}_m(\eta) h_m(\sqrt{|\eta|x}) d\eta, \quad (2.3)$$

where  $h_m(x)$  is the  $m$ -th Hermite function, eigenfunction of the harmonic oscillator  $-\partial_x^2 + x^2$ . As explained in [GG12], there exists an orthogonal decomposition

$$L^2(\mathbb{R}^2) = \oplus_{\pm} \oplus_{m=0}^{\infty} V_m^{\pm}, \quad \Delta_G|_{V_m^{\pm}} = \pm i(2m+1)\partial_y,$$

hence (2.3) can be viewed as a coupled system of transport equations:

$$i(\partial_t \pm (2m+1)\partial_y)u_m^{\pm} = 0.$$

The  $m$ -th wave  $u_m^{\pm}$  propagates on the vertical direction of velocity  $2m+1$ . Actually, the bottom spectral portion  $m=0$  is responsible for the restriction  $T_0 > \mathcal{L}(\omega)$  for the observability. For larger  $m$ , the propagation speed of the portion  $u_m^{\pm}$  is larger, and this can be interpreted as “dispersion”. In the section 9, we will construct concrete example to disprove the observability inequality if  $T_0 \leq \mathcal{L}(\omega)$ . Unlike the equation (2.3) posed on the whole space, we have no explicit formula for the first eigenfunction of the semi-classical harmonic oscillator  $-\partial_x^2 + \eta^2 x^2$  with Dirichlet boundary condition. The construction is then based on the careful quantitative analysis of the first eigenvalue and eigenfunction of the semi-classical harmonic oscillator.

**2.2. Outline of the proof of Observability if  $T_0 > \mathcal{L}(\omega)$ .** For simplicity, we shall first study the case where there is a single band in  $\omega^c$ , and by translation in  $y$  we can assume that  $\omega^c = (-1, 1)_x \times (-a, a)$ , hence  $\mathcal{L}(\omega) = 2a$ , and  $a_0 \in (a, T_0/2)$ . Assume that  $\psi \in C_c^{\infty}(\mathbb{R}; [0, 1])$  such that

$$\psi(\zeta) \equiv 1, \text{ if } \frac{1}{2} \leq |\zeta| \leq 2 \text{ and } \psi(\zeta) \equiv 0 \text{ if } |\zeta| < \frac{1}{4} \text{ or } |\zeta| > 4.$$

From standard compactness argument, one can reduce the proof of Proposition 4 to the following spectral-localized version:

**Proposition 2.1.** *Let  $T > a_0$ , then for any  $u_0 \in L^2(\Omega)$ , we have for all  $0 < h \ll 1$ ,*

$$\|\psi(h^2\Delta_G)u_0\|_{L^2(\Omega)}^2 \leq C_{T,a_0} \int_{-T}^T \|\phi(y)e^{it\Delta_G}\psi(h^2\Delta_G)u_0\|_{L^2(\Omega)}^2 dt.$$

Due to the anisotropic feature, we need to consider different spectral regimes. A basic observation is that  $\partial_y$  commutes with  $-\Delta_G$ . If we localize the spectral  $\sqrt{-\Delta_G} \sim h^{-1}$ , then by the hypoellipticity, the vertical frequency  $|D_y|$  is constrained to  $|D_y| \lesssim h^{-2}$ . In section 3, we will prove this simple coercivity estimate. In the sequel, our analysis will require a second microlocalization with respect to the variable  $D_y$ , which will in turn require a second microlocalization with respect to the variables  $(x, D_x)$ . Fortunately this will remain at a quite basic level and appear only through the elementary elliptic regularity result Proposition 4.4.

In section 4, we will deal with the regime:

$$\sqrt{-\Delta_G} = \sqrt{D_x^2 + x^2 D_y^2} \sim h^{-1} \ll |D_y| \lesssim h^{-2}.$$

We introduce another scale  $|D_y| \sim \hbar^{-1}$ , and the half wave regime corresponds to such that  $h^2 \lesssim \hbar \ll h$ . The main constraints in this regime come from the subregime (proper half-wave regime) when  $\hbar \sim h^2$ , and  $|x| \lesssim \hbar^{1/2}$ , since  $|x| \gg \hbar^{1/2}$  is the classical forbidden region where the contributions are of order  $O(h^\infty)$ , if we write

$$i\partial_t + \Delta_G = i\partial_t + \partial_x^2 + \underbrace{\left(\frac{1}{i}\hbar\partial_y\right)(\hbar^{-1/2}x)^2}_{\text{uniformly bounded operator}} i\partial_y.$$

It turns out that the operator  $\frac{1}{i}\hbar\partial_y(\hbar^{-1/2}x)^2$  is also bounded from below, in an averaged sense. In this regime, the key tool will be the positive commutator relation

$$[-\Delta_G, x\partial_x + 2y\partial_y] = -2\Delta_G$$

Roughly speaking, the commutator relation

$$[i\partial_t + \Delta_G, \varphi_T(t)(x\partial_x + 2y\partial_y)] = 2\Delta_G + i\varphi_T'(t)(x\partial_x + 2y\partial_y)$$

allows one to control the energy norm  $2T \times 2(\|x\partial_y u\|_{L^2}^2 + \|\partial_x u\|_{L^2}^2)$  by the contributions inside the control region  $\omega$  and  $(\varphi_T'(t)2y\partial_y u, u)$ . The difficulty is the fact that in the half-wave regime,  $|x| \lesssim \hbar^{1/2}$ , one cannot absorb the norm  $\|\partial_y u\|_{L^2}$  by the energy norm. To overcome this difficulty, we observe that, heuristically, at  $t = 0$ , the solutions are concentrated near  $y = 0$ , then on the support of  $\varphi_T'(t)$ ,  $t \sim T$ , the solutions should be propagated near  $|y| = T > a$ . With appropriate time cut-off  $\varphi_T(t)$ , this term  $|(\varphi_T'(t)2y\partial_y u, u)|$  is essentially

$$2a \times 2 \times \hbar^{-1} \times \|u\|_{L^2}^2 + \text{errors}.$$

By using the hypoellipticity, for the well-localized solution  $u$ , the main contribution turns out to be

$$4a \times (\|x\partial_y u\|_{L^2}^2 + \|\partial_x u\|_{L^2}^2).$$

This allows us to conclude, provided that  $T > a$ .



In the section 5, we discuss the semi-classical dispersive regime, in which the geometric control condition is fulfilled:

$$\sqrt{-\Delta_G} \sim |D_y| \sim h^{-1}.$$

From the time rescaling trick introduced in [Le92], it is then reduced to prove the observability estimate for the semi-classical Grushin Schrödinger equation

$$ih\partial_s w + (h\partial_x)^2 w + x^2(h\partial_y^2)w = 0. \quad (2.4)$$

In the considered regime, the Hamiltonian flow is never degenerated and all geodesics enter the control region in finite time. By a standard defect-measure based argument, we are able to prove the semi-classical propagation observability for (2.4).

In the sections 6 and 7, we will deal with the non-semiclassical dispersive regime. If

$$h^\epsilon \lesssim |D_y| \ll \sqrt{-\Delta_G} \sim h^{-1},$$

thanks to the rapid propagation in the horizontal variable  $x$ , we are still able to use the positive commutator method to detect the vertical propagation, but in this case the arguments are simpler and any  $T > 0$  would work. Finally, if

$$|D_y| \ll h^\epsilon \ll \sqrt{-\Delta_G} \sim h^{-1},$$

we will use a normal form transformation method, inspired by the work of the first author and M. Zworski [BZ12], to convert  $i\partial_t + \Delta_G$  to  $i\partial_t + \partial_x^2 + M^2\partial_y^2$  in this regime, modulo errors. Then an application of the observability theorem for the classical Schrödinger equation allows us to conclude.

### 3. COERCIVITY ESTIMATE

From hypoellipticity, we have the following simple coercivity estimate:

**Lemma 3.1.** *For every  $f \in D(\Delta_G)$ , there holds*

$$\|f\|_{L^2(\Omega)}^2 + \||D_y|^{1/2}f\|_{L^2(\Omega)}^2 \leq ((-\Delta_G + 1)f, f)_{L^2(\Omega)}.$$

*Proof.* By doing integration by part, we have

$$(-\Delta_G f, f)_{L^2(\Omega)} = \|\partial_x f\|_{L^2(\Omega)}^2 + \|x\partial_y f\|_{L^2(\Omega)}^2.$$

Denote by  $\Pi_+$ ,  $\Pi_-$ , the projection to the strictly positive and the strictly negative frequencies in  $y$  variable. Obviously,  $\Pi_\pm$  commute with  $\Delta_G$ ,  $\partial_x$ ,  $x\partial_y$ . From the commutator relation  $[\partial_x, x\partial_y] = \partial_y$ , we have

$$\||D_y|^{1/2}\Pi_\pm f\|_{L^2(\Omega)}^2 = \mp i(\partial_y \Pi_\pm f, \Pi_\pm f)_{L^2(\Omega)} = \mp i([\partial_x, x\partial_y]\Pi_\pm f, \Pi_\pm f)_{L^2(\Omega)}.$$

After integration by part, we have

$$\||D_y|^{1/2}\Pi_\pm f\|_{L^2(\Omega)}^2 = \pm 2 \operatorname{Im}(\partial_x \Pi_\pm f, x\partial_y \Pi_\pm f)_{L^2(\Omega)}.$$

Thus

$$\| |D_y|^{1/2} \Pi_{\pm} f \|_{L^2(\Omega)} \leq \| \partial_x \Pi_{\pm} f \|_{L^2(\Omega)}^2 + \| x \partial_y \Pi_{\pm} f \|_{L^2(\Omega)}^2.$$

The proof of Lemma 3.1 is now complete.  $\square$

Let  $\chi_0 \in C_c^\infty(\mathbb{R}; [0, 1])$  such that

$$\chi_0(\zeta) \equiv 1, \text{ if } |\zeta| \leq 2 \text{ and } \chi_0(\zeta) \equiv 0, \text{ if } |\zeta| > 3.$$

**Corollary 3.2.** For  $0 < h < 1$  and  $e_0 > \frac{1}{2}$ ,

$$\psi(h^2 \Delta_G) (1 - \chi_0(e_0 h^2 D_y)) = 0.$$

*Proof.* Take  $f \in D(\Delta_G)$ , and we expand  $f_h := \psi(h^2 \Delta_G) (1 - \chi_0(e_0 h^2 D_y)) f$

$$f_h = \sum_{m,n} a_{m,n} \psi(h^2 \lambda_{m,n}^2) (1 - \chi_0(e_0 h^2 n)) \varphi_{m,n} e^{iny}.$$

Applying Lemma 3.1 to  $f_h$ , we obtain that

$$\sum_{m,n} |n| |a_{m,n}|^2 \psi(h^2 \lambda_{m,n}^2)^2 (1 - \chi_0(e_0 h^2 n))^2 \leq \sum_{m,n} \lambda_{m,n}^2 |a_{m,n}|^2 \psi(h^2 \lambda_{m,n}^2)^2 (1 - \chi_0(e_0 h^2 n))^2.$$

For the non-zero contributions on the right side,  $\lambda_{m,n}^2 \leq \frac{4}{h^2}$ , while for the non-zero contributions on the left side, we have  $|n| > \frac{2}{e_0 h^2}$ . Therefore, if  $e_0 < \frac{1}{2}$ , the only possibility is that  $f_h = 0$ . This completes the proof of Corollary 3.2.  $\square$

#### 4. HALF-WAVE REGIME:

In this section, the numerical constant  $e_0 > \frac{1}{2}$  is fixed, and the cut-offs  $\chi_0$  and  $\psi$  are also fixed as in the previous section. Denote by  $\tilde{\psi}$  another cut-off function with similar support property as  $\psi$ , such that

$$\psi \tilde{\psi} \equiv \psi.$$

For  $0 < h < 1, b_0 > 0$ , we define the semi-classical spectral projector

$$\Pi_h^{b_0 h} := \psi(-h^2 \Delta_G) (\chi_0(e_0 h^2 D_y) - \chi_0(b_0 h D_y)).$$

This allows us to localize the spectral into the regime:  $h^{-1} \ll |D_y| \lesssim h^{-2}$ , and the half-wave regime formally corresponds to  $h^2 |D_y| \sim 1, h \sqrt{-\Delta_G} \sim 1$ .

The goal of this section is to prove the following observability.

**Proposition 4.1.** Fix  $T > a_0$ . There exist  $b_0 > 0, h_0 > 0$ , such that for all  $h < h_0$  we have

$$\| \Pi_h^{b_0 h} u_0 \|_{L^2(\Omega)}^2 \leq C_{T,a} \int_{-T}^T \| \phi(y) e^{it \Delta_G} \Pi_h^{b_0 h} u_0 \|_{L^2(\Omega)}^2 dt. \quad (4.1)$$

We will prove the inequality above by proving firstly a localized version for  $h^{-1} \ll |D_y| \sim \hbar^{-1} \leq c_0 h^{-2}$ , and gluing such inequalities together. The constant  $b_0 > 0$  will be chosen in the proof as a small parameter. Note that  $\hbar$  can be viewed as a second semi-classical parameter.

For any  $\hbar^{-1} > (b_0 h)^{-1}$ ,  $0 < \delta < 1$ , we denote by

$$u_{h,\hbar,\delta} := \tilde{\psi}(-h^2 \Delta_G) \tilde{\psi} \left( \frac{\hbar |D_y| - 1}{\delta} \right) u. \quad (4.2)$$

**4.1. Elliptic regularity.** Denote by  $\mathcal{L}_n = -\partial_x^2 + n^2 x^2$ , the semi-classical harmonic oscillator associated with the Dirichlet boundary condition. Note that  $\varphi_{m,n}(x)$  is the  $m$  th eigenfunction of  $\mathcal{L}_n$  with eigenvalue  $\lambda_{m,n}^2$ . Denote by  $H_n = -\partial_x^2 + n^2 x^2$ , the semi-classical harmonic oscillator on the whole line with domain  $D(H_n) = H^2(\mathbb{R}) \cap \mathcal{H}^1(\mathbb{R})$ , where  $\mathcal{H}^1(\mathbb{R})$  is defined via the norm

$$\|f\|_{\mathcal{H}^1(\mathbb{R})}^2 := \|\partial_x^2 f\|_{L^2(\mathbb{R})}^2 + \|xf\|_{L^2(\mathbb{R})}^2.$$

Denote by  $\omega_{j,n}^2$ , the  $j$  th eigenvalue of  $H_n$ . It is well-known that  $\omega_{j,n}^2 = (2j+1)|n|$ . We need the following comparison principle.

**Lemma 4.2.** *Let  $n \in \mathbb{N}$ , then for any  $j \in \mathbb{N}$ , we have*

$$\omega_{j,n}^2 \leq \lambda_{j,n}^2.$$

*Proof.* This is a simple consequence of the max-min principle. Recall that from max-min principle,

$$\lambda_{j,n}^2 = \sup_{\psi_1, \dots, \psi_{j-1} \in L^2((-1,1))} \inf_{\substack{f \in \text{span}(\psi_1, \dots, \psi_n)^\perp \\ f \neq 0, f \in H_0^1((-1,1))}} \frac{(\mathcal{L}_n f, f)_{L^2((-1,1))}}{\|f\|_{L^2((-1,1))}^2},$$

and

$$\omega_{j,n}^2 = \sup_{\psi_1, \dots, \psi_{j-1} \in L^2(\mathbb{R})} \inf_{\substack{f \in \text{span}(\psi_1, \dots, \psi_n)^\perp \\ f \neq 0, f \in \mathcal{H}^1(\mathbb{R})}} \frac{(H_n f, f)_{L^2(\mathbb{R})}}{\|f\|_{L^2(\mathbb{R})}^2}.$$

Define  $\iota f := f \mathbf{1}_{(-1,1)}$ , the zero-extension of  $f \in H_0^1(I_R)$ , then obviously  $\iota f \in \mathcal{H}^1(\mathbb{R})$ . Thus by definition, we have

$$\begin{aligned} \omega_{j,n}^2 &\leq \sup_{\psi_1, \dots, \psi_{j-1} \in L^2(\mathbb{R})} \inf_{\substack{f \in \text{span}(\psi_1, \dots, \psi_n)^\perp \\ f \neq 0, f \in H_0^1((-1,1))}} \frac{(H_n(\iota f), \iota f)_{L^2(\mathbb{R})}}{\|\iota f\|_{L^2(\mathbb{R})}^2} \\ &= \sup_{\psi_1, \dots, \psi_{j-1} \in L^2(\mathbb{R})} \inf_{\substack{f \in \text{span}(\psi_1, \dots, \psi_n)^\perp \\ f \neq 0, f \in H_0^1((-1,1))}} \frac{(\mathcal{L}_n f, f)_{L^2(\mathbb{R})}}{\|f\|_{L^2(\mathbb{R})}^2} \\ &\leq \sup_{\psi_1, \dots, \psi_{j-1} \in L^2((-1,1))} \inf_{\substack{f \in \text{span}(\psi_1, \dots, \psi_n)^\perp \\ f \neq 0, f \in H_0^1((-1,1))}} \frac{(\mathcal{L}_n f, f)_{L^2(\mathbb{R})}}{\|f\|_{L^2(\mathbb{R})}^2} \\ &= \lambda_{j,n}^2. \end{aligned}$$

This completes the proof of Lemma 4.2. □

Using that by comparison principle, we have  $N_{n,\tau} \leq \#\{j : \omega_{j,n}^2 \leq \tau^2\} \leq \frac{\tau^2}{2|n|}$ , we get

**Corollary 4.3.** *For fixed  $n$ ,*

$$N_{n,\tau} := \#\{j : \lambda_{j,n}^2 \leq \tau^2\} \leq \frac{\tau^2}{2|n|}.$$

Let  $\chi_1 \in C_c^\infty(\mathbb{R})$  such that  $\chi_1(\zeta)$  vanishes if  $|\zeta| > C_1$  or  $|\zeta| < c_1$ , for some  $C_1 > c_1 > 0$ . We have the following

**Proposition 4.4** (Elliptic regularity). *Let  $\sigma > 0$ . There exist small constants  $0 < h_0 \ll 1, 0 < b_0 \ll 1$ , such that for all  $0 < h < h_0$  and  $\frac{\epsilon_0 h^2}{4} < \hbar < b_0 h$ , the following statement is true. For  $w_{h,\hbar} = \psi(h^2 \Delta_G) \chi_1(\hbar D_y) w_{h,\hbar}$  and  $\chi \in C_c^\infty(-1, 1)$  such that  $\chi(x) \equiv 1$  near 0, we have with  $\epsilon = \max(h^{1-\sigma}, 5c_1^{-1} \hbar h^{-1})$*

$$\|(1 - \chi)\left(\frac{x}{\epsilon}\right) h \partial_x w_{h,\hbar}\|_{L^2(\Omega)} + \|(1 - \chi)\left(\frac{x}{\epsilon}\right) w_{h,\hbar}\|_{L^2(\Omega)} \leq C_N h^N, \quad (4.3)$$

for all  $N \in \mathbb{N}$ .

*Proof.* We first prove the estimate for eigenfunctions

**Lemma 4.5.** *For all  $\chi \in C^\infty(-1, 1)$  and all  $N$ , there exists  $C > 0$  such that for all eigenfunction  $\Phi_{m,n}(x, y) = e^{iny} \varphi_{m,n}(x)$ , where  $|n| \in [c_1 \hbar^{-1}, C_1 \hbar^{-1}]$ ,  $\|\Phi_{m,n}\|_{L^2(-1,1)} = 1$ ,*

$$\|(1 - \chi)\left(\frac{x}{\epsilon}\right) h \partial_x \Phi_{m,n}\|_{L^2(\Omega)}^2 + \|(1 - \chi)\left(\frac{x}{\epsilon}\right) \Phi_{m,n}\|_{L^2(\Omega)}^2 \leq C_N h^N, \quad (4.4)$$

for all  $N \in \mathbb{N}$ .

*Proof.* Thanks to the support property of  $\chi_1$ . The eigenfunction  $\varphi_{m,n}$  of the operator  $\mathcal{L}_n = -\partial_x^2 + n^2 x^2$  satisfies

$$\mathcal{L}_n \varphi_{m,n} = \lambda_{m,n}^2 \in \left[ \frac{1}{16h^2}, \frac{16}{h^2} \right], \quad \varphi_{m,n}|_{x=\pm 1} = 0. \quad (4.5)$$

Taking the Fourier transform with respect to the  $y$  variable, multiplying (4.5) by  $(1 - \chi)^2\left(\frac{x}{\epsilon}\right) \bar{\varphi}_{m,n}$  and doing the integration by part, we obtain that

$$\begin{aligned} \int_{-1}^1 (1 - \chi)^2\left(\frac{x}{\epsilon}\right) [|\varphi'_{m,n}|^2 + n^2 x^2 |\varphi_{m,n}|^2] dx - \int_{-1}^1 \frac{2}{\epsilon} (1 - \chi) \chi' \left(\frac{x}{\epsilon}\right) \varphi'_{m,n} \bar{\varphi}_{m,n} dx \\ = \lambda_{m,n}^2 \int_{-1}^1 (1 - \chi)^2\left(\frac{x}{\epsilon}\right) |\varphi_{m,n}|^2 dx. \end{aligned} \quad (4.6)$$

This yields

$$\int_{-1}^1 (1 - \chi)^2\left(\frac{x}{\epsilon}\right) [|\varphi'_{m,n}|^2 + (n^2 x^2 - \lambda_{m,n}^2) |\varphi_{m,n}|^2] dx \leq \frac{2}{\epsilon} \int_{-1}^1 |(1 - \chi) \chi' \varphi'_{m,n} \varphi_{m,n}| dx.$$

Note that on the support of  $(1 - \chi)(\frac{x}{\epsilon})$ ,  $n^2 x^2 - \lambda_{m,n}^2 \geq 9h^{-2}$ , which implies

$$\begin{aligned} h^2 \left\| (1 - \chi)\left(\frac{x}{\epsilon}\right) \varphi'_{m,n} \right\|_{L^2(-1,1)}^2 + 9 \left\| (1 - \chi)\left(\frac{x}{\epsilon}\right) \varphi_{m,n} \right\|_{L^2(-1,1)}^2 \\ \leq \frac{2h^2}{\epsilon} \left\| (1 - \chi)\left(\frac{x}{\epsilon}\right) \varphi'_{m,n} \right\|_{L^2(-1,1)} \left\| \chi'\left(\frac{x}{\epsilon}\right) \varphi_{m,n} \right\|_{L^2(-1,1)}. \end{aligned} \quad (4.7)$$

We now show (4.4) for  $N = k\sigma$  by induction on  $k$ . Applying the induction assumption we get

$$\left\| (1 - \chi)\left(\frac{x}{\epsilon}\right) \varphi'_{m,n} \right\|_{L^2(-1,1)} \left\| \chi'\left(\frac{x}{\epsilon}\right) \varphi_{m,n} \right\|_{L^2(-1,1)} \leq Ch^{k\sigma-1}$$

which, from (4.7) implies (using  $\epsilon \geq h^{1-\sigma}$ ),

$$\left\| (1 - \chi)\left(\frac{x}{\epsilon}\right) \varphi'_{m,n} \right\|_{L^2(-1,1)} + \left\| (1 - \chi)\left(\frac{x}{\epsilon}\right) \varphi_{m,n} \right\|_{L^2(-1,1)} \leq Ch^{(k+1)\sigma}.$$

□

For  $w_{h,\hbar}$ , we expand it as

$$w_{h,\hbar} = \sum_{|n| \sim \hbar^{-1}, \lambda_{m,n} \sim \hbar^{-1}} a_{m,n} \Phi_{m,n}.$$

Note that for fixed  $n$ , applying Cauchy-Schwartz and Corollary 4.3, we have

$$\begin{aligned} \left\| \chi(x) \sum_{m: \lambda_{m,n} \sim \hbar^{-1}} a_{m,n} \varphi_{m,n}(x) \right\|_{L^2(-1,1)} + \hbar \left\| \chi(x) \sum_{m: \lambda_{m,n} \sim \hbar^{-1}} a_{m,n} \varphi'_{m,n}(x) \right\|_{L^2(-1,1)} \\ \leq C_N \hbar^N \left( \sum_{m: \lambda_{m,n} \sim \hbar^{-1}} |a_{m,n}|^2 \right)^{1/2} (\#\{m : \lambda_{m,n} \sim \hbar^{-1}\})^{1/2} \leq C_N \hbar^N \hbar^{-1} \hbar^{-1/2} \leq C'_N \hbar^{N-2}. \end{aligned} \quad (4.8)$$

Finally, using Plancherel's identity, we obtain that

$$\left\| \chi(x) \hbar \partial_x w_{h,\hbar} \right\|_{L^2(\Omega)} + \left\| \chi(x) w_{h,\hbar} \right\|_{L^2(\Omega)} \leq C_N \hbar^N,$$

for all  $N \in \mathbb{N}$ . This completes the proof of Proposition 4.4.

□

**4.2. Half-wave regime.** We shall use the following elementary lemma.

**Lemma 4.6.** *For any  $0 < \epsilon_0 < 1$ , there exists a  $C^1$  function  $\varphi$ , compactly supported in  $[-1, 1]$ , such that*

- (1)  $\varphi(t) \equiv 1, \forall |t| \leq 1 - \epsilon_0$ ,  $0 \leq \varphi(t) \leq 1$  and  $\varphi(t) \equiv 0, \forall |t| > 1$ .
- (2)

$$\int_{1-\epsilon_0 \leq |t| \leq 1} |\varphi'(t)| dt = 2.$$

Introducing the Grushin energy norm

$$\|f\|_{\dot{H}_G^1(\Omega)}^2 := \|\partial_x f\|_{L^2(\Omega)}^2 + \|x \partial_y f\|_{L^2(\Omega)}^2.$$

The following proposition is crucial in the half-wave regime.

**Proposition 4.7.** *Assume that  $T > a_0$ . Let  $u_{h,\hbar,\delta}$  be the solution of (1.3) with the localization property (4.2). Then there exist small constants  $0 < h_0, \hbar_0, \delta_0, \sigma_0 < 1$ , and  $C_{T,a_0} > 0$  such that for all  $0 < h < h_0, 0 < \delta < \delta_0, 0 < \hbar < \hbar_0, 2e_0 < \rho \leq \sigma_0 \hbar^{-1/2}$  obeying  $\frac{\epsilon_0 \hbar^2}{2} < \hbar < \rho \hbar^2$ , we have*

$$\begin{aligned} \|u_{h,\hbar,\delta}(0)\|_{\dot{H}_G^1(\Omega)}^2 &\leq C_{T,a_0} \int_{-T}^T \left( \|\phi(y)x\partial_y u_{h,\hbar,\delta}(t)\|_{L^2(\Omega)}^2 + \|\phi(y)\partial_x u_{h,\hbar,\delta}(t)\|_{L^2(\Omega)}^2 \right) dt \\ &+ C_{T,a_0} \int_{-T}^T \hbar^{-1} \|\phi(y)u_{h,\hbar,\delta}(t)\|_{L^2(\Omega)}^2 dt + C_{T,a_0} (1 + \delta \hbar^{-1}) \|u_{h,\hbar,\delta}(0)\|_{L^2(\Omega)}^2. \end{aligned} \quad (4.9)$$

(recall that numerical constant  $e_0$  satisfies  $e_0 > \frac{1}{2}$ .)

*Proof.* We first give the proof in the simpler case where there is a single band in  $\omega^c$ , and by translation in  $y$  we can assume that  $\omega^c = (-1, 1)_x \times (-a, a)$ , hence  $\mathcal{L}(\omega) = 2a$ . In this case, the cut-off function  $\phi$  defined in (2.1) satisfies

$$\phi(y) = 0 \text{ if } |y| \leq a; \quad \phi(y) = 1 \text{ if } a_0 < |y| \leq \pi.$$

To simplify the notation, we denote by  $u = u_{h,\hbar,\delta}$ . Pick  $\varphi(t)$  as in the Lemma 4.6, with a small parameter  $\epsilon_0 > 0$  to be chosen later. For  $T > 0$ , we define  $\varphi_T(t) := \varphi\left(\frac{t}{T}\right)$ . Take  $\chi_1(x) = \chi(\epsilon^{-1}x)$  for some function  $\chi \in C_c^\infty(\mathbb{R})$  with the property that  $\chi(\zeta) \equiv 1$  near 0, where  $\epsilon = \max\{h^{1-\sigma}, 5c_1^{-1}\hbar h^{-1}\}$  as in Proposition 4.4.

Take  $\chi_2 \in C_c^\infty((-\pi, \pi))$  equal to 1 on  $(-a, a)$  and such that  $\text{supp}(\chi_2') \subset \text{supp}(\phi_2)$ . We define another cut-off  $\phi_1(y)$  such that

$$\text{supp}(\chi_2') \subset \text{supp}(\phi_1) \subset \text{supp}(\phi), \quad \phi_1|_{\text{supp}(\chi_2')} \equiv 1, \quad \phi|_{\text{supp}(\phi_1)} \equiv 1. \quad (4.10)$$

Moreover, we require that  $\text{supp}(1 - \chi_2) \subset \text{supp}(\phi_1)$ . The proof is an application of positive commutator method, based on the simple commutator relation

$$[\Delta_G, x\partial_x + 2y\partial_y] = \Delta_G. \quad (4.11)$$

More precisely, with cutoffs  $\varphi_T(t), \chi_1(x), \chi_2(y)$ , we have

$$\begin{aligned} &[i\partial_t + \Delta_G, \varphi_T(t)\chi_1(x)\chi_2(y)(x\partial_x + 2y\partial_y)] \\ &= 2\varphi_T(t)\chi_1(x)\chi_2(y)\Delta_G + i\varphi_T'(t)\chi_1(x)\chi_2(y)(x\partial_x + 2y\partial_y) \\ &+ \varphi_T(t) [\chi_2(y)(\chi_1''(x) + 2\chi_1'(x)\partial_x) + x^2\chi_1(x)(\chi_2''(y) + 2\chi_2'(y)\partial_y)] (x\partial_x + 2y\partial_y). \end{aligned} \quad (4.12)$$

By developing the commutator and using the equation, we have

$$([i\partial_t + \Delta_G, \varphi_T(t)\chi_1\chi_2(x\partial_x + 2y\partial_y)]u, u)_{L^2(\mathbb{R} \times \Omega)} = 0.$$

On the other hand, using (4.12), we obtain that

$$\begin{aligned}
-2(\varphi_T(t)\chi_1\chi_2\Delta_G u, u)_{L^2(\mathbb{R}\times\Omega)} &= \underbrace{i(\varphi'_T(t)\chi_1\chi_2(x\partial_x + 2y\partial_y)u, u)}_{\text{I}}_{L^2(\mathbb{R}\times\Omega)} \\
&\quad + \underbrace{(\varphi_T(t)\chi_2(\chi_1'' + 2\chi_1'\partial_x)(x\partial_x + 2y\partial_y)u, u)}_{\text{II}}_{L^2(\mathbb{R}\times\Omega)} \\
&\quad + \underbrace{(\varphi_T(t)x^2\chi_1(\chi_2'' + 2\chi_2'\partial_y)(x\partial_x + 2y\partial_y)u, u)}_{\text{III}}_{L^2(\mathbb{R}\times\Omega)}.
\end{aligned} \tag{4.13}$$

First, we observe that on the support of  $1 - \chi_1(x)$ ,  $\chi_1'(x)$ ,  $\chi_1''(x)$ , we have  $|x| > \epsilon$ . Therefore, from Proposition 4.4, together with the fact that  $\|\partial_y u\|_{L^2(\Omega)} \sim \hbar^{-1}\|u\|_{L^2(\Omega)} \leq Ch^{-2}\|u\|_{L^2(\Omega)}^2$ ,

$$|\text{II}| \leq \mathcal{R}_1 := C_{T,N}h^N\|u(0)\|_{L^2(\Omega)}^2.$$

Next, we observe that the supports of  $1 - \chi_2(y)$ ,  $\chi_2'(y)$ ,  $\chi_2''(y)$  are both contained in  $\{y : \phi_1(y) > 0\}$ . Thus from integration by part, we obtain that

$$|\text{III}| \leq \mathcal{R}_2 := C \int_{\mathbb{R}} \int_{\Omega} \varphi_T(t)\phi_1(y)^2 [|\partial_x u|^2 + |x\partial_y u|^2 + |u|^2] dx dy dt.$$

Similarly, doing integration by part, we deduce that the left hand side of (4.13) equals to

$$2 \int_{\mathbb{R}} \varphi_T(t)\chi_1(x)\chi_2(y) (|\partial_x u|^2 + |x\partial_y u|^2) dy dx dt + O(\mathcal{R}_1) + O(\mathcal{R}_2).$$

Using again Proposition 4.4 we can eliminate the  $\chi_1$  cut off and get (adding another  $\mathcal{R}_1$  error)

$$2 \int_{\mathbb{R}} \varphi_T(t)\chi_2(y) (|\partial_x u|^2 + |x\partial_y u|^2) dy dx dt + O(\mathcal{R}_1) + O(\mathcal{R}_2).$$

Moreover,

$$|(\varphi'_T(t)\chi_1\chi_2x\partial_x u, u)_{L^2(\mathbb{R}\times\Omega)}| \leq C_T\epsilon\|u(0)\|_{\dot{H}_G^1(\Omega)}\|u(0)\|_{L^2(\Omega)},$$

since on the support of  $\chi_1(x)$ ,  $|x| \leq C\epsilon$ . This term is in lower order and can be bounded by

$$\mathcal{R}_3 \leq \epsilon\|u(0)\|_{\dot{H}_G^1(\Omega)}^2 + C_{T,a'}\epsilon\|u(0)\|_{L^2(\Omega)}^2.$$

Thus, using that  $\chi_2 + \phi \geq 1$  on  $(-\pi, \pi)$ , (because we can add  $\mathcal{R}_2$  on the l.h.s.) we obtain from (4.13) that

$$2 \int_{\mathbb{R}} \varphi_T(t)\|u(t)\|_{\dot{H}_G^1(\Omega)}^2 dt \leq 2|(\varphi'_T(t)\chi_2(y)y\partial_y u, u)_{L^2(\mathbb{R}\times\Omega)}| + \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3. \tag{4.14}$$

Pick  $a' \in (a_0, T)$ , such that  $\phi_1(y) > C_{a'}^{-1} > 0$  for all  $a' \leq |y| \leq \pi$ . Therefore, the first term on the right side of (4.14) can be majorized by

$$\begin{aligned} & \underbrace{2a' \int_{(1-\epsilon_0)T \leq |t| \leq T} |\varphi'_T(t)| \|\partial_y u(t)\|_{L^2(|y| \leq a')} \|u(t)\|_{L^2(|y| \leq a')} dt}_{\text{IV}} \\ & + \underbrace{C_{a'}^2 \int_{\mathbb{R}} |\varphi'_T(t)| \|\phi_1(y) \partial_y u(t)\|_{L^2(\Omega)} \|\phi_1(y) u(t)\|_{L^2(\Omega)} dt}_{\text{V}}. \end{aligned} \quad (4.15)$$

Note that by our choice of  $\phi_1$  and the localization property of  $u = \psi(\hbar D_y)u$ , we can write

$$\phi_1(y) \partial_y u = \phi_1(y) \partial_y (\phi(y) u) = \phi_1(y) \partial_y \psi(\hbar D_y) (\phi(y) u) + \phi_1(y) \partial_y [\phi(y), \psi(\hbar D_y)] u.$$

Using the fact that  $\|\hbar \partial_y [\phi, \psi(\hbar D_y)]\|_{L^2 \rightarrow L^2} = O(\hbar)$ , we can majorize the second term of (4.19) by

$$V \leq \mathcal{R}_4 := C_{T, a'} \int_{-T}^T \hbar^{-1} \|\phi(y) u(t)\|_{L^2(\Omega)}^2 dt + C_{T, a'} \|u(0)\|_{L^2(\Omega)}^2.$$

It remains to treat IV.

Using the spectral localization  $\|\hbar D_y - 1\| \leq \delta$  of  $u$ , we can majorize IV by

$$\frac{2a'(1+3\delta)}{\hbar} \int_{(1-\epsilon_0)T \leq |t| \leq T} |\varphi'_T(t)| \|u(t)\|_{L^2(\Omega)}^2 dt. \quad (4.16)$$

From the conservation of mass and Lemma 4.6, we have that

$$(4.16) \leq \frac{2a'(1+3\delta_0)}{\hbar T(1-\epsilon_0)} \int_{\mathbb{R}} \varphi_T(t) \|u(t)\|_{L^2(\Omega)}^2 dt.$$

Denote by  $\Pi_{\pm}$  be the projector to positive(negative) Fourier modes in  $y$ . We write

$$\|u(t)\|_{L^2(\Omega)}^2 = (\Pi_+ u(t), \Pi_+ u(t))_{L^2(\Omega)} + (\Pi_- u(t), \Pi_- u(t))_{L^2(\Omega)}.$$

Hence we have

$$|((1 \mp \hbar D_y) \Pi_{\pm} u(t), \Pi_{\pm} u(t))_{L^2(\Omega)}| \leq \delta \|u(t)\|_{L^2(\Omega)}^2.$$

Now using the fact that  $[\partial_x, x \partial_y] = \partial_y$  and  $[\partial_x, \Pi_{\pm}] = 0$ , we obtain that

$$\begin{aligned} \|u(t)\|_{L^2(\Omega)}^2 & \leq 2\delta \|u(t)\|_{L^2(\Omega)}^2 + 2\hbar \|\partial_x \Pi_+ u(t)\|_{L^2(\Omega)} \|x \partial_y \Pi_+ u(t)\|_{L^2(\Omega)} \\ & \quad + 2\hbar \|\partial_x \Pi_- u(t)\|_{L^2(\Omega)} \|x \partial_y \Pi_- u(t)\|_{L^2(\Omega)} \\ & \leq 2\delta \|u(t)\|_{L^2(\Omega)}^2 + \hbar \|u(t)\|_{\dot{H}_G^1(\Omega)}^2. \end{aligned}$$

Plugging into (4.14), we have

$$2 \|\varphi_T^{1/2} u\|_{L^2(\mathbb{R}; \dot{H}_G^1(\Omega))}^2 \leq \frac{2a'(1+3\delta)}{T(1-\epsilon_0)} \|\varphi_T^{1/2} u\|_{L^2(\mathbb{R}; \dot{H}_G^1(\Omega))}^2 + C_{T, a'} \hbar^{-1} \delta \|u(0)\|_{L^2(\Omega)}^2 + \sum_{k=1}^4 \mathcal{R}_k. \quad (4.17)$$



Note that if  $T > a_0$ , we can choose  $\delta_0 > 0$ ,  $\epsilon_0 > 0$  sufficiently small, such that

$$\frac{a'(1+3\delta)}{T(1-\epsilon_0)} < 1.$$

Then we choose  $\hbar_0, h_0 > 0, \sigma_0 > 0$  sufficiently small, then for any  $0 < h < h_0, 0 < \hbar < \hbar_0$ ,  $2e_0 < \rho < \sigma_0 \hbar^{-1/2}$ , and  $\frac{\epsilon_0 h^2}{2} < \hbar < \rho h^2$ . Substituting the expression of  $\mathcal{R}_k$ , the proof of Proposition 4.7 is then complete in the particular case where there is only one band in  $\omega^c$ .

In the general case, there may be several bands (all with width at most  $\mathcal{L}(\omega)$ ). In the previous proof, we replace the cut-off  $\chi_2$  by a family of cut-off  $\chi_2^j(y)$  equal to 1 on  $(a_j, b_j)$  and satisfying all the other assumptions of  $\chi_2$  and we can perform the same estimates with the commutator relation (4.11) replaced by

$$[\Delta_G, x\partial_x + 2(y - \frac{b_j + a_j}{2})\partial_y] = \Delta_G, \quad (4.18)$$

(i.e. centering our estimates on the middle of the segment  $(a_j, b_j)$ ) leading to the following version of (4.14)

$$2 \int_{\mathbb{R}} \varphi_T(t) \|u(t)\|_{\dot{H}_G^1(\Omega)}^2 dt \leq 2 \sum_{j=1}^N \left| \left( \varphi_T'(t) \chi_2^j(y) \left( y - \frac{b_j + a_j}{2} \right) \partial_y u, u \right)_{L^2(\mathbb{R} \times \Omega)} \right| + \sum_{k=1}^4 \mathcal{R}_k, \quad (4.19)$$

and we conclude the proof as previously.  $\square$

Next, we use a simple integration by part argument to pass to the  $L^2$  observability estimate.

**Corollary 4.8.** *Under the same assumption as Proposition 4.7, we have*

$$\|u_{h,\hbar,\delta}(0)\|_{L^2(\Omega)}^2 \leq C_{T,a_0,b_0} \int_{-T}^T \|\phi(y)u_{h,\hbar,\delta}(t)\|_{L^2(\Omega)}^2 dt.$$

*Proof.* The right side of (4.9) is

$$\|u(0)\|_{\dot{H}_G^1(\Omega)}^2 \sim h^{-2} \|u(0)\|_{L^2(\Omega)}^2.$$

Since  $\hbar^{-1} < \frac{2}{\epsilon_0 h^2}$ , the left side of (4.9) can be majorized by

$$C_{T,a_0} \int_{-T}^T \int_{\Omega} \phi(y)^2 [|\partial_x u|^2 + |x\partial_y u|^2] dx dy dt + C'_{T,a_0} \left( 1 + \delta h^{-2} + h^{-2} \int_{-T}^T \int_{\Omega} \phi(y)^2 |u|^2 dt dx dy \right).$$

Doing integration by part, we have

$$\int_{-T}^T \int_{\Omega} \phi(y)^2 [|\partial_x u|^2 + |x\partial_y u|^2] dx dy dt = - \int_{-T}^T \int_{\Omega} [\phi(y)^2 \Delta_G u \bar{u} dx dy dt + 2\phi\phi'(y)x^2\partial_y u \bar{u}] dx dy dt.$$

By Cauchy-Schwartz, the right hand side can be bounded by

$$\|\phi(y)u\|_{L^2((-T,T)\times\Omega)} \|\Delta_G u\|_{L^2((-T,T)\times\Omega)} + \|\phi(y)u\|_{L^2((-T,T)\times\Omega)} \|u\|_{\dot{H}_G^1(\Omega)}.$$

From the localization of the spectral, we have

$$\int_{-T}^T \int_{\Omega} \phi(y)^2 [|\partial_x u|^2 + |x\partial_y u|^2] dx dy dt \leq \frac{C_T}{\hbar^2} \|\phi(y)u\|_{L^2((-T,T)\times\Omega)} \|u(0)\|_{L^2(\Omega)}.$$

Finally, applying Young's inequality of the form

$$ab \leq \epsilon a^2 + C(\epsilon)b^2,$$

the proof of Corollary 4.8 is complete.  $\square$

The following proposition shows that it is the regime for which  $\hbar$  is close to  $h^2$  that induces the condition  $T > a_0$ .

**Proposition 4.9.** *Assume that  $T > 0$ , then there exist  $b_0 > 0, \rho_0 > 0, h_0 > 0, \hbar_0 > 0, \delta_0 > 0$ , and  $C_T > 0$ , such that for all  $0 < h < h_0, 0 < \hbar < \hbar_0, 0 < \delta < \delta_0$ , and  $\rho \geq \rho_0$ , obeying  $\rho h^2 < \hbar \leq b_0 h$ , we have*

$$\|u_{h,\hbar,\delta}(0)\|_{L^2(\Omega)}^2 \leq C_T \int_{-T}^T \|\phi(y)u_{h,\hbar,\delta}(t)\|_{L^2(\Omega)}^2 dt. \quad (4.20)$$

*Proof.* We use the same positive commutator method as in the proof of Proposition 4.7, but the proof is much simpler in the present situation. We adapt the notation there. Indeed, from (4.13), the term |II|, |III| can be majorized by the same bound  $\mathcal{R}_1, \mathcal{R}_2$ . For  $I = i(\varphi'_T(t)\chi_1\chi_2(x\partial_x + 2y\partial_y)u, u)_{L^2(\mathbb{R}\times\Omega)}$ , we estimate it as

$$|(\varphi'_T(t)\chi_1\chi_2x\partial_x u, u)_{L^2(\mathbb{R}\times\Omega)}| \leq C_T \epsilon \|u(0)\|_{\dot{H}_G^1(\Omega)} \|u(0)\|_{L^2(\Omega)},$$

where  $\epsilon = \max\{h^{1-\sigma}, 5c_1^{-1}\hbar h^{-1}\}$ , and

$$|(\varphi'_T(t)\chi_1\chi_2y\partial_y u, u)_{L^2(\mathbb{R}\times\Omega)}| \leq C_T \|\partial_y u(0)\|_{L^2(\Omega)} \|u(0)\|_{L^2(\Omega)} \leq C_T \hbar^{-1} \|u(0)\|_{L^2(\Omega)}^2.$$

Therefore,

$$\begin{aligned} \int_{\mathbb{R}} \varphi_T(t) \|u(t)\|_{\dot{H}_G^1(\Omega)}^2 dt &\leq C_T \int_{-T}^T \int_{\Omega} \phi(y)^2 [|\partial_x u|^2 + |x\partial_y u|^2] dx dy dt \\ &\quad + C_T \epsilon \|u(0)\|_{\dot{H}^1(\Omega)} \|u(0)\|_{L^2(\Omega)} + C_T \hbar^{-1} \|u(0)\|_{L^2(\Omega)}^2. \end{aligned} \quad (4.21)$$

Note that  $\|u(t)\|_{\dot{H}_G^1(\Omega)} = \|u(0)\|_{\dot{H}_G^1(\Omega)} \sim h^{-2}$ , after doing integration by part as in the proof of Corollary 4.8, we obtain that

$$\|u(0)\|_{L^2(\Omega)}^2 \leq C_T \int_{-T}^T \|\phi(y)u(t)\|_{L^2(\Omega)}^2 dt + C_T \epsilon \|u(0)\|_{L^2(\Omega)}^2 + C_T h^2 \hbar^{-1} \|u(0)\|_{L^2(\Omega)}^2.$$

Note that  $\hbar h^{-1} \leq b_0$ ,  $h^2 \hbar^{-1} \leq \rho^{-1}$ . We choose  $\rho_0 > 0$  large enough and  $b_0 > 0$  small enough, such that

$$C_T b_0 + C_T \rho_0^{-1} < \frac{1}{2}.$$

This completes the proof of Proposition 4.9.  $\square$

### 4.3. $L^2$ observability in the half-wave regime.

*Proof of Proposition 4.1.* Fix  $R > 1$ , close enough to 1 such that

$$\max\{R - 1, 1 - R^{-1}\} < \delta,$$

for some  $\delta$  fixed as in the Corollary 4.8 and Proposition 4.9. Choose a cut-off function  $\beta_R \in C_c^\infty((R^{-1}, R))$ ,  $\beta_R \geq 0$  and

$$\beta_0(s) + \sum_{k=1}^{\infty} \beta_{R,k}(s) = 1, \quad \forall s > 0.$$

where  $\beta_{R,k}(s) := \beta_R(R^{-k}s)$  and  $\beta_0(s) \in C_c^\infty([0, \frac{R+1}{2R}))$ .

We will apply Proposition 4.8 for the semi-classical parameter  $\hbar = \lambda = R^{-k}$  with large  $k$ , and denote by  $\rho = \rho_0$  as in Proposition 4.9. Fix the small parameters  $\sigma_0 > 0$  as in Proposition 4.7, and we choose  $h_0 > 0$  smaller than the one required in Proposition 4.7, Proposition 4.9, and  $\sigma_0 \rho^{-3/2}$ . Note that in this case, if  $\frac{e_0 h^2}{2} < \hbar < \rho h^2$ , the constraint  $\rho < \sigma_0 \hbar^{-1/2}$  to apply Proposition 4.7 is satisfied automatically. Denote simply by  $u_h = \Pi_h^{b_0 h} u$ , we know that  $\beta_{R,k}(|D_y|)u_h = 0$  unless

$$N_h := -\frac{\log(b_0 h)}{\log R} \leq k \leq M_h := -\frac{\log(e_0 h^2/2)}{\log R}.$$

Now from almost orthogonality

$$C_R^{-1} \|u_h\|_{L^2(\Omega)}^2 \leq \sum_{k \geq N_h} \|\beta_{R,k}(|D_y|)u_h\|_{L^2(\Omega)}^2 \leq C_R \|u_h\|_{L^2(\Omega)}^2$$

and Corollary 4.8, Proposition 4.9, we have that

$$\begin{aligned} \|u_h(0)\|_{L^2(\Omega)}^2 &\leq C_R \sum_{N_h \leq k \leq M_h} \|\beta_{R,k}(|D_y|)u_h(0)\|_{L^2(\Omega)}^2 \\ &\leq C_{T,R} \sum_{N_h \leq k \leq M_h} \int_{-T}^T \|\phi(y) \beta_{R,k}(|D_y|)u_h(t)\|_{L^2(\Omega)}^2 dt. \end{aligned} \tag{4.22}$$

Note that  $\|[\phi(y), \beta_{R,k}(|D_y|)]\|_{L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})} \leq C_R R^{-k}$ , we have

$$(4.22) \leq C_{T,R} \int_{-T}^T \|\Pi_h^{b_0 h}(\phi(y)u_h(t))\|_{L^2(\Omega)}^2 dt + C_{T,R} \sum_{N_h \leq k \leq M_h} R^{-2k} \int_{-T}^T \|u_h(t)\|_{L^2(\Omega)}^2 dt.$$

The second term on the right hand side can be bounded from above by

$$C_{T,R} R^{-2N_h} \|u_h(0)\|_{L^2(\Omega)}^2 \leq C_{T,R} (b_0 h)^2 \|u_h(0)\|_{L^2(\Omega)}^2.$$

By taking  $h < h_0$  small enough, it can be absorbed to the left side. The proof of Proposition 4.1 is now complete.  $\square$

5. DISPERSIVE REGIME I:  $b_0 h^{-1} \leq |D_y| \leq b_0^{-1} h^{-1}$ : SEMI-CLASSICAL CONTROL

Fix  $0 < b_0 < 1$  from the last section. Denote by

$$\Pi_{h,b_0} = \psi(h^2 \Delta_G) \chi_0(b_0 h D_y) (1 - \chi_0(b_0^{-1} h D_y)).$$

We will prove the following proposition in this section.

**Proposition 5.1.** *Given  $T > 0$ , there exists  $h_0 > 0$ , such that for all  $0 < h < h_0$ , the following inequality holds true for all  $u_0 \in L^2(\Omega)$ :*

$$\|\Pi_{h,b_0} u_0\|_{L^2(\Omega)}^2 \leq C_T \int_{-2T}^{2T} \|\phi(y) e^{it\Delta_G} \Pi_{h,b_0} u_0\|_{L^2(\Omega)}^2 dt. \quad (5.1)$$

*Proof.* The proof follows essentially the strategy of Lebeau ([Le92]). Making the change of variable  $t = hs$ ,  $v(s, x, y) = e^{it\Delta_G} \Pi_{h,b_0} u_0(x, y)$  and  $v_0 = \Pi_{h,b_0} u_0$ , the function  $v$  satisfies the semi-classical Schrödinger equation

$$ih\partial_s v + h^2 \Delta_G v = 0. \quad (5.2)$$

By using the conservation of  $L^2$  norm, the proof of (5.1) reduces to the semi-classical observability: there exists  $T_1 = T_1(b_0) > 0$ , such that

$$\|v_0\|_{L^2(\Omega)}^2 \leq C_{T_1} \int_{-T_1}^{T_1} \|\phi(y) v(s, \cdot)\|_{L^2(\Omega)}^2 ds. \quad (5.3)$$

Indeed, to see how (5.3) implies (5.1), we change back to  $t$  variable and rewrite (5.3) as

$$\|\Pi_{h,b_0} u_0\|_{L^2(\Omega)}^2 \leq \frac{C_{T_1}}{h} \int_{-hT_1}^{hT_1} \|\phi(y) \Pi_{h,b_0} u(t, \cdot)\|_{L^2(\Omega)}^2 dt.$$

For any  $k \in \mathbb{Z}$ , applying the inequality above to the initial data  $\Pi_{h,b_0} u_0(khT_1, \cdot)$  and using the  $L^2$  conservation, we obtain that

$$\|\Pi_{h,b_0} u_0\|_{L^2(\Omega)} = \|\Pi_{h,b_0} u(khT_1, \cdot)\|_{L^2(\Omega)} \leq \frac{C_{T_1}}{h} \int_{(k-1)hT_1}^{(k+1)hT_1} \|\phi(y) \Pi_{h,b_0} u(t, \cdot)\|_{L^2(\Omega)}^2 dt.$$

Summing over  $-\lfloor \frac{T}{T_1 h} \rfloor + 1 \leq k \leq \lfloor \frac{T}{T_1 h} \rfloor - 1$ , we obtain (5.1).

Now we prove (5.3). With a little abuse of the notation, we denote by  $v = \Pi_{h,b_0} v$ . Again by conservation of the  $L^2$  norm, it is sufficient to show that

$$\|v\|_{L^2((-2T_1, 0) \times \Omega)}^2 \leq C_{T_1} \|\phi(y) v\|_{L^2((-T_1, T_1) \times \Omega)}^2. \quad (5.4)$$

Denote by  $X = \mathbb{R}_t \times \Omega$  with boundary  $\partial X = \mathbb{R}_s \times \partial\Omega$ . Take  $\chi \in C_c^\infty(\mathbb{R})$  such that  $\chi|_{\text{supp}(\psi)} \equiv 1$ . Then, using that if  $u = e^{ish\Delta_G} u_0$ ,  $(1 - \chi)(-hD_s)u = (1 - \chi)(-h^2 \Delta_G)u$ , we get

$$(1 - \chi)(hD_s) e^{ish\Delta_G} \psi(h^2 \Delta_G) = 0.$$

Therefore, it suffices to prove (5.4) for time-frequency localized functions  $\chi(hD_s)v$ . We argue by contradiction. If (5.4) is not true, then we obtain sequences  $h_k \rightarrow 0$  and  $v_k = \chi(h_k D_s)\Pi_{h,b_0}v_k$ , satisfying

$$(ih_k\partial_s + h_k^2\Delta_G)v_k = 0, \quad \|v_k(0)\|_{L^2(\Omega)} = 1, \quad \text{and} \quad \|\phi(y)v_k\|_{L^2((-T_1,T_1)\times\Omega)} \rightarrow 0 \quad (5.5)$$

as  $k \rightarrow \infty$ . From [Ge91]<sup>1</sup>, after substracting a subsequence, still denoted by  $(v_k)$ , there exists a Radon measure on  $Z = j(\text{Ch}(P))$ , such that for any sum of the interior and tangential operators  $A_h = A_{i,h} + A_{\partial,h}$ , compactly supported in  $s$ , we have

$$\lim_{k \rightarrow \infty} (A_{h_k}v_k, v_k)_{L^2(X)} = \langle \mu, \kappa(\sigma(A_h)) \rangle.$$

Away from the boundary  $\partial X = \mathbb{R} \times \partial\Omega$ , this measure is invariant along the characteristic flow of  $H_p = \partial_s - 2\xi\partial_x + 2x\eta^2\partial_\xi - 2x^2\eta\partial_y$ , where  $p = -\sigma - \xi^2 - x^2\eta^2$  is the principal symbol. Near the boundary, the cotangent bundle can be decomposed as  $T^*\partial X = \tilde{\mathcal{E}} \cup \tilde{\mathcal{H}} \cup \tilde{\mathcal{G}}$ , where

$$\begin{aligned} \tilde{\mathcal{E}} &= \{(x, s, y; \sigma, \eta) : -\sigma - x^2\eta^2 < 0\}, & \tilde{\mathcal{H}} &= \{(x, t, y; \tau, \eta) : -\sigma - x^2\eta^2 > 0\}, \\ \tilde{\mathcal{G}} &:= \{(x, s, y; \sigma, \eta) : \sigma = -x^2\eta^2\}. \end{aligned}$$

Note that for any point  $\rho \in \tilde{\mathcal{G}}$ ,  $(H_p^2x)|_\rho > 0$  if  $x(\rho) = -1$  and  $(H_p^2x)|_\rho < 0$  if  $x(\rho) = 1$  where  $\rho \mapsto x(\rho)$  is the boundary defining function. Therefore  $\tilde{\mathcal{G}} = \widetilde{\mathcal{G}^{2,+}}$ , namely it consists only the diffractive points. It follows from [Bu04] that  $\mu(\tilde{\mathcal{H}}) = \mu(\tilde{\mathcal{E}}) = \mu(\widetilde{\mathcal{G}^{2,+}}) = 0$ . Moreover,  $\mu$  is invariant along the generalized bicharacteristic flow. Note that away from the reflexion points at the boundary  $\{x = \pm 1\}$ , the flow is given by

$$\dot{s} = -\sigma = \tau_0, \quad \dot{y} = 2x^2\eta, \quad \dot{\eta} = 0 \Rightarrow \tau = \tau_0, \dot{x} = 2\xi, \dot{\xi} = -2x\eta^2$$

with initial data  $-\sigma_0 \in \text{supp}(\psi) \subset (\frac{1}{4}, 4)$ ,  $b_0 \leq |\eta_0| \leq b_0^{-1}$ . Integrating the flow, in the  $x, \xi$  variables gives an ellipse away from the boundary  $\{x = \pm 1\}$ . As a consequence, the variable  $x$  is bounded away from 0 for long sequences of time, and hence  $y$  is strictly increasing for these sequences of time (and increasing otherwise). This shows that there exists  $T_1 = T_1(b_0) > 0$ , such that for any  $\rho$  on phase space,  $y(\pm T_1, \rho) \in \text{supp}(\phi)$ , the controlled region. As in the assumption (5.5),  $\mu|_{(-T_1, T_1) \times \text{supp}(\phi)} = 0$  hence  $\mu|_{(-2T_1, 0) \times \Omega} = 0$ . This contradicts to  $\|v_k\|_{L^2((-2T_1, 0) \times \Omega)} = 2T_1$ , since  $\|v_k(0)\|_{L^2(\Omega)} = 1$  and  $t \mapsto \|v_k(t)\|_{L^2(\Omega)}$  is a constant function. The proof of Proposition 5.1 is now complete. □

## 6. DISPERSIVE REGIME II: $h^{-\epsilon/2} \leq |D_y| \leq b_0 h^{-1}$

We fix  $0 < b_0 < 1$  from the last sections. Denote by

$$\underline{\Pi}_{h,b_0}^c := \psi(h^2\Delta_G)\chi_0(b_0^{-1}hD_y)(1 - \psi_0(h^\epsilon D_y))$$

In this section, we prove the following proposition:

---

<sup>1</sup>here we adopt the notation from [Bu04]

**Proposition 6.1.** *Given  $T > 0$ , there exist  $h_0 > 0$ ,  $C_T > 0$ , such that the following observability holds true for all  $0 < h < h_0$  and  $\epsilon > 0$ :*

$$\|\underline{\Pi}_{h,b_0}^\epsilon u_0\|_{L^2(\Omega)}^2 \leq C_T \int_0^T \|\phi(y)e^{it\Delta_G} \underline{\Pi}_{h,b_0}^{\epsilon_0} u_0\|_{L^2(\Omega)}^2 dt + C_T h^\epsilon \|u_0\|_{L^2(\Omega)}^2.$$

The proof of this theorem will be decomposed into several lemmas.

**6.1. Control of vertical propagation.** We will use the positive commutator method to control the vertical propagation. The goal is to control the norm  $\|\partial_y u\|_{L^2}$ . In the half-wave regime, this norm can be controlled by  $\|x\partial_y u\|_{L^2}$  and  $\|\partial_x u\|_{L^2}$  by spending some explicit classical time, thanks to the hypoellipticity. Away from half-wave regime, we can control  $\|\partial_y u\|_{L^2}$  directly by  $\|x\partial_y u\|_{L^2}$ , exploiting the horizontal propagation in semi-classical time, which is small in the classical time scale.

**Lemma 6.2** (Horizontal propagation). *Given  $0 < b_0 < \frac{1}{10}$  and  $0 < r_0 < 1$ , there exists  $h_0 > 0$ , such that the following is true for all  $0 < h < h_0$ : If  $(w_h)_{h>0}$  is a family of solutions of semi-classical equations*

$$ih\partial_s w_h + h^2 \Delta_G w_h = 0, \quad w_h|_{\partial\Omega} = 0$$

with spectral-localized property:

$$w_h = \psi_1(-h^2 \Delta_G) \chi_0(b_0^{-1} h D_y) w_h,$$

where  $\psi_1 \in C_c^\infty(1/2 \leq r \leq 2)$  and  $\chi_0 \in C_c^\infty(r \leq 2)$  are two cutoff functions. Then there exists a uniform constants  $\kappa > 0$ , independent of  $b_0$  and  $h$  such that for all  $T_0 > \kappa r_0$ , we have

$$\|w_h(0)\|_{L^2(\Omega)}^2 \leq C_{T_0} \int_0^{T_0} \|w_h(t)\|_{L^2(r_0 < |x| < 1)}^2 dt. \quad (6.1)$$

*Proof.* This is again a consequence of propagation arguments. Indeed, as in Section 5 we can insert a cutoff  $\chi(-hD_s)$ ,  $\chi = 1$  on the support of  $\psi$ ,  $\chi$  supported in  $(\frac{1}{4}, 4)$  and arguing by contradiction and defining defect measures, we get a contradiction if we can ensure that for all  $(t_0 = 0, -\sigma_0 \in \text{supp}(\chi), x_0 \in [-1, 1], y_0 \in \mathbb{T}^1, \xi_0, \eta_0 \in [0, b_0])$  such that  $-\sigma_0 - \xi_0^2 - x_0^2 \eta_0^2$ , the point on the bicharacteristic i.e. the solution of the system

$$\dot{s} = -\sigma_0, \dot{x} = 2\xi, \dot{\xi} = -2x\eta_0, \dot{y} = 2x^2\eta_0 \quad (6.2)$$

reaches the region  $\{|x| \in (r_0, 1)\}$  at a time  $s \in (0, T_0)$ . Notice that

$$\xi_0^2 = -\sigma_0 - x_0^2 \eta_0^2 \in \left(-\frac{1}{4} - \frac{1}{100}, 4 + \frac{1}{100}\right),$$

and consequently the  $(x, \xi)$  integration of (6.2) gives again an ellipse and the geometric assumption is satisfied. This completes the proof of Lemma 6.2. □

Denote by  $u_h(t) = \underline{\Pi}_{h,b_0}^\epsilon e^{it\Delta_G} u_0$ , we have the following

**Lemma 6.3.** *Let  $T > 0$ , there exists  $C_T > 0$ , and  $h_0 > 0$ , such that for all  $0 < h < h_0$ , we have*

$$\begin{aligned} \|x\partial_y u_h(t)\|_{L^2((-T,T);L^2(\Omega))}^2 + \|u_h(0)\|_{H_y^1(\Omega)}^2 \\ \leq C_T \int_{-3T}^{3T} \|\phi(y)x\partial_y u_h(t)\|_{L^2(\Omega)}^2 dt + C_T \|u_h(0)\|_{L^2(\Omega)}^2. \end{aligned} \quad (6.3)$$

*Proof.* From minor modification as in the proof of Proposition 4.7, it suffices to deal with the case where  $\phi(y) = 0$  on  $(-a, a)$ . Denote by  $\varphi_T(t) := \varphi(T^{-1}t)$  the time cutoff. Take  $\chi \in C_c^\infty(\mathbb{T}_y)$  such that  $\chi|_{\mathbb{T}_y \setminus \omega} \equiv 1$  and  $\text{supp}(\chi') \subset \text{supp}(\phi) \subset \omega$ . By direct calculation, one verifies that

$$\begin{aligned} [i\partial_t + \Delta_G, \varphi_T(t)\chi(y)y\partial_y] = 2\varphi_T(t)\chi(y)(x\partial_y)^2 + i\varphi_T'(t)\chi(y)y\partial_y \\ + x^2(y\chi''(y) + 2\chi'(y))\varphi_T(t)\partial_y + 2\varphi_T(t)\chi'(y)y(x\partial_y)^2. \end{aligned} \quad (6.4)$$

Taking the inner product of  $[i\partial_t + \Delta_G, \varphi_T(t)\chi(y)y\partial_y]u_h$  and  $u_h$ , using the equation, we have

$$\begin{aligned} 2 \int_{\mathbb{R} \times M} \varphi_T(t) \|\chi(x)x\partial_y u_h(t, x, y)\|_{L^2(\Omega)}^2 dt \\ \leq a \int_{\mathbb{R} \times M} |\varphi_T'(t)| \|u_h(t)\|_{L^2(\Omega)} \|\partial_y u_h(t)\|_{L^2(\Omega)} dt \\ + C \int_{\mathbb{R}} \varphi_T(t) \|\phi(y)x\partial_y u_h(t)\|_{L^2(\Omega)}^2 + C \int_{\mathbb{R}} \varphi_T(t) \|u_h(t)\|_{L^2(\mathbb{R} \times \Omega)}^2. \end{aligned} \quad (6.5)$$

Next, we change the time scale by setting  $w_h(s, x, y) := \partial_y u_h(t, x, y)$ ,  $t = sh$ . Note that  $w_h$  satisfies the semi-classical equation  $(ih\partial_s + h^2\Delta_G)w_h = 0$ . Applying Lemma 6.2, we obtain that for any  $0 < r_0 < 1$ ,  $T_0 > \kappa r_0$ ,

$$\|w_h(0)\|_{L^2(\Omega)}^2 \leq C_{T_0} \int_0^{T_0} \|w_h(s)\|_{L^2(r_0 < |x| < 1)}^2 ds.$$

Back to the time variable  $t$  and the original function  $u_h$ , exploiting again the conservation of  $L^2$  norm, we obtain that for any  $T > 0$ ,

$$\|\partial_y u_h(0)\|_{L^2(\Omega)}^2 \leq C_T \int_{-T}^T \|\partial_y u_h(t)\|_{L^2(r_0 < |x| < 1)}^2 dt.$$

Note that

$$\|\partial_y u_h(t)\|_{L^2(r_0 < |x| < 1)}^2 \leq \frac{C_T}{r_0^2} \int_{\mathbb{R}} \varphi_T(t) \|x\partial_y u_h(t)\|_{L^2(\Omega)}^2 dt.$$

Plugging into (6.5), and using the conservation of the norms  $\|\partial_y u_h(t)\|_{L^2(\Omega)}^2$ ,  $\|u_h(t)\|_{L^2(\Omega)}^2$  as well as the Young's inequality<sup>2</sup>, we have

$$\|\partial_y u_h(0)\|_{L^2(\Omega)}^2 \leq C \int_{-2T}^{2T} \|\phi(y)x\partial_y u_h(t)\|_{L^2(\Omega)}^2 + C_T \|u_h(0)\|_{L^2(\Omega)}^2.$$

<sup>2</sup>We use  $2AB \leq \epsilon A^2 + \epsilon^{-1} B^2$ .

Replacing  $T$  to  $3T/2$  in the definition of  $\varphi_T$ , we obtain from (6.5) that:

$$\|x\partial_y u_h(t)\|_{L^2((-T,T)\times\Omega)}^2 \leq C \int_{-3T}^{3T} \|\phi(y)x\partial_y u_h(t)\|_{L^2(\Omega)}^2 dt + C_T \|u_h(0)\|_{L^2(\Omega)}^2, \quad (6.6)$$

This completes the proof of Lemma 6.3.  $\square$

## 6.2. $L^2$ observability for the rapid propagation regime.

*Proof of Proposition 6.1.* For  $u_h(t) = \underline{\Pi}_{h,b_0}^\epsilon e^{it\Delta_G} u_0$ , without loss of generality, we may assume that

$$u_h = (1 - \psi_0(h^\epsilon D_y)) \chi_0(b_0^{-1} h D_y) u_h.$$

We write

$$\phi(y)xh\partial_y u_h = xh\partial_y \chi_0((2b_0)^{-1} h D_y) (\phi(y)u_h) + O_{L^2(\Omega)}(h).$$

Hence from Lemma 6.3, we obtain that

$$\|h\partial_y u_h(0)\|_{L^2(\Omega)}^2 \leq C_T \int_{-3T}^{3T} \|\phi(y)u_h(t)\|_{L^2(\Omega)}^2 dt + C_T h^2 \|u_h(0)\|_{L^2(\Omega)}^2 \quad (6.7)$$

By assumption,  $\mathcal{F}_y u_h(\cdot, k) = 0$  for all  $|k| \leq Ch^{-\epsilon}$ , thus we have that

$$\begin{aligned} 2C_T h^2 \|u_h(0)\|_{L^2(\Omega)}^2 &\leq 2C_T (Ch)^{2\epsilon} \|h\partial_y u_h(0)\|_{L^2(\Omega)}^2 \\ &\leq C_T h^{2\epsilon} \int_{-3T}^{3T} \|\phi(y)u_h(t)\|_{L^2(\Omega)}^2 dt + C_T h^{2+2\epsilon} \|u_h(0)\|_{L^2(\Omega)}^2. \end{aligned}$$

Absorbing the second term on the right hand side to the left, we complete the proof in this case, provided that  $h < h_0$  is small enough. The proof of Proposition 6.1 is then complete.  $\square$

## 7. NON-SEMICLASSICAL DISPERSIVE REGIME II: $h^{-\epsilon/2} \leq |D_y| \leq b_0 h^{-1}$ : BOTTOM VERTICAL FREQUENCIES

$\psi \in C_c^\infty(1/2, 2)$ ,  $\psi_0 \in C_c^\infty(|\xi| \leq 1)$ . Let

$$u_h = \psi(h^2 \Delta_G) \psi_0(h^\epsilon D_y) u.$$

In this section, we prove the following result:

**Proposition 7.1.** *For any  $T > 0$ , there exist  $C_T > 0$ ,  $h_0 > 0$ , such that for sufficiently small  $\epsilon > 0$  and all  $0 < h < h_0$ , we have*

$$\|u_h(0, \cdot)\|_{L^2(\Omega)}^2 \leq C_T \int_{-T}^T \|\phi(y)e^{it\Delta_G} u_h(0, \cdot)\|_{L^2(\Omega)}^2 dt + C_T h^{1-4\epsilon} \|u(0, \cdot)\|_{L^2(\Omega)}^2.$$



Inspired by [BZ12], we use normal form method. The key point is to search for a microlocal transformation

$$v = (1 + hQD_y^2)u$$

for some suitable semi-classical pseudo-differential operator  $Q = q(x, hD_x)$ , such that the conjugated equation (satisfied by  $v$ ) is

$$i\partial_t v + \partial_x^2 v + M\partial_y^2 v = \text{errors} ,$$

where

$$M = \frac{1}{2} \int_{-1}^1 x^2 dx$$

is the mean value of  $x^2$ . Then we will be able to use the following theorem:

**Theorem 5** ([Ja90],[BZ12],[AM14]). *Let  $\Delta_M = \partial_x^2 + M\partial_y^2$ . Then for any non-empty open set  $\omega_0 \subset \mathbb{T}^2$  and  $T > 0$ , the observability*

$$\|f\|_{L^2(\mathbb{T}^2)}^2 \leq C_T \int_{-T}^T \|e^{it\Delta_M} f\|_{L^2(\omega_0)}^2 dt \quad (7.1)$$

holds true for any  $f \in L^2(\mathbb{T}^2)$ .

However, dealing with Dirichlet boundary value problem induces difficulties and consequently, we preferred to extend the analysis to the periodic setting.

**7.1. Periodic extension.** Let us introduce several notations. Denote by

$$\tilde{\mathbb{T}} := [-1, 3]/\{-1, 3\} \text{ and } \tilde{\mathbb{T}}^2 := \tilde{\mathbb{T}}_x \times \mathbb{T}_y, \Omega_* = (-1, 3)_x \times \mathbb{T}_y.$$

Define

$$a(x) = x, \text{ if } |x| \leq 1 \text{ and } a(x) = 2 - x, \text{ if } 1 \leq x \leq 3,$$

and the operator

$$P_a := \partial_x^2 + a(x)^2 \partial_y^2.$$

Note that  $a(x)$  and  $a^2(x)$  are Lipschitz functions on  $\tilde{\mathbb{T}}$ . Denote by

$$H_a^k(\tilde{\mathbb{T}}^2) := \{f \in \mathcal{D}'(\tilde{\mathbb{T}}^2) : P_a^j f \in L^2(\tilde{\mathbb{T}}^2), \forall 0 \leq j \leq k\}$$

the associated function spaces and the domain of  $P_a$  is  $D(P_a) = H_a^2(\tilde{\mathbb{T}}^2)$ . Note that  $D(\Delta_G) = H_{G,0}^1(\Omega) \cap H_G^2(\Omega)$ . Consider the extension map:

$$\iota_1 : f \mapsto \tilde{f},$$

with

$$\tilde{f}(x, y) = f(x, y), \text{ if } |x| \leq 1, \text{ and } \tilde{f}(x, y) = -f(2 - x, y), \text{ if } 1 \leq x \leq 3.$$

The mapping  $\iota_1$  is the odd extension with respect to  $x = 1$ . Note that for  $f \in C^\infty(\bar{\Omega})$ , we have

$$\partial_x f|_{x=1-} = \partial_x(\iota_1 f)|_{x=1+}.$$

**Lemma 7.2.** *The extension map  $\iota_1 : D(\Delta_G) \rightarrow D(P_a)$  is continuous. Moreover, for all  $f \in D(\Delta_G)$ ,  $\|\iota_1 f\|_{L^2(\tilde{\mathbb{T}}^2)} = \sqrt{2}\|f\|_{L^2(\Omega)}$ .*

*Proof.* Assume that  $u \in D(\Delta_G)$ , we extend it to  $\tilde{u} = \iota_1 u$ . Denote by  $\Omega' = (1, 3)_x \times \mathbb{T}_y$ . By definition,  $(P_a \tilde{u})|_\Omega \in L^2(\Omega)$  and  $(P_a \tilde{u})|_{\Omega'} \in L^2(\Omega')$ . To check that  $P_a \tilde{u} \in L^2(\tilde{\mathbb{T}}^2)$ , we have to be careful near  $x = 1$ , where we glue  $\Omega$  and  $\Omega'$ . Take a test function  $F \in C^\infty(\tilde{\mathbb{T}}^2)$ , we calculate

$$\begin{aligned} -\langle P_a \tilde{u}, F \rangle &= \int_{\Omega \cup \Omega'} (\partial_x \tilde{u} \partial_x F + a(x)^2 \partial_y \tilde{u} \partial_y F) dx dy \\ &= \int_{\Omega \cup \Omega'} (-\partial_x^2 \tilde{u} \cdot F - a(x)^2 \partial_y^2 \tilde{u} F) dx dy + \int_{x=1-} \partial_x \tilde{u} F dy - \int_{x=-1+} \partial_x \tilde{u} F dy \\ &\quad + \int_{x=3-} \partial_x \tilde{u} F dy - \int_{x=1+} \partial_x \tilde{u} F dy. \end{aligned}$$

From the definition of  $\tilde{u}$ , we have  $\partial_x \tilde{u}|_{x=1+} = \partial_x u|_{x=1-}$ , and  $\partial_x \tilde{u}|_{x=3-} = \partial_x \tilde{u}|_{x=-1+}$ . Thus all the boundary terms vanish. This implies that

$$\|P_a \iota_1 u\|_{L^2(\tilde{\mathbb{T}}^2)} \leq \|\Delta_G u\|_{L^2(\Omega)} + \|(P_a \tilde{u})_{\Omega'}\|_{L^2(\Omega')} = 2\|\Delta_G u\|_{L^2(\Omega)}.$$

The last assertion  $\|\iota_1 f\|_{L^2(\tilde{\mathbb{T}}^2)} = \sqrt{2}\|f\|_{L^2(\Omega)}$  is obvious. This completes the proof of Lemma 7.2.  $\square$

**Lemma 7.3.** *Let  $S_1, S_2$  be two self-adjoint operators on Banach spaces  $E_1, E_2$  with domains  $D(S_1), D(S_2)$ , with respectively. Assume that  $j : D(S_1) \rightarrow D(S_2)$  is a continuous embedding. Suppose that there holds  $j \circ S_1 = S_2 \circ j$ , then for any Schwartz function  $g \in \mathcal{S}(\mathbb{R})$ , we have*

$$j \circ g(S_1) = g(S_2) \circ j$$

*Proof.* Thanks to Helffer-Sjöstrand's formula, it suffices to check the validity for resolvent, namely

$$j \circ (z - S_1)^{-1} = (z - S_2)^{-1} \circ j. \quad (7.2)$$

For  $f_1 \in E_1$ , we denote by  $u_1 = (z - S_1)^{-1} f_1 \in D(S_1)$ , and  $u_2 = j(u_1) \in E_2$ . We directly check that

$$(z - S_2)u_2 = (z - S_2) \circ j(u_1) = j((z - S_1)u_1) = j(f_1).$$

This completes the proof of Lemma 7.3.  $\square$

Lemma 7.3 ensures the preservation of the spectral localization property by odd extension procedure. Another interesting fact is that the extended eigenfunctions are still smooth. Indeed, since  $\partial_y$  commutes with  $\Delta_G$  as well as  $P_a$ , the eigenfunctions of  $P_a$  can be taken of the form  $\tilde{\varphi}_n(x)e^{iny}$ , where  $\tilde{\varphi}_n(x)$  is an eigenfunction of  $\tilde{L}_n := -\partial_x^2 + a(x)^2 n^2$  with domain  $D(\tilde{L}_n) = H^2(\tilde{\mathbb{T}})$ . Moreover, the extension  $\iota_1$  can be viewed as a continuous map from  $D(L_n) \rightarrow D(\tilde{L}_n)$ , where  $L_n = -\partial_x^2 + n^2 x^2$  is defined on its domain  $D(L_n) = H_0^1((-1, 1)) \cap H^2((-1, 1))$ . Recall that  $(\varphi_{m,n})_{m \in \mathbb{N}}$  is a sequence of eigenfunctions of  $L_n$  with associated eigenvalues  $(\lambda_{m,n}^2)_{m \in \mathbb{N}}$ . Therefore, the extension  $\tilde{\varphi}_{m,n} := \iota_1(\varphi_{m,n})$  is eigenfunction of  $\tilde{L}_n$  with the same eigenvalue  $\lambda_{m,n}^2$ .

Note that  $\tilde{\varphi}_{m,n}$  is  $C^\infty$  for  $x \in (-1, 1) \cup (1, 3)$ . From  $L_n \varphi_{m,n} = \lambda_{m,n}^2 \varphi_{m,n}$ , we have  $\varphi_{m,n}''(x) = (x^2 n^2 - \lambda_{m,n}^2) \varphi_{m,n}$ . By induction, we have that  $\varphi_{m,n}^{(2k)}|_{x=\pm 1} = 0$ , for all  $k \in \mathbb{N}$ . Since  $\tilde{\varphi}_{m,n}$  is odd with respect to  $x = 1$ , we deduce that  $\tilde{\varphi}_{m,n}^{(2k+1)}|_{x=1-} = \tilde{\varphi}_{m,n}^{(2k+1)}|_{x=1+}$ ,  $\tilde{\varphi}_{m,n}^{(2k+1)}|_{x=3-} = \tilde{\varphi}_{m,n}^{(2k+1)}|_{x=-1+}$ . Therefore,  $\tilde{\varphi}_{m,n} \in C^\infty(\tilde{\mathbb{T}})$ . Moreover, the orthogonality condition holds

$$(\tilde{\varphi}_{m,n}, \tilde{\varphi}_{m',n})_{L^2(\tilde{\mathbb{T}})} = C_n \delta_{m,m'}.$$

Now we extend  $(\tilde{\varphi}_{m,n})_{m \in \mathbb{N}}$  to a orthonormal eigenbasis of  $\tilde{L}_n$ . This new basis will be denoted by  $(e_{m,n})_{m \in \mathbb{N}}$ . Let  $f_n(x) = \sum_{m \in \mathbb{N}} c_{m,n} \varphi_{m,n}(x)$ , then for any Schwartz function  $g : \mathbb{R} \rightarrow \mathbb{C}$ , we have

$$(\iota_1 \circ g(h^2 L_n) f_n)(x) = \sum_{m \in \mathbb{N}} c_{m,n} g(h^2 \lambda_{m,n}^2) \iota_1(\varphi_{m,n})(x). \quad (7.3)$$

Using (7.3), since the coefficient corresponding to the new added eigenfunctions  $e_{m,n}$  is zero, we deduce that for any Schwartz function  $g : \mathbb{R} \rightarrow \mathbb{C}$ ,

$$\iota_1 \circ g(h^2 L_n) = g(h^2 \tilde{L}_n) \circ \iota_1, \text{ and } \iota_1 \circ g(h^2 \Delta_G) = g(h^2 P_a) \circ \iota_1. \quad (7.4)$$

Consequently, we have the following lemma, reducing the proof of Proposition 7.1 to the observability of the extended solutions:

**Lemma 7.4.** *Let  $T > 0$ , assume that for any  $0 < h < h_0 \ll 1$ ,  $0 < \epsilon \ll 1$ , the following observability holds true for all  $\tilde{u}_0 \in L^2(\tilde{\mathbb{T}}^2)$ :*

$$\begin{aligned} & \|\psi(h^2 P_a) \psi_0(h^\epsilon D_y) \tilde{u}_0\|_{L^2(\tilde{\mathbb{T}}^2)}^2 \\ & \leq C_T \int_{-T}^T \|\phi(y) \psi(h^2 P_a) \psi_0(h^\epsilon D_y) \tilde{u}(t)\|_{L^2(\tilde{\mathbb{T}}^2)}^2 dt + C_T h \|\tilde{u}_0\|_{L^2(\tilde{\mathbb{T}}^2)}^2, \end{aligned} \quad (7.5)$$

where  $\tilde{u}(t) = e^{itP_a} \tilde{u}_0$ , the solution of Schrödinger equation  $i\partial_t \tilde{u} + P_a \tilde{u} = 0$  with initial data  $\tilde{u}|_{t=0} = \tilde{u}_0$ . Then Proposition 7.1 is true. More precisely, with the same constant  $C_T > 0$ , for all  $0 < h < h_0$ ,  $0 < \epsilon \ll 1$ , the observability

$$\begin{aligned} & \|\psi(h^2 \Delta_G) \psi_0(h^\epsilon D_y) u_0\|_{L^2(\Omega)}^2 \\ & \leq C_T \int_{-T}^T \|\phi(y) \psi(h^2 \Delta_G) \psi_0(h^\epsilon D_y) e^{it\Delta_G} u_0\|_{L^2(\Omega)}^2 dt + C_T h \|u_0\|_{L^2(\Omega)}^2 \end{aligned} \quad (7.6)$$

holds true for all  $u_0 \in L^2(\Omega)$ .

*Proof.* With a little abuse of notation, we assume that  $u_0 = \psi(h^2 \Delta_G) \psi_0(h^\epsilon D_y) u_0 \in D(\Delta_G)$ . Denote by  $\tilde{u}_0 = \iota_1 u_0$ . Thanks to Lemma 7.2,  $\tilde{u}_0 = \psi(h^2 P_a) \psi_0(h^\epsilon D_y) \tilde{u}_0$ . Using (7.4) with  $g(r) = e^{itr} \psi(h^2 r)$ , we deduce that

$$\phi(y) e^{itP_a} \tilde{u}_0 = \iota_1(\phi(y) e^{it\Delta_G} u_0).$$

From the observability for  $\tilde{u}_0$ , we obtain that

$$\|\tilde{u}_0\|_{L^2(\tilde{\mathbb{T}}^2)}^2 \leq C_T \int_{-T}^T \|\phi(y)e^{itP_a}\tilde{u}_0\|_{L^2(\tilde{\mathbb{T}}^2)}^2 dt,$$

which implies that

$$\|u_0\|_{L^2(\Omega)}^2 \leq C_T \int_{-T}^T \|\phi(y)e^{it\Delta_G}u_0\|_{L^2(\Omega)}^2 dt,$$

thanks to the fact that  $\|\iota_1 f\|_{L^2(\tilde{\mathbb{T}}^2)}^2 = 2\|f\|_{L^2(\Omega)}^2$ . This completes the proof of Lemma 7.4.  $\square$

*Proof of Proposition 7.1.* From Lemma 7.4, it is sufficient to prove (7.5). With a little abuse of notation, we denote by  $u_0 \in D(P_a)$  such that  $u_0 = \psi(h^2 P_a)\psi(h^\epsilon D_y)u_0$  and  $u(t) = e^{itP_a}u_0$ . We are now in the periodic setting. Yet, we should pay an extra attention to the fact that  $P_a = \partial_x^2 + a(x)^2 \partial_y^2$  is a hypoelliptic operator with only Lipschitz coefficient. More precisely,  $a \in \text{Lip}(\tilde{\mathbb{T}}^2)$  which is not  $C^1$  at  $x = 1$ .

Before proceeding, we need a lemma which, modulo errors allows us to replace  $\psi(h^2 P_a)\psi_0(h^\epsilon D_y)$  by  $\psi_1(hD_x)\psi(h^\epsilon D_y)$ .

**Lemma 7.5.** *Let  $\psi_1 \in C_c^\infty(\frac{1}{4} < |\xi| < 4)$  such that  $\psi_1 = 1$  on  $\text{supp}(\psi)$ . Then, as bounded operator on  $L^2(\tilde{\mathbb{T}}^2)$ , we have*

$$(1 - \psi_1(hD_x))\psi(h^2 P_a)\psi_0(h^\epsilon D_y) = O_{L^2 \rightarrow L^2}(h^{\frac{3}{2} - \frac{5}{2}\epsilon}).$$

*Proof.* Since  $D_y$  commutes with everything, it suffices to show that, uniformly in  $|n| \leq Ch^{-\epsilon}$ , as an operator on  $L^2(\tilde{\mathbb{T}})$ ,

$$(1 - \psi_1(hD_x))\psi(h^2 \tilde{L}_n) = O_{L^2 \rightarrow L^2}(h^{\frac{3}{2} - \frac{5}{2}\epsilon}). \quad (7.7)$$

Let  $f = \psi(h^2 \tilde{L}_n)f$ ,  $f(x) = \sum_j \psi(h^2 \lambda_{j,n}^2) c_j e_{j,n}(x)$ . Note that

$$-(h^2 \partial_x^2 + h^2 \lambda_{j,n}^2) e_{j,n} = -h^2 n^2 a(x)^2 e_{j,n} = O_{L^2}(h^{2-2\epsilon}).$$

Observe that on the support of  $(1 - \psi_1(\xi))$ , the semi-classical operator  $-(h^2 \partial_x^2 + h^2 \lambda_{j,n}^2)$  is elliptic, uniformly in  $j, n$  such that  $|n| \leq Ch^{-\epsilon}$  and  $\lambda_{j,n}^2 h^2 \in \text{supp}(\psi)$ . Therefore,  $\|(1 - \psi_1(hD_x))e_{j,n}\|_{L^2(\tilde{\mathbb{T}})} = O(h^{2-2\epsilon})$ . By Cauchy-Schwartz,

$$\|(1 - \psi_1(hD_x))f\|_{L^2(\tilde{\mathbb{T}})} \leq \sup_j \|(1 - \psi_1(hD_x))e_{j,n}\|_{L^2(\tilde{\mathbb{T}})} \cdot \#\{j : \lambda_{j,n}^2 \in \text{supp}(\psi)\}^{1/2} \|f\|_{L^2(\tilde{\mathbb{T}})}.$$

From Weyl's law,

$$\#\{j : \lambda_{j,n}^2 \in \text{supp}(\psi)\} \leq Ch^{-1-\epsilon} \leq \#\{(j, n) : |n| \leq Ch^{-\epsilon}, \lambda_{j,n}^2 \in \text{supp}(\psi)\} \leq Ch^{-1-\epsilon}.$$

Therefore,

$$\|(1 - \psi_1(hD_x))f\|_{L^2(\tilde{\mathbb{T}})} \leq Ch^{\frac{3}{2} - \frac{5}{2}\epsilon} \|f\|_{L^2(\tilde{\mathbb{T}})}.$$

Applying Plancherel in  $y$ , we complete the proof of Lemma 7.5.  $\square$

Modulo an error  $O_{L^2}(h^{\frac{3}{2}-\frac{5}{2}\epsilon})\|u_0\|_{L^2(\tilde{\mathbb{T}}^2)}$ , we may assume that  $u = \psi_1(hD_x)\psi_0(h^\epsilon D_y)u$ . Now we search for the function

$$v = (1 + hQD_y^2)u$$

with a operator  $Q$  acting only in  $x$ , to be chosen later. Let  $M = \frac{1}{4} \int_{-1}^3 a(x)^2 dx = \frac{1}{2} \int_{-1}^1 x^2 dx$  be the average of  $a(x)^2$  along the horizontal trajectory  $y = \text{const}$ . Using the equation  $(i\partial_t + P_a)u = 0$ , we have

$$\begin{aligned} (i\partial_t - D_x^2 - MD_y^2)v &= (1 + hQD_y^2)(a(x)^2 - M)D_y^2u - h[D_x^2, Q]D_y^2u \\ &= (a(x)^2 - M)D_y^2u - h[D_x^2, Q]D_y^2u + hQD_y^2(a(x)^2 - M)D_y^2u \\ &= (a(x)^2 - M)D_y^2u - h[D_x^2, Q]D_y^2u + O_{L^2(\tilde{\mathbb{T}}^2)}(h^{1-4\epsilon}), \end{aligned}$$

since  $\|hQD_y^2(a(x)^2 - M)D_y^2u\|_{L^2(\tilde{\mathbb{T}}^2)} = O(h^{1-4\epsilon})$ , due to the localization property of  $u$ . Take  $\psi_2 \in C_c^\infty(1/8 \leq |\xi| \leq 8)$ , such that  $\psi_2\psi_1 = \psi_1$ . We define the operator

$$Q = \frac{1}{2i} \left( \int_{-1}^x (a(z)^2 - M) dz \right) (hD_x)^{-1} \psi_2(hD_x),$$

and denote by  $b(x) = \frac{1}{2i} \int_{-1}^x (a(z)^2 - M) dz$ ,  $m(hD_x) = (hD_x)^{-1} \psi_2(hD_x)$ . Since  $a(x)^2 - M$  has zero average, the function  $b$  is well-defined as a periodic function in the space  $C^1(\tilde{\mathbb{T}}) \cap W^{2,\infty}(\tilde{\mathbb{T}})$ . From direct calculation, we have

$$-h[D_x^2, Q] = 2ib'(x)m(hD_x)hD_x + i[hD_x, b'(x)]m(hD_x).$$

Note that  $[hD_x, b'(x)] = hb''(x)$ , and  $b'' \in L^\infty(\tilde{\mathbb{T}})$ , thus

$$(i\partial_t + \Delta_M)v = r_h = O_{L^2(\tilde{\mathbb{T}}^2)}(h^{1-4\epsilon})\|u_0\|_{L^2(\tilde{\mathbb{T}}^2)},$$

where  $\Delta_M = \partial_x^2 + M\partial_y^2$ . Applying Theorem 5, we obtain that

$$\begin{aligned} \|v(0, \cdot)\|_{L^2(\tilde{\mathbb{T}}^2)}^2 &\leq C_T \int_{-T}^T \|\phi(y)v(t, \cdot)\|_{L^2(\tilde{\mathbb{T}}^2)}^2 dt + C_T \int_{-T}^T \left\| \phi(y) \int_0^t e^{i(t-t')\Delta_M} r_h(t') dt' \right\|_{L^2(\tilde{\mathbb{T}}^2)}^2 dt \\ &\leq C_T \int_{-T}^T \|\phi(y)v(t, \cdot)\|_{L^2(\tilde{\mathbb{T}}^2)}^2 dt + C_T h^{2(1-4\epsilon)} \|u_0\|_{L^2(\tilde{\mathbb{T}}^2)}^2. \end{aligned}$$

Since  $v = u + O_{L^2(\tilde{\mathbb{T}}^2)}(h^{1-2\epsilon})\|u_0\|_{L^2(\tilde{\mathbb{T}}^2)}$ , the proof of Proposition 7.1 is now complete.  $\square$

## 8. COMPACTNESS ARGUMENT AND THE PROOF OF THEOREM 1

**Lemma 8.1.** *The embeddings  $H_{G,0}^1(\Omega) \hookrightarrow L^2(\Omega)$  and  $L^2(\Omega) \hookrightarrow H_{G,0}^{-1}(\Omega)$  are compact.*

*Proof.* By duality, we only need to prove that  $H_{G,0}^1(\Omega) \hookrightarrow L^2(\Omega)$  is compact. Denote by  $\Pi_0$ , the projection to the zero mode of  $y$ . Define several closed linear subspaces:

$$\mathcal{H}_1 := \{f \in L^2(\Omega) : \mathcal{F}_y f(\cdot, 0) = 0\}, \quad \mathcal{H}_2 := \Pi_0(L^2(\Omega)),$$

and

$$X_1 := \{f \in H_{G,0}^1(\Omega) : \mathcal{F}_y f(\cdot, 0) = 0\}, \quad X_2 := \Pi_0(H_{G,0}^1(\Omega)).$$

Then

$$L^2(\Omega) = \mathcal{H}_1 \oplus \mathcal{H}_2, \quad H_{G,0}^1(\Omega) = X_1 \oplus X_2.$$

Moreover,  $X_j \subset \mathcal{H}_j, j = 1, 2$  and  $\mathcal{H}_1 \perp \mathcal{H}_2$ , with respect to the  $L^2(\Omega)$  inner product. The compact embedding will follow, if we prove that the each embedding  $i_j : \mathcal{H}_j \hookrightarrow X_j$  is compact. Note that  $X_2$  is isometric to  $H_0^1((-1, 1))$ , while  $\mathcal{H}_2$  is isometric to  $L^2((-1, 1))$ . By Rellich theorem,  $i_2$  is compact.

It remains to show that  $i_1$  is compact. Denote again by  $\Pi_+$  and  $\Pi_-$  the projection to strictly positive and negative frequencies, with respectively. For  $f \in X_1$ , we can write

$$\| |D_y|^{1/2} u \|_{L^2(\Omega)}^2 = (-i\partial_y \Pi_+ f, f)_{L^2(\Omega)} - (-i\partial_y \Pi_- f, f)_{L^2(\Omega)}.$$

Using  $[\partial_x, x\partial_y] = \partial_y$ , we deduce that

$$(\partial_y \Pi_{\pm} f, f)_{L^2(\Omega)} \leq \|\partial_x f\|_{L^2(\Omega)} \|x\partial_y f\|_{L^2(\Omega)} \leq \|u\|_{H_G^1(\Omega)}^2.$$

Since  $\|\partial_x f\|_{L^2(\Omega)}$  can be controlled by  $\|f\|_{H_G^1(\Omega)}$ , we deduce that  $H_{G,0}^1(\Omega) \hookrightarrow H^{1/2}(\Omega)$  is continuous. Decomposing  $i_1$  as  $H_{G,0}^1(\Omega) \hookrightarrow H^{1/2}(\Omega) \hookrightarrow L^2(\Omega)$ , the compactness of  $i_1$  is implied by the compactness of  $H^{1/2}(\Omega) \hookrightarrow L^2(\Omega)$ . This completes the proof of Lemma 8.1.  $\square$

*Proof of Theorem 1.* Combining with Proposition 4.1, Proposition 5.1, Proposition 6.1 and Proposition 7.1, for some  $h_0 > 0$ , for  $T > a$  and all  $0 < h < h_0$ , there holds

$$\|\psi(h^2 \Delta_G) u_0\|_{L^2(\Omega)}^2 \leq C_T \int_{-T}^T \|\phi(y) e^{it\Delta_G} \psi(h^2 \Delta_G) u_0\|_{L^2(\Omega)}^2 dt + C_T h^{1/2} \|u_0\|_{L^2(\Omega)}^2.$$

The rest argument is now standard. We follow the approach of Bardos-Lebeau-Rauch [BLR92]. Choosing  $h = 2^{-j}$  and summing over the inequality above for  $j \geq j_0 = \log_2(h_0^{-1})$ , after standard manipulations, we obtain that

$$\|u_0\|_{L^2(\Omega)}^2 \leq C_T \int_{-T}^T \|\phi(y) e^{it\Delta_G} u_0\|_{L^2(\Omega)}^2 dt + C_T \|\psi_0(c2^{-2j_0} \Delta_G) u_0\|_{L^2(\Omega)}^2, \quad (8.1)$$

where  $\psi_0 \in C_c^\infty(|\xi| \leq 1)$ . Note that the second term on the right side is controlled by  $\|u_0\|_{H_G^{-1}(\Omega)}^2$ .

For  $T' > 0$ , defining the set

$$\mathcal{N}_{T'} := \{u_0 \in L^2(\Omega) : e^{it\Delta_G} u_0|_{[-T', T'] \times \omega} = 0\}$$

Take  $T' \in (a, T)$ , (8.1) implies that any function  $u_0 \in \mathcal{N}_{T'}$  satisfies

$$\|u_0\|_{L^2(\Omega)} \leq C_T \|u_0\|_{H_G^{-1}(\Omega)}.$$

Thanks to Lemma 8.1, we deduce that  $\dim \mathcal{N}_{T'} < \infty$ . Note that for any  $T_1 < T_2$ ,  $\mathcal{N}_{T_2} \subset \mathcal{N}_{T_1}$ . Consider the mapping  $S(\delta) := \delta^{-1} (e^{i\delta\Delta_G} - \text{Id}) : \mathcal{N}_{T'} \rightarrow \mathcal{N}_{T'-\delta}$ . For fixed  $T' \in (a, T)$ , when  $\delta < T' - a$ ,  $\dim \mathcal{N}_{T'-\delta} < \infty$ . Since the dimension is an integer, there exists  $\delta_0 > 0$ , such that for all  $0 < \delta < \delta_0$ ,  $\mathcal{N}_{T'-\delta} = \mathcal{N}_{T'}$ . Therefore,  $S(\delta)$  is a linear map on  $\mathcal{N}_{T'}$ . Let  $\delta \rightarrow 0$ , we obtain

that  $\partial_t : \mathcal{N}_{T'} \rightarrow \mathcal{N}_{T'}$  is a well-defined linear operator. Take any eigenvalue  $\lambda \in \mathbb{C}$  of  $\partial_t$ , and assume that  $u_* \in \mathcal{N}_{T'}$  is a corresponding eigenfunction (if it exists). There holds

$$-\Delta_G u_* = i\lambda u_*.$$

This implies that  $u_*$  is an eigenfunction of  $-\Delta_G$ . However,  $u_*|_\omega \equiv 0$ , hence  $u_* \equiv 0$ . Therefore,  $\mathcal{N}_{T'} = \{0\}$ .

Now we choose  $T_0 = T'$  as above. By contradiction, assume that Theorem (1) is untrue. Then there exists a sequence  $(u_{k,0})_{k \in \mathbb{N}}$ , such that

$$\|u_{k,0}\|_{L^2(\Omega)} = 1, \quad \lim_{k \rightarrow \infty} \int_{-T_0}^{T_0} \|\phi(y) e^{it\Delta_G} u_{k,0}\|_{L^2(\Omega)}^2 = 0.$$

Up to a subsequence, we may assume that  $u_{k,0} \rightharpoonup u_0$ . Thus from Lemma 8.1,  $u_{k,0} \rightarrow u_0$ , strongly in  $H_G^{-1}(\Omega)$ . Passing  $k \rightarrow \infty$  of (8.1),  $1 \leq \|u_0\|_{H_G^{-1}(\Omega)}$ . In particular,  $u_0 \neq 0$ . However, since  $e^{it\Delta_G} u_{k,0} \rightarrow 0$  in  $L^2([-T_0, T_0] \times \omega)$ . This implies that  $u_0 \in \mathcal{N}_{T_0} = \{0\}$ . This is a contradiction. The proof of Theorem 1 is now complete.  $\square$

## 9. COUNTEREXAMPLE TO THE OBSERVABILITY IF $T \leq \mathcal{L}(\omega)$

The goal of this section is to prove Theorem 3. This is based on detailed asymptotic analysis for the first eigenvalue and eigenfunction of the harmonic oscillator  $\mathcal{L}_w = -\partial_x^2 + w^2 x^2$  with respect to the large parameter  $w$ . Some of the estimates can be also found in [HS84]. We will present an self-contained ODE approach in this section. First, we note that it is sufficient to consider the case  $T < a$ . Since if the interior observability holds true for  $T = a$ , then there exists  $\delta_1 > 0$ , such that the observability is also true for  $T = a - \delta$ .<sup>3</sup>

By changing of variable  $z = \sqrt{w}x$ , it is more convenient to deal with the harmonic oscillator  $\mathcal{P} = -\partial_z^2 + z^2$  on  $L^2((-\sqrt{w}, \sqrt{w}))$  with Dirichlet boundary condition.

Let us consider the  $\mu$  dependent equations

$$\begin{cases} -f'' + (z^2 - \mu)f = 0, \\ f(0) = 1, \quad f'(0) = 0. \end{cases} \quad (9.1)$$

We will denote by  $f_\mu(z) = f(\mu, z)$  the unique solution to (9.1).

Denote by

$$Z_0(\mu) := \inf\{z > 0 : f_\mu(z) = 0\}.$$

$Z_0(\mu)$  is the first zero of  $f(\mu, z)$ . If  $f_\mu$  does not have zero, then  $Z_0(\mu) = +\infty$ . By inverse function theorem and Sturm-Liouville theorem, it is known that  $Z_0(\mu) < 1$  for all  $\mu > 1$ , and the function  $\mu \mapsto Z_0(\mu)$  is strictly decreasing for  $\mu > 1$ . Moreover,  $Z_0(1) = +\infty$ . Denote by

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<sup>3</sup>This comment would not apply for a boundary observability problem.

$w \mapsto \mu(w)$  the inverse function of  $Z_0(\mu)$ . It is also known that the function  $\mu(w)$  is smooth, strictly decreasing, and for all  $w > 0$ ,

$$\mu(w) > 1, \quad \lim_{w \rightarrow \infty} \mu(w) = 1.$$

Since  $\mu(w) - 1$  tends to 0 as  $w \rightarrow \infty$ , it is more convenient to work with

$$\nu(w) := \mu(w) - 1$$

and denote by  $f(\nu, z) = f(\mu, z)$ , the solution of (9.1), by abusing the notation.

**9.1. Lack of observability for  $T < \mathcal{L}(\omega)$ .** Let  $I = (-1, 1)$ , and denote by

$$p(w, x) := \frac{f(\nu(w), \sqrt{w}x)}{\|f(\nu(w), \sqrt{w}\cdot)\|_{L^2(I)}}$$

the first normalized eigenfunction of  $\mathcal{L}_w := -\frac{d^2}{dx^2} + w^2x^2$  on  $L^2((-1, 1))$ , which is real-valued, with the least eigenvalue  $\lambda(w) = w(1 + \nu(w))$ . For  $m \in \mathbb{N}, y \in \mathbb{R}$ , we define the phase function

$$\Phi_m(t, y, w) = wy - \lambda(w)t - 2\pi mw.$$

The following estimate for the eigenvalue  $\lambda(w)$ , eigenfunction  $p(w, x)$  and phase function  $\Phi_m(t, y, w)$  is crucial for the proof of Theorem 3.

**Proposition 9.1.** *There exists  $w_0 > 0$ , large enough, such that for all  $w \geq w_0, |x| \leq \frac{1}{2}$ ,*

- (1)  $|\lambda(w) - w| \leq C_0 w^{3/2} e^{-w}, \nu^{(k)}(w) \leq C_k w^{A_k} e^{-w}$ , for  $k \geq 0$ ;
- (2)  $|\lambda^{(k)}(w)| \leq C_k w^{A_k} e^{-w}$ , for  $k \geq 2$ ;
- (3)  $\left| \frac{\partial^k \Phi_m}{\partial w^k}(t, y, w) \right| \leq C_0(1 + |t|) w^{A_k} e^{-w}$ , for  $k \geq 2, t, y \in \mathbb{R}, m \in \mathbb{N}$ ;
- (4)  $cw^{1/4} e^{-\frac{wx^2}{2}} \leq |p(w, x)| \leq w^{1/4} e^{-\frac{wx^2}{2}}$  and  $\left| \frac{\partial^k p}{\partial w^k}(w, x) \right| \leq Cw^{1/4-k}$ , for  $k = 1, 2$ ;
- (5)  $|p(w, x_0)| \leq e^{-\frac{w}{10}}$ , for  $\frac{1}{2} < |x_0| \leq 1$ .

where  $A_k, C_k$  are positive constants depending on  $k$  and  $C_0, C, c > 0$  are uniform constants.

**Remark 9.2.** The content of Proposition 9.1 is not new. Indeed, estimates for the eigenvalues and eigenfunctions of more general semi-classical harmonic oscillator can be found in [HS84]. However, for our specific purpose, and to have a self contained exposition, we will present an elementary by-hand proof of the Proposition 9.1, which may be of its own interest.

We postpone the proof of Proposition 9.1 and use it to prove Theorem 3. It is sufficient to consider the case where  $\omega^c$  consists of a single band. Thus a reformulation of 3 is the following:



**Proposition 9.3.** *Let  $T < a$ . Then there exists a sequence of solutions  $(u_n)_{n \in \mathbb{N}}$  of equation (1.3), such that*

$$\liminf_{n \rightarrow \infty} \int_{\Omega} |u_n(0, x, y)|^2 dx dy > 0,$$

while

$$\lim_{n \rightarrow \infty} \int_{-T}^T \int_{(-1,1)_x \times ((a,\pi) \cup (-\pi,-a))_y} |u_n(t, x, y)|^2 dx dy dt = 0. \quad (9.2)$$

*Proof.* We use the similar Gaussian beam construction as in [RS18]. Let  $h_n = 2^{-n}$  and define

$$g_n(w) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-\frac{y^2}{2h_n^2} - iwy} \frac{dy}{\sqrt{h_n}}, \quad \chi \in C_c^\infty((1/2, 2)).$$

By changing variables we observe that  $\|g_n\|_{L^\infty} = O(h_n^{1/2})$  and  $\|\partial_w^k g_n\|_{L^\infty} = O(h_n^{k+\frac{1}{2}})$ . Define

$$u_{n,0}(x, y) = \sum_{k \in \mathbb{Z}} g_n(k) \chi(h_n k) p(k, x) e^{iky}.$$

We first claim that  $\|u_{n,0}\|_{L^2(\Omega)} \sim 1$ . Indeed,

$$g_n(w) = \frac{h_n^{1/2}}{2\pi} \int_{-\frac{\pi}{h_n}}^{\frac{\pi}{h_n}} e^{-\frac{z^2}{2} - izh_n w} dz = \frac{h_n^{1/2}}{2\pi} \widehat{(e^{-|\cdot|^2})}(h_n w) - \frac{h_n^{1/2}}{2\pi} \int_{|z| > \frac{\pi}{h_n}} e^{-\frac{z^2}{2} - izh_n w} dz,$$

hence  $g_n(w) = \frac{h_n^{1/2}}{\sqrt{2\pi}} e^{-\frac{h_n^2 w^2}{2}} + O(e^{-\frac{\pi^2}{4h_n^2}})$ . By Plancherel and (4) of Proposition 9.1, for fixed  $x \in (-1/2, 1/2)$ ,

$$\begin{aligned} \|u_{n,0}(x, \cdot)\|_{L^2(\mathbb{T})}^2 &= \sum_{k \in \mathbb{Z}} |g_n(k) \chi(h_n k)|^2 |p(k, x)|^2 \\ &\geq c \sum_{h_n^{-1}/2 \leq |k| \leq 2h_n^{-1}} h_n e^{-h_n^2 k^2} k^{1/2} e^{-\frac{kx^2}{2}}. \end{aligned}$$

Integrating with respect to  $x \in (-1, 1)$ , we obtain that

$$\|u_{n,0}\|_{L^2(\Omega)}^2 \geq c e^{-4} \int_{-1/2}^{1/2} e^{-\frac{kx^2}{2}} k^{1/2} dx \geq c_1 > 0,$$

uniformly in  $n \in \mathbb{N}$ .

Next, we prove (9.2). Note that the solutions to (1.3) with initial data  $u_{n,0}$  are given by

$$u_n(t, x, y) = \sum_{k \in \mathbb{Z}} g_n(k) \chi(h_n k) p(k, x) e^{iky - i\lambda(k)t}.$$

If  $|x| > \frac{1}{2}$ , from (5) of Proposition 9.1,

$$|u_n(t, x, y)| \leq C h_n^{-1/2} e^{-\frac{1}{20h_n}}. \quad (9.3)$$

For  $|x| \leq \frac{1}{2}$ , we estimate  $|u_n(t, x, y)|$  in another way. From Poisson summation formula, we have

$$u_n(t, x, y) = \sum_{m \in \mathbb{Z}} \widehat{K_{t,x,y}^{(n)}}(2\pi m),$$

where

$$\widehat{K_{t,x,y}^{(n)}}(2\pi m) = \int_{\mathbb{R}} g_n(w) \chi(h_n w) p(w, x) e^{i\Phi_m(t,y,w)} dw.$$

Writting  $e^{i\Phi_m} = \frac{1}{i\partial_w \Phi_m} \frac{\partial}{\partial w} (e^{i\Phi_m})$ , we have from the integration by part that

$$\widehat{K_{t,x,y}^{(n)}}(2\pi m) = - \int_{\mathbb{R}} e^{i\Phi_m} \frac{\partial}{\partial w} \left[ \frac{\partial_w (g_n(w) \chi(h_n w) p(w, x)) \partial_w \Phi_m - g_n(w) \chi(h_n w) p(w, x) \partial_w^2 \Phi_m}{|\partial_w \Phi_m(t, y, w)|^3} \right] dw.$$

Thanks to Proposition 9.1,  $|\partial_w \Phi_m(t, y, w)| = |y - t - 2\pi m| + o(e^{-w/2})$ ,  $|\partial_w^k \Phi_m| = O(e^{-w/2})$ , for  $k \geq 2$ . Therefore, if  $|t| \leq T < a$  and  $y \in (a, \pi) \cup (-\pi, -a)$ , we have  $|y - t - 2\pi m| \geq |m - c_0|$  for some  $0 < c_0 < 1$ , thus  $|\partial_w \Phi_m| \neq 0$ , since  $w \sim h_n^{-1}$  is large. The main contributions inside the integral is

$$\frac{\partial_w^2 (g_n(w) \chi(h_n w) p(w, x))}{|\partial_w \Phi_m|^2}$$

and all other terms can be bounded by  $C e^{-\frac{c}{h_n}} |y - t - 2\pi m|^{-3}$ .

The contributions of  $\partial_w^2 (g_n(w) \chi(h_n w) p(w, x))$  are

$$O(h_n^{5/2} w^{1/4}), O(h_n^{3/2} w^{-3/4}), O(h_n^{1/2} w^{-7/4}).$$

The integration is taken over  $w \sim h_n^{-1}$ , we have that

$$\sup_{(t,x,y) \in (0,T) \times \omega} |\widehat{K_{t,x,y}^{(n)}}(2\pi m)| \leq \frac{C h_n^{5/4}}{|m - c_0|^2}. \quad (9.4)$$

Since  $\sum_{m \in \mathbb{Z}} |m - c_0|^2 < \infty$ , combining (9.3) and (9.4), we conclude that

$$\lim_{n \rightarrow \infty} \|u_n(t, x, y)\|_{L^2((-T,T) \times \omega)} = 0.$$

The proof of Proposition 9.3 is now complete.  $\square$

In the rest subsections, we analyze the solution of the Hermite equation (9.1) and prove Proposition (9.1).

**9.2. Formal power expansions.** To solve the Hermite equation (9.1), we make the ansatz of the power series expansion

$$f(\nu, z) \simeq \sum_{j=0}^{\infty} \nu^j f_j(z)$$

Plugging into (9.1) and comparing the degree of  $\nu^j$  in both side, we have

$$\begin{aligned} f_0'' &= (z^2 - 1)f_0, & f_0(0) &= 1, f_0'(0) = 0, \\ f_{n+1}'' &= (z^2 - 1)f_{n+1} - f_n, & f_{n+1}(0) &= f_{n+1}'(0) = 0, \forall n \geq 0. \end{aligned}$$

It is easy to verify that  $f_0(z) = e^{-\frac{z^2}{2}}$ . Moreover, the formally expansion formula holds

$$f(\nu, z) \simeq \sum_{j=0}^{\infty} \nu^j f_j(z) \simeq e^{-\frac{z^2}{2}} + \sum_{j=1}^{\infty} (-1)^j c_j \nu^j \frac{(\log z)^{j-1}}{z} e^{\frac{z^2}{2}},$$

where  $c_j > 0$  are numerical constants. However, for our need, we only need expand to the first order term  $f_1(z)$ .

$$f(\nu, z) = f_0(z) + \nu f_1(z) + \nu^2 R_2(\nu, z).$$

The remainder term  $R_2$  solves the equation

$$\begin{cases} R_2'' = (z^2 - \nu - 1)R_2 - f_1, \\ R_2(0) = R_2'(0) = 0. \end{cases} \quad (9.5)$$

For  $f_1(z)$ , we have

**Proposition 9.4.**  $f_1(z) < 0$  for all  $z > 0$  and

$$f_1(z) = e^{\frac{z^2}{2}} \left( -\frac{\sqrt{\pi}}{4z} + O(z^{-3}) \right), \quad z \rightarrow +\infty.$$

Furthermore, for all  $k \geq 0$ ,

$$|f_1^{(k)}(z)| \leq C_k |z|^{k-1} e^{\frac{z^2}{2}}, \quad \text{as } z \rightarrow \infty.$$

**Lemma 9.5.** Assume that  $f \in C^1([0, \infty))$  and  $|f(z)| + |f'(z)| \leq C|z|^N$  for some  $N \in \mathbb{N}^*$ . Let  $I_f(z) = \int_0^z f(y) e^{-(z^2-y^2)} dy$ , then

$$I_f(z) = \frac{f(z)}{2z} + O\left(\frac{1 + |zf'(z)|}{|z|^3}\right), \quad z \rightarrow +\infty.$$

*Proof.* By changing of variable:  $u = y^2$ , we have

$$I_f(z) = \int_0^{z^2} f(\sqrt{u}) \frac{e^{-(z^2-u)}}{2\sqrt{u}} du = \int_0^{z^2} \frac{f(\sqrt{z^2-v}) e^{-v}}{2\sqrt{z^2-v}} dv.$$

By further changing of variable  $v \mapsto z^2 s^2$ , we have

$$I_f(z) = z \int_0^1 \frac{s f(z\sqrt{1-s^2}) e^{-z^2 s^2}}{\sqrt{1-s^2}} ds = z \int_0^{\frac{1}{2}} \frac{s f(z\sqrt{1-s^2})}{\sqrt{1-s^2}} e^{-z^2 s^2} ds + o(e^{-z^2/8})$$

since  $\frac{s}{\sqrt{1-s^2}}$  is integrable on  $(1/2, 1)$  and  $f$  is polynomially growth in  $z$ . Writting the first integral on the right hand side as  $-\frac{1}{z} \int_0^{\frac{1}{2}} \frac{f(z\sqrt{1-s^2})}{2\sqrt{1-s^2}} \frac{d}{ds} (e^{-z^2 s^2}) ds$  and doing integration by part,

we have

$$\begin{aligned}
& -\frac{1}{z} \int_0^{\frac{1}{2}} \frac{f(z\sqrt{1-s^2})}{2\sqrt{1-s^2}} \frac{d}{ds} (e^{-z^2 s^2}) ds \\
&= \frac{f(z)}{2z} + o(e^{-z^2/8}) + \frac{1}{2z} \int_0^{\frac{1}{2}} e^{-z^2 s^2} \left( f(z\sqrt{1-s^2})(1-s^2)^{-\frac{1}{2}} \right)' ds \\
&= \frac{f(z)}{2z} + O(e^{-z^2/4}) - \frac{1}{4z^3} \int_0^{\frac{1}{2}} \frac{d}{ds} (e^{-z^2 s^2}) \frac{1}{s} \left( f(z\sqrt{1-s^2})(1-s^2)^{-\frac{1}{2}} \right)' ds.
\end{aligned}$$

Since the third term on the right hand side is  $O(z^{-3})$ , the proof of Lemma 9.5 is complete.  $\square$

*Proof of Proposition 9.4.* Denote by  $g_1(z) = e^{-\frac{z^2}{2}} f_1(z) := f_1 f_0$ , then  $g_1$  solves the equation

$$\begin{cases} g_1'' + 2(zg_1)' = -e^{-z^2}, \\ g_1(0) = g_1'(0) = 0, \quad g_1''(0) < 0. \end{cases} \quad (9.6)$$

Integrating (9.6) and then solving a first order ODE, we obtain that

$$g_1(z) = - \int_0^z \Phi(y) e^{-(z^2-y^2)} dy < 0, \quad \text{where } \Phi(z) = \int_0^z e^{-y^2} dy.$$

Since  $\Phi(z)$  is bounded as well as its derivatives and  $g_1(z) = -I_\Phi(z)$ , applying Lemma 9.5, we obtain that  $g_1(z) = -\frac{\Phi(z)}{2z} + O(|z|^{-3})$ , as  $z \rightarrow +\infty$ . From the fact that  $|\frac{\sqrt{\pi}}{2} - \Phi(z)| = |\Phi(+\infty) - \Phi(z)| = O(e^{-z^2/2})$ , as  $z \rightarrow +\infty$ , we obtain that  $g_1(z) = -\frac{\sqrt{\pi}}{4z} + O(|z|^{-3})$ . The estimate for the derivatives can be deduced by induction. Since  $f_1' = e^{\frac{z^2}{2}}(g_1' + zg_1)$  and  $g_1'(z) = -\Phi(z) + \int_0^z \Phi(z) z e^{-(z^2-y^2)} dy$ , we have  $|f_1'(z)| \leq C_1 e^{\frac{z^2}{2}}$ , thanks to Lemma 9.5. Finally, from the equation of  $f_1$ , we deduce that  $|f^{(k)}(z)| \leq C|z|^{k-1} e^{\frac{z^2}{2}}$  and this completes the proof of Proposition 9.4.  $\square$

**9.3. Estimate for remainders: Laplace method.** Our goal is to obtain estimates for  $R_2$  and its derivatives as  $z \rightarrow \infty$ . It is difficult to write the solution of (9.5) explicitly. Fortunately, since (9.5) is a (inhomogeneous) Hermite type equation, we could obtain the asymptotics by using the Laplace transform. For this standard method, we follow the book [WG88].

Let  $P_2(\nu, z) = R_2(\nu, z) e^{\frac{z^2}{2}}$ ,  $Q_2(z) = -f_1(z) e^{\frac{z^2}{2}}$ , then (9.5) is transformed to the following equation:

$$\begin{cases} P_2'' - 2zP_2' + \nu P_2 = Q_2, \\ P_2(0) = P_2'(0) = 0. \end{cases} \quad (9.7)$$

We first look for two independent solutions of the homogeneous equation

$$q'' - 2zq' + \nu q = 0. \quad (9.8)$$

We split the domain  $\Omega = \mathbb{C} \setminus (-\infty, 0]$  and choose a branch

$$\log s = \log |s| + i \arg s, \quad \arg s \in (-\pi, \pi).$$

Then we consider the contour  $C$  in the  $s$  plane: we start from  $-\infty + i0^+$  and turn around  $s = 0$ , then turn to  $-\infty + i0^-$ . Assume that the solution is of the form

$$q(z) = \int_C e^{sz} Y(s) ds.$$

Plugging into the equation (9.8), we have (assuming legality for changing the differentiation and the integration)

$$q'' - 2zq' + \nu q = \int_C Y(s) (s^2 + \nu - 2s\partial_s) e^{sz} ds.$$

After Integration by part, we have that

$$\begin{aligned} q'' - 2zq' + \nu q &= \int_C [(s^2 + \nu)Y(s) + 2(sY(s))'] e^{sz} ds - \{2sY(s)e^{sz}\}_C \\ &= \int_C [(s^2 + \nu + 2)Y + 2sY'(s)] e^{sz} dz - \{2sY(s)e^{sz}\}_C, \end{aligned}$$

where

$$\{2sY(s)e^{sz}\}_C = [2sY(s)e^{sz}]_{s=-\infty+i0^-}^{s=-\infty+i0^+},$$

if both the limits exist. We require that

$$(s^2 + \nu + 2)Y + 2sY'(s) = 0,$$

and a simple choice is

$$Y(s) = e^{-\frac{s^2}{4}} s^{-\frac{\nu+2}{2}} = e^{-\frac{s^2}{4} - \frac{\nu+2}{2} \log s}.$$

From the choice of the contour  $C$ ,  $\{2sY(s)e^{sz}\}_C = 0$ , for all  $|\arg z| \leq \frac{\pi}{2} - \delta$  ( $\delta > 0$ ). Moreover, the interchange of the order of integration and differentiation is legality. Thus, we obtained one solution of the homogeneous equation of (9.8)

$$q_1(\nu, z) := \int_C e^{sz - \frac{s^2}{4} - \frac{\nu+2}{2} \log s} ds. \quad (9.9)$$

For the convenience of the estimates, we will rewrite  $q_1$  as an integral of another contour  $C_+$  in the  $s$  plane: Choosing a branch of  $\log s = \log |s| + i\arg(s)$ ,  $\arg(s) \in (0, 2\pi)$  by splitting  $\Omega_+ := \mathbb{C} \setminus [0, +\infty)$ . Then start from  $+\infty + i0^+$  and turn around  $s = 0$ , then turn to  $+\infty + i0^-$ . Changing  $s$  to  $-s$  in the expression (9.9), we have

$$q_1(\nu, z) = - \int_{C_+} (-s)^{-\frac{\nu+2}{2}} e^{-\frac{s^2}{4} - sz} ds = e^{-\frac{i\pi\nu}{2}} \int_{C_+} s^{-\frac{\nu+2}{2}} e^{-\frac{s^2}{4} - sz} ds. \quad (9.10)$$

**Lemma 9.6** (Watson's Lemma[[Wa1918](#)]). *Let  $g(s)$  be an analytic function in  $0 < \arg s < 2\pi$ . Assume that  $g(s)$  satisfies the following asymptotics:*

$$g(s) = O(e^{b|s|}), s \rightarrow \infty, \text{ for some } b \in \mathbb{R},$$

$$sg(s) \sim_{s \rightarrow 0} \sum_{n=1}^{\infty} a_n s^{\lambda_n}$$

in the sense that for all  $N \in \mathbb{N}$ ,

$$\left| sg(s) - \sum_{n=1}^N a_n s^{\lambda_n} \right| = o(|s|^{\lambda_N}), \quad s \rightarrow 0, \quad (9.11)$$

where  $0 < \lambda_1 < \lambda_2 < \dots$ , and  $\lambda_n \rightarrow \infty$ . Then the function

$$F(z) = \int_{C_+} e^{-sz} g(s) ds, \quad |\arg z| \leq \frac{\pi}{2} - \delta, \quad \delta > 0$$

satisfies the asymptotics:

$$F(z) \sim 2i \sum_{n=1}^{\infty} a_n \Gamma(\lambda_n) \sin(\lambda_n \pi) e^{i\lambda_n \pi} z^{-\lambda_n}, \quad |z| \rightarrow \infty,$$

in the same sense as (9.11).

*Proof.* The proof can be found in [WG88]. For the completeness, we present the proof for  $z \in \mathbb{R}, z \rightarrow \infty$ , which is enough for our need. From the asymptotic behaviour of  $g(s)$  near  $\infty$ , for any  $N$ , there exists a bounded function  $\sigma_N(s) > 0$  with the property  $\sigma_N(s) = o(1), s \rightarrow 0$ , such that

$$\left| g(s) - \sum_{n=1}^N a_n s^{\lambda_n - 1} \right| \leq \sigma_N(s) |s|^{\lambda_N - 1} e^{b|s|}. \quad (9.12)$$

By changing variables and shrinking the contour if necessary, we have

$$\int_{C_+} s^{\lambda_n - 1} e^{-sz} ds = z^{-\lambda_n} \left( \int_{+\infty+i0^+}^{0+i0^+} + \int_{0+i0^-}^{+\infty+i0^-} \right) v^{\lambda_n - 1} e^{-v} dv$$

Since on  $[0+i0^+, +\infty+i0^+)$ ,  $v^{\lambda_n - 1} = |v|^{\lambda_n - 1}$ , while on  $[0+i0^-, +\infty+i0^-)$ ,  $v^{\lambda_n - 1} = |v|^{\lambda_n - 1} e^{2\pi i \lambda_n}$ , the integral above equals to

$$z^{-\lambda_n} \Gamma(\lambda_n) (e^{2\pi i \lambda_n} - 1) = z^{-\lambda_n} \Gamma(\lambda_n) 2i \sin(\lambda_n \pi) e^{i\lambda_n \pi}.$$

It remains to estimate the remainder term

$$R_N(z) = \int_{C_+} e^{-sz} g(s) ds - \int_{C_+} \sum_{n=1}^N a_n s^{\lambda_n - 1} e^{-sz} ds.$$

From (9.12), we have

$$|R_N(z)| \leq C z^{-\lambda_N} \int_0^\infty e^{-u} \sigma_N\left(\frac{u}{z}\right) u^{\lambda_N - 1} e^{\frac{bu}{z}} du.$$

By dominating convergence,  $|z^{\lambda_N} R_N(z)| = o(1)$ , as  $z \rightarrow +\infty$ . This completes the proof of Lemma 9.6.  $\square$

**Corollary 9.7.** *Fix  $0 < \nu < 1$ . Then*

$$q_1(\nu, z) \sim_{z \rightarrow +\infty} z^{\frac{\nu}{2}} \sum_{k=0}^{\infty} c_{2k}(\nu) z^{-2k},$$

in the sense that for any  $N \in \mathbb{N}$ ,

$$\left| z^{-\frac{\nu}{2}} q_1(\nu, z) - \sum_{k=0}^N c_{2k}(\nu) z^{-2k} \right| = o(|z|^{-2N}), \quad z \rightarrow +\infty,$$

uniformly with respect to  $0 < \nu < 1$ . Moreover,  $c_0(\nu) \rightarrow 2\pi i$  as  $\nu \rightarrow 0$ , and there exist  $0 < \nu_0 < 1$ ,  $z_0 > 1$ , such that for all  $0 < \nu < \nu_0$ ,  $z \geq z_0$ ,  $|q_1(\nu, z)| \geq \pi |z|^{\frac{\nu}{2}}$ . Furthermore, we have

$$|(\partial_\nu^k q_1)(\nu, z)| \leq C_k |z|^{\nu/2} |\log z|^k, \quad |(\partial_z^k q_1)(\nu, z)| \leq C_k |z|^{\frac{\nu}{2}-k}. \quad (9.13)$$

for all  $k \in \mathbb{N}^*$  as  $z \rightarrow +\infty$ , where the constants  $C_k$  are independent of  $\nu$ .

*Proof.* Denote by  $G(s) = e^{-\frac{i\pi\nu}{2}} s^{-\frac{\nu+2}{2}} e^{-\frac{s^2}{4}}$  which is bounded near  $\infty$ . Then by definition,  $q_1(\nu, z) = \int_{C_+} G(s) e^{-sz} ds$ . Note that

$$s \left( G(s) - e^{-\frac{i\pi\nu}{2}} s^{-\frac{\nu+2}{2}} \right) \sim_{s \rightarrow 0} e^{-\frac{i\pi\nu}{2}} \sum_{k=1}^{\infty} \frac{(-1)^k}{4^k \cdot k!} s^{2k-\frac{\nu}{2}}.$$

Applying Watson's Lemma 9.6 for  $g(s) = G(s) - e^{-\frac{i\pi\nu}{2}} s^{-\frac{\nu+2}{2}}$  with  $\lambda_k = 2k - \frac{\nu}{2}$ ,  $a_k = e^{-\frac{i\pi\nu}{2}} \frac{(-1)^k}{4^k \cdot k!}$ , we have

$$q_1(z) - e^{-\frac{i\pi\nu}{2}} \int_{C_+} s^{-\frac{\nu+2}{2}} e^{-sz} ds \sim 2iz^{\frac{\nu}{2}} \sin\left(\frac{\nu+2}{2}\pi\right) \sum_{k=1}^{\infty} \frac{(-1)^k e^{-i\pi\nu}}{4^k \cdot k!} \Gamma\left(2k - \frac{\nu}{2}\right) z^{-2k}.$$

Note that the bound on the remainder terms is independent of  $0 < \nu < 1$ , since for  $k \geq 1$ ,  $\sin((\nu+2)\pi/2)\Gamma(2k - \nu/2)e^{-i\pi\nu}$  is bounded with respect to  $\nu$ . In particular, the coefficient  $c_k(\nu)$  is bounded in  $\nu$ .

By changing of variables and applying Cauchy's theorem, we have

$$e^{-\frac{i\pi\nu}{2}} \int_{C_+} s^{-\frac{\nu+2}{2}} e^{-sz} ds = e^{-\frac{i\pi\nu}{2}} z^{\frac{\nu}{2}} \int_{\tilde{C}_+} u^{-\frac{\nu+2}{2}} e^{-u} du.$$

where  $\tilde{C}_+$  is an enlargement of the contour  $C_+$  so that  $|s| \geq 1$  along  $\tilde{C}_+$ . Hence

$$c_0(\nu) = e^{-\frac{i\pi\nu}{2}} \int_{\tilde{C}_+} u^{-\frac{\nu+2}{2}} e^{-u} du.$$

Next we show that  $c_0 \neq 0$  for sufficiently small  $\nu$ . Note that  $\Gamma(s)$  is analytic for  $\operatorname{Re} s > 0$ , therefore, by Cauchy's theorem, we have

$$\Gamma(s) = \frac{e^{-i\pi s}}{2i \sin(\pi s)} \int_{\tilde{C}_+} e^{-u} u^{s-1} du, \quad \text{for } \operatorname{Re} s > 0. \quad (9.14)$$

The right hand side and left hand side are both analytic functions for  $s \in \mathbb{C}$  except for simple poles  $s = 0, -1, -2, \dots$ , therefore, (9.14) holds true for all  $s \in \mathbb{C}$ ,  $s \notin -\mathbb{N}$ . Therefore,

$$c_0(\nu) = e^{-\frac{i\pi\nu}{2}} \int_{C_+} u^{-\frac{\nu}{2}-1} e^{-u} du = -2ie^{-i\pi\nu} \sin\left(\frac{\pi\nu}{2}\right) \Gamma\left(-\frac{\nu}{2}\right) \neq 0$$

for all  $0 < \nu < 1$ . Clearly,  $c_0(\nu) \rightarrow 2\pi i$  as  $\nu \rightarrow 0$ , thanks to the formula  $\Gamma(s+1) = s\Gamma(s)$ . Consequently,  $|q_1(\nu, z)|$  has a uniform lower bound for small  $\nu > 0$  and large  $z > 1$ .

Finally, we estimate the derivatives. From direct calculation and using Cauchy's theorem,

$$(\partial_\nu^k q_1)(\nu, z) = e^{-\frac{i\pi\nu}{2}} \int_{\tilde{C}_+} P_k(\log s) s^{-\frac{\nu+2}{2}} e^{-\frac{s^2}{4}-sz} ds,$$

where  $P_k(\cdot)$  is a polynomial of degree  $k$ . Note that on  $\tilde{C}_+$  such that  $|s| \geq 1$  around the origin, we deduce from Lemma 9.6 that

$$|(\partial_\nu^k q_1)(\nu, z)| \leq C_k |z|^{\frac{\nu}{2}} |\log z|^k.$$

For the derivatives on  $z$ , we calculate

$$(\partial_z^k q_1)(\nu, z) = - \int_{C_+} (-s)^{k-\frac{\nu+2}{2}} e^{-\frac{s^2}{4}-sz} ds.$$

Using Watson's Lemma 9.6 we have that  $|\partial_z^k q_1| \leq C_k z^{\frac{\nu}{2}-k}$ . The proof of Corollary 9.7 is complete.  $\square$

The Wronskian of (9.8) satisfies  $W'(z) = 2zW$ , thus another independent solution of (9.8) is given by

$$q_2(z) = q_1(z) \left[ \int_{z_0}^z \frac{e^{w^2}}{(q_1(w))^2} dw + C \right]$$

for  $z > z_0$ , where  $z_0 > 0$  is chosen as in Corollary 9.7. Note that

$$W(z) = q_1 q_2' - q_2 q_1' = e^{z^2}. \quad (9.15)$$

**Corollary 9.8.** *There exist  $0 < \nu_0 < 1$ ,  $z_0 > 1$ , such that for all  $k \in \mathbb{N}$ ,  $0 < \nu < \nu_0$  and  $z \geq z_0$ , we have*

$$|q_2(\nu, z)| \leq C z^{\frac{\nu}{2}-1} e^{z^2}, |\partial_\nu^k q_2(\nu, z)| \leq C_k z^{\frac{\nu}{2}-1} (\log z)^k e^{z^2}, |\partial_z^k q_2(\nu, z)| \leq C_k (z^{\frac{\nu}{2}-k-1} + z^{k-1-\frac{\nu}{2}}) e^{z^2},$$

where the constants  $C_k$  are independent of  $\nu$ .

*Proof.* First we remark that we only need estimate the term  $q_1(z) \int_{z_0}^z \frac{e^{w^2}}{(q_1(w))^2} dw$ . Let  $\nu_0, z_0$  as in Corollary 9.7. Denote by  $I(z) = \int_0^z e^{-(z^2-x^2)} dx$ . From Lemma 9.5,  $I(z) \sim \frac{1}{2z}$  for large  $z$ . Applying Corollary 9.7, we have

$$|q_2(\nu, z)| e^{-z^2} \leq \frac{C z^{\nu/2}}{z_0^\nu} \int_{z_0}^z e^{x^2-z^2} dx \leq \frac{C z^{\nu/2}}{z_0^\nu} I(z) \leq C (z_0) z^{\frac{\nu}{2}-1}.$$



To estimate the derivatives, observe that for each  $k \in \mathbb{N}^*$ ,  $\partial_\nu^k(q_2)$  can be written as linear combinations of

$$\partial_\nu^{k_1}(q_1) \cdot \int_{z_0}^z e^{w^2} \partial_\nu^{k_2}(q_1^{-2}) dw.$$

where  $k_1 + k_2 = k$ . From Faà di Bruno's formula,  $\partial_\nu^{k_2}(q_1^{-2})$  is linear combinations of

$$q_1^{-(2+m_1+m_2+\dots+m_{k_2})} (\partial_\nu q_1)^{m_1} (\partial_\nu^2 q_1)^{m_2} \dots (\partial_\nu^{k_2} q_1)^{m_{k_2}}, \text{ where } \sum_{j=1}^{k_2} j m_j = k_2.$$

From Corollary 9.7,

$$|\partial_\nu^{k_2}(q_2)^{-2}| \leq C_{k_2} z^{-\nu} |\log z|^{k_2}.$$

Hence

$$|\partial_\nu^k q_2| \leq \sum_{k_1+k_2=k} C_k z^{\frac{\nu}{2}} (\log z)^{k_1} \int_{z_0}^z e^{w^2} (\log w)^{k_2} w^{-\nu} dw \leq C'_k z^{\frac{\nu}{2}} (\log z)^k e^{z^2} I(z).$$

Applying Lemma 9.5, we obtain the desired bound. For  $\partial_z^k q_2$ , it can be written as linear combinations of

$$\partial_z^k q_1 \cdot \int_{z_0}^z \frac{e^{w^2}}{q_1(w)^2} dw \text{ and } (\partial_z^{k_1} q_1) \cdot (\partial_z^{k_2}(q_1^{-2})) \cdot (\partial_z^{k-k_1-k_2}(e^{z^2})), k_1 < k.$$

The first term can be majorized by  $C_k |z|^{\frac{\nu}{2}-k-1}$ , thanks to Corollary 9.7. Again, from Faà di Bruno's formula,  $\partial_z^{k_2}(q_1^{-2})$  is the linear combination of the terms

$$q_1^{-(2+m_1+m_2+\dots+m_{k_2})} (\partial_z q_1)^{m_1} (\partial_z^2 q_1)^{m_2} \dots (\partial_z^{k_2} q_1)^{m_{k_2}}, \text{ where } \sum_{j=1}^{k_2} j m_j = k_2,$$

and

$$|\partial_z^{k_2}(q_1^{-2})| \leq C_k |z|^{-\nu-k_2}.$$

Using Corollary 9.7, we obtain that

$$|\partial_z^{k_1} q_1 \cdot \partial_z^{k_2}(q_1^{-2}) \cdot \partial_z^{k-k_1-k_2-1}(e^{z^2})| \leq C_k |z|^{\frac{\nu}{2}-k_1} \cdot |z|^{-\nu-k_2} \cdot |z|^{k-k_1-k_2-1} e^{z^2} \leq C_k |z|^{k-\frac{\nu}{2}-2(k_1+k_2)-1} e^{z^2}.$$

The worst case is  $k_1 = k_2 = 0$ . This completes the proof of Corollary 9.8.  $\square$

**Corollary 9.9.** *As  $z \rightarrow +\infty$ , we have for all  $k \in \mathbb{N}$ ,*

$$|R_2(\nu, z)| \leq C z^{\nu-1} e^{\frac{z^2}{2}}, |\partial_\nu^k R_2(\nu, z)| \leq C_k z^{\nu-1} |\log z|^k e^{\frac{z^2}{2}}, |\partial_z^k R_2(\nu, z)| \leq C_k z^{\nu+k-1} e^{\frac{z^2}{2}}.$$

*In particular, if  $\nu \leq A w^{1/2} e^{-w}$ , then for all  $1 \leq |z| \leq w^{1/2}$ , all  $k \in \mathbb{N}$ , we have*

$$|\partial_z^k f(\nu, z)| \leq C_k w^{1/2} z^{k-1} e^{\frac{z^2}{2}},$$

*where  $f(\nu, z)$  is the solution of (9.1).*

*Proof.* The solution of the inhomogeneous ODE (9.7) is given by

$$P_2(\nu, z) = -q_1(\nu, z) \int_0^z \frac{q_2(\nu, w)Q_2(w)}{W(w)} dw + q_2(z) \int_0^z \frac{q_1(\nu, w)Q_2(w)}{W(w)} dw.$$

In particular,

$$P_2(\nu, z) = -q_1(\nu, z) \int_0^z e^{-w^2} q_2(\nu, w)Q_2(w) dw + q_2(\nu, w) \int_0^z e^{-w^2} q_1(\nu, w)Q_2(w) dw,$$

where  $Q_2(z) = -f_1(z)e^{\frac{z^2}{2}}$ . From Proposition 9.4, Corollary 9.7 and Corollary 9.8 and the fact that  $I(z) \leq \frac{C}{z}$ , we have

$$|P_2(\nu, z)| \leq Cz^{\nu-1}e^{z^2}.$$

Recalling that  $R_2(\nu, z) = e^{-\frac{z^2}{2}}P_2(\nu, z)$ , we obtain that desired bound for  $|R_2|$ . For the derivatives, we observe that  $\partial_z^k P_2$  consists of linear combinations of

$$(\partial_z^k q_1) \cdot \int_0^z e^{-\frac{w^2}{2}} f_1(w)q_2(\nu, w) dw, \quad (\partial_z^k q_2) \cdot \int_0^z e^{-\frac{w^2}{2}} f_1(w)q_1(\nu, w) dw,$$

and

$$(\partial_z^{k_1} q_1) \cdot \partial_z^{k_2-1} (e^{-\frac{z^2}{2}} f_1(z)q_2), \quad (\partial_z^{k_1} q_2) \cdot \partial_z^{k_2-1} (e^{-\frac{z^2}{2}} f_1(z)q_1) \text{ where } k_1 + k_2 = k \text{ if } k_2 \geq 1.$$

Thus from Proposition 9.4, Corollary 9.7, Corollary 9.8, the dominating part can be bounded by  $C_k |z|^{\nu+k-1} e^{z^2}$ . Since  $R_2 = e^{-\frac{z^2}{2}} P_2$ , we obtain that  $|\partial_z^k R_2| \leq C_k |z|^{\nu+k-1} e^{\frac{z^2}{2}}$ . The derivatives  $\partial_\nu^k P_2$  consists of linear combinations of

$$(\partial_\nu^{k_1} q_1) \cdot \int_0^z e^{-\frac{w^2}{2}} f_1(w) (\partial_\nu^{k_2} q_2)(\nu, w) dw, \quad (\partial_\nu^{k_1} q_2) \cdot \int_0^z e^{-\frac{w^2}{2}} f_1(w) (\partial_\nu^{k_2} q_1)(\nu, w) dw,$$

where  $k_1 + k_2 = k$ . Again, from Proposition 9.4, Corollary 9.7, Corollary 9.8, these terms are dominated by  $|z|^{\nu-1} |\log z|^k e^{z^2}$ . Thus  $|\partial_\nu^k P_2| \leq C_k |z|^{\nu-1} |\log z|^k e^{\frac{z^2}{2}}$ . The last assertion is the direct consequence of the estimates above and Proposition 9.4. The proof of Corollary 9.9 is complete.  $\square$

**9.4. Estimate for the first eigenvalue and eigenfunction.** For fixed  $w > 0$ , consider the function

$$F(\nu, w) := f(\nu, \sqrt{w}) = f_0(\sqrt{w}) + \nu f_1(\sqrt{w}) + \nu^2 R_2(\nu, \sqrt{w}).$$

**Proposition 9.10.** *There exists  $w_0 > 0, A > 0$ , such that for all  $w \geq w_0$ , there exists a unique solution  $\nu(w) \in (0, Aw^{1/2}e^{-w})$  of the equation*

$$F(\nu(w), w) = 0.$$

Moreover, the application  $w \mapsto \nu(w)$  is smooth.

*Proof.* For fixed  $w > 0$  large, we define the iteration scheme as follows:

$$f_0(\sqrt{w}) + \nu_1 f_1(\sqrt{w}) = 0, \quad f_0(\sqrt{w}) + \nu_{n+1} f_1(\sqrt{w}) = -\nu_n^2 R_2(\nu_n, \sqrt{w}), \quad n \geq 1.$$

Since  $f_1(z)$  never vanish for large  $w$ , we have

$$\nu_{n+1} - \nu_n = \frac{G_2(\nu_{n-1}, w) - G_2(\nu_n, w)}{f_1(\sqrt{w})}$$

where

$$G_2(\nu, w) = \nu^2 R_2(\nu, \sqrt{w}).$$

Applying mean-value theorem and Proposition 9.4, we obtain that

$$|\nu_{n+1} - \nu_n| \leq \frac{|\partial_\nu G_2(\theta \nu_{n-1} + (1-\theta)\nu_n, w)|}{e^{w/2} c_1^{-1} w^{-1/2}} \cdot |\nu_n - \nu_{n-1}|.$$

Assume that  $\nu_j \leq Aw^{1/2}e^{-w}$ , for all  $j \leq n$ , then from Corollary 9.9, the right hand side can be bounded by

$$2Ac_1 e^{-w} w^{\frac{\nu}{4}+1} |\nu_n - \nu_{n-1}|.$$

Since  $\nu_1 = -f_0(\sqrt{w})/f_1(\sqrt{w}) \in [c_1 w^{1/2} e^{-w}, c_1^{-1} w^{1/2} e^{-w}]$ , we have

$$\nu_{n+1} \leq \sum_{j=0}^n (2Ac_1 e^{-w} w^{\frac{\nu}{4}+1})^j \nu_1 \leq c_1^{-1} w^{1/2} e^{-w} (1 - 2Ac_1 e^{-w} w^{\frac{\nu}{4}+1})^{-1} < Aw^{1/2} e^{-w},$$

provided that  $A > 2c_1^{-1}$  and  $w \geq w_0$  are large enough. Fixing such  $A$  and  $w_0$ , by induction, we have that  $\nu_n \leq Aw^{1/2}e^{-w}$  for all  $n \in \mathbb{N}$ . Now the absolute convergence of the sequence  $\sum_{n=0}^{\infty} |\nu_{n+1} - \nu_n|$  (with the convention  $\nu_0 = 0$ ) implies the existence of the limit

$$\nu(w) = \lim_{n \rightarrow \infty} \nu_n(w) \in [0, Aw^{1/2}e^{-w}].$$

Next we show that  $F(\nu(w), w) = 0$ . From the continuity of  $F(\cdot, w)$ , we have  $F(\nu_n, w) \rightarrow F(\nu(w), w)$  as  $n \rightarrow \infty$ . By definition, we write

$$f_0(\sqrt{w}) + \nu_n f_1(\sqrt{w}) = -F(\nu_{n-1}, w) + f_0(\sqrt{w}) + \nu_{n-1} f_1(\sqrt{w}).$$

Passing to the limit for both sides, we have  $F(\nu(w), w) = 0$ . The uniqueness follows from similar argument, and the smoothness of the mapping  $w \mapsto \nu(w)$  is guaranteed by the inverse function theorem. This completes the proof of Proposition 9.10.  $\square$

*Proof of Proposition 9.1.* From Proposition 9.4, Corollary 9.9, we know that

$$|f_1^{(k)}(\sqrt{w})| \leq C_k w^{\frac{k-1}{2}} e^{\frac{w}{2}}, \quad |\partial_\nu^k R_2(\nu, \sqrt{w})| \leq C_k w^{\frac{\nu-1}{2}} |\log w|^k e^{\frac{w}{2}}, \quad |\partial_z^k R_2(\nu, \sqrt{w})| \leq C_k w^{\frac{\nu+k-1}{2}} e^{\frac{w}{2}}.$$

We remark that the exact power of  $w$  is of no importance, since the dominated factors are always exponentially in  $w$ . To simplify the notation, we will replace all the power of  $w$  as well as  $\log w$  by some polynomials  $P_k$  in variable  $\sqrt{w}$ , which may change line by line in the calculations below.

The first estimate of (1) is a direct consequence of Proposition 9.10. To estimate the derivative  $\nu^{(k)}(w)$ , we first take one derivative in  $w$  to the equation

$$F(\nu(w), w) = f_0(\sqrt{w}) + \nu(w)f_1(\sqrt{w}) + \nu(w)^2 R_2(\nu(w), \sqrt{w}) = 0.$$

Since  $\nu(w) \leq Cw^{1/2}e^{-w}$ , the dominating coefficient of  $\nu'(w)$  is  $f_1(\sqrt{w})$ , since  $f_1$  and  $R_2$  have the same  $e^{w/2}$  factor. Moreover, the  $|\partial_w f_0(\sqrt{w})|$  is always much larger than  $|2\nu(w)R_2(\nu(w), \sqrt{w}) + \nu(w)^2 \partial_w (R_2(\nu(w), \sqrt{w}))|$ . Therefore,  $|\nu'(w)| \leq C_1 w^{A_1} e^{-w}$  holds true with some power  $A_1$ . Taking  $k$  derivatives in  $w$ , the dominating coefficient of  $\nu^{(k)}(w)$  is still  $f_1(\sqrt{w})$  while  $|\partial_w^k (f_0(\sqrt{w}))|$  is always much larger than all other terms, therefore,  $|\nu^{(k)}(w)| \leq C_k w^{A_k} e^{-w}$ . Then (2) and (3) can be deduced from the bound of  $\nu^{(k)}(w)$  easily.

Finally, we estimate  $p$  and  $\partial_w^k p$ . We first claim that

$$\|f(\nu(w), \sqrt{w}\cdot)\|_{L^2((-1,1))}^2 = w^{-1/2} \int_{-w^{1/2}}^{w^{1/2}} |f(\nu(w), z)|^2 dz \sim w^{-1/2}.$$

Indeed, the upper bound follows directly from Proposition 9.4 and Corollary 9.9. For the lower bound, note that for  $|z| < w^{1/2}/2$ ,  $\nu < Aw^{1/2}e^{-w}$ , Proposition 9.4 and Corollary 9.9 yield

$$|f(\nu, z)| \geq e^{-\frac{z^2}{2}} - Cw^{1/2}e^{-w}z^{-1/2}e^{z^2/2} \geq e^{-\frac{z^2}{2}} \left(1 - Cw^{1/2}z^{-1/2}e^{-(w-z^2)}\right) > e^{-\frac{z^2}{2}}/2,$$

if  $w$  is large enough. Hence  $\int_{-w^{1/2}}^{w^{1/2}} |f(\nu, z)|^2 dz$  is uniformly bounded from below.

Assume that  $|x| \leq \frac{1}{2}$ . Observe that

$$|\nu f_1(\sqrt{wx}) + \nu^2 R_2(\nu, \sqrt{wx})| \leq Cw^{A_1} e^{-\frac{wx^2}{2}} e^{-(w-wx^2)} \leq Cw^{A_1} e^{-\frac{3w}{4}} e^{-\frac{wx^2}{2}},$$

hence  $|p(w, x)| \sim w^{1/4} e^{-\frac{wx^2}{2}}$ . Next we estimate the derivatives of  $p(w, x)$ . Direct calculation shows

$$\partial_w (\|f(\nu(w), \sqrt{w}\cdot)\|_{L^2((-1,1))}^2) = O(w^{-3/2}).$$

Consequently, the absolute value of the second term of

$$\partial_w p(w, x) = \frac{\partial_w (f(\nu(w), \sqrt{wx}))}{\|f(\nu(w), \sqrt{w}\cdot)\|_{L^2((-1,1))}} - \frac{f(\nu(w), \sqrt{wx}) \partial_w (\|f(\nu(w), \sqrt{w}\cdot)\|_{L^2}^2)}{2\|f(\nu(w), \sqrt{w}\cdot)\|_{L^2((-1,1))}^3},$$

can be bounded by  $O(w^{-3/4})$ . To control the other term, we write

$$\begin{aligned} \partial_w (f(\nu(w), \sqrt{wx})) &= -\frac{x^2}{2} e^{-\frac{wx^2}{2}} + \nu' f_1(\sqrt{wx}) + 2\nu \nu' R_2(\nu, \sqrt{wx}) \\ &\quad + \nu \frac{x}{2\sqrt{w}} f_1'(\sqrt{wx}) + \frac{\nu^2 x}{2\sqrt{w}} (\partial_z R_2)(\nu, \sqrt{wx}) + \nu^2 \nu' (\partial_\nu R_2)(\nu, \sqrt{wx}). \end{aligned}$$

From Proposition 9.4 and Corollary 9.9, all the terms except for the first term can be bounded by  $O(w^A e^{-\frac{3w}{4}})$ . The first term  $x^2 e^{-wx^2/2} = (wx^2) e^{-wx^2/2} = O(w^{-1})$ . Therefore,  $|\partial_w p(w, x)| = O(w^{-3/4})$ . Taking one more derivatives, one easily verifies that  $|\partial_w^2 p(w, x)| = O(w^{-7/4})$ .

Finally, for  $\frac{1}{2} < |x_0| \leq 1$ ,

$$|p(w, x)| \leq Cw^{1/4}|f(\nu(w), \sqrt{wx})| \leq w^{1/4}e^{-\frac{w}{8}} + Cw^Ae^{-\frac{w}{2}} \leq e^{-\frac{w}{10}},$$

provided that  $w$  is large enough. This completes the proof of Proposition 9.1.  $\square$

## REFERENCES

- [AM14] N. Anantharaman and F. Macià, *Semiclassical measures for the Schrödinger equation on the torus*, J. Eur. Math. Soc., 16 (2014), 1253–1288.
- [ALM16] N. Anantharaman, M. Léautaud, F. Macià, *Wigner measures and observability for the Schrödinger equation on the disk*, Invent. Math., 206 (2016), 485–599.
- [BLR92] C. Bardos, G. Lebeau and J. Rauch. *Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary*, SIAM J. Control Optim., 30 (1992), 1024–1065.
- [BCG14] K. Beauchard, P. Cannarsa, R. Guglielmi. *Null controllability of Grushin-type operators in dimension two*, J. Eur. Math. Soc., 16 (2014), 67101.
- [BBZ13] J. Bourgain, N. Burq and M. Zworski, *Control for Schrödinger operators on 2-tori: rough potentials*. J. Eur. Math. Soc. 15 (2013), 1597–1628.
- [Bu04] N. Burq. *Smoothing effect for Schrödinger boundary value problems*. Duke Math. Jour., 123 (2004), 403–427.
- [BL03] N. Burq, G. Lebeau, *Mesures de défaut de compacité, application au système de Lamé*, Ann. Scient. Ec. Norm. Sup., (4) 34 (2001), 817–870.
- [BG96] N. Burq and P. Gérard, *Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes*, Comptes Rendus de L’Académie des Sciences, 325 (1997), 749–752.
- [BZ12] N. Burq and M. Zworski. *Control for Schrödinger equations on tori*, Math. Research Letters, 19 (2012), 309–324.
- [Ge91] P. Gérard, *Microlocal defect measures*, Comm. Partial Differential Equations, 16.11 (1991), 1761–1794.
- [GG12] P. Gérard, S. Grellier, *The Cubic Szegő equation*, Ann. Scient. c. Norm. Sup., (4) 43 (2010), 761–810.
- [HS84] B. Helffer and J. Sjostrand. *Multiple Wells in the Semi-Classical Limit I*, Comm. Partial Differential Equations, 9.4 (1984), 337–408.
- [Ja90] S. Jaffard. *Contrôle interne exact des vibrations d’une plaque rectangulaire*, Portugal. Math., 47 (1990), 423–429.
- [Ji17] L. Jin, *Control for Schrödinger equation on hyperbolic surfaces*, Math. Research Letters, 25 (2018), 1865–1877.
- [Ko17] A. Koenig, *Non-null-controllability of the Grushin operator in 2D*, Comptes Rendus Mathématiques, 355.12 (2017), 1215–1235.
- [DuKo18] M. Duprez, A. Koenig, *Control of the Grushin equation: non-rectangular control region and minimal time*, preprint, 2018, arxiv:1807.01241.
- [Mel] R.B. Melrose, *The wave equation for a hypoelliptic operator with symplectic characteristics of codimension two*, J. Anal. Math., 44 (1984) 134–182.
- [Le92] G. Lebeau *Contrôle de l’équation de Schrödinger*, J. Math. Pures Appl., (9) 71 (1992), 267–291.
- [Li88] J.L. Lions. *Contrôlabilité exacte. Perturbation et stabilisation des systèmes distribués*, Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués, Tome 1, vol. 8 (1988), RMA.
- [RS18] I. Rivas, C-M. Sun, *Internal controllability of non-localized solution for the Kodomtsev-Petviashvili II equation*, preprint, arXiv:1711.09359.
- [WG88] Z-X. Wang, D-R. Guo, *Introduction to special functions*, Beijing Science Press, p 354 (1979): 383.

- [Wa1918] G. Watson, *The harmonic functions associated with the parabolic cylinder*, Proceedings of the London Mathematical Society, (17) 2 (1918), 116-148.
- [Zw12] M. Zworski. *Semiclassical analysis*, Graduate Studies in Mathematics, (2012) AMS.

UNIVERSITÉ PARIS-SUD, UNIVERSIT PARIS-SACLAY AND INSTITUT UNIVERSITAIRE DE FRANCE, LABORATOIRE DE MATHÉMATIQUES D'ORSAY, UMR 8628 DU CNRS, BÂTIMENT 307, 91405 ORSAY CEDEX, FRANCE

*E-mail address:* nicolas.burq@math.u-psud.fr

UNIVERSITÉ DE CERGY-PONTOISE, LABORATOIRE DE MATHÉMATIQUES AGM, UMR 8088 DU CNRS, 2 AV. ADOLPHE CHAUVIN 95302 CERGY-PONTOISE CEDEX, FRANCE

*E-mail address:* chenmin.sun@u-cergy.fr