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# Ultralocal Lax connection for para-complex $\mathbb{Z}_T$ -cosets

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## Abstract

We consider  $\sigma$ -models on para-complex  $\mathbb{Z}_T$ -cosets, which are analogues of those on complex homogeneous target spaces considered recently by D. Bykov. For these models, we show the existence of a gauge-invariant Lax connection whose Poisson brackets are ultralocal. Furthermore, its light-cone components commute with one another in the sense of Poisson brackets. This extends a result of O. Brodbeck and M. Zagermann obtained twenty years ago for hermitian symmetric spaces.

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## 1 Introduction

In a classical integrable  $(1 + 1)$ -dimensional field theory, the integrals of motion in involution can be extracted from the monodromy of its Lax connection along a constant-time curve. For this reason, the spatial component of the Lax connection, known as the Lax matrix, plays a central role in establishing the property of integrability. In particular, the involution of the integrals of motion is deduced from the specific form of the Poisson brackets of the Lax matrix. In integrable  $\sigma$ -models the latter are non-ultralocal [1, 2], in the sense that they contain a term proportional to the derivative of the Dirac  $\delta$ -distribution. Yet the presence of such a term has posed a serious obstacle, for over 30 years, in the problem of quantising such theories from first principles.

Indeed, the most effective and powerful known way to quantise a classical integrable field theory is to use the Quantum Inverse Scattering Method (QISM) [3–5]. Unfortunately, the central assumption behind this method is that the Poisson bracket of the Lax matrix of the classical integrable field theory one starts with is ultralocal, *i.e.* does not depend on derivatives of the Dirac  $\delta$ -distribution.

More precisely, a standard way of applying the QISM is to start by putting the theory on the lattice, which first requires constructing a discretisation of the classical Lax matrix. There are two important properties which such a discretised Lax matrix should have. Firstly, just as in the continuum, one would like its Poisson brackets to have a form which ensures the existence of sufficiently many integrals of motion in involution. A very general

family of Poisson algebras with this property is given by the Freidel-Maillet quadratic algebras [6, 7]. Secondly, we should also recover the Lax matrix of the field theory from it in the continuum limit. In an ultralocal theory, these two requirements are fulfilled by defining the discretised Lax matrix as the path-ordered exponential of the continuum Lax matrix between two sites. In the non-ultralocal setting, however, the Poisson bracket of the path-ordered exponential of the Lax matrix, on adjacent or overlapping intervals, is not well defined [1, 2, 8] due to the presence of  $\delta'$ -terms in the Poisson bracket of the Lax matrix.

Faced with the problem of non-ultralocality in any given integrable field theory, it is natural to seek an alternative Lax matrix for this theory which would not suffer from the presence of  $\delta'$ -terms in its Poisson brackets. Such an alternative has not been found for a generic integrable  $\sigma$ -model. Let us recall that in some cases a different strategy may be applied. It consists in discretising and quantising *à la* Faddeev-Reshetikhin. This was first developed for the Principal Chiral Model [9] (see [10–12] for other recent applications of this approach). This way of treating non-ultralocality relies however on an ultralocal Lax matrix which is associated with a modified canonical structure.

Among classical integrable non-linear  $\sigma$ -models, there are the ones on  $\mathbb{Z}_T$ -cosets [13–15]. The Poisson brackets of their Lax matrix are non-ultralocal [16–19]. In this article, we show that classical para-complex  $\mathbb{Z}_T$ -cosets also admit an ultralocal Lax connection. Moreover, the light-cone components of this Lax connection Poisson commute with one another. These results generalise the ones obtained in [20, 21] for the  $O(3)$  non-linear  $\sigma$ -model and in [22] for hermitian symmetric space  $\sigma$ -models. The complex structure of the latter target spaces plays an important role in the construction of the ultralocal Lax pair. Such an interplay between integrability and the para-complex structure is also crucial in our analysis. Furthermore, the para-complex target spaces we shall consider are analogues of complex target spaces considered by D. Bykov in [23–25] (see also [26, 27]). The reason why we depart from the case of target spaces having a complex structure is the following. For complex  $\mathbb{Z}_T$ -cosets with  $T > 2$  and a worldsheet with Minkowski signature, one would encounter known (see for instance [15]) problems with reality conditions already at the level of the action. Furthermore, even when  $T = 2$ , the construction of the ultralocal Lax connection for complex target spaces would spoil reality conditions. Let us note that such problems have already been pointed out in [21] for the ultralocal Lax connection of the  $O(3)$  non-linear  $\sigma$ -model considered there. This is the reason why each Lie algebra we shall consider is the split real form of a complex Lie algebra and why we shall deal with para-complex instead of complex cosets.

The plan of this article is the following. In section 2, we describe the para-complex  $\mathbb{Z}_T$ -cosets  $G/H$  we shall consider. Their para-complex structure and the  $\mathbb{Z}_T$ -grading are

both defined from a particular element of the Lie algebra  $\mathfrak{g}$  of  $G$ . We explain how these three characteristics are related to each other.

We proceed in section 3 with the Lagrangian analysis. We first explain how the action of generic  $\mathbb{Z}_T$ -cosets may be greatly simplified, in the case of para-complex  $\mathbb{Z}_T$ -cosets, by adding to it a total derivative. The main advantage of such a procedure is that it enables to find easily a conserved and gauge-invariant current  $\mathcal{K}_\pm$ , which is also flat. This current is associated with the isometry of the para-complex  $\mathbb{Z}_T$ -cosets. The existence of this conserved and flat current allows one to define a Lax connection,  $\mathcal{L}_\pm$ , which is of the Zakharov-Mikhailov [28] type. One important property of this Lax connection is its gauge invariance, which is inherited from that of the current. We end this section by explaining how the Lax connection  $\mathcal{L}_\pm$  is related to the ordinary Lax connection of  $\mathbb{Z}_T$ -coset  $\sigma$ -models by a formal gauge transformation depending on the spectral parameter. Section 3 generalises results obtained in [23–25] for some complex target spaces.

Section 4 is devoted to the Hamiltonian analysis. We start by giving the canonical expression of the conserved and flat current. Since the action admits a gauge symmetry, we recall that there is a freedom to add to the Hamiltonian expression of any quantity a term proportional to the first-class constraint associated with the gauge invariance. We explain how we use this freedom in order to have a strongly vanishing Poisson bracket between  $\mathcal{K}_+$  and  $\mathcal{K}_-$ . We also give details of the computation of the Poisson bracket of  $\mathcal{K}_\pm$  with itself. All these Poisson brackets are ultralocal. It is then immediate that the Poisson brackets of the Lax connection are ultralocal. Furthermore, they take the standard  $R$ -matrix form. This implies that the monodromy matrix satisfies a Poisson algebra which is the classical analogue of a Yangian. Finally, we make some comments in the conclusion.

## 2 Para-complex $\mathbb{Z}_T$ -cosets

In this section, we describe the particular class of  $\mathbb{Z}_T$ -cosets which we shall consider. Let  $G$  be a semisimple real Lie group whose Lie algebra  $\mathfrak{g}$  is assumed to be the split real form of a complex Lie algebra  $\mathfrak{g}^{\mathbb{C}}$ .

**The  $\mathbb{Z}$ -gradation.** An important role in the whole analysis is played by an element  $u$  in the Cartan subalgebra of  $\mathfrak{g}$  whose eigenvalues in the adjoint representation are integers between  $-T + 1$  and  $T - 1$ . This defines a  $\mathbb{Z}$ -gradation

$$\mathfrak{g} = \bigoplus_{k=-T+1}^{T-1} \mathfrak{g}^{[k]}, \quad (2.1)$$

where  $\mathfrak{g}^{[k]}$  is the eigenspace of  $\text{ad}_u$  corresponding to the eigenvalue  $k$  with  $-T < k < T$ . Note that this  $\mathbb{Z}$ -gradation is not cyclic. In particular, we have

$$\forall m \in \mathfrak{g}^{[k]}, \forall n \in \mathfrak{g}^{[k']}, \quad [m, n] = 0 \quad \text{if} \quad |k + k'| \geq T. \quad (2.2)$$

**The  $\mathbb{Z}_T$ -gradation.** Before explaining how to construct the distinguished element  $u$ , let us first describe how it also induces a  $\mathbb{Z}_T$ -gradation on  $\mathfrak{g}$ . Let  $\omega = e^{2i\pi/T}$  and define the automorphism  $\sigma$  of  $\mathfrak{g}^{\mathbb{C}}$  by

$$\sigma = \omega^{\text{ad}_u} = \exp\left(\frac{2i\pi}{T}\text{ad}_u\right). \quad (2.3)$$

This is, by construction, an automorphism of order  $T$ . It defines a  $\mathbb{Z}_T$ -gradation

$$\mathfrak{g} = \bigoplus_{k=0}^{T-1} \mathfrak{g}^{(k)} \quad (2.4)$$

of the Lie algebra  $\mathfrak{g}$ , where  $\mathfrak{g}^{(k)}$  is the eigenspace of  $\sigma$  corresponding to the eigenvalue  $\omega^k$ . In particular, we have

$$\forall m \in \mathfrak{g}^{(k)}, \forall n \in \mathfrak{g}^{(k')}, \quad [m, n] \in \mathfrak{g}^{(k+k' \bmod T)}$$

for any  $k, k' = 0, \dots, T-1$ , which is to be compared with (2.2) for the  $\mathbb{Z}$ -gradation. In fact, by using the property that  $\omega^T = 1$ , we see that the relation between the  $\mathbb{Z}$ -gradation (2.1) and the  $\mathbb{Z}_T$ -gradation (2.4) is

$$\mathfrak{g}^{(0)} = \mathfrak{g}^{[0]} \quad \text{and} \quad \mathfrak{g}^{(k)} = \mathfrak{g}^{[k]} \oplus \mathfrak{g}^{[-T+k]}. \quad (2.5)$$

We shall decompose any  $m^{(k)} \in \mathfrak{g}^{(k)}$ , using the direct sum decomposition (2.5), as

$$m^{(k)} = m^{[k]} + m^{[-T+k]}$$

with  $m^{[k]} \in \mathfrak{g}^{[k]}$  and  $m^{[-T+k]} \in \mathfrak{g}^{[-T+k]}$ .

Let us introduce the notation  $\mathfrak{h} \equiv \mathfrak{g}^{(0)} = \mathfrak{g}^{[0]}$ . The subgroup  $H$  of  $G$  with Lie algebra  $\mathfrak{h}$  is the centralizer of  $u$  under the adjoint action of  $G$ . Note that  $H$  has a non-trivial center, which contains at least the abelian subgroup of  $G$  generated by  $u$ .

**Para-complex structure.** For any element  $Y = \sum_{k=-T+1}^{T-1} Y^{[k]}$  of the Lie algebra  $\mathfrak{g}$ , it will be convenient to use the notations

$$Y^< = P^<(Y) = \sum_{k=-T+1}^{-1} Y^{[k]}, \quad Y^> = P^>(Y) = \sum_{k=1}^{T-1} Y^{[k]}, \quad Y^{\geq} = P^{\geq}(Y) = \sum_{k=0}^{T-1} Y^{[k]},$$

where  $P^<$ ,  $P^>$  and  $P^{\geq}$  are projectors on the subalgebras of  $\mathfrak{g}$  with respectively negative, positive and non-negative grades. We denote by  $\mathfrak{g}^<$  and  $\mathfrak{g}^>$  the images of  $P^<$  and  $P^>$ .

Let us then define the map  $J = P^< - P^>$  acting on  $\mathfrak{g}$ . Its restriction to  $\mathfrak{g}^< \oplus \mathfrak{g}^>$  satisfies the two properties

$$J^2(X) = \mathbf{1},$$

$$[J(X), J(Y)] - J([X, J(Y)] + [J(X), Y]) + [X, Y] = 0,$$

for any  $X, Y \in \mathfrak{g}^< \oplus \mathfrak{g}^>$ . The latter equation may be interpreted as the vanishing of the Nijenhuis tensor associated with  $J$ , which means that  $J$  defines a para-complex structure on  $G/H$  [29].

**Construction of  $u$ .** The distinguished element  $u$  which defines the  $\mathbb{Z}$ -gradation in (2.1) and the para-complex structure may be constructed as follows. Let  $\{\alpha_i\}_{i=1}^l$  denote a set of positive simple roots of the Lie algebra  $\mathfrak{g}$ . The longest positive root is  $\theta = \sum_{i=1}^l a_i \alpha_i$ , where  $a_i$  are positive integers. We denote by  $\{\check{\omega}_i\}_{i=1}^l$  the basis of the Cartan subalgebra of  $\mathfrak{g}$  formed of fundamental co-weights defined by  $\alpha_j(\check{\omega}_i) = \delta_{ij}$ . We then choose

$$u = \sum_{i=1}^l b_i \check{\omega}_i,$$

where  $b_i$  are non-negative integers to be fixed shortly. If  $\alpha = \sum_{i=1}^l m_i \alpha_i$  is a positive root, with  $E_\alpha, F_\alpha$  denoting the corresponding root vectors in  $\mathfrak{g}$ , then

$$[u, E_\alpha] = \left( \sum_{i=1}^l b_i m_i \right) E_\alpha, \quad [u, F_\alpha] = - \left( \sum_{i=1}^l b_i m_i \right) F_\alpha.$$

We shall therefore fix the  $b_i$  by requiring that  $T - 1 = \sum_{i=1}^l b_i a_i$ . Let  $N_0 \subset \{1, \dots, l\}$  be such that  $b_i = 0$  if and only if  $i \in N_0$ . We then have that  $E_\alpha, F_\alpha \in \mathfrak{g}^{[0]} = \mathfrak{h}$  whenever the root  $\alpha$  is of the form  $\alpha = \sum_{i \in N_0} m_i \alpha_i$ . Notice that for a generic choice of the  $b_i$ 's, some of the subspaces  $\mathfrak{g}^{[k]}$  may be trivial.

Let us finally note that the definition of the  $\mathbb{Z}_T$ -automorphism in (2.3) is such that the root vector associated with the negative of the longest root has grade 1 with respect to the  $\mathbb{Z}_T$ -gradation, namely

$$[u, F_\theta] = (1 - T)F_\theta \implies \sigma(F_\theta) = \omega F_\theta.$$

**Decomposition of the quadratic Casimir.** Let  $\{I_a\}$  be a basis of  $\mathfrak{g}$  and  $\{I^a\}$  be its dual basis with respect to the opposite of the Killing form  $\kappa$ . The ad-invariance of  $\kappa$  implies that  $\kappa(m^{[k]}, m^{[p]}) = 0$  unless  $k = -p$ . This implies that the subalgebras  $\mathfrak{g}^>$  and  $\mathfrak{g}^<$  of  $\mathfrak{g}$  are isotropic.

Let us also fix a basis  $\{I_a^{[k]}\}$  of  $\mathfrak{g}^{[k]}$ , for each  $k = -T + 1, \dots, T - 1$ , and let  $\{I^{a[-k]}\}$  denote its dual basis. A basis of  $\mathfrak{g}^{(0)} = \mathfrak{g}^{[0]}$  is then given by  $\{I_a^{(0)}\} = \{I_a^{[0]}\}$  and its dual

basis is given by  $\{I^{a(0)}\} = \{I^{a[0]}\}$ . The quadratic Casimir can be written as

$$C_{\mathbf{12}} = \sum_a I_a \otimes I^a = C_{\mathbf{12}}^{\langle \rangle} + C_{\mathbf{12}}^{\geq \leq},$$

with

$$C_{\mathbf{12}}^{\langle \rangle} = \sum_{k=-T+1}^{-1} \sum_a I_a^{[k]} \otimes I^{a[-k]} \quad \text{and} \quad C_{\mathbf{12}}^{\geq \leq} = \sum_{k=0}^{T-1} \sum_a I_a^{[k]} \otimes I^{a[-k]}.$$

**Examples.** In the case  $T = 2$ , one could have relaxed the condition that  $\mathfrak{g}$  is the split real form of a complex Lie algebra. For compact real forms, the  $\mathbb{Z}_2$ -cosets constructed in the previous paragraphs correspond to Kählerian symmetric spaces. These are the cosets considered in [22]. We shall, however, not consider these cases because their ultralocal Lax connection is not compatible with reality conditions. The reason for this may be illustrated in the case of the coset  $\text{SU}(3)/(\text{SU}(2) \times \text{U}(1)) \simeq \mathbb{C}\mathbb{P}^2$ . Indeed, taking  $u = \text{diag}(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$  is fine in order for the subalgebra  $\mathfrak{su}(2) \oplus \mathfrak{u}(1)$  to correspond to the eigenspace of the adjoint action of  $u$  with null eigenvalue. However, it is then clear that the two other eigenspaces are not subspaces of  $\mathfrak{su}(3)$ .

For  $T \geq 2$ , and to fix the ideas, the pseudo-Riemannian manifolds such as

$$\frac{\text{SL}(p_1 + \cdots + p_T)}{\text{S}(\text{GL}(p_1) \times \cdots \times \text{GL}(p_T))}$$

are para-complex  $\mathbb{Z}_T$ -cosets and non-symmetric whenever  $T > 2$ .

## 3 Lagrangian analysis

### 3.1 Action

We start with the action [15] of  $\mathbb{Z}_T$ -cosets,

$$S[g] = K \iint dx^+ dx^- \sum_{k=1}^{T-1} k \kappa(j_+^{(k)}, j_-^{(T-k)}). \quad (3.1)$$

The field  $g(x, t)$  takes values in the Lie group  $G$  and  $j_{\pm} = g^{-1} \partial_{\pm} g$  with  $x^{\pm} = \frac{1}{2}(t \pm x)$  and  $\partial_{\pm} = \partial_t \pm \partial_x$ . The target space is the coset  $G/H$  since the action is invariant under the gauge transformation

$$g(x, t) \longmapsto g(x, t)h(x, t) \quad (3.2)$$

with  $h(x, t)$  taking values in  $H$ .

A short computation shows that the action (3.1) may be rewritten in terms of the  $\mathbb{Z}$ -graded components of the current  $j_{\pm}$  as

$$S[g] = K \iint dx^+ dx^- \sum_{k=1}^{T-1} \left( k \kappa(j_+^{[k]}, j_-^{[-k]}) + (T-k) \kappa(j_+^{[-k]}, j_-^{[k]}) \right). \quad (3.3)$$



It may further be separated into a metric part and a  $B$ -field part as follows

$$S[g] = K \iint dx^+ dx^- \sum_{k=1}^{T-1} \left( \frac{T}{2} \left( \kappa(j_+^{[k]}, j_-^{[-k]}) + \kappa(j_+^{[-k]}, j_-^{[k]}) \right) + \frac{2k-T}{2} \left( \kappa(j_+^{[k]}, j_-^{[-k]}) - \kappa(j_+^{[-k]}, j_-^{[k]}) \right) \right). \quad (3.4)$$

Aside from the fact that the grade zero is absent, the metric part is clearly independent of the  $\mathbb{Z}$ -gradation. Indeed, two  $\mathbb{Z}$ -gradations with the same zero grade component  $\mathfrak{g}^{[0]}$  give the same metric. The  $B$ -field part may, at first sight, seem to depend on it. However, using the Maurer-Cartan equations, invariance of the Killing form and the definition of the  $\mathbb{Z}$ -gradation one has

$$\kappa(u, \partial_- j_+ - \partial_+ j_-) = \sum_{k=1}^{T-1} k \left( \kappa(j_+^{[k]}, j_-^{[-k]}) - \kappa(j_+^{[-k]}, j_-^{[k]}) \right).$$

Thus, the term in the  $B$ -field proportional to  $k$  is in fact a total derivative. This means that the  $\sigma$ -model may be defined by the action

$$S[g] = KT \iint dx^+ dx^- \sum_{k=1}^{T-1} \kappa(j_+^{[-k]}, j_-^{[k]}) = KT \int dx^+ dx^- \kappa(j_+^<, j_-^>). \quad (3.5)$$

The  $B$ -field part of the action (3.5) may simply be written as

$$\frac{KT}{2} \iint dx^+ dx^- \kappa(j_+, J(j_-)).$$

This is fully analogous, in the split framework, to the models considered in the compact case in [22] and in [23, 24].

### 3.2 Flat and conserved current

The action (3.5) is invariant under the global symmetry  $g(x, t) \rightarrow g_0 g(x, t)$  with  $g_0 \in G$ . A conserved current  $\mathcal{K}_\pm$  associated with this symmetry is obtained by applying Noether's theorem. Furthermore, the equations of motion correspond to the equation of conservation

$$\partial_+ \mathcal{K}_- + \partial_- \mathcal{K}_+ = 0$$

of this current whose light-cone components are given explicitly by

$$\mathcal{K}_+ = -2g j_+^< g^{-1}, \quad \mathcal{K}_- = -2g j_-^> g^{-1}. \quad (3.6)$$

The current  $\mathcal{K}_\pm$  is also gauge-invariant. This is immediate since under a gauge transformation (3.2), we have, for  $k \neq 0$ ,

$$j_\pm^{[k]}(x, t) \longmapsto h^{-1}(x, t) j_\pm^{[k]}(x, t) h(x, t).$$

The overall factor in this conserved current  $\mathcal{K}_\pm$  has been fixed in order for it to also be flat, namely we have

$$\partial_+ \mathcal{K}_- - \partial_- \mathcal{K}_+ + [\mathcal{K}_+, \mathcal{K}_-] = 0.$$

However, we postpone the proof of this flatness property until the next subsection, where we will establish this result in an indirect way.

The Noether current is not unique. In fact, starting from the action (3.1), one would have naturally found

$$K_+ = \sum_{k=1}^{T-1} k g j_+^{(k)} g^{-1} \quad \text{and} \quad K_- = \sum_{k=1}^{T-1} k g j_-^{(T-k)} g^{-1}.$$

It is then clear from the analysis of the previous section that the existence of the element  $u \in \mathfrak{g}$  allows one to introduce an improvement term relating the two currents

$$\mathcal{K}_\pm = -\frac{2}{T} (K_\pm \pm \partial_\pm (g u g^{-1})).$$

Let us note that for symmetric space  $\sigma$ -models, that is when  $T = 2$ , the conserved current  $K_\pm$  can be made flat after an overall re-scaling to  $-2K_\pm$ . However, for  $T > 2$ , it is not possible to make the conserved current  $K_\pm$  also be flat in this way. The existence of the real, flat and conserved current (3.6) is thus a characteristic of para-complex  $\mathbb{Z}_T$ -cosets. Furthermore, as we shall prove in section 4, its Poisson brackets with itself are ultralocal.

### 3.3 Lax connection

If one has a flat and conserved current, one can define the Lax connection which is of Zakharov-Mikhailov [28] type,

$$\mathcal{L}_\pm(\lambda) = \frac{\mathcal{K}_\pm}{1 \mp \lambda}, \quad (3.7)$$

where  $\lambda$  denotes the spectral parameter. This Lax connection is flat on-shell, *i.e.* the conservation and flatness of  $\mathcal{K}_\pm$  is equivalent to the zero-curvature equation

$$\partial_+ \mathcal{L}_-(\lambda) - \partial_- \mathcal{L}_+(\lambda) + [\mathcal{L}_+(\lambda), \mathcal{L}_-(\lambda)] = 0. \quad (3.8)$$

Let us discuss a few simple properties of this Lax connection before showing, in section 4, the ultralocality of its Poisson brackets.

**Gauge invariance.** A crucial property of  $\mathcal{L}_\pm(\lambda)$  is its gauge invariance. This follows from the gauge invariance of the current itself. This property has a very important consequence at the Hamiltonian level. Indeed, when the gauge invariance is fixed, Poisson

brackets have to be replaced by Dirac brackets. However, the Dirac bracket of two gauge invariant quantities is equal to their Poisson bracket (see for instance [30]). This implies that the ultralocal structure computed in the next section is unchanged when the gauge invariance is fixed.

**Link with the ordinary Lax connection of  $\mathbb{Z}_T$ -cosets.** The ordinary Lax connection  $L_{\pm}(z)$  of  $\mathbb{Z}_T$ -cosets is [15]

$$L_+(z) = \sum_{k=0}^{T-1} z^k j_+^{(k)}, \quad L_-(z) = \sum_{k=0}^{T-1} z^{-k} j_-^{(T-k)}, \quad (3.9)$$

where the spectral parameter is denoted here by  $z$ . Let us then define

$$\alpha(z) = \exp(u \ln z),$$

which is valued in  $G^{\mathbb{C}}$ . It satisfies the property

$$\alpha(z)^{-1} m \alpha(z) = z^{-k} m, \quad \forall m \in \mathfrak{g}^{[k]}$$

for every  $k = -T+1, \dots, T-1$ . Recall that the zero curvature equation (3.8) is invariant under formal gauge transformations. We apply the formal gauge transformation

$$L_{\pm}^U(z) = U(z) L_{\pm}(z) U(z)^{-1} + U(z) \partial_{\pm} U(z)^{-1} \quad (3.10)$$

depending on the spectral parameter  $z$ , where

$$U(z, x, t) = g(x, t) \alpha(z)^{-1}.$$

Let us work out the expression for the gauge transformed Lax connection  $L_{\pm}^U(z)$ . We first observe that

$$U \partial_{\pm} U^{-1} = -g j_{\pm} g^{-1} \quad (3.11)$$

and  $U L_{\pm} U^{-1} = g (\alpha^{-1} L_{\pm} \alpha) g^{-1}$ . Focusing on  $L_+(z)$ , we obtain successively:

$$\begin{aligned} \alpha(z)^{-1} L_+(z) \alpha(z) &= \sum_{k=0}^{T-1} z^k \alpha(z)^{-1} j_+^{(k)} \alpha(z) \\ &= \alpha(z)^{-1} j_+^{(0)} \alpha(z) + \sum_{k=1}^{T-1} z^k \alpha(z)^{-1} (j_+^{[k]} + j_+^{[k-T]}) \alpha(z) \\ &= j_+^{(0)} + \sum_{k=1}^{T-1} (j_+^{[k]} + z^T j_+^{[k-T]}). \end{aligned} \quad (3.12)$$

It therefore follows from (3.10), (3.11) and (3.12) that

$$L_+^U(z) = g \left( j_+^{(0)} + \sum_{k=1}^{T-1} (j_+^{[k]} + z^T j_+^{[k-T]}) - j_+ \right) g^{-1} = (z^T - 1) \sum_{k=1}^{T-1} g j_+^{[k-T]} g^{-1}.$$

Proceeding in the same way for  $L_-(z)$ , and recalling the expressions (3.6) of  $\mathcal{K}_\pm$ , we obtain

$$L_\pm^U(z) = -\frac{1}{2}(z^{\pm T} - 1)\mathcal{K}_\pm. \quad (3.13)$$

Finally, performing also the following change of spectral parameter

$$\lambda \mapsto z(\lambda) = \left( \frac{\lambda + 1}{\lambda - 1} \right)^{1/T},$$

we arrive at the relation

$$L_\pm^U(z(\lambda)) = \mathcal{L}_\pm(\lambda).$$

In other words, the ultralocal Lax connection  $\mathcal{L}_\pm(\lambda)$  coincides, up to a change of spectral parameter, with a formal gauge transformation of the ordinary Lax connection  $L_\pm(z)$ . An immediate consequence of this is that the Lax connection  $\mathcal{L}_\pm(\lambda)$  is flat, since we know that  $L_\pm(z)$  is flat and that formal gauge transformations preserve the flatness property. Moreover, since  $\mathcal{L}_\pm(\lambda)$  is of the Zakharov-Mikhailov form, this proves indirectly that the current  $\mathcal{K}_\pm$  is also flat on-shell.

## 4 Hamiltonian analysis and ultralocality

### 4.1 Result of the canonical analysis

The phase space is parameterised by fields  $g(x)$  and  $X(x)$  taking values in  $G$  and  $\mathfrak{g}$ , respectively, the pair of which describes a field valued in the cotangent bundle  $T^*G$ . They satisfy the canonical Poisson brackets, which written in tensorial notation read

$$\{g_1(x), g_2(y)\} = 0, \quad (4.1a)$$

$$\{X_1(x), g_2(y)\} = g_2(x)C_{12}\delta_{xy}, \quad (4.1b)$$

$$\{X_1(x), X_2(y)\} = -[C_{12}, X_2(x)]\delta_{xy}. \quad (4.1c)$$

The canonical analysis associated with the action (3.5) is standard. We shall not reproduce its details here, which lead to the following relation

$$X = \frac{KT}{2}(j_-^> + j_+^<).$$

There is a first-class constraint  $X^{[0]} = 0$ , which corresponds to the gauge invariance (3.2) of the action.

Using (3.6), we immediately obtain the phase space expressions of the flat current:

$$\mathcal{K}_+ = -\frac{4}{KT}gX^<g^{-1}, \quad \mathcal{K}_- = -\frac{4}{KT}gX^\geq g^{-1}. \quad (4.2)$$

Note that here we have added to  $\mathcal{K}_-$  the extra term  $-\frac{4}{KT}gX^{[0]}g^{-1}$  which is proportional to the constraint. This freedom to add terms proportional to the constraint is a standard procedure in integrable field theories with gauge symmetry (see [17, 18]). Indeed, as we shall see, the coefficient of this extra term has been fixed in order for the Poisson bracket between  $\mathcal{K}_+$  and  $\mathcal{K}_-$  to vanish strongly, that is without making use of the constraint. Note, however, that the chosen value of this coefficient also makes sense for the following reason. Let us consider the temporal component of the current  $\mathcal{K}_\pm$ , namely

$$\frac{1}{2}(\mathcal{K}_+ + \mathcal{K}_-) = -\frac{2}{KT}gXg^{-1}.$$

For any  $\epsilon \in \mathfrak{g}$  we then have

$$\left\{ -\frac{KT}{2} \int dy \kappa\left(\epsilon, \frac{1}{2}(\mathcal{K}_+(y) + \mathcal{K}_-(y))\right), g(x) \right\} = \epsilon g(x).$$

This is what we expect in order for the time component of the current to generate the symmetries corresponding to left multiplication on  $g$ .

## 4.2 Computation of the Poisson brackets of $\mathcal{K}_\pm$

**Ultralocality.** A key property of the expression (4.2) of the current is that it depends on the fields  $X$  and  $g$ , but not on their spatial derivatives. This ensures *de facto* that all Poisson brackets of the current, and thus of the Lax pair (3.7), are ultralocal! This property alone explains why the para-complex  $\mathbb{Z}_T$ -cosets are so special. Indeed, the fact that there is no spatial derivative in (4.2) is a consequence of the form of the simplified action (3.5).

In the remainder of this subsection we compute all the Poisson brackets of the current  $\mathcal{K}_\pm$ . It is clear that they take the following form,

$$\{\mathcal{K}_{a1}(x), \mathcal{K}_{b2}(x')\} = \frac{16}{K^2T^2}g_1(x)g_2(x')\alpha_{ab}(x)g_1^{-1}(x)g_2^{-1}(x')\delta_{xx'}, \quad (4.3)$$

where  $\alpha_{ab}$  belongs to the tensor product of two copies of the Lie algebra  $\mathfrak{g}$ , and  $a, b = \pm$ . Since the Poisson brackets are ultralocal, we shall not indicate the spatial dependence in intermediate computations. Each  $\alpha_{ab}$  is the sum of three terms:

$$\begin{aligned} \alpha_{ab} &= g_1^{-1}g_2^{-1}\{g_1P_1^{s(a)}X_1g_1^{-1}, g_2P_2^{s(b)}X_2g_2^{-1}\}g_1g_2 \\ &= P_1^{s(a)}P_2^{s(b)}\{X_1, X_2\} + P_2^{s(b)}[g_1^{-1}\{g_1, X_2\}, P_1^{s(a)}X_1] + P_1^{s(a)}[g_2^{-1}\{X_1, g_2\}, P_2^{s(b)}X_2] \\ &= -P_1^{s(a)}P_2^{s(b)}[C_{12}, X_2] + P_2^{s(b)}[C_{12}, P_2^{s(a)}X_2] + [P_1^{s(a)}C_{12}, P_2^{s(b)}X_2], \end{aligned} \quad (4.4)$$

where  $s(+)$  = ‘<’ and  $s(-)$  = ‘ $\geq$ ’. To establish this result, we have made use of (4.1c), (4.1b), the antisymmetry of the Poisson bracket and the identity  $[C_{12}, M_1 + M_2] = 0$  valid for any  $M \in \mathfrak{g}$ . It remains then to compute  $\alpha_{ab}$  for each possibility.

**Poisson bracket  $\{\mathcal{K}_+, \mathcal{K}_-\}$ .** In this case,  $a = '+'$ ,  $b = '-'$  and thus  $s(a) = '<'$  and  $s(b) = ' \geqslant '$ . We have therefore

$$\alpha_{+-} = -P_2^{\geqslant}[C_{12}^{<}, X_2] + P_2^{\geqslant}[C_{12}, X_2^{<}] + [C_{12}^{<}, X_2^{\geqslant}]. \quad (4.5)$$

For the second term in the r.h.s. of (4.5), the projector  $P_2^{\geqslant}$  forces the grading of the commutator in the second tensorial space to be greater than or equal to zero. However, since the grading of  $X_2^{<}$  is negative, we have:

$$P_2^{\geqslant}[C_{12}, X_2^{<}] = P_2^{\geqslant}[C_{12}^{<}, X_2^{<}].$$

The grading in the second tensorial space of the third term in the r.h.s. of (4.5) is strictly positive. One has therefore the identity

$$[C_{12}^{<}, X_2^{\geqslant}] = P_2^{\geqslant}[C_{12}^{<}, X_2^{\geqslant}].$$

It is then clear that the sum (4.5) vanishes, and thus that  $\{\mathcal{K}_{+1}(x), \mathcal{K}_{-2}(x')\} = 0$ .

**Poisson bracket  $\{\mathcal{K}_-, \mathcal{K}_-\}$ .** In this case,  $a = b = '-'$  and thus  $s(a) = s(b) = ' \geqslant '$ . We proceed in the same way as for the previous computation. We obtain:

$$\begin{aligned} \alpha_{--} &= -P_2^{\geqslant}[C_{12}^{\leqslant}, X_2] + P_2^{\geqslant}[C_{12}, X_2^{\geqslant}] + [C_{12}^{\leqslant}, X_2^{\geqslant}] \\ &= -P_2^{\geqslant}[C_{12}^{\leqslant}, X_2^{\geqslant}] + P_2^{\geqslant}[C_{12}, X_2^{\geqslant}] + (P_2^{<} + P_2^{\geqslant})[C_{12}^{\leqslant}, X_2^{\geqslant}] \\ &= P_2^{\geqslant}[C_{12}, X_2^{\geqslant}] + P_2^{<}[C_{12}^{\leqslant}, X_2^{\geqslant}] = P_2^{\geqslant}[C_{12}, X_2^{\geqslant}] + P_2^{<}[C_{12}, X_2^{\geqslant}] = [C_{12}, X_2^{\geqslant}]. \end{aligned}$$

To conclude the computation, we use the property  $g_1(x)g_2(x)C_{12}g_1^{-1}(x)g_2^{-1}(x) = C_{12}$  and obtain  $\{\mathcal{K}_{-1}(x), \mathcal{K}_{-2}(x')\} = -\frac{4}{KT}[C_{12}, \mathcal{K}_{-2}(x)]\delta_{xx'}$ .

**Poisson bracket  $\{\mathcal{K}_+, \mathcal{K}_+\}$ .** In this case,  $a = b = '+'$  and thus  $s(a) = s(b) = '<'$ . There are only minor differences with the previous computation since we obtain

$$\begin{aligned} \alpha_{++} &= -P_2^{<}[C_{12}^{>}, X_2] + P_2^{<}[C_{12}, X_2^{<}] + [C_{12}^{>}, X_2^{<}] \\ &= -P_2^{<}[C_{12}^{>}, X_2^{<}] + P_2^{<}[C_{12}, X_2^{<}] + [C_{12}^{>}, X_2^{<}] = P_2^{<}[C_{12}, X_2^{<}] + P_2^{\geqslant}[C_{12}^{>}, X_2^{<}] \\ &= P_2^{<}[C_{12}, X_2^{<}] + P_2^{\geqslant}[C_{12}, X_2^{<}] = [C_{12}, X_2^{<}]. \end{aligned}$$

This gives the last Poisson bracket,  $\{\mathcal{K}_{+1}(x), \mathcal{K}_{+2}(x')\} = -\frac{4}{KT}[C_{12}, \mathcal{K}_{+2}(x)]\delta_{xx'}$ .

In conclusion, we have shown that

$$\{\mathcal{K}_{+1}(x), \mathcal{K}_{-2}(x')\} = 0, \quad (4.6a)$$

$$\{\mathcal{K}_{\pm 1}(x), \mathcal{K}_{\pm 2}(x')\} = -\frac{4}{KT}[C_{12}, \mathcal{K}_{\pm 2}(x)]\delta_{xx'}. \quad (4.6b)$$

### 4.3 Poisson brackets of the Lax connection and Yangian

It is then straightforward to compute all the Poisson brackets of the Lax connection from its definition in (3.7) and the above Poisson brackets (4.6). The result is:

$$\{\mathcal{L}_{+1}(\lambda, x), \mathcal{L}_{-2}(\mu, x')\} = 0, \quad (4.7a)$$

$$\{\mathcal{L}_{\pm 1}(\lambda, x), \mathcal{L}_{\pm 2}(\mu, x')\} = \mp \frac{4}{KT} \left[ \frac{C_{12}}{\mu - \lambda}, \mathcal{L}_{\pm 1}(\lambda, x) + \mathcal{L}_{\pm 2}(\mu, x) \right] \delta_{xx'}. \quad (4.7b)$$

The Poisson bracket of the Lax matrix  $\mathcal{L} = \frac{1}{2}(\mathcal{L}_+ - \mathcal{L}_-)$  is then

$$\{\mathcal{L}_1(\lambda, x), \mathcal{L}_2(\mu, x')\} = -\frac{2}{KT} \left[ \frac{C_{12}}{\mu - \lambda}, \mathcal{L}_1(\lambda, x) + \mathcal{L}_2(\mu, x) \right] \delta_{xx'}. \quad (4.8)$$

We then define the monodromy

$$T(\lambda) = P\overleftarrow{\exp} \left( - \int_W dx \mathcal{L}(\lambda, x) \right)$$

where  $W$  is either the circle  $S^1$  or  $\mathbb{R}$ . It is a consequence of the zero-curvature equation (3.8) that  $T(\lambda)$  is conserved in time when  $W = \mathbb{R}$ , provided the Lax connection decays sufficiently fast at  $\pm\infty$ , or that the invariants of  $T(\lambda)$  are conserved in time when  $W = S^1$ . In particular, this provides an indirect proof that the monodromy matrix  $T(\lambda)$  (or rather its invariants in the case  $W = S^1$ ) Poisson commutes with the Hamiltonian.

The Poisson brackets of the monodromy take [31] the form of a Poisson algebra corresponding to a Yangian (see also [32, 33] and the reviews [34–36]),

$$\{T_1(\lambda), T_2(\mu)\} = \frac{2}{KT} \left[ \frac{C_{12}}{\mu - \lambda}, T_1(\lambda)T_2(\mu) \right].$$

Note that because of the ultralocality of the Poisson bracket (4.8), there is no ambiguity in the computation of this Poisson algebra.

## 5 Conclusion

We have shown that classical integrable  $\sigma$ -models on para-complex  $\mathbb{Z}_T$ -coset target spaces admit an ultralocal Lax connection, which is related to the standard one by a spectral parameter dependent formal gauge transformation. The most important open problem relating to this class of models is therefore to apply the Quantum Inverse Scattering Method to them. A natural related question is also to determine whether the approach developed by V. Bazhanov, G. Kotousov and S. Lukyanov in [21] can be extended to integrable  $\sigma$ -models on para-complex  $\mathbb{Z}_T$ -coset target spaces.

One may also wonder if there are other integrable  $\sigma$ -models which admit an ultralocal Lax connection. For instance, there is [21] a generalisation of the ultralocal Lax connection

of the  $O(3)$  non-linear  $\sigma$ -model for the sausage model, which is a deformation [37] of the former. It would therefore be interesting to investigate if the result of the present article could be extended to one-parameter deformations [38–40] of para-complex  $\mathbb{Z}_T$ -cosets.

In general, the question of whether or not the classical integrability of a  $\sigma$ -model is preserved at the quantum level is a difficult one [41–45, 25, 46–48]. Having an ultralocal description of the para-complex analogues of certain problematic models like the  $\mathbb{C}\mathbb{P}^N$   $\sigma$ -model, with  $N > 1$ , at the classical level is then quite appealing as it could provide another way to investigate the fate of their integrability at the quantum level.

Very recently, a general formalism for describing classical integrable field theories was proposed in [49], which is based on a certain four-dimensional variant of Chern-Simons theory. In this setting, integrable field theories are constructed from the four-dimensional gauge theory by inserting different surface defects. In particular, it was shown in [49] that non-linear  $\sigma$ -models whose target space is a Kähler manifold can be constructed by using so-called order defects. The analysis of [50] suggests that such order defects can be used more generally to describe ultralocal integrable field theories. It would therefore be interesting to construct the class of  $\sigma$ -models with para-complex  $\mathbb{Z}_T$ -coset target spaces considered in the present paper within the framework of [49].

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