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# Measures from infinite-dimensional gauge integration

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## Abstract

We construct and investigate an integration process for infinite products of compact metrizable spaces that generalizes the standard Henstock–Kurzweil gauge integral. The integral we define here relies on gauge functions that are valued in the set of divisions of the space.

We show in particular that this integration theory provides a unified setting for the study of non-absolute infinite-dimensional integrals such as the gauge integrals on  $\mathbb{R}^T$  of Muldowney and the construction of several type of measures, such as the Lebesgue measure, the Gaussian measures on Hilbert spaces, the Wiener measure, or the Haar measure on the infinite dimensional torus. Furthermore, we characterize Lebesgue-integrable functions for those measures as measurable absolutely gauge integrable functions.

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# 1 Introduction

## 1.1 Summary

The main concept behind gauge (or Henstock–Kurzweil, or generalized-Riemann) integration is the notion of *gauge function*. The role of those functions is to control the size of the blocks of the partition we use to compute a given integral. In the case of classical Riemann integration on a segment  $[a, b]$ , that role is played by a number  $\delta > 0$ , and the size control by  $\delta$  is made through the condition which says that all the intervals of a partition have to be of length smaller than  $\delta$ .

In Henstock–Kurzweil integration, the idea is to replace this global strictly positive constant by a gauge function  $\delta : [a, b] \rightarrow \mathbb{R}_+^*$ , and to allow the size of the intervals of a partition to be locally controlled by the condition which says that the length of a given interval is smaller than  $\delta(x)$  where  $x$  is the point associated to the interval.

This simple and natural generalization has far-reaching consequences. One of them is that gauge integration on  $\mathbb{R}$  is a *non-absolute* generalization of the standard Lebesgue integration in the sense that any Lebesgue integrability implies gauge integrability, and gauge integrable functions whose absolute value are gauge integrable are actually Lebesgue integrable.

For example, the Fresnel function  $x \mapsto e^{ix^2}$  is gauge integrable but not Lebesgue integrable. This observation is at the origin of the gauge integration approach to construct a rigorous theory of Feynman path integrals [2–4, 8, 9].

Some links between measure theory and the infinite-dimensional gauge integration presented by Henstock in [2–4] have been studied in [6]. However, it turns out that the notion of gauge associated to that theory is too restrictive for applications to infinite-dimensional integrals that appear in quantum mechanics or in the theory of stochastic processes.

In [8, 9], Muldowney developed a theory of infinite-dimensional gauge integration with more flexible gauges that allow the size control of the partition blocks to be dependent on the set of the dimensions where the block is restricted, and those gauges are adapted to the construction of Feynman path integrals and other non-absolute infinite-dimensional integrals.

In this article, we aim at providing a general setting of infinite-dimensional gauge integration that contains the gauge integral over  $\mathbb{R}^T$  (for infinite  $T$ ) as described in [9], and the Lebesgue integrals associated to standard measures, such as the Lebesgue measure, the Gaussian measures over Hilbert spaces, the Wiener measure and the Haar measure on the infinite-dimensional torus. Ideally, the framework we want to develop has to be general enough to contain those examples but concrete enough so that it remains easy to verify the conditions to obtain new examples.

The setting we are working with here is described in Subsection 2.1 and is given by the product  $X$  (called *product division space*) of a family  $(X_t)_{t \in T}$  (called *division family*) of topological spaces, each of them being equipped with a semialgebra (called *division structure*) of sets (called *cells*) that generates the topology (see Definitions 2.6 and 2.7 for more details). The goal of Section 2 is to construct a general theory of gauge integration on a product division space  $X$ , where each  $X_t$  is a compact metrizable space.

Examples of such product division spaces include  $\overline{\mathbb{R}}^T$ , the infinite-dimensional torus  $\mathbb{T}^{\mathbb{N}}$ , the Cantor set  $2^{\mathbb{N}}$ , the extended Baire set  $\overline{\mathbb{N}}^{\mathbb{N}}$  and the profinite integers  $\widehat{\mathbb{Z}}$  (see Example 2.16 for details).

In Subsection 2.2, we study some basic properties of divisions of  $X$ , which are defined as finite sets of pairwise disjoint nonempty cells.

In Subsection 2.3, we define and study the notion of gauge and their associated resolutions. A gauge is essentially a pair  $(L, \delta)$  where  $L$  is a function valued in the indexing set (or dimension set)  $T$  of the product division space  $X$  and  $\delta$  is a family of maps indexed by the finite sets of  $T$ , each of them valued in the set of divisions of the finite product of  $(X_t)_{t \in N}$  (see Definition 2.28).

We decided to give the name of *pixel* to what is usually called a *pointed cell* (a couple  $(J, x)$  where  $J$  is a cell and where  $x \in \overline{J}$ ) and the name of *resolution* to what could be also called a *pointed division* (a finite set of pointed cells with pairwise disjoint associated cells)– see Definition 2.33 for details. This choice of terminology was mainly made in order to avoid repeating the adjective “pointed” for such fundamental objects of the theory, and also because it suggests a visually possibly useful analogy with actual screen resolutions and pixels.

A cell  $J = \prod_t J_t$  of  $X$  is such that there is a finite set  $N$  of  $T$  where  $J_t$  is restricted (i.e. different from  $X_t$ ) and such that for any  $t \notin N$ ,  $J_t = X_t$ . This finite set, which is denoted  $N_J$  (and called the *dimension set* of  $J$ , see Definition 2.14), plays a crucial role in the theory. Indeed, gauges will control the dimension set of a cell to make it big enough, and the restricted cell  $\underline{J} := \prod_{t \in N_J} J_t$  to make it small enough. To be more precise, given a gauge  $\gamma = (L, \delta)$ , a pixel  $(J, x)$  is said  $\gamma$ -fine if  $L(x) \subseteq N_J$ , and if  $\underline{J}$  is finer than the division  $\delta_{N_J}(x)$ . A resolution is said  $\gamma$ -fine if so are all of its pixels (see Definition 2.35 for details).

Thus, in a gauge  $(L, \delta)$ ,  $L$  will provide the control over the dimension sets of cells in a resolution, and  $\delta$  will provide the control over the size of the associated restricted cell, and over the way it can fit in a given division. This notion of gauge essentially contains the type of gauge that was described in [9, p. 103] for the product division space  $\overline{\mathbb{R}}^T$  (see Remark 2.44 and Lemma 2.43).

An important consequence of working with this type of gauge is that for any division  $D$ , there is a gauge  $\gamma$  such that any  $\gamma$ -fine resolution will be finer than  $D$  (see Corollary 2.40). In particular, that fact is used to prove that real absolutely gauge integrable pixel functions form a Riesz space (see Proposition 2.75, item 8). It was not investigated here, but it is possible that another approach where we only consider pixels  $(J, x)$  where  $x$  is a vertex of  $J$  (in a sense to be defined) could lead to similar results.

Subsections 2.4, 2.5 and 2.6 are respectively devoted to basic definitions and properties of the gauge integral, additivity properties, and variation functionals. Since the notion of division space we develop here is an example of the general theory of division systems of Henstock [4], the results of Subsections 2.4, 2.5 and 2.6 could also be obtained by checking the relevant axioms associated to those results, instead of proving them directly. However, we decided here to systematically give direct proofs for the sake of self-containment and argument simplicity.

In Subsection 2.7, we prove Theorem 2.78, which gives a sufficient continuity condition over a point function  $f$  so that  $fh$  is integrable, where  $h$  is an integrable positive pixel function.

In Subsection 2.8, we prove a version of the Hake theorem for our gauge integral (Theorem 2.79). This result provides a sufficient condition over partial integrals of a pixel function to obtain full integrability, showing that improper gauge integrals are, in fact, proper.

Section 3 is devoted to the construction of a measure on  $X$  associated to any positive pixel function (i.e. a function defined on all pixels  $(J, x)$ ) satisfying the right conditions.

In Subsection 3.1, we establish the main tool for this goal : the monotonous convergence theorem for the gauge integral (Theorem 3.9), which is proven here through a result about the integrability of series of integrable positive pixel functions (Theorem 3.8).

In Subsection 3.2, we establish the first main result (Definition-Theorem 3.13). This result shows that if  $h$  is a positive and projectively integrable (see Definition 3.2) pixel function, then :

$$\mu_h : \mathcal{M}_h \rightarrow \overline{\mathbb{R}}_+$$

$$A \mapsto \begin{cases} \int \mathbf{1}_A h, & \text{if } A \in \mathcal{G}_h \\ \infty & \text{otherwise} \end{cases}$$

is a complete measure on  $(X, \mathcal{M}_h)$ , where  $\mathcal{G}_h$  is the set of all  $h$ -integrable subsets of  $X$ , and  $\mathcal{M}_h$  is the  $\sigma$ -algebra of all  $h$ -measurable subsets of  $X$  (see Definition 3.10 and Proposition 3.12 for details). This measure  $\mu_h$  is a Borel (and even Radon if  $T$  is countable, see Theorem 3.16) when  $h$  is  $\sigma$ -cellular, which means that any open cell is a countable union of  $h$ -integrable measurable subsets (see Definition 3.15).

We establish the second main result in Subsection 3.3, which gives a characterization of Lebesgue integrable functions for the measure  $\mu_h$  as absolutely  $h$ -integrable measurable functions (Theorem 3.21). More precisely, it is shown that if  $h$  is  $\sigma$ -cellular, the space  $L^1(X, \mu_h)$  of Lebesgue-integrable functions for the measure  $\mu_h$  coincide with the space of all measurable functions  $f$  such that  $|f|h$  is gauge integrable, and that for any such function  $f$ ,

$$\int f d\mu_h = \int fh.$$

Section 4 is devoted to the study of four different examples where the previous results can be applied. In Subsection 4.1, we show that we can recover the classical Lebesgue measure from the gauge integral (a fact that is already well-known, see for example [11, p. 64]), and in Subsection 4.2 we apply the theory

to the case of  $\overline{\mathbb{R}}^{\mathbb{N}}$  to construct Gaussian measures via gauge integrals. The same is done for the Wiener measure in Subsection 4.3. We note that similar results for the Wiener measure were obtained in [10]. Finally, we study in Subsection 4.4 the infinite-dimensional torus as a product division space. We show in particular that its canonical Haar measure can also be recovered from the gauge integral.

## 1.2 Notations and conventions

On numbers :

- We denote  $\mathbb{N}$  the set of natural numbers, including zero, and  $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ . The word "positive" will mean greater or equal to zero, and the term "strictly positive" will mean positive and different than zero.
- The extended real line is denoted  $\overline{\mathbb{R}}$ . It is a compact metrizable space with the order topology, and a complete lattice. With the induced order and topology,  $\overline{\mathbb{R}}_+$  is also a compact space and a complete lattice. We naturally extend the addition  $+$  from  $\mathbb{R}_+$  to  $\overline{\mathbb{R}}_+$  by saying that  $+\infty + a = a + +\infty = +\infty$  for all  $a \in \overline{\mathbb{R}}_+$  (this makes  $\overline{\mathbb{R}}_+$  a totally ordered commutative monoid).
- We use  $\Re z$  and  $\Im z$  to denote the real part and the imaginary part of a complex number  $z$ , and if  $x$  is a real number, we denote  $x_+ := \max(x, 0)$  and  $x_- := -\min(x, 0)$ .

On sets :

- The characteristic function of a subset  $S$  of a set  $X$  is denoted  $\mathbf{1}_S$ .
- For any given set  $X$ ,  $\mathcal{F}(X)$  denotes the lattice (for inclusion) of all finite subsets of  $X$ .
- The axiom of dependent choice (DC) will be freely used in this paper (its use will be mentioned though). The full axiom of choice (AC) will not be used.
- If  $T$  is a set, we will say that a proposition  $P(t)$  holds for **cofinitely almost all**  $t \in T$  if there is a finite subset  $S \subseteq T$  such that the proposition  $P(t)$  holds for all  $t \in T \setminus S$ .
- If  $T$  is a countable set, an **enumeration** of  $T$  is a sequence  $(t_n)_{n \in \mathbb{N}}$  such that  $T = \{t_n : n \in \mathbb{N}\}$ .
- When  $x \in \prod_{t \in T} S_t$  is an element of a product of a family of sets, we will use the notation  $x_U$  for any  $U \subseteq T$  to denote the restriction of  $x$  over  $U$ , so that  $x_U \in \prod_{t \in U} S_t$ .

On orders :

- The real vector subspace  $\mathbb{R}^S$  is a Riesz space partially ordered by the relation  $f \leq g$  defined as: for any  $x \in S$ ,  $f(x) \leq g(x)$ . The join (maximum) of  $f, g$  is denoted  $f \vee g$ , and the meet (minimum) is denoted  $f \wedge g$ .
- If  $f, g : A \rightarrow B$  are maps from a set  $A$  into a semilattice  $(B, \leq, \wedge)$ , we will denote by  $f \leq g$  the relation : for all  $x \in A$ ,  $f(x) \leq g(x)$ . Equipped with that relation,  $B^A$  is a meet semilattice where the meet of  $f$  and  $g$  is  $f \wedge g : A \rightarrow B, x \mapsto f(x) \wedge g(x)$ .
- We will view  $\mathcal{F}(X)$ , where  $X$  is a set, as a bounded meet semilattice for the relation :  $N \leq N'$  if and only if  $N \supseteq N'$ . The meet of  $N$  and  $N'$  is thus  $N \cup N'$ . For functions valued in  $\mathcal{F}(X)$ , we will therefore use the notation  $f \cup g$  and  $f \supseteq g$  for respectively the meet of  $f$  and  $g$  the order relation  $f \leq g$ .

On topology and  $\sigma$ -algebras :

- For any given family  $(X_t)_{t \in T}$  of topological space, we always provide  $\prod_{t \in T} X_t$  with the product topology.
- If  $X$  is a topological space we will denote  $\mathcal{B}_X$  its Borel  $\sigma$ -algebra.

- If  $X = \prod_t X_t$  is a product of topological space, the  $\sigma$ -algebra generated by products of the form  $\prod_t A_t$  where for all  $t$ ,  $A_t \in \mathcal{B}_{X_t}$  and for cofinitely almost all  $t$ ,  $A_t = X_t$ , is called the **product  $\sigma$ -algebra** on  $X$  and will be denoted  $\mathcal{E}_X$ . The inclusion  $\mathcal{E}_X \subseteq \mathcal{B}_X$  holds, but the equality doesn't hold in general (but holds if we are dealing with a countable product of second countable spaces). Unless stated otherwise, we will always provide a product  $X$  with its product  $\sigma$ -algebra  $\mathcal{E}_X$ .

## 2 Integration on product division spaces

### 2.1 Division spaces, families and products

This section is dedicated to the definition of division spaces. We will investigate some examples and properties of division spaces. The basic ingredient in the division space structure are finite partitions and their order structure :

**Definition 2.1.** Let  $X$  be a set.

1. A **partition** of  $X$  is a set  $P$  of nonempty subsets of  $X$  that are pairwise disjoint and such that  $X = \cup P$ .
2. A **finite partition** of  $X$  is a partition  $P$  of  $X$  such that the set  $P$  is finite. Note that in this terminology, the elements of  $P$  are not required to be finite. We denote  $\text{FPart}(X)$  the set of all finite partitions of  $X$ .
3. If  $J \subseteq X$  and  $P$  is a finite partition of  $X$ , we say that  $J$  is **finer than**  $P$  if there is  $J' \in P$  such that  $J \subseteq J'$ .
4. If  $P$  and  $P'$  are two finite partitions of  $X$ , we say that  $P$  is **finer than**  $P'$ , and we write  $P \leq P'$ , if for all  $J \in P$ ,  $J \leq P'$ .
5. If  $P, P'$  are two finite partitions of  $X$ , we denote  $P \wedge P'$  the following finite partition of  $X$  :

$$\{J \cap J' : (J, J') \in P \times P', J \cap J' \neq \emptyset\}.$$

We note the following basic facts about finite partitions :

**Proposition 2.2.** Let  $X$  be a set.

1. The relation  $\leq$  is a partial order on  $\text{FPart}(X)$ .
2. If  $X$  is nonempty,  $\{X\}$  is the maximum of  $\text{FPart}(X)$ , and  $\text{FPart}(\emptyset) = \{\emptyset\}$ .
3. For any  $P, P' \in \text{FPart}(X)$ ,  $P \wedge P'$  is the greatest lower bound of  $\{P, P'\}$ .

Thus,  $(\text{FPart}(X), \leq, \wedge)$  is a meet-semilattice with  $\{X\}$  as top element if  $X$  is nonempty.

*Proof.* Items 1. and 2. are straightforward. For 3., we first see that  $P \wedge P'$  is actually a finite partition of  $X$  if  $P, P' \in \text{FPart}(X)$ . Indeed, if  $C_1, C_2 \in P \wedge P'$ , then there is  $(J_1, J'_1) \in P \times P'$  such that  $J_1 \cap J'_1 \neq \emptyset$  and  $C_1 = J_1 \cap J'_1$ , and there is  $(J_2, J'_2) \in P \times P'$  such that  $J_2 \cap J'_2 \neq \emptyset$  and  $C_2 = J_2 \cap J'_2$ . Thus, if  $C_1 \neq C_2$ , then  $J_1 \neq J_2$ , or  $J'_1 \neq J'_2$ . In the first case, we have  $J_1 \cap J_2 = \emptyset$  and thus,  $C_1 \cap C_2 = \emptyset$ , and similarly in the second case. As a consequence,  $P \wedge P'$  is a finite set of pairwise disjoint nonempty subsets of  $X$ . Since  $\cup P \wedge P' = \cup_{J \in P} \cup_{J' \in P'} J \cap J' = \cup_{J \in P} J \cap X = X$ , we see then that  $P \wedge P' \in \text{FPart}(X)$ .

Let us now check that  $P \wedge P'$  is the greatest lower bound of  $\{P, P'\}$ . Clearly,  $P \wedge P' \leq P$  and  $P \wedge P' \leq P'$ . Let  $P'' \in \text{FPart}(X)$  be such that  $P'' \leq P$  and  $P'' \leq P'$ . Then for any  $J'' \in P''$ , there is  $J \in P$  such that  $J'' \subseteq J$  and there is  $J' \in P'$  such that  $J'' \subseteq J'$ . Thus,  $J'' \subseteq J \cap J'$ , and finally,  $P'' \leq P \wedge P'$ .  $\square$

**Proposition 2.3.** Let  $P, P' \in \text{FPart}(X)$  with  $P \leq P'$  and let  $J' \in P'$ . Then  $\{J \in P : J \subseteq J'\}$  is a finite partition of  $J'$  made of elements of  $P$ .

*Proof.* Denote  $Q := \{J \in P : J \subseteq J'\}$ . Clearly,  $Q \subseteq P$ , and  $\cup Q \subseteq J'$ . Let us check that  $J' \subseteq \cup Q$ . Let  $x \in J'$ . There is a unique  $J \in P$  such that  $x \in J$ . Since  $P \leq P'$ , there is  $J'' \in P'$  such that  $J \subseteq J''$ , and thus  $x \in J''$ . Since  $J' \cap J'' \neq \emptyset$ ,  $J' = J''$  and thus  $x \in \cup Q$ .  $\square$

**Definition 2.4.** Let  $X$  be a set. A **semialgebra over  $X$**  is a collection  $\mathcal{A}$  of subsets of  $X$  such that

1.  $\mathcal{A}$  contains  $X$ .
2.  $\mathcal{A}$  is stable by finite intersection.
3. The complement of any element of  $\mathcal{A}$  can be partitioned by elements of  $\mathcal{A}$  (in other words, for any  $A \in \mathcal{A}$ , there is a finite partition  $P$  of  $A^c := X \setminus A$  such that  $P \subseteq \mathcal{A}$ ).

**Remark 2.5.** The definition we use here of a semialgebra  $\mathcal{A}$  does not require the empty set to be in  $\mathcal{A}$ . Note that  $\{\emptyset\}$  is the unique semialgebra over  $\emptyset$ . Over a singleton  $\{x\}$ , we have two possible semialgebras:  $\{\{x\}\}$  and  $\{\emptyset, \{x\}\}$ . A semialgebra that contains a nonempty proper subset of  $X$ , also contains  $\emptyset$ .

**Definition 2.6.** 1. Let  $X$  be a topological space. A **division structure** on  $X$  is a semialgebra  $\mathcal{C}$  over  $X$  such that  $\{J \in \mathcal{C} : J \text{ is open}\}$  is a base for the topology of  $X$ . Elements of  $\mathcal{C}$  are called **cells** of  $X$ .

2. A pair  $(X, \mathcal{C})$  where  $X$  a topological space and  $\mathcal{C}$  a division structure is called a **division space**.
3. A division space  $(X, \mathcal{C})$  is said **nontrivial** if  $X$  contains at least two elements.
4. A family  $(X_t)_{t \in T}$ , where  $T$  is any set and where each  $X_t$  is a nontrivial division space, such that  $\prod_{t \in T} X_t \neq \emptyset$  will be called a **division family**.
5. A division family  $(X_t)$  is called **compact metrizable** if each  $X_t$  is a compact metrizable topological space.

**Definition 2.7.** Let  $(X_t)_{t \in T}$  be a division family, and denote  $\mathcal{C}_t$  the division structure on  $X_t$  for each  $t \in T$ . The **product** of  $(X_t)_t$  is by definition  $X := \prod_{t \in T} X_t$ , and the **cell semialgebra** of  $(X_t)_t$  is the set  $\otimes_{t \in T} \mathcal{C}_t$ , also denoted  $\mathcal{C}$  in the following, of all subsets of  $X$  of the form

$$\prod_{t \in T} J_t$$

such that for any  $t \in T$ ,  $J_t \in \mathcal{C}_t$  and for cofinitely almost all  $t \in T$ ,  $J_t = X_t$ . The pair  $(X, \mathcal{C})$  is called the **product division space** of the division family  $(X_t)_{t \in T}$ .

**Theorem 2.8.** Let  $(X_t)_{t \in T}$  be a division family. Then

1. The product division space  $(X, \mathcal{C})$  is itself a division space. It is nontrivial if and only if  $T$  is nonempty.
2. If  $T$  is countable and  $(X_t)$  is a compact metrizable division family, then  $(X, \mathcal{C})$  is a compact metrizable division space.

*Proof.* 1. If  $T = \emptyset$ , we see that  $X = \{\emptyset\}$  (the singleton containing the empty map) and  $\mathcal{C} = \{X\}$ . This is a (trivial) division space.

If  $T \neq \emptyset$ , then since  $X \neq \emptyset$ , there is a function  $x$  defined on  $T$  such that for any  $t \in T$ ,  $x(t) \in X_t$ . Fix  $t_0 \in T$ . Since  $X_{t_0}$  contains at least two elements, there is  $y_0$  in  $X_{t_0}$  such that  $x(t_0) \neq y_0$ . The map  $x'$  such that  $x'|_{T \setminus \{t_0\}} = x|_{T \setminus \{t_0\}}$  and  $x'(t_0) = y_0$  is different from  $x$  and is also in  $X$ .

Denote  $\tau_t$  the topology of  $X_t$  for each  $t \in T$  and  $\tau$  the product topology on  $X$ . It is straightforward to check that  $\mathcal{C} \cap \tau$  is equal to

$$\otimes_t (\mathcal{C}_t \cap \tau_t) := \left\{ \prod_{t \in T} J_t : J_t \in \mathcal{C}_t \cap \tau_t, \text{ and } J_t = X_t \text{ for cofinitely almost all } t \in T \right\}.$$

Since  $\mathcal{C}_t \cap \tau_t$  is a base for  $\tau_t$  for each  $t$ ,  $\mathcal{C} \cap \tau$  is a base for the (product) topology of  $X$ .

It remains to check that  $\mathcal{C}$  is indeed a semialgebra on  $X$ . We will denote  $p_t : X \rightarrow X_t$  the canonical projections. First, it is clear that  $X$  and  $\emptyset$  are in  $\mathcal{C}$ . Let  $J, J' \in \mathcal{C}$ . We have  $J = \bigcap_{t \in F} p_t^{-1}(J_t)$  and  $J' = \bigcap_{t \in F'} p_t^{-1}(J'_t)$  for finite sets  $F, F'$  and cells  $J_t, J'_t$  such that  $J_t \in \mathcal{C}_t$  and  $J'_t \in \mathcal{C}_t$ . We extend  $J_t$  for all  $t \in F \cup F'$  so that  $J_t := X_t$  for  $t \notin F$ , and we do the same for  $J'_t$ . Thus,  $J \cap J' = \bigcap_{t \in F \cup F'} p_t^{-1}(J_t) \cap p_t^{-1}(J'_t) = \bigcap_{t \in F \cup F'} p_t^{-1}(J_t \cap J'_t)$ , and therefore  $J \cap J'$  is in  $\mathcal{C}$ .

Moreover, given  $J \in \mathcal{C}$ , of the form  $\bigcap_{t \in F} p_t^{-1}(J_t)$ , for all  $t \in F$  there are disjoint cells  $K_{t,1}, \dots, K_{t,n_t}$  of  $\mathcal{C}_t$  such that  $X_t = \bigcup_{0 \leq j \leq n_t} K_{t,j}$ , where we denote  $K_{t,0} := J_t$ . Since  $X = \bigcap_{t \in F} p_t^{-1}(X_t)$ , we obtain the equality  $X = \bigcup_{j \in \prod_{t \in F} \{0, \dots, n_t\}} \bigcap_{t \in F} p_t^{-1}(K_{t,j(t)})$ . Thus, with  $J = \bigcap_{t \in F} p_t^{-1}(K_{t,0})$  we see that  $J^c$  is a finite union of disjoint elements of  $\mathcal{C}$ .

2. Since we assume in this paper the principle of dependent choices (DC), Tychonoff theorem for a countable product of compact spaces holds [1, p.88]. As a consequence,  $X = \prod_t X_t$  is a compact metrizable space. □

**Assumption 2.9.** The goal of this chapter is to construct and describe an integration process over the product division space associated to a given compact metrizable division family. Thus, from now on, we fix a compact metrizable division family  $(X_t)_{t \in T}$  and we denote  $(X := \prod_{t \in T} X_t, \mathcal{C} := \otimes_{t \in T} \mathcal{C}_t)$  the associated product division space.

**Notation 2.10.** If  $U \subseteq T$ , we will denote  $(X_U, \mathcal{C}_U)$  the product division space of the subfamily  $(X_t)_{t \in U}$ , which is also a compact metrizable division family. If  $J \in \mathcal{C}$ , we denote  $J_U := \prod_{t \in U} J_t$ . Note that  $\mathcal{C}_U = \{J_U : J \in \mathcal{C}\}$ .

**Remark 2.11.** The product division space  $(X_\emptyset, \mathcal{C}_\emptyset)$  of the empty division family is equal to the division space  $(\{\emptyset\}, \{\{\emptyset\}\})$ . It will be called **the singleton division space**. It will be used in the following mainly as a way to simplify the presentation and statement of a few results, so that we don't have to exclude the case of the empty indexing set.

**Remark 2.12.** If  $t \in T$ , we can (and will) identify naturally  $X_{\{t\}}$  with  $X_t$  and  $\mathcal{C}_{\{t\}}$  with  $\mathcal{C}_t$ . Thus, a nontrivial compact metrizable division space is a particular case of a product division space of a compact metrizable division family indexed by a singleton.

**Notation 2.13.** If  $(x_t)_{t \in U}$  and  $(x'_t)_{t \in U'}$  are families such that  $U \cap U' = \emptyset$ , the overriding family  $x \oplus x'$  is the family indexed by  $U \cup U'$  that coincide with  $x$  on  $U$  and with  $x'$  on  $U'$ . This operation is associative and commutative, and has the empty function as an identity element. If  $A$  and  $B$  are sets containing families indexed by disjoint sets, we denote  $A \oplus B$  the set of all  $x \oplus x'$  where  $x \in A$  and  $x' \in B$ . This operation is associative and commutative and has the singleton  $\{\emptyset\}$  as identity element.

Observe that if  $U \subseteq T$ , then  $X = X_U \oplus X_{T \setminus U}$ .

**Definition 2.14.** If  $J \in \mathcal{C}$ , we denote  $N_J$ , and call this subset of  $T$  the **dimension set** of  $J$ , the unique element of  $\mathcal{F}(T)$  such that for any  $t \in N_J$ ,  $J_t \neq X_t$ , and for any  $t \notin N_J$ ,  $J_t = X_t$ .

We will use the notation :

$$\underline{J} := J_{N_J}$$

**Remark 2.15.** Note that for any cell  $J \in \mathcal{C}$ ,  $J = \underline{J} \oplus X_{T \setminus N_J}$ , and more generally for any  $N \in \mathcal{F}(T)$  such that  $N \supseteq N_J$ ,  $J = J_N \oplus X_{T \setminus N}$ .

**Example 2.16.** Here are some examples of nontrivial compact metrizable division spaces, and associated product division spaces :

1. Any finite set  $X$  with at least two elements with discrete topology is a nontrivial compact metrizable division space, where a cell is either  $X$ ,  $\emptyset$ , or a singleton. Considering now the compact metrizable division family  $(\{0, 1\})_{n \in \mathbb{N}}$ , we obtain  $2^{\mathbb{N}}$  (the topological Cantor set) as a particular example of a product division space.

2. The space  $\overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$  with the induced topology from  $\overline{\mathbb{R}}_+$  is a division space with the cells given by the empty set, all the singletons  $\{n\}$  and the subsets of the form  $\llbracket n, \infty \rrbracket := [n, \infty] \cap \overline{\mathbb{N}}$ , where  $n \in \mathbb{N}$ , is a compact metrizable division space.

This example gives the following product division spaces :  $\overline{\mathbb{N}}^n$  and  $\overline{\mathbb{N}}^{\mathbb{N}}$ .

3. The extended real line  $\overline{\mathbb{R}}$  is a division space where the cells are all the intervals. This example is the fundamental prototype of a compact metrizable division spaces.

Thus, we obtain a compact metrizable division family  $(\overline{\mathbb{R}})_{t \in T}$  for any set  $T$ , and  $\overline{\mathbb{R}}^T$  is an example of a product division space, that is compact metrizable when  $T$  is countable.

4. The circle  $\mathbb{T} = \mathbb{S}^1$  with the standard topology induced from  $\mathbb{R}^2$  is a compact metrizable division space where a cell is either the whole circle  $\mathbb{S}^1$  or an arc of length less than  $\pi$ , i.e. of the form  $\{e^{i\theta} : \theta \in I, I \text{ interval of } \mathbb{R} \text{ of length } < \pi\}$ .

This shows that the  $n$ -torus  $\mathbb{T}^n$  and the infinite torus  $\mathbb{T}^{\mathbb{N}}$  are examples of compact metrizable product division spaces.

5. If  $X$  is a compact ultrametric space, the family of all open balls of  $X$  together with the empty set, is a division structure on  $X$ . For example, the space of all  $p$ -adic integers  $\mathbb{Z}_p$  (where  $p$  is a prime number) has a natural structure of division space, and thus the space of all profinite integers  $\prod_{p \text{ prime}} \mathbb{Z}_p$  is an example of a compact metrizable product division space.

## 2.2 Divisions and elementary sets

As previously said (see Assumption 2.9), we fix in the following section and in the rest of the chapter a compact metrizable division family  $(X_t)_{t \in T}$  and we denote  $(X, \mathcal{C})$  its associated product division space.

**Definition 2.17.** 1. A finite set of pairwise disjoint nonempty cells of  $X$  is called a **division**.

2. A subset of  $X$  of the form  $\cup D$  where  $D$  is a division is called a **elementary set** of  $X$ . We denote  $\mathcal{E}$  the set of all elementary sets.

3. If  $E$  is an elementary set of  $X$ , a **division of** (resp. **in**)  $E$  is a division  $D$  such that  $E = \cup D$  (resp.  $E \subseteq \cup D$ ). We denote  $\mathcal{D}_E$  the set of all divisions of  $E$ .

**Remark 2.18.** Since  $\mathcal{C}$  is a semialgebra over  $X$ , the set of all elementary sets  $\mathcal{E}$  is an *algebra* over  $X$ , that means that  $\mathcal{E}$  contains  $\emptyset, X$ , and is stable by finite intersections, finite unions, and by complementation (see for instance [13, Theorem 21.4]).

**Remark 2.19.** For each  $t \in T$ , since  $X_t$  is nontrivial,  $X_t \setminus \{x\}$  is an open proper nonempty subset of  $X_t$ , if  $x \in X_t$ . Thus,  $X_t \setminus \{x\}$  contains a nonempty open cell  $J$ . As a consequence, there is a division of  $X_t$  that is different from the trivial division  $\{X_t\}$ .

**Proposition 2.20.** *Let  $E$  be an elementary set. If  $D, D' \in \mathcal{D}_E$ , then  $D \wedge D' \in \mathcal{D}_E$ . In particular,  $(\mathcal{D}_E, \leq, \wedge)$  is a meet semilattice.*

*Proof.* Let  $D, D' \in \mathcal{D}_E$ . By definition,  $D \wedge D' = \{J \cap J' : (J, J') \in D \times D', J \cap J' \neq \emptyset\}$ . We see directly that  $D \wedge D'$  is thus a finite partition of  $E$  made of cells.  $\square$

**Remark 2.21.** The semilattice  $\mathcal{D}_X$  is bounded, with  $\{X\}$  as top element.

**Example 2.22.** Here are some examples of divisions :

1. If  $X$  is the division space associated to a finite set with at least two elements,  $\mathcal{D}_X$  has two elements :  $\{X\}$  and  $\{\{x\} : x \in X\}$ .
2. The divisions of  $\overline{\mathbb{N}}$  are either  $\{\overline{\mathbb{N}}\}$  or of the following form, where  $n \in \mathbb{N}$  :

$$\{\{0\}, \dots, \{n\}, \llbracket n+1, \infty \rrbracket\}.$$

3. The set  $\{[-\infty, -1], ]-1, 1[, [1, \infty]\}$  is an example of a division of  $\overline{\mathbb{R}}$ .

4. In  $\overline{\mathbb{R}}^{\mathbb{N}}$ , the set  $\{J, K, L\}$  is a division of  $\overline{\mathbb{R}}^{\mathbb{N}}$ , where we define  $J, K, L$  by setting :  $N_J = \{0\}$ ,  $N_K = N_L = \{0, 1\}$ ,  $J_0 = \overline{\mathbb{R}}_-$ ,  $K_0 = L_0 = \overline{\mathbb{R}}_+^*$ ,  $K_1 = [1, \infty]$ ,  $L_1 = [-\infty, 1[$ .

**Definition 2.23.** If  $D$  is a division of  $X$ , then the **dimension set** of  $D$  is  $N_D := \cup_{J \in D} N_J$ . We will denote  $\underline{D} := \{J_{N_D} : J \in D\}$ , and for any  $N \supseteq N_D$ ,  $D_N := \{J_N : J \in D\}$ .

**Remark 2.24.** If  $D$  is a division of  $X$ , then for any  $J \in D$ , we have  $J = J_{N_D} \oplus X_{T \setminus N_D}$ , and thus,  $\underline{D}$  is a division of  $X_{N_D}$ . More generally, if  $N \supseteq N_D$ ,  $D_N$  is a division of  $X_N$ .

The following definition deals with a special type of division that will play a important role in the proof of the generalized Cousin's theorem 2.42.

**Definition 2.25.** Let  $N \in \mathcal{F}(T)$ . If for any  $t \in N$ ,  $D_t$  is a division of  $X_t$ , the **division product**  $\otimes_{t \in N} D_t$  is the set of all products  $\prod_{t \in N} J_t$ , where for each  $t$ ,  $J_t$  is an element of  $D_t$ .

A division  $D$  of  $X_N$  is said **regular** if it is of the form of  $\otimes_{t \in N} D_t$  for a family  $(D_t)_{t \in N}$  of divisions (such that  $D_t$  is a division of  $X_t$ ).

Note that for  $N = \emptyset$ , the unique division  $\{X_\emptyset\}$  of the singleton division space  $X_\emptyset = \{\emptyset\}$  is regular.

**Proposition 2.26.** Let  $N \in \mathcal{F}(T)$ .

1. A division product  $\otimes_{t \in N} D_t$  is indeed a division of  $X_N$ .
2. If  $D$  is a regular division of  $X_N$ , then the family  $(D_t)_{t \in N}$  of divisions such that  $D = \otimes_{t \in N} D_t$  is unique. We will keep using that notation in the following.
3. Let  $D$  be a division of  $X_N$ . Then there is a regular division  $D'$  of  $X_N$  such that  $D' \leq D$ .

*Proof.* 1. It is clear that the elements of  $\otimes D_t$  are pairwise disjoint. Moreover,

$$X_N = \prod_t X_t = \prod_t (\cup D_t) = \cup_{J_t \in D_t} \prod J_t = \cup \otimes_t D_t.$$

2. Suppose that  $\otimes_t D_t = \otimes_t D'_t = D$  where  $(D_t)$  and  $(D'_t)$  are families (indexed by  $N$ ) of divisions, such that  $D_t$  and  $D'_t$  is a division of  $X_t$  for all  $t \in N$ . Fix  $t \in N$  and take  $J \in D_t$ . Take now elements  $J_k \in D_k$  for any  $k \in N \setminus \{t\}$ . Thus,  $J \oplus \prod_k J_k \in D$ . As a consequence, there exists  $J'_k \in D'_k$  for each  $k$  and  $J'$  in  $D'_t$  such that  $J \oplus \prod_k J_k = J' \oplus \prod_k J'_k$  and this implies  $J = J'$  and thus  $J \in D'_t$ .
3. The case of  $N = \emptyset$  is trivial. Suppose then  $N \neq \emptyset$ . Take  $J \in D$  and  $t \in N$ . There is a division  $D(J, t)$  of  $X_t$  such that  $J_t \in D(J, t)$ . Define now  $D_t := \wedge_{J \in D} D(J, t)$ . This is a division of  $X_t$ . Take now  $D' := \otimes_t D_t$ . This is clearly a regular division of  $X_N$ . To prove that  $D' \leq D$ , let  $J' \in D'$ . Since  $D$  is a division of  $X_N$ , there is a cell  $J \in D$  such that  $J \cap J' \neq \emptyset$ , which implies that for any  $t \in N$ ,  $J_t \cap J'_t \neq \emptyset$ . Since  $J'_t \in D_t \leq D(J, t)$ , and  $J_t \in D(J, t)$ , it follows that  $J'_t \subseteq J_t$  and therefore  $J' \subseteq J$ . This shows that  $D' \leq D$ . □

**Notation 2.27.** If  $D$  is a regular division of  $X_N$ , and  $K \subseteq N$ , we will denote  $D|_K$  the regular division  $\otimes_{t \in K} D_t$  of  $X_K$ .

If  $D$  is a regular division of  $X_N$  and  $D'$  is a regular division of  $X_M$  where  $N$  and  $M$  are disjoint, then we denote  $D \otimes D'$  the regular division on  $X_{N \cup M}$  given by  $\otimes_{t \in N \cup M} D_t$  where  $D_t := D'_t$  whenever  $t \in M$ .

Remark that if  $J \leq D$  and  $J' \leq D'$ , then  $J \oplus J' \leq D \otimes D'$ .

## 2.3 Gauges and resolutions

**Definition 2.28.** 1. A **gauge** is a pair  $\gamma := (L, \delta := (\delta_N)_{N \in \mathcal{F}(T)})$  such that for all  $N \in \mathcal{F}(T)$ ,

$$\begin{cases} L : X \rightarrow \mathcal{F}(T) \\ \delta_N : X \rightarrow \mathcal{D}_{X_N} \end{cases}$$

and the maps  $L$  and  $\delta_N$  have countable images (i.e.  $L(X)$  and  $\delta_N(X)$  are countable sets). A gauge is said **regular** if for any  $N \in \mathcal{F}(T)$ ,  $x \in X$ ,  $\delta_N(x)$  is a regular division of  $X_N$ .

2. Given two gauges  $\gamma = (L, \delta)$  and  $\gamma' = (L', \delta')$ , we say that  $\gamma$  is **finer than**  $\gamma'$  and write  $\gamma \leq \gamma'$ , if  $L \supseteq L'$  and  $\delta_N \leq \delta'_N$  for all  $N \in \mathcal{F}(T)$ .
3. If  $\gamma$  and  $\gamma'$  are two gauges, we denote  $\gamma \wedge \gamma'$  the gauge  $(L \cup L', \delta \wedge \delta')$  where  $(\delta \wedge \delta')_N := \delta_N \wedge \delta'_N$  for all  $N \in \mathcal{F}(T)$ .

Here are fundamental examples of gauges :

**Definition 2.29.** Let  $D$  be a division of  $X$ . The **gauge associated to**  $D$  is  $\gamma_D := (L_D, \delta_D)$ , where  $L_D$  is the constant function  $x \mapsto N_D$ , and for each  $N$ ,  $(\delta_D)_N : X \rightarrow \mathcal{D}_{X_N}$  is the constant function defined by :

$$(\delta_D)_N := \begin{cases} D_N & \text{if } N \supseteq N_D \\ \{X_N\} & \text{otherwise.} \end{cases}$$

More generally, let  $D_\bullet : X \rightarrow \mathcal{D}_X$ ,  $x \mapsto D_x$  be a function from  $X$  into the set of divisions of  $X$ , with countable image. Then  $\gamma_{D_\bullet} = (L_{D_\bullet}, \delta_{D_\bullet})$  defined by  $L_{D_\bullet} : x \mapsto N_{D_x}$  and

$$(\delta_{D_\bullet})_N : x \mapsto \begin{cases} (D_x)_N & \text{if } N \supseteq N_{D_x} \\ \{X_N\} & \text{otherwise.} \end{cases}$$

is a gauge on  $X$ , called the **gauge associated to**  $D_\bullet$ .

We can also create a gauge in a trivial way from a map  $L : X \rightarrow \mathcal{F}(T)$  :

**Definition 2.30.** Let  $L : X \rightarrow \mathcal{F}(T)$  with countable image. The **gauge associated to**  $L$  is the gauge  $\gamma_L := (L, \delta_X)$  where  $(\delta_X)_N$  is the constant function  $x \mapsto \{X_N\}$ .

**Proposition 2.31.** *The set of gauges is a bounded (meet) semilattice with the partial order  $\leq$ , and the greatest lower bound of  $\gamma, \gamma'$  is  $\gamma \wedge \gamma'$ , and the top element is  $\gamma_{\{X\}}$ .*

*Proof.* The relation  $\leq$  is clearly a partial order on the set of gauges. Let  $\gamma, \gamma'$  be two gauges. We see that  $\gamma \wedge \gamma'$  is indeed a gauge (which was already claimed in the previous definition). Indeed, it is clear that  $L \cup L'$  has a countable image and the same is true for  $\delta_N \wedge \delta'_N$  for any  $N$ . We see now that  $\gamma \wedge \gamma'$  is the greatest lower bound of  $\gamma, \gamma'$  since  $L \cup L'$  and  $\delta_N \wedge \delta'_N$  are the greatest lower bounds of respectively  $\{L, L'\}$  and  $\{\delta_N, \delta'_N\}$ .  $\square$

**Proposition 2.32.** *If  $\gamma$  is a gauge, then there is a regular gauge  $\gamma'$  such that  $\gamma' \leq \gamma$ .*

*Proof.* Let  $\gamma = (L, \delta)$  be a gauge and fix  $N \in \mathcal{F}(T)$ . Let  $(\delta_{n,N})_{n \in \mathbb{N}}$  be an enumeration of  $\delta_N(X_N)$  and for each  $x \in X$ , let  $n_x$  be the smallest number  $n$  for which  $\delta_{n,N} = \delta_N(x)$ . For each  $n \in \mathbb{N}$ , let  $D_{n,N}$  be a regular division of  $X_N$  such that  $D_{n,N} \leq \delta_{n,N}$ . Note that such regular division exist by Proposition 2.26, item 3. Therefore,  $\gamma' := (L, (x \mapsto D_{n_x, N})_N)$  is a regular gauge finer than  $\gamma$ .  $\square$

**Definition 2.33.** 1. A **pixel** (or a **pointed cell**) is a pair  $(J, x)$  where  $J$  is a nonempty cell, and  $x \in \bar{J}$ . The data  $J$ , and  $x$  are respectively called the **cell**, and **point of** the pixel  $(J, x)$ .

2. A **resolution** is a finite set of pixels such that their cells are pairwise disjoint.

3. If  $R$  is a resolution, the **domain** of  $R$  is the division  $D_R$  made by all the cells of the elements of  $R$ .
4. If  $E$  is an elementary set, a **resolution of** (resp. **resolution in**)  $E$  is a resolution  $R$  such that  $\cup D_R = E$  (resp.  $\cup D_R \subseteq E$ ).

**Remark 2.34.** Note also that a pixel singleton  $\{(J, x)\}$  is a particular resolution of a nonempty cell  $J$ . Note also that if  $(J, x)$  is a pixel, then for any  $N \in \mathcal{F}(T)$ ,  $(J_N, x_N)$  is a pixel in the division space product  $X_N$ .

**Definition 2.35.** Let  $\gamma = (L, \delta)$  be a gauge.

1. A pixel  $(J, x)$  is  $\gamma$ -**fine** if the following holds :

- (a)  $N_J \supseteq L(x)$ .
- (b)  $\underline{J} \leq \delta_{N_J}(x)$ , that is to say : there is  $J' \in \delta_{N_J}(x)$  such that  $\underline{J} \subseteq J'$ .

2. A resolution  $R$  is said  $\gamma$ -**fine** or is a  $\gamma$ -**resolution** if the elements of  $R$  are all  $\gamma$ -fine. We will also use the notation  $R \leq \gamma$  for this.

**Notation 2.36.** We denote  $\mathcal{R}_E^\gamma$  the set of all  $\gamma$ -resolutions of  $E$ , where  $E$  is an elementary set, and  $\mathcal{R}_E$  the set of all resolutions of  $E$ .

**Remark 2.37.** If  $R$  is a  $\gamma$ -resolution and if  $\gamma \leq \gamma'$ , then  $R$  is a  $\gamma'$ -resolution. Note also that the empty resolution is always  $\gamma$ -fine trivially.

In the case of a gauge associated to a division or a map from  $X$  into  $\mathcal{D}_X$  with countable image, we obtain the following :

**Proposition 2.38.** 1. Let  $D$  be a division of  $X$ . A pixel  $(J, x)$  is  $\gamma_D$ -fine if and only if  $N_J \supseteq N_D$  and  $J \leq D$ . As a consequence, if  $R$  is a  $\gamma_D$ -resolution of  $X$ , then  $D_R \leq D$ .

2. Let  $D_\bullet : X \rightarrow \mathcal{D}_X$  be a map with countable image. A pixel  $(J, x)$  is  $\gamma_{D_\bullet}$ -fine if and only if  $N_J \supseteq N_{D_x}$  and  $J \leq D_x$ .

*Proof.* We prove 2., since 1. follows from 2. directly.

Let us denote  $(L, \delta)$  the components of the gauge  $\gamma_{D_\bullet}$ . Suppose that the pixel  $(J, x)$  is  $\gamma_{D_\bullet}$ -fine. Then we have  $N_J \supseteq N_{D_x}$ , since  $L(x) = N_{D_x}$  by definition. Moreover, there is  $J' \in \delta_{N_J}(x)$  such that  $\underline{J} \subseteq J'$ . Since  $\delta_{N_J}(x) = (D_x)_{N_J}$  here (because  $N_J \supseteq N_{D_x}$ ), we have  $J' = J''_{N_J}$  for a  $J'' \in D_x$ , so that  $J'' = J' \oplus \overline{X}_{T \setminus N_J}$  (since  $N_{J''} \subseteq N_{D_x} \subseteq N_J$ ). This implies  $J \subseteq J''$  and therefore  $J \leq D_x$ .

Inversely, suppose that the pixel  $(J, x)$  is such that  $N_J \supseteq N_{D_x}$  and there is  $J' \in D_x$  such that  $J \subseteq J'$ . This implies directly that  $N_J \supseteq L(x)$ . Moreover, we have  $\underline{J} \subseteq J'_{N_J}$ . And since  $J'_{N_J} \in (D_x)_{N_J}$  (because  $N_J \supseteq N_{D_x}$ ), we have the result.  $\square$

**Definition 2.39.** Let  $D$  be a division in  $X$ . A  $D$ -**gauge**, is a gauge  $\gamma$  such that for any  $\gamma$ -fine resolution  $R$  of  $\cup D$ ,  $D_R \leq D$ .

It is possible to generalize item 1 of the previous proposition by considering division of elementary sets :

**Corollary 2.40.** For any division  $D$  in  $X$ , there exists a  $D$ -gauge.

*Proof.* Fix a division  $D'$  of  $X \setminus E$ , and consider the gauge  $\gamma_{D \cup D'}$ . Let  $R$  be a  $\gamma_{D \cup D'}$ -resolution of  $E$ . Fix  $R'$  a  $\gamma_{D \cup D'}$ -resolution of  $X \setminus E$ . By Proposition 2.38, since  $R \cup R'$  is a  $\gamma_{D \cup D'}$ -resolution of  $X$ ,  $D_{R \cup R'} \leq D \cup D'$ . This implies  $D_R \leq D$ .  $\square$

In the case of a regular gauge, we have the following characterization :

**Proposition 2.41.** Let  $\gamma$  be a regular gauge. Then a pixel  $(J, x)$  is  $\gamma$ -fine if and only if  $N_J \supseteq L(x)$  and for any  $t \in N_J$ ,  $J_t \leq \delta_{N_J}(x)_t$ .

*Proof.* Suppose that  $D = \otimes D_t$  is a regular division of  $X_N$  where  $N \in \mathcal{F}(T)$ . Then we see that a cell  $J$  of  $X_N$  is such that  $J \leq D$  if and only if for any  $t \in N$ ,  $J_t \leq D_t$ . The result follows straightforwardly.  $\square$

The next result is crucial for the construction of the gauge integral, it is essentially a generalization of Cousin's lemma. We state the result here but we postpone of the proof in Appendix 5.1.

**Theorem 2.42.** *For any gauge  $\gamma$  and elementary set  $E$ , there exists a  $\gamma$ -resolution of  $E$ , i.e.  $\mathcal{R}_E^\gamma \neq \emptyset$ .*

We saw previously that any division  $D$  gives rise to a gauge. We now show that we can obtain gauges from a family of strictly positive functions  $(\eta_N)_N$  indexed by the dimension set, which correspond to a maximal size  $\eta_N(x)$  of the pixel whose point is  $x$ . More accurately, we have :

**Lemma 2.43.** *Fix for each  $N \in \mathcal{F}(T)$  a distance  $d_N$  on  $X_N$  that is compatible with the topology of  $X_N$ . We denote  $B_N(y, r)$  the open ball in  $X_N$  of center  $y$  and radius  $r$  for this distance.*

*Let  $(\eta_N)_{N \in \mathcal{F}(T)}$  be a family such that  $\eta_N : X \rightarrow \mathbb{R}_+^*$  and let  $L : X \rightarrow \mathcal{F}(T)$  with countable image. Then there is a gauge  $\gamma$  such that any  $\gamma$ -fine pixel  $(J, x)$  is such that :*

1.  $N_J \supseteq L(x)$ .
2.  $\underline{J} \subseteq B_{N_J}(x_{N_J}, \eta_{N_J}(x))$ .

*Proof.* For any  $N \in \mathcal{F}(T)$ , since  $X_N$  is compact and metrizable, it is second countable. Thus,  $\mathcal{C}_N \cap \tau_N$  (where  $\tau_N$  is the topology of  $X_N$ ) being a base for  $\tau_N$  (see Theorem 2.8), there is a countable set in  $\mathcal{C}_N \cap \tau_N$  that is a base for  $\tau_N$ . Let us fix  $(J_{n,N})_{n \in \mathbb{N}}$  an enumeration of this countable set. For each  $n \in \mathbb{N}$ , fix a division  $D_{n,N}$  of  $X_N$  that contains  $J_{n,N}$ .

For any  $y \in X$ , the open ball  $B_{y,N} := B_N(y_N, \eta_N(y))$  is equal to the countable union

$$\bigcup_{n \in \mathbb{N} : J_{n,N} \subseteq B_{y,N}} J_{n,N}$$

since  $(J_{n,N})_{n \in \mathbb{N}}$  is an enumeration of a countable base of  $\tau_N$ . For any  $y \in X$ , it is clear that  $y_N \in B_{y,N}$ . Let then  $n_{y,N}$  be the smallest number  $n$  such that  $y_N \in J_{n,N}$  and  $J_{n,N} \subseteq B_{y,N}$ . Denote  $J_{y,N} := J_{n_{y,N},N}$  and  $D_{y,N} := D_{n_{y,N},N}$ .

Thus, we have for any  $y \in X$ ,  $y_N \in J_{y,N} \subseteq B_{y,N}$ . Define now  $\gamma := (L, (\delta_N)_N)$  where

$$\delta_N : X \rightarrow \mathcal{D}_{X_N}, \quad x \mapsto D_{x,N}.$$

Let  $(J, x)$  be a  $\gamma$ -fine pixel. We have obviously  $N_J \supseteq L(x)$  and  $\underline{J} \leq \delta_{N_J}(x) = D_{x,N_J}$ . Thus, there is  $J' \in D_{x,N_J}$  such that  $\underline{J} \subseteq J'$ . This implies that  $x_{N_J} \in \overline{J'}$  and therefore  $\overline{J'} \cap J_{x,N_J} \neq \emptyset$ . Suppose that  $J'$  is different from  $J_{x,N_J}$ . Then, they are disjoint, and since  $J_{x,N_J}$  is open, we would have  $\overline{J'} \cap J_{x,N_J} = \emptyset$ , which gives a contradiction. As a consequence,  $J' = J_{x,N_J}$ , and therefore,  $\underline{J} \subseteq J_{x,N_J} \subseteq B_{x,N_J}$ .  $\square$

**Remark 2.44.** For the case of the division family  $(\overline{\mathbb{R}})_{t \in T}$  where  $T$  is any indexing set, a notion of gauge was defined in [9, p. 103] (see also [5, p. 797]). It takes the  $(L, (\eta_N)_N)$  data described in the previous lemma as definition for a gauge that will control locally the size of the cells. The previous lemma implies that our notion of gauge essentially contains as a particular case the  $(L, (\eta_N)_N)$  type of gauge, and therefore the gauge integral we will obtain is a generalization of the gauge integral associated to the  $(L, (\eta_N)_N)$  gauges.

**Definition 2.45.** Let  $S$  be a set and  $f : S \rightarrow \mathbb{C}$  be a function, and  $A \subseteq S$  nonempty. The **oscillation** of  $f$  on  $A$  is the quantity :

$$\text{osc}(f, A) := \sup_{x, y \in A} |f(x) - f(y)|.$$

The previous lemma can be applied to show the following result which essentially says that we can obtain gauges  $\gamma$  from a family of continuous functions  $(f_N)$  that will make the oscillation of  $f_N$  on the  $\gamma$ -fine pixels (restricted on  $N$ ) as small as we want.

**Theorem 2.46.** Let  $f = (f_N)_{N \in \mathcal{F}(T)}$  be a family of continuous functions  $f_N : X_N \rightarrow \mathbb{C}$ , let  $(e_N)$  be a family such that for any  $N \in \mathcal{F}(T)$ ,  $e_N : X \rightarrow \mathbb{R}_+^*$  and let  $L : X \rightarrow \mathcal{F}(T)$  with countable image. Then there is a gauge  $\gamma$  such that for any  $\gamma$ -fine pixel  $(J, x)$ ,

1.  $N_J \supseteq L(x)$ .
2.  $\text{osc}(f_{N_J}, \bar{J}) \leq e_{N_J}(x)$ .

*Proof.* We keep the notation  $B_N(y, r)$  given in the previous lemma for the open balls of  $X_N$  for a fixed compatible distance  $d_N$  on  $X_N$ . We will denote  $B_{\mathbb{C}}(z, a)$  the open ball in  $\mathbb{C}$  of radius  $a$  and center  $z \in \mathbb{C}$ . Since  $f_N : X_N \rightarrow \mathbb{C}$  is continuous, for any  $y \in X$ , there is  $\eta > 0$  such that  $f_N(B_N(y_N, \eta)) \subseteq B_{\mathbb{C}}(f_N(y_N), e_N(y)/2)$ . We define for any  $y \in X$ , the number  $n_{y,N}$  as the smallest nonzero natural number  $n$  such that  $f_N(B_N(y_N, 1/n)) \subseteq B_{\mathbb{C}}(f_N(y_N), e_N(y)/2)$ .

Lemma 2.43 applied to the family  $(\eta_N)$  where  $\eta_N : y \mapsto 1/n_{y,N}$  shows that there is a gauge  $\gamma = (L, \delta)$  on  $X$  such that for any  $\gamma$ -fine pixel  $(J, x)$ ,  $N_J \supseteq L(x)$  and  $\bar{J} \subseteq B_{N_J}(x_{N_J}, 1/n_{x,N_J})$ . As a consequence, if  $(J, x)$  is a  $\gamma$ -fine pixel,  $f_{N_J}(\bar{J}) \subseteq B_{\mathbb{C}}(f_{N_J}(x_{N_J}), e_{N_J}(x)/2) =: B$ . Since  $f_{N_J}$  is continuous, we get  $f_{N_J}(\bar{J}) \subseteq \bar{B}$ . Thus, if  $y, y' \in \bar{J}$ , we obtain

$$|f_{N_J}(y) - f_{N_J}(y')| \leq |f_{N_J}(y) - f_{N_J}(x_{N_J})| + |f_{N_J}(x_{N_J}) - f_{N_J}(y')| \leq e_{N_J}(x)/2 + e_{N_J}(x)/2 = e_{N_J}(x)$$

which concludes the proof.  $\square$

When  $T$  is countable and we deal directly with a point function on  $X$  and not a function on  $X \times \mathcal{F}(T)$ , we can obtain similar results :

**Lemma 2.47.** Suppose that  $T$  is countable and fix a distance on  $X$  that is compatible with its topology. We denote  $B(a, r)$  the open ball in  $X$  for this distance.

Let  $\eta : X \rightarrow \mathbb{R}_+^*$  and  $L : X \rightarrow \mathcal{F}(T)$ . Then there is a gauge  $\gamma$  such that for any  $\gamma$ -fine pixel  $(J, x)$ ,

1.  $N_J \supseteq L(x)$ .
2.  $J \subseteq B(x, \eta(x))$ .

*Proof.* Since  $T$  is supposed countable, the division space  $X$  is compact and metrizable and thus second countable. Thus,  $\mathcal{C} \cap \tau$  (where  $\tau$  is the topology of  $X$ ) being a base for  $\tau$  (by Theorem 2.8), there is a countable set in  $\mathcal{C} \cap \tau$  that is a base for  $\tau$ . Let us fix  $(J_n)_{n \in \mathbb{N}}$  an enumeration of this countable set. For each  $n \in \mathbb{N}$ , fix a division  $D_n$  of  $X$  that contains  $J_n$ .

For any  $x \in X$ , the open ball  $B_x := B(x, \eta(x))$  is equal to the countable union

$$\cup_{n \in \mathbb{N}} : J_n \subseteq B_x J_n$$

since  $(J_n)_{n \in \mathbb{N}}$  is an enumeration of a countable base of  $\tau$ . For any  $x \in X$ , let  $n_x$  be the smallest number  $n$  such that  $x \in J_n$  and  $J_n \subseteq B_x$ . Denote  $J_x := J_{n_x}$  and  $D_x := D_{n_x}$ . Thus, we have for any  $x \in X$ ,  $x \in J_x \subseteq B_x$ .

Consider now the gauge  $\gamma_{D_x}$  associated to the map  $x \mapsto D_x$ , and define  $\gamma := \gamma_{D_x} \wedge \gamma_L$ .

Let  $(J, x)$  be a  $\gamma$ -fine pixel. We have obviously  $N_J \supseteq L(x)$  and by Proposition 2.38, item 2,  $J \subseteq D_x$ . Thus, there is  $J' \in D_x$  such that  $J \subseteq J'$ .

This implies that  $x \in \bar{J}'$  and therefore  $\bar{J}' \cap J_x \neq \emptyset$ . Suppose that  $J'$  is different from  $J_x$ . Then, they are disjoint, and since  $J_x$  is open, we would have  $\bar{J}' \cap J_x = \emptyset$ , which gives a contradiction. As a consequence,  $J' = J_x$ , and therefore,  $J \subseteq J_x \subseteq B_x$ .  $\square$

**Theorem 2.48.** Suppose that  $T$  is countable. Let  $f : X \rightarrow \mathbb{C}$  be a continuous function,  $e : X \rightarrow \mathbb{R}_+^*$ , and  $L : X \rightarrow \mathcal{F}(T)$ . Then there is a gauge  $\gamma$  such that for any  $\gamma$ -fine pixel  $(J, x)$ ,

1.  $N_J \supseteq L(x)$ .
2.  $\text{osc}(f, \bar{J}) \leq e(x)$ .

*Proof.* Since  $f : X \rightarrow \mathbb{C}$  is continuous, for any  $x \in X$ , there is  $\eta > 0$  such that

$$f(B(x, \eta)) \subseteq B_{\mathbb{C}}(f(x), e(x)/2).$$

We define for any  $x \in X$ , the number  $n_x$  as the smallest nonzero natural number  $n$  such that

$$f(B(x, 1/n)) \subseteq B_{\mathbb{C}}(f(x), e(x)/2).$$

Lemma 2.47 applied to the function  $\eta : x \mapsto 1/n_x$  shows that there is a gauge  $\gamma = (L, \delta)$  on  $X$  such that for any  $\gamma$ -fine pixel  $(J, x)$ ,  $N_J \supseteq L(x)$  and  $J \subseteq B(x, 1/n_x)$ . As a consequence, if  $(J, x)$  is a  $\gamma$ -fine pixel,  $f(J) \subseteq B_{\mathbb{C}}(f(x), e(x)/2) =: B$ . Since  $f$  is continuous, we get  $f(\bar{J}) \subseteq \bar{B}$ . Thus, if  $y, y' \in \bar{J}$ , we obtain

$$|f(y) - f(y')| \leq |f(y) - f(x)| + |f(x) - f(y')| \leq e(x)/2 + e(x)/2 = e(x)$$

which concludes the proof. □

## 2.4 Integrable pixel functions

In all this section, we fix  $E \in \mathcal{E}$  an elementary set in  $X$ .

**Notation 2.49.** If  $A$  is a subset of  $X$ , we will denote  $\mathcal{P}A$  the set of all pixels  $(J, x)$  that are such that  $J \subseteq A$ .

**Definition 2.50.** A function that is defined on pixels will be called a **pixel function** and a function that is defined on nonempty cells will be called a **cell function**. We will identify a cell function  $f$  as the obvious pixel function  $(J, x) \mapsto f(J)$ .

A function that is defined on a subset of  $X$  will be also called a **point function**. Such function  $f$  is naturally identified as the following pixel function  $(J, x) \mapsto f(x)$ .

A **point-dimension function** over  $X \times \mathcal{F}(T)$  is a family  $(f_N)_{N \in \mathcal{F}(T)}$  such that  $f_N : X_N \rightarrow \mathbb{C}$ . We can identify such function as the following pixel function  $(J, x) \mapsto f_{N_J}(x_{N_J})$ .

**Definition 2.51.** For any pixel function  $h : \mathcal{P}E \rightarrow \mathbb{C}$ , and  $R$  a resolution in  $E$ , we define the **Riemann sum** of  $(h, R)$  as:

$$S(h, R) := \sum_{(J, x) \in R} h(J, x).$$

**Remark 2.52.** The Riemann sum  $S(h, R)$  is linear in  $h$ . Note also that if  $R$  and  $R'$  are resolutions in  $E$  such that  $\cup D_R$  and  $\cup D_{R'}$  are disjoint, then  $R \cup R'$  is a resolution in  $E$  and for any  $h : \mathcal{P}E \rightarrow \mathbb{C}$ , we have  $S(h, R \cup R') = S(h, R) + S(h, R')$ .

We are now in position to define the generalized gauge integral:

**Definition 2.53.** Let  $h : \mathcal{P}E \rightarrow \mathbb{C}$  and  $\alpha \in \mathbb{C}$ . We say that  $h$  is **(gauge) integrable to  $\alpha$** , if for any  $\varepsilon > 0$ , there is a gauge  $\gamma$  such that for any  $\gamma$ -resolution  $R$  of  $E$ ,

$$|S(h, R) - \alpha| < \varepsilon.$$

A pixel function  $h : \mathcal{P}E \rightarrow \mathbb{C}$  is **(gauge) integrable** if there is  $\alpha \in \mathbb{C}$  such that  $h$  is integrable to  $\alpha$  and we denote  $\mathcal{H}(E)$  the set of all pixel functions in  $\mathbb{C}^{\mathcal{P}E}$  which are integrable.

As the following shows, the gauge integral is the limit of a particular net of complex numbers :

**Proposition 2.54.** 1. The set  $GR(E)$  of all pairs  $(\gamma, R)$  where  $\gamma$  is a gauge and  $R$  a  $\gamma$ -resolution of  $E$  is a (downward) directed set with the following preorder :  $(\gamma, R) \leq (\gamma', R')$  if and only if  $\gamma \leq \gamma'$ .

2. A function  $h : \mathcal{P}E \rightarrow \mathbb{C}$  is integrable to  $\alpha \in \mathbb{C}$  if and only if the net  $(S(h, R_\beta))_{\beta \in GR(E)}$  converges to  $\alpha$ , where  $\beta =: (\gamma_\beta, R_\beta)$ .

3. If  $h$  is integrable, then there is a unique  $\alpha \in \mathbb{C}$  such that  $h$  is integrable to  $\alpha$ .

*Proof.* 1. The relation  $\leq$  on  $GR(E)$  is clearly a preorder. Let  $(\gamma, R)$  and  $(\gamma', R')$  be two elements of  $GR(E)$ . By Theorem 2.42, there exists a  $\gamma \wedge \gamma'$ -resolution  $R''$ . Then it is clear that  $(\gamma \wedge \gamma', R'') \leq (\gamma, R)$  and  $(\gamma \wedge \gamma', R'') \leq (\gamma', R')$ , which proves that  $(GR(E), \leq)$  is a downward directed set.

2. Suppose  $h : \mathcal{P}E \rightarrow \mathbb{C}$  is integrable to  $\alpha$ . We want to prove that  $(S(h, R_\beta))_\beta$  converges to  $\alpha$ . Let  $\varepsilon > 0$ . There is a gauge  $\gamma_0$  such that for all  $\gamma_0$ -resolution  $R$  of  $E$ ,  $|S(h, R) - \alpha| < \varepsilon$ . Thus, for any gauge  $\gamma \leq \gamma_0$ , and any  $\gamma$ -resolution  $R$  of  $E$ ,  $|S(h, R) - \alpha| < \varepsilon$ . Fixing a  $\gamma_0$ -resolution  $R_0$  of  $E$ , we see that for all  $\beta \leq \beta_0 := (\gamma_0, R_0)$ , we have  $|S(h, R_\beta) - \alpha| < \varepsilon$ , which shows that  $(S(h, R_\beta))_\beta$  converges to  $\alpha$ . Inversely, suppose that  $(S(h, R_\beta))_\beta$  converges to  $\alpha$  and let  $\varepsilon > 0$ . There is  $\beta_0 \in GR(E)$  such that for all  $\beta \leq \beta_0$ ,  $|S(h, R_\beta) - \alpha| < \varepsilon$ . In particular, for all  $\gamma_{\beta_0}$ -resolution  $R$ , we have  $(\gamma_{\beta_0}, R) \leq \beta_0$ , and thus  $|S(h, R) - \alpha| < \varepsilon$ .

3. Since the scalar  $\alpha$  is the limit of a net of  $\mathbb{C}$ , and  $\mathbb{C}$  is Hausdorff, the result follows immediately.  $\square$

**Definition 2.55.** Let  $h : \mathcal{P}E \rightarrow \mathbb{C}$  an integrable pixel function.

1. We denote  $\int h$  the unique scalar  $\alpha \in \mathbb{C}$  such that  $h$  is integrable to  $\alpha$  and we call it the **integral** of  $h$ .
2. Given  $\varepsilon > 0$ , a gauge  $\gamma$  such that for all  $\gamma$ -resolution  $R$ , we have  $|S(h, R) - \int h| \leq \varepsilon$  is called a  **$(h, \varepsilon)$ -gauge**.

We will also use the following notion :

**Definition 2.56.** A sequence of integrable pixel functions  $(h_n)_{n \in \mathbb{N}}$  in  $\mathcal{H}(E)$  is **uniformly integrable** if for any  $\varepsilon > 0$  there is a gauge  $\gamma$  such that for any  $n \in \mathbb{N}$  and any  $\gamma$ -resolution  $R$  of  $E$ ,

$$|S(h_n, R) - \int h_n| \leq \varepsilon.$$

Being the limit of a net of scalars, the integral has automatically a few basic properties that we mention here :

**Proposition 2.57.** 1. (**Linearity**) The set  $\mathcal{H}(E)$  is a complex vector subspace of  $\mathbb{C}^{\mathcal{P}E}$  and the map  $h \mapsto \int h$  is a linear form on  $\mathcal{H}(E)$ .

2. (**Adherence**) If  $h \in \mathcal{H}(E)$ , then  $\int h \in \overline{S(h)}$ , where  $S(h)$  is the set of all  $S(h, R)$  where  $R$  is a resolution of  $E$ . In particular, if  $h$  is real valued, then  $\int h \in \mathbb{R}$ , and if  $h$  is positive, then  $\int h \in \mathbb{R}_+$ .

3. (**Order preservation**) If  $h, g \in \mathcal{H}(E)$  are real valued and such that  $h \leq g$ , then  $\int h \leq \int g$ .

4. (**Module estimation**) If  $h, |h| \in \mathcal{H}(E)$ , then  $|\int h| \leq \int |h|$ .

5. (**Cauchy criterion**) Let  $h : \mathcal{P}E \rightarrow \mathbb{C}$ . Then  $h \in \mathcal{H}(E)$  if and only if for any  $\varepsilon > 0$ , there exists a gauge  $\gamma$  such that for any  $\gamma$ -resolutions  $R_1, R_2$  of  $E$ ,

$$|S(h, R_1) - S(h, R_2)| \leq \varepsilon.$$

Such gauge will be called a  **$(h, \varepsilon)$ -Cauchy gauge**.

6. (**Uniformly integrable convergence**) Let  $(h_n)$  be a uniformly integrable sequence of pixel functions in  $\mathcal{H}(E)$  that converges pointwisely to  $h : \mathcal{P}E \rightarrow \mathbb{C}$ . Then  $h \in \mathcal{H}(E)$  and

$$\int h = \lim_{n \rightarrow \infty} \int h_n.$$

*Proof.* 1. Since  $(S(h, R_\beta))_{\beta \in GR(E)}$  is linear in  $h$  and the limit of the sum of two nets in  $\mathbb{C}$  is the sum of the limit, we obtain directly the result.

2. Let  $h \in \mathcal{H}(E)$  and  $\varepsilon > 0$ . Thus, there is a resolution  $R$  of  $E$  such that  $|S(h, R) - \int_E h| < \varepsilon$ .
3. Direct consequence of 1. and 2.
4. We have  $|\int h| = |\lim_\beta S(h, R_\beta)| = \lim_\beta |S(h, R_\beta)|$  by continuity of the module. Since  $|S(h, R_\beta)| \leq S(|h|, R_\beta)$ , we then get  $\lim_\beta |S(h, R_\beta)| \leq \lim_\beta S(|h|, R_\beta) = \int |h|$ .
5. Only the "if" part is non-trivial. Suppose that  $h$  satisfies the Cauchy criterion, that is to say for all  $\varepsilon > 0$ , there is a  $(h, \varepsilon)$ -Cauchy gauge. We want to prove that  $h$  is integrable. In other words, we want to prove that  $\lim_\beta S(h, R_\beta)$  exists. Since  $\mathbb{C}$  is complete, it is enough to prove that  $(S(h, R_\beta))_\beta$  is a Cauchy net in  $\mathbb{C}$ . Let  $\varepsilon > 0$  be fixed, and let  $\gamma_0$  be a  $(h, \varepsilon)$ -Cauchy gauge. Fix  $R_0$  a  $\gamma_0$ -resolution, and define  $\beta_0 := (\gamma_0, R_0)$ . Then for all  $\beta$  and  $\beta'$  in  $GR(E)$  such that  $\beta \leq \beta_0$  and  $\beta' \leq \beta_0$ , we have  $\gamma_\beta \leq \gamma_0$  and  $\gamma_{\beta'} \leq \gamma_0$  and thus both  $R_\beta$  and  $R_{\beta'}$  are  $\gamma_0$ -resolutions of  $E$ , which implies that  $|S(h, R_\beta) - S(h, R_{\beta'})| \leq \varepsilon$ . This is what we wanted to prove.
6. If  $h_n \rightarrow h$  pointwisely, then for any resolution  $R$  of  $E$ ,  $S(h_n, R) \rightarrow S(h, R)$  in  $\mathbb{C}$ . On the other hand,  $S(h_n, R_\beta) \rightarrow_\beta \int h_n$ . We are thus dealing with a double net  $(N_{n,\beta}) := (S(h_n, R_\beta))_{n,\beta}$ , such that for any  $n \in \mathbb{N}$ ,  $(N_{n,\beta})_\beta$  converges, and for any  $\beta \in GR(E)$ ,  $(N_{n,\beta})_n$  converges. We are looking for a condition for which we can exchange the limits (for  $n \in \mathbb{N}$  and for  $\beta \in GR(E)$ ). By the Moore–Smith double limit theorem (see Theorem 5.6 in Appendix), a sufficient condition for this is the uniform convergence with respect to one of the two parameters  $n, \beta$ . For instance, if for any  $\varepsilon > 0$ , there is a  $\beta_0 \in GR(E)$ , such that for all  $n \in \mathbb{N}$ , and all  $\beta \leq \beta_0$ ,  $|N_{n,\beta} - \int h_n| < \varepsilon$ , then the following double limits :  $\lim_n \lim_\beta N_{n,\beta}$ ,  $\lim_\beta \lim_n N_{n,\beta}$  exist and are equal, which is the expected conclusion. But this condition of convergence of  $(N_{n,\beta})_\beta$  uniformly in  $n$  is nothing but a reformulation of the uniform integrability condition. □

The Cauchy criterion can be slightly improved :

**Proposition 2.58. (Nested Cauchy criterion)** *Let  $h : \mathcal{P}E \rightarrow \mathbb{C}$ . Then  $h \in \mathcal{H}(E)$  if and only if for any  $\varepsilon > 0$ , there exists a gauge  $\gamma$  such that for any  $\gamma$ -resolutions  $R_1, R_2$  of  $E$  such that  $D_{R_1} \leq D_{R_2}$ ,  $|S(h, R_1) - S(h, R_2)| \leq \varepsilon$ .*

*Proof.* By the Cauchy criterion, only the "if" part is non-trivial. Suppose then that for any  $\varepsilon > 0$ , there exists a gauge  $\gamma$  such that for any  $\gamma$ -resolutions  $R_1, R_2$  of  $E$  such that  $D_{R_1} \leq D_{R_2}$ ,  $|S(h, R_1) - S(h, R_2)| \leq \varepsilon$ . Fix  $\varepsilon > 0$ . By hypothesis, there is a gauge  $\gamma$  such that for any  $\gamma$ -resolutions  $R_1, R_2$  of  $E$  such that  $D_{R_1} \leq D_{R_2}$ ,  $|S(h, R_1) - S(h, R_2)| < \varepsilon/2$ . Take  $R_1, R_2$  two  $\gamma$ -resolution of  $E$ .

Fix now  $\gamma_1$  a  $D_{R_1}$ -gauge, and  $\gamma_2$  a  $D_{R_2}$ -gauge (see Definition 2.39). Consider now  $R_3$  a  $\gamma \wedge \gamma_1 \wedge \gamma_2$ -resolution of  $E$ . Thus,  $D_{R_3} \leq D_{R_1}$  and  $D_{R_3} \leq D_{R_2}$ , and therefore

$$|S(h, R_1) - S(h, R_2)| \leq |S(h, R_1) - S(h, R_3)| + |S(h, R_3) - S(h, R_2)| \leq \varepsilon.$$

The Cauchy criterion now implies that  $h$  is integrable. □

## 2.5 Additivity and Saks-Henstock lemma

We continue to fix an elementary set  $E$ .

**Definition 2.59.** If  $h : \mathcal{P}E \rightarrow \mathbb{C}$  is a pixel function and  $E'$  is an elementary set included in  $E$ , then the **restriction** of  $h$  to  $E'$  is the pixel function  $h_{E'} : \mathcal{P}E' \rightarrow \mathbb{C}$ ,  $(J, x) \mapsto h(J, x)$ .

**Proposition 2.60. (Restrictability of integrability)** *If  $h \in \mathcal{H}(E)$  and  $E'$  is an elementary set included in  $E$ , then  $h_{E'} \in \mathcal{H}(E')$ .*

*Proof.* Let  $h \in \mathcal{H}(E)$  and let  $E'$  be an elementary set included in  $E$ . We will apply the Cauchy criterion (Proposition 2.57, item 5) to the pixel function  $h_{E'}$ .

Let  $\varepsilon > 0$ . Since  $h \in \mathcal{H}(E)$ , there exists a  $(h, \varepsilon)$ -Cauchy gauge  $\gamma$ . Let  $R'_1$  and  $R'_2$  be  $\gamma$ -resolutions of  $E'$ . Take a  $\gamma$ -resolution  $R'_3$  of  $E \setminus E'$ . Thus,  $R_1 := R'_1 \cup R'_3$  and  $R_2 := R'_2 \cup R'_3$  are  $\gamma$ -resolutions of  $E$ . Therefore,

$|S(h, R_1) - S(h, R_2)| \leq \varepsilon$ . But it is clear that  $S(h, R_1) - S(h, R_2) = S(h, R'_1) - S(h, R'_2)$ , and thus we obtain  $|S(h, R'_1) - S(h, R'_2)| \leq \varepsilon$  which shows that  $h_{E'}$  satisfies the Cauchy criterion.  $\square$

**Notation 2.61.** Let  $h \in \mathcal{H}(E)$ .

1. The integral of  $h_{E'}$  where  $E'$  is an elementary set included in  $E$ , will be denoted  $\int_{E'} h$ .
2. The cell function  $\mathcal{P}E \rightarrow \mathbb{C}, J \mapsto \int_J h$  will be denoted  $\int_{\bullet} h$ .

**Theorem 2.62. (Additivity)** Let  $h : \mathcal{P}E \rightarrow \mathbb{C}$  and suppose that  $E = E_1 \cup E_2$  where  $E_1$  and  $E_2$  are disjoint elementary sets. The following are equivalent.

1.  $h \in \mathcal{H}(E)$
2.  $h_{E_1} \in \mathcal{H}(E_1)$  and  $h_{E_2} \in \mathcal{H}(E_2)$ .

In this case,  $\int_E h = \int_{E_1} h + \int_{E_2} h$ .

*Proof.* The first implication follows directly from Proposition 2.60. We prove the second implication.

Let  $h : \mathcal{P}E \rightarrow \mathbb{C}$  be such that  $h_{E_1} \in \mathcal{H}(E_1)$  and  $h_{E_2} \in \mathcal{H}(E_2)$ . We want to prove that  $h$  is integrable to  $\int h_{E_1} + \int h_{E_2}$ . Fix  $\varepsilon > 0$  and let  $\gamma_1$  (resp.  $\gamma_2$ ) be a  $(h_{E_1}, \varepsilon/2)$ -gauge (resp. a  $(h_{E_2}, \varepsilon/2)$ -gauge). Let  $D_1, D_2$  be divisions of respectively  $E_1, E_2$ , and fix  $\gamma'$  a  $D_1 \cup D_2$ -gauge. Define now

$$\gamma := \gamma_1 \wedge \gamma_2 \wedge \gamma'$$

and let  $R$  be a  $\gamma$ -resolution of  $E$ . Thus,  $D_R \leq D_1 \cup D_2$ . As a consequence,  $R$  is the disjoint union of the following two resolutions  $R_1 := \{(J, x) \in R : J \subseteq E_1\}$  and  $R_2 := \{(J, x) \in R : J \subseteq E_2\}$ . This implies  $S(h, R) = S(h, R_1) + S(h, R_2)$ . It is clear also that  $R_1$  is  $\gamma_1$ -resolution of  $E_1$  and  $R_2$  is a  $\gamma_2$ -resolution of  $E_2$ . Thus,

$$|S(h, R) - \int h_{E_1} - \int h_{E_2}| \leq |S(h_{E_1}, R_1) - \int h_{E_1}| + |S(h_{E_2}, R_2) - \int h_{E_2}| \leq \varepsilon. \quad \square$$

$\square$

As a direct consequence we obtain the following :

**Corollary 2.63.** If  $h \in \mathcal{H}(E)$ , and  $E'$  is a an elementary set included in  $E$ , then the pixel function  $\mathbf{1}_{\mathcal{P}E'} h$  is in  $\mathcal{H}(E)$  and

$$\int \mathbf{1}_{\mathcal{P}E'} h = \int_{E'} h.$$

*Proof.* Since  $(\mathbf{1}_{\mathcal{P}E'} h)_{E'} = h_{E'}$ , and  $(\mathbf{1}_{\mathcal{P}E'} h)_{E \setminus E'} = 0$ , and the zero function is integrable to 0, the result follows directly from the previous theorem.  $\square$

**Definition 2.64.** A cell function  $h : \mathcal{P}E \rightarrow \mathbb{C}$  is said **additive** if for any cell  $J$  included in  $E$  and any division  $D$  of  $J$ ,  $h(J) = \sum_{J' \in D} h(J')$ .

The basic fact concerning additive cell function is that they are integrable :

**Proposition 2.65.** Let  $h : \mathcal{P}E \rightarrow \mathbb{C}$  be an additive cell function. Then  $h$  is integrable,  $h = \int_{\bullet} h$  and  $\int_E h = \sum_{J \in D} h(J)$  for any division  $D$  of  $E$ .

*Proof.* Suppose that  $h : \mathcal{P}E \rightarrow \mathbb{C}$  is an additive cell function. Let us check that the quantity  $I(D) = \sum_{J \in D} h(J)$  does not depend on the choice of the division  $D$  of  $E$ . Take  $D$  and  $D'$  two divisions of  $E$ . Since  $D'$  is a division of  $E$ , for each  $J \in D$ ,  $J = \cup \Delta_J$  where  $\Delta_J = \{J' \cap J : J' \cap J \neq \emptyset, J' \in D'\}$ . By additivity of  $h$ , this implies that  $h(J) = \sum_{J' \in \Delta_J} h(J')$ , and therefore  $I(D) = \sum_{J \in D} \sum_{J' \in D' : J' \cap J \neq \emptyset} h(J' \cap J) = I(D \wedge D')$ . The same argument shows that  $I(D') = I(D \wedge D')$ , and thus  $I(D) = I(D')$ .

Denote  $I$  the quantity previous quantity  $I(D)$  (which is independent of  $D$ ). Now we can see that  $h$  is integrable to  $I$ , since, for any  $\varepsilon > 0$ , if we consider any resolution  $R$  of  $E$ , then  $|S(h, R) - I| = |\sum_{J \in D_R} h(J) - I| = 0 < \varepsilon$ . This shows that  $\int_E h = I$ .

To check that  $h = \int_{\bullet} h$ , note that if  $J$  is a cell included in  $E$ , then  $h_J$  is an additive cell function and therefore, by what we just proved,  $\int h_J = h(J)$  since  $\{J\}$  is a division of  $J$ .  $\square$

We can then obtain as consequence of the additivity theorem :

**Theorem 2.66.** *Let  $h \in \mathcal{H}(E)$ . Then :*

1. *The cell function  $\int_{\bullet} h$  is additive and therefore integrable.*
2.  *$\int_{\bullet} \int_{\bullet} h = \int_{\bullet} h$  and  $\int_E \int_{\bullet} h = \int_E h$ .*

*Proof.* 1. By induction, it follows from the additivity of the integral (Theorem 2.62) that  $\int_{\bullet} h$  is additive, and thus integrable by Proposition 2.65.

2. We apply Proposition 2.65 to the additive cell function  $\int_{\bullet} h$ .

□

The following result, usually called the Saks–Henstock lemma in the case of the Henstock–Kurzweil integral, is crucial for the derivation of the convergence properties of the gauge integral.

**Lemma 2.67. (Saks–Henstock lemma)** *Let  $h \in \mathcal{H}(E)$ . Fix  $\varepsilon > 0$ ,  $\gamma$  a  $(h, \varepsilon)$ -gauge and  $E'$  an elementary set included in  $E$ . Then for any  $\gamma$ -resolution  $R$  of  $E'$ ,*

1.  $|S(h, R) - \int_{E'} h| \leq \varepsilon$ .
2.  $|S(h - \int_{\bullet} h, R)| \leq \varepsilon$ .
3.  $S(|h - \int_{\bullet} h|, R) \leq 4\varepsilon$ .

*Proof.* 1. Let  $R$  be a  $\gamma$ -resolution of  $E'$ . Fix  $\eta > 0$ . Since  $h_{E \setminus E'}$  is integrable (by Proposition 2.60), there exists a  $(h_{E \setminus E'}, \eta)$ -gauge  $\gamma'$ , and a  $\gamma \wedge \gamma'$ -resolution  $R'$  of  $E \setminus E'$ . Thus,  $R \cup R'$  is a  $\gamma$ -resolution of  $E$ . By additivity, we get  $S(h, R \cup R') - \int h = S(h_{E'}, R) - \int h_{E'} + S(h_{E \setminus E'}, R') - \int h_{E \setminus E'}$  which implies, since  $\gamma$  is a  $(h, \varepsilon)$ -gauge,

$$|S(h_{E'}, R) - \int h_{E'}| \leq \varepsilon + |S(h_{E \setminus E'}, R') - \int h_{E \setminus E'}| \leq \varepsilon + \eta.$$

Since  $\eta$  was arbitrary, we get  $|S(h_{E'}, R) - \int h_{E'}| \leq \varepsilon$ . We are done.

2. This follows from 1. and the additivity of the integral.

3. Let  $R$  be a  $\gamma$ -resolution of  $E'$ . Denote for any  $J \in D_R$ ,  $z_J := h(J, p(J)) - \int h_J$ , where  $p(J)$  is the point such that  $(J, p(J)) \in R$ . Denote also  $x_J := \Re(z_J)$ ,  $y_J := \Im(z_J)$ . Since  $\sum_J |z_J| \leq \sum_J |x_J| + \sum_J |y_J|$ , it is enough to check that  $\sum_J |x_J| \leq 2\varepsilon$  (resp.  $\sum_J |y_J| \leq 2\varepsilon$ ).

To prove this, we partition  $R$  the following way :  $R = R_+ \cup R_-$  where  $R_+$  (resp.  $R_-$ ) is the subset of  $R$  consisting of all pixels  $(J, p(J))$  such that  $x_J \geq 0$  (resp.  $x_J < 0$ ). It is clear that  $R_+$  (resp.  $R_-$ ) is a  $\gamma$ -resolution of  $\cup D_{R_+}$  (resp.  $\cup D_{R_-}$ ). Thus, by applying item 2 to  $R_+$  and  $R_-$ , we obtain the estimates :  $|\sum_{J \in D_{R_+}} z_J| \leq \varepsilon$  and  $|\sum_{J \in D_{R_-}} z_J| \leq \varepsilon$ . Therefore,  $|\sum_{J \in D_{R_+}} x_J| \leq \varepsilon$  and  $|\sum_{J \in D_{R_-}} x_J| \leq \varepsilon$ . We get

$$\sum_J |x_J| = \sum_{J \in D_{R_+}} x_J - \sum_{J \in D_{R_-}} x_J = \left| \sum_{J \in D_{R_+}} x_J \right| + \left| \sum_{J \in D_{R_-}} x_J \right| \leq \varepsilon + \varepsilon = 2\varepsilon.$$

The same argument can be written for the  $y_J$ .

□

**Remark 2.68.** We can reformulate the previous lemma by saying that if  $E'$  is an elementary set in  $E$ , and  $h \in \mathcal{H}(E)$ , then for any  $(h, \varepsilon)$ -gauge  $\gamma$ ,

1.  $\gamma$  is a  $(h_{E'}, \varepsilon)$ -gauge,
2.  $\gamma$  is a  $(h_{E'} - \int_{\bullet} h_{E'}, \varepsilon)$ -gauge, and  $h_{E'} - \int_{\bullet} h_{E'}$  is integrable to zero,
3.  $\gamma$  is a  $(|h_{E'} - \int_{\bullet} h_{E'}|, 4\varepsilon)$ -gauge and  $|h_{E'} - \int_{\bullet} h_{E'}|$  is integrable to zero.

## 2.6 Variation and integration

The concept of variation is strongly related to the gauge integral. In a way, the lower (resp. upper) variation is to the integral what the limit inferior (resp. superior) is to the concept of limit. We investigate in this section some properties of those concepts.

As before, we fix an elementary set  $E$  in  $X$ .

**Definition 2.69.** Let  $h : \mathcal{P}E \rightarrow \mathbb{C}$  be a pixel function.

1. The **(upper) variation**  $V(h)$  and the **lower variation**  $\underline{V}(h)$  of  $h$  are defined as elements of  $\overline{\mathbb{R}}_+$  by

$$V(h) := \inf_{\gamma} \sup_{R \in \mathcal{R}_E^\gamma} S(|h|, R), \quad \underline{V}(h) := \sup_{\gamma} \inf_{R \in \mathcal{R}_E^\gamma} S(|h|, R).$$

where  $\gamma$  runs over all gauges.

2. The **total variation** of  $h$  is  $TV(h) := \sup_{R \in \mathcal{R}_E} S(|h|, R) \in \overline{\mathbb{R}}_+$ .
3. The function  $h$  is said to have **bounded variation** if  $V(h) < \infty$ , **zero variation** if  $V(h) = 0$ , **bounded total variation** if  $TV(h) < \infty$ , and **regular** if  $V(h) = \underline{V}(h)$ .

Remark that  $TV(h) = \sup_{\gamma} \sup_{R \in \mathcal{R}_E^\gamma} S(|h|, R)$ .

**Notation 2.70.** The set of all bounded variations pixel functions (in  $E$ ) is denoted  $\mathcal{V}(E)$ , the set of all zero variation pixel functions (in  $E$ ) is denoted  $\mathcal{Z}(E)$  and the set of bounded total variation pixel functions (in  $E$ ) is denoted  $\mathcal{T}(E)$ .

We first record some basic properties of variation functionals and their relation to absolute integrability.

**Lemma 2.71.** 1. We have  $0 \leq \underline{V}(h) \leq V(h) \leq TV(h)$  for any pixel function  $h : \mathcal{P}E \rightarrow \mathbb{C}$ .

2.  $\mathcal{V}(E)$  is a vector subspace of  $\mathbb{C}^{\mathcal{P}E}$ , and the sets  $\mathcal{Z}(E)$ ,  $\mathcal{T}(E)$  are vector subspaces of  $\mathcal{V}(E)$ . Moreover,  $V$  is a seminorm on  $\mathcal{V}(E)$ , which is zero on  $\mathcal{Z}(E)$ , and  $TV$  is a norm on  $\mathcal{T}(E)$ .
3. If  $f, g : \mathcal{P}E \rightarrow \mathbb{C}$  satisfy  $|f| \leq |g|$ , then  $V(f) \leq V(g)$  and  $TV(f) \leq TV(g)$ .
4. Let  $h : \mathcal{P}E \rightarrow \mathbb{C}$ . Then  $|h| \in \mathcal{H}(E)$  if and only if  $h$  is regular and has bounded variation. In this case  $\int |h| = V(h)$ .
5. Let  $h : \mathcal{P}E \rightarrow \mathbb{C}$  be an additive cell function. Then  $|h| \in \mathcal{H}(E)$  if and only if  $h$  has bounded variation, and if and only if  $h$  has bounded total variation. In this case,  $\int |h| = V(h) = TV(h)$ .

*Proof.* 1. The inequality  $V(h) \leq TV(h)$  is clear. Let us check that  $\underline{V}(h) \leq V(h)$ . Let  $\gamma$  and  $\delta$  be two gauges. We have

$$\inf_{R \in \mathcal{R}_E^\gamma} S(|h|, R) \leq \inf_{R \in \mathcal{R}_E^{\gamma \wedge \delta}} S(|h|, R) \leq \sup_{R \in \mathcal{R}_E^{\gamma \wedge \delta}} S(|h|, R) \leq \sup_{R \in \mathcal{R}_E^\delta} S(|h|, R).$$

Thus, for any gauges  $\gamma, \delta$ ,  $\inf_{R \in \mathcal{R}_E^\gamma} S(|h|, R) \leq \sup_{R \in \mathcal{R}_E^\delta} S(|h|, R)$ , which implies  $\underline{V}(h) \leq V(h)$ .

2. Suppose that  $f, g \in \mathcal{V}(E)$ . Let us check that  $V(f + g) \leq V(f) + V(g)$ . Fix  $\varepsilon > 0$ . There exist gauges  $\gamma_1$  and  $\gamma_2$  such that  $a_1 := \sup_{R \in \mathcal{R}_E^{\gamma_1}} S(|f|, R) \leq V(f) + \varepsilon/2$  and  $a_2 := \sup_{R \in \mathcal{R}_E^{\gamma_2}} S(|g|, R) \leq V(g) + \varepsilon/2$ . Therefore, we have

$$V(f + g) \leq \sup_{R \in \mathcal{R}_E^{\gamma_1 \wedge \gamma_2}} S(|f + g|, R) \leq a_1 + a_2 \leq V(f) + V(g) + \varepsilon.$$

Since  $\varepsilon > 0$  was arbitrary, we get  $V(f + g) \leq V(f) + V(g)$ , which implies  $f + g \in \mathcal{V}(E)$ . Clearly, if  $\lambda \in \mathbb{C}$ ,  $V(\lambda f) = |\lambda|V(f)$ , which implies  $\lambda f \in \mathcal{V}(E)$ . This shows that  $\mathcal{V}(E)$  is a vector space with seminorm  $V$ , and that it contains  $\mathcal{Z}(E)$  a subspace.

Clearly,  $TV$  is a seminorm on  $\mathcal{T}(E)$ , and thus  $\mathcal{T}(E)$  is a vector subspace of  $\mathcal{V}(E)$ . Let us check that it is a norm. Let  $h \in \mathcal{T}(E)$  and suppose that  $TV(h) = 0$ . Let  $(J, x) \in \mathcal{P}E$ . Clearly, there exists a resolution  $R$  of  $E$  such that  $(J, x) \in R$ . We have  $S(|h|, R) = 0$ , and in particular,  $|h(J, x)| = 0$ .

3. Suppose that  $f, g$  satisfy  $|f| \leq |g|$ . The inequality  $TV(f) \leq TV(g)$  is clear. Let us check that  $V(f) \leq V(g)$ . If  $V(g) = +\infty$ , we are done. Suppose then that  $g \in \mathcal{V}(E)$ . Fix  $\varepsilon > 0$ . There is a gauge  $\gamma$  such that  $\sup_{R \in \mathcal{R}_E^\gamma} S(|g|, R) \leq V(g) + \varepsilon$ . Thus, we have  $V(f) \leq \sup_{R \in \mathcal{R}_E^\gamma} S(|f|, R) \leq \sup_{R \in \mathcal{R}_E^\gamma} S(|g|, R) \leq V(g) + \varepsilon$ . Since  $\varepsilon > 0$  was arbitrary, we get the desired inequality.
4. Suppose first that  $|h| \in \mathcal{H}(E)$ . We prove that  $V(h) = \int |h|$  and  $\underline{V}(h) = \int |h|$ . Let  $\gamma$  be a gauge,  $\varepsilon > 0$  and fix a  $(|h|, \varepsilon)$ -gauge  $\gamma_\varepsilon$ . Take  $R$  a  $\gamma_\varepsilon \wedge \gamma$ -resolution of  $E$ . We have  $|\int |h| - S(|h|, R)| < \varepsilon$  which implies that

$$\inf_{R \in \mathcal{R}_E^{\gamma_\varepsilon}} S(|h|, R) - \varepsilon \leq \int |h| \leq \sup_{R \in \mathcal{R}_E^\gamma} S(|h|, R) + \varepsilon.$$

Since  $\varepsilon > 0$  was arbitrary, we just proved that for any gauge  $\gamma$ ,

$$\inf_{R \in \mathcal{R}_E^\gamma} S(|h|, R) \leq \int |h| \leq \sup_{R \in \mathcal{R}_E^\gamma} S(|h|, R),$$

and therefore  $\underline{V}(h) \leq \int |h| \leq V(h)$ . Note also that

$$\sup_{R \in \mathcal{R}_E^{\gamma_\varepsilon}} S(|h|, R) - \varepsilon \leq \int |h| \leq \inf_{R \in \mathcal{R}_E^{\gamma_\varepsilon}} S(|h|, R) + \varepsilon,$$

which implies that  $V(h) - \varepsilon \leq \int |h| \leq \underline{V}(h) + \varepsilon$ , and since  $\varepsilon$  is arbitrary, this gives the equalities  $V(h) = \int |h|$  and  $\underline{V}(h) = \int |h|$ .

Suppose now that  $h$  is regular and has bounded variation. Denote  $\alpha$  the value  $V(h) = \underline{V}(h)$ . We claim that  $|h|$  is integrable to  $\alpha$ . Let  $\varepsilon > 0$ . There exist gauges  $\gamma_1$  and  $\gamma_2$  such that

$$\sup_{R \in \mathcal{R}_E^{\gamma_1}} S(|h|, R) - \varepsilon \leq \alpha \leq \inf_{R \in \mathcal{R}_E^{\gamma_2}} S(|h|, R) + \varepsilon.$$

This implies that for any  $\gamma_1 \wedge \gamma_2$ -resolution  $R$  of  $E$ ,  $|S(|h|, R) - \alpha| \leq \varepsilon$ . We are done.

5. Suppose first that  $h$  has bounded variation. Thus, there exists a gauge  $\gamma_0$  such that

$$s_0 := \sup_{R \in \mathcal{R}_E^{\gamma_0}} S(|h|, R) < \infty.$$

We claim that  $|h| \in \mathcal{H}(E)$  and  $\int |h| = V(h)$ . Let  $\varepsilon > 0$ . There exists a  $\gamma_0$ -resolution  $R_0$  of  $E$  such that  $s_0 - S(|h|, R_0) \leq \varepsilon$ . Let  $\gamma_1$  be a  $D_{R_0}$ -gauge, and let  $R$  be a  $\gamma_0 \wedge \gamma_1$ -resolution of  $E$ . Thus,  $R \leq \gamma_0$ , and  $D_R \leq D_{R_0}$ . By additivity of  $h$ , this implies  $S(|h|, R_0) \leq S(|h|, R)$ . We thus obtain  $0 \leq s_0 - S(|h|, R) \leq \varepsilon$ , which proves that  $|h|$  is integrable to  $s_0$ . Now, 4. shows that  $\int |h| = V(h)$ .

Suppose now that  $|h|$  is integrable. We claim that  $h$  has bounded total variation and  $\int |h| = TV(h)$ . Let  $R \in \mathcal{R}_E$ ,  $\varepsilon > 0$  and  $\gamma$  be a  $(|h|, \varepsilon)$ -gauge, and let  $\gamma_1$  be a  $D_R$ -gauge. Take  $R'$  a  $\gamma \wedge \gamma_1$ -resolution of  $E$ . We have then  $D_{R'} \leq D_R$  and  $S(|h|, R) \leq S(|h|, R')$ . As a consequence, since  $R' \leq \gamma$ ,  $S(|h|, R) \leq \int |h| + \varepsilon$ , which shows that  $TV(h) \leq \int |h| + \varepsilon$ . Since  $\varepsilon > 0$  is arbitrary here, this proves that  $h$  has bounded total variation and  $TV(h) \leq \int |h|$ .

It remains to check the other inequality  $\int |h| \leq TV(h)$  which follows from 4., since  $V(h) \leq TV(h)$ .  $\square$

**Definition 2.72.** Two pixel functions  $f, g$  are **variationally equivalent**, and we denote it  $f \sim g$ , if  $f - g \in \mathcal{Z}(E)$  (i.e.  $V(f - g) = 0$ ).

**Remark 2.73.** The relation of variational equivalence is an equivalence relation since  $\mathcal{Z}(E)$  is a vector space.

**Definition 2.74.** A pixel function  $f : \mathcal{P}E \rightarrow \mathbb{C}$  is said to be **absolutely integrable** if  $f \in \mathcal{H}(E)$  and  $|f| \in \mathcal{H}(E)$ . We denote  $\mathcal{A}(E)$  the set of all absolutely integrable pixel functions (from  $\mathcal{P}E$  to  $\mathbb{C}$ ).

The following proposition list a few basic properties that are consequences of the previous results about variation and integrability and the Saks–Henstock lemma (Lemma 2.67). We note in particular (item 6 and 8) the remarkable non-trivial facts that  $\mathcal{A}(E)$  is a vector space, and that the real valued functions in  $\mathcal{A}(E)$  are stable by the operations  $\wedge$  and  $\vee$  (maximum and minimum) of the Riesz space  $\mathbb{R}^{\mathcal{P}E}$ .

**Proposition 2.75.** 1. If  $h \in \mathcal{Z}(E)$ , then  $h, |h| \in \mathcal{H}(E)$  and  $\int h = \int |h| = 0$ .

2. If  $f \sim g$ , then  $V(f) = V(g)$  and  $|f| \sim |g|$ . In particular, if  $f \in \mathcal{V}(E)$  and  $f \sim g$ , then  $g \in \mathcal{V}(E)$ . If  $f \in \mathcal{H}(E)$  and  $f \sim g$ , then  $g \in \mathcal{H}(E)$  and  $\int f = \int g$ . If  $|f| \in \mathcal{H}(E)$  and  $f \sim g$ , then  $|g| \in \mathcal{H}(E)$  and  $\int |f| = \int |g|$ .

3. For any  $h \in \mathcal{H}(E)$ , the function  $h - \int_{\bullet} h$  has zero variation. In other words,  $h \sim \int_{\bullet} h$ .

4. If  $f \sim g$  and  $h$  is bounded, then  $fh \sim gh$ .

5. Let  $h \in \mathcal{H}(E)$ . Then  $|h| \in \mathcal{H}(E)$  if and only if  $\int_{\bullet} h \in \mathcal{T}(E)$  and if and only if  $h \in \mathcal{V}(E)$ . In this case  $\int |h| = V(h) = TV(\int_{\bullet} h)$ .

6.  $\mathcal{A}(E) = \mathcal{H}(E) \cap \mathcal{V}(E)$ . In particular  $\mathcal{A}(E)$  is a vector subspace of  $\mathcal{H}(E)$  and of  $\mathcal{V}(E)$ .

7. The map  $f \mapsto \int |f|$  is a seminorm on  $\mathcal{A}(E)$ . Moreover, if  $|f| \leq |g|$ ,  $f \in \mathcal{H}(E)$  and  $g \in \mathcal{A}(E)$ , then  $f \in \mathcal{A}(E)$ .

8. The subspace  $\mathcal{A}_{\mathbb{R}}(E)$  of real-valued  $\mathcal{A}(E)$  function is a Riesz subspace of  $\mathbb{R}^{\mathcal{P}E}$ . That is to say : for any  $f, g \in \mathcal{A}_{\mathbb{R}}(E)$ ,  $f \wedge g$  and  $f \vee g$  are in  $\mathcal{A}_{\mathbb{R}}(E)$ .

*Proof.* 1. Let  $h \in \mathcal{Z}(E)$ . By Lemma 2.71 1. and 4.,  $|h| \in \mathcal{H}(E)$  and  $\int |h| = 0$ . We now check that  $h$  is integrable to 0. Let  $\varepsilon > 0$  and  $\gamma$  a  $(|h|, \varepsilon)$ -gauge. For any  $\gamma$ -resolution  $R$  of  $E$ , we have  $|S(h, R)| \leq S(|h|, R) \leq \varepsilon$ .

2. If  $f \sim g$  then  $V(f) \leq V(g) + V(f - g) = V(g)$ . Similarly,  $V(g) \leq V(f)$ , so that  $V(f) = V(g)$ . Moreover, from  $||f| - |g|| \leq |f - g|$ , we see that  $|f| \sim |g|$ .

If  $f \in \mathcal{H}(E)$  and  $f \sim g$ , then by 1.,  $g \in \mathcal{H}(E)$  and  $\int f = \int g$ . If  $|f| \in \mathcal{H}(E)$  and  $f \sim g$ , then we have  $|f| \sim |g|$  and therefore  $|g| \in \mathcal{H}(E)$  and  $\int |g| = \int |f|$ .

3. This is direct consequence of the Saks-Henstock Lemma 2.67 3. applied to  $E' = E$ .

4. If  $f$  is bounded and  $g \in \mathcal{Z}(E)$ , then for any resolution  $R$  of  $E$ ,  $S(|fg|, R) \leq MS(|g|, R)$ , where  $M$  is a constant in  $\mathbb{R}_+$  such that  $|f| \leq M$ . It follows then that  $V(fg) \leq MV(g)$  and therefore  $fg \in \mathcal{Z}(E)$ .

5. By Theorem 2.66,  $\int_{\bullet} h$  is an additive cell function. Thus, Lemma 2.71 5 and items 2 and 3 imply that  $|h| \in \mathcal{H}(E)$  if and only if  $|h| \in \mathcal{V}(E)$  and if and only if  $|\int_{\bullet} h| \in \mathcal{T}(E)$ , and when it is the case, we have  $\int |\int_{\bullet} h| = V(\int_{\bullet} h) = TV(\int_{\bullet} h)$ , which imply the desired equalities with items 2 and 3.

6. This follows straightforwardly from item 5.

7. This follows from the fact that  $V$  is seminorm and by item 5,  $\int |h|$  coincide with  $V(h)$  for all  $h \in \mathcal{A}(E)$ . If  $|f| \leq |g|$  and  $g \in \mathcal{A}(E)$ , and  $f \in \mathcal{H}(E)$ , then  $V(f) \leq V(g)$ , so  $f \in \mathcal{A}(E)$  by item 6.

8. It is obvious that  $\mathcal{A}_{\mathbb{R}}(E)$  is stable by the absolute value operation. Thus, this result follows item 6 and the equalities :  $f \wedge g = \frac{1}{2}(f + g - |f - g|)$  and  $f \vee g = \frac{1}{2}(f + g + |f - g|)$ . □

**Remark 2.76.** The existence of  $D$ -gauges (where  $D$  is a division) (see Corollary 2.40) was used in a crucial way in 2.71 5., and this property is the main result behind the proof of Proposition 2.75 5,6,7 and 8.

## 2.7 Continuity and integrability

As an application of previous results, we investigate integrability of a product  $fh$  where  $f$  is a continuous point function on a closed elementary set  $E$  and  $h$  is positive and integrable on  $E$ .

**Lemma 2.77.** *Let  $E$  be a closed elementary set,  $h : \mathcal{P}E \rightarrow \mathbb{R}_+$  a positive additive cell function, and  $f : E \rightarrow \mathbb{C}$  a point function. Then for any resolutions  $R, R'$  of  $E$  such that  $D_{R'} \leq D_R$ ,*

$$|S(fh, R) - S(fh, R')| \leq \sum_{J \in D_R} \text{osc}(f, \bar{J})h(J).$$

*Proof.* Suppose that  $R$  and  $R'$  are such that  $D_{R'} \leq D_R$ . We denote  $x_J$  the point of the pixel of cell  $J$  in  $R$ , and  $x'_{J'}$  the point of the pixel of cell  $J'$  in  $R'$ . Thus, by Proposition 2.3, for any  $J \in D_R$ , the set  $D_J := \{J' \in D_{R'} : J' \subseteq J\}$  is a division of  $J$ . By additivity of  $h$ , this implies that  $h(J) = \sum_{J' \in D_J} h(J')$ , and  $S(fh, R) = \sum_{J \in D_R} f(x_J)h(J) = \sum_{J \in D_R} \sum_{J' \in D_J} f(x_J)h(J')$ . Since  $S(fh, R') = \sum_{J' \in D_{R'}} f(x'_{J'})h(J') = \sum_{J \in D_R} \sum_{J' \in D_J} f(x'_{J'})h(J')$ , we obtain

$$|S(fh, R) - S(fh, R')| \leq \sum_{J \in D_R} \sum_{J' \in D_J} |f(x_J) - f(x'_{J'})|h(J') \leq \sum_{J \in D_R} \text{osc}(f, \bar{J})h(J).$$

□

**Theorem 2.78.** *Suppose here that  $T$  is countable. Let  $E$  be a closed elementary set,  $h : \mathcal{P}E \rightarrow \mathbb{R}_+$  a positive integrable pixel function, and  $f : E \rightarrow \mathbb{C}$  a continuous function. Then  $fh \in \mathcal{H}(E)$ .*

*Proof.* We can suppose that  $h$  is an additive cell function. Indeed, if  $h$  is a positive integrable pixel function, then  $\int_{\bullet} h$  is a positive additive cell function by Theorem 2.66, item 1. Moreover, by Proposition 2.75 items 3 and 4, and the fact that  $f$  is bounded (since it is continuous on the compact  $E$ ),  $fh \sim f \int_{\bullet} h$ , which implies that  $fh$  is integrable if and only if  $f \int_{\bullet} h$  is.

Suppose then that  $h$  is a positive additive cell function. We can suppose that  $I := \int_E h$  is nonzero, because if it is, then  $h = 0$ . We apply the nested Cauchy criterion (Proposition 2.58) to prove integrability of  $fh$ .

Let  $\varepsilon > 0$ . By Tietze extension theorem we can extend  $f$  to a continuous function  $\tilde{f}$  defined on the whole space  $X$ . By Theorem 2.48 applied to the function  $\tilde{f}$  and the constant function  $e : x \mapsto \varepsilon/I$ , there is a gauge  $\gamma$  such that for any  $\gamma$ -resolution  $R$  of  $E$ , and any  $J \in D_R$ ,  $\text{osc}(\tilde{f}, \bar{J}) \leq \varepsilon/I$ . If  $R$  and  $R'$  are  $\gamma$ -resolutions of  $E$  such that  $D_{R'} \leq D_R$ , we know by Lemma 2.77 that  $|S(fh, R) - S(fh, R')| \leq \sum_{J \in D_R} \text{osc}(f, \bar{J})h(J)$ . Thus

$$|S(fh, R) - S(fh, R')| \leq (\varepsilon/I) \sum_{J \in D_R} h(J) = \varepsilon.$$

This concludes the proof. □

## 2.8 A Hake-type theorem

We conclude this chapter with an application of the Saks–Henstock lemma, which shows that in some sense, there are no "improper" gauge integrals, a result usually called "Hake's theorem" in the case of the one-variable Henstock–Kurzweil integral :

**Theorem 2.79.** *Let  $(J_n)$  be a sequence of cells and let  $h : \mathcal{P}X \rightarrow \mathbb{C}$  be a pixel function that is integrable on each  $J_n$ . Suppose the following conditions :*

1. For any  $n$ ,  $\bar{J}_n \subseteq J_{n+1}^\circ$ .
2. We have  $h(p) = 0$  for any pixel  $p$  whose point is not in  $\cup_n J_n$ .
3. There exists  $L \in \mathbb{C}$ , such that for any  $\varepsilon > 0$ , there is  $n_0 \in \mathbb{N}$  such that for any elementary set  $E$  included in one of the  $J_n$ , and containing  $J_{n_0}$ , we have  $|\int_E h - L| < \varepsilon$ .

Then  $h$  is integrable (on  $X$ ) and  $\int h = L$ .

*Proof.* We adapt to our situation the strategy of the proof of Hake's theorem in one real variable [11, p.37]. Let  $\varepsilon > 0$  and take  $n_0 \in \mathbb{N}$  such that  $|\int_E h - L| < \varepsilon/2$  for any elementary set  $E$  containing  $J_{n_0}$  and included in one of the  $J_n$ .

We define the following sequence of elementary sets  $E_n := J_n \setminus J_{n-1}$  (if  $n \geq 1$ ) and  $E_0 := J_0$ . The elementary sets  $(E_n)$  form a partition of  $S := \cup_n J_n$ . As a consequence, for any  $x \in S$ , there is a unique natural number  $n_x$  such that  $x \in E_{n_x}$ . When  $x \in X \setminus S$ , we set  $n_x := n_0$ .

For any  $n \in \mathbb{N}$ , pick a division  $D_n$  of  $E_n$  and a division  $D'_n$  of  $X \setminus (E_0 \cup \dots \cup E_n) = X \setminus J_n$ . For each  $n \in \mathbb{N}$ , we define the division  $\Delta_n := D_0 \cup \dots \cup D_n \cup D'_n$  of  $X$ . Let us denote  $\gamma_c$  the gauge associated to the map with countable image  $: X \rightarrow \mathcal{D}_X, x \mapsto \Delta_{n_x+1}$ .

Denote also  $F_n := J_{n+1} \setminus J_{n-1}$  for all  $n \geq 1$  and  $F_0 := J_1$ . Since for any  $n \in \mathbb{N}$ ,  $h$  is integrable on  $F_n$ , pick a  $(h_{F_n}, \varepsilon/2^{n+2})$ -gauge, and denote it  $\gamma_n = (L_n, \delta_n)$ . Define then  $\gamma_h := (L_h, \delta_h)$  where

$$\begin{aligned} L_h : x &\mapsto L_{n_x}(x), \\ (\delta_h)_N : x &\mapsto (\delta_{n_x})_N(x). \end{aligned}$$

Clearly,  $L_h$  and  $(\delta_h)_N$  are functions with countable image, and thus,  $\gamma_h$  is a gauge. We now take a  $\gamma_h \wedge \gamma_c$  resolution  $R$  of  $X$ . The goal is to prove that  $|S(h, R) - L| \leq \varepsilon$ .

We write  $R = R_{S^c} \cup R_S$  where  $R_S$  is the subset of  $R$  made of all pixels whose point is in  $S$ , and  $R_{S^c}$  is  $R \setminus R_S$ , that is to say the subset of all pixels in  $R$  whose point is outside  $S$ .

We first claim that each pixel  $(J, x)$  in  $R_S$  is such that  $J \subseteq F_{n_x}$ . Let  $(J, x) \in R_S$ . Since  $(J, x)$  is  $\gamma_c$ -fine, by Proposition 2.38 2., there is  $J' \in \Delta_{n_x+1}$  such that  $J \subseteq J'$ . If such  $J'$  is in  $D'_{n_x+1}$ , we have  $J' \subseteq X \setminus J_{n_x+1}$ . Moreover, since  $x \in S$ ,  $x \in E_{n_x}$ , so  $x \in \bar{J}_{n_x} \subseteq J_{n_x+1}^\circ$ . But  $x \in \bar{J} \subseteq \bar{J}' \subseteq X \setminus (J_{n_x+1})^\circ$ , which gives a contradiction. Therefore, such  $J'$  cannot be in  $D'_{n_x+1}$ . If such  $J'$  is in  $\cup_{k \leq n_x-2} D_k$ , then we have in that case  $J' \subseteq J_{n_x-2}$  and thus  $x \in \bar{J}_{n_x-2} \subseteq J_{n_x-1}^\circ$ . But we also have  $x \notin J_{n_x-1}$ , so there is a contradiction again. As a consequence,  $J'$  must belong in one of the  $D_k$  for  $k \in \{n_x-1, n_x, n_x+1\}$  (with the convention  $D_{-1} = \emptyset$ ). This proves the first claim.

Define  $s := \max\{n_x : (J, x) \in R_S\}$ . Thus, we obtain the decomposition  $R_S = \cup_{n=0}^s R_S^{(n)}$  where  $R_S^{(n)} := \{(J, x) \in R_S : n_x = n\}$ . Let us denote  $E_{S^c} := \cup D_{R_{S^c}}, E_S := \cup D_{R_S}, E_S^{(n)} := \cup D_{R_S^{(n)}}$ .

We now claim that each pixel  $(J, x) \in R_{S^c}$  is such that  $J \subseteq X \setminus J_{n_0}$ . Indeed, if  $(J, x)$  is such a pixel, then  $x \notin S$ , and since  $(J, x)$  is  $\gamma_c$ -fine, there is  $J' \in \Delta_{n_0+1}$  such that  $J \subseteq J'$ . If  $J'$  is in one of the  $D_k$  where  $k \leq n_0 + 1$ , then this implies that  $x \in \bar{J} \subseteq \bar{J}' \subseteq \bar{J}_{n_0+1} \subseteq J_{n_0+2}$ , which is impossible since  $x \notin S$ . Thus, we must have  $J' \in D'_{n_0+1}$ , and therefore  $J \subseteq X \setminus J_{n_0+1} \subseteq X \setminus J_{n_0}$ . which proves the claim.

Both of the claims we proved now imply that  $E_S \subseteq J_{s+1}$  and  $E_{S^c} \subseteq X \setminus J_{n_0}$ . Thus,  $J_{n_0} \subseteq E_S \subseteq J_{s+1}$ . As a consequence,  $|\int_{E_S} h - L| < \varepsilon/2$ , and since  $S(h, R_{S^c}) = 0$  (by hypothesis 2.), we obtain the following estimate :

$$|S(h, R) - L| \leq |S(h, R_S) - \int_{E_S} h| + |\int_{E_S} h - L| < \sum_{n=0}^s |S(h, R_S^{(n)}) - \int_{E_S^{(n)}} h| + \varepsilon/2.$$

Since  $R$  is a  $\gamma_h$ -resolution, each  $R_S^{(n)}$  is a  $\gamma_n$ -resolution, and since  $E_S^{(n)}$  is included in  $F_n$  (by the first claim we proved), we can apply the Saks–Henstock lemma (item 1) to  $\gamma_n$  which is a  $(h_{F_n}, \varepsilon/2^{n+2})$ -gauge for each  $n \in \{0, \dots, s\}$ . This implies for all  $n \in \{0, \dots, s\}$  :  $|S(h, R_S^{(n)}) - \int_{E_S^{(n)}} h| \leq \varepsilon/2^{n+2}$ . As a conclusion, we obtain the estimate :

$$|S(h, R) - L| < \sum_{n=0}^s \varepsilon/2^{n+2} + \varepsilon/2 = \varepsilon$$

which proves that  $h$  is integrable to  $L$ . □

### 3 Gauge integration and measures

As before, we assume in this chapter that  $(X, \mathcal{C})$  is the product division space of a fixed compact metrizable division family  $(X_t)_{t \in T}$ .

### 3.1 Series integration and monotonous convergence

**Definition 3.1.** Let  $h : \mathcal{P}X \rightarrow \mathbb{C}$  be a pixel function. The set of all pixel functions  $f : \mathcal{P}X \rightarrow \mathbb{C}$  such that  $fh \in \mathcal{H}(X)$  will be denoted  $\mathcal{H}(X, h)$ . Those functions are called  $h$ -**integrable**.

**Definition 3.2.** Let  $h$  be positive pixel function. We say that  $h$  is **projectively integrable** if there is a positive pixel function  $g$  such that :

1.  $g \in \mathcal{H}(X, h)$  and  $\int gh > 0$ .
2. There is a gauge  $\gamma$  and a strictly positive function  $m : X \rightarrow \mathbb{R}_+^*$  such that for any  $\gamma$ -fine pixel  $(J, x)$ , we have  $g(J, x) \geq m(x)$ .

**Remark 3.3.** Note that any integrable positive pixel function with nonzero integral is projectively integrable (by taking  $g = m = 1$ ).

**Definition 3.4.** A sequence  $(f_n)$  of pixel functions is said to be **cu-convergent** (or **cell uniformly convergent**) to a pixel function  $f$  if for any map  $e : X \rightarrow \mathbb{R}_+^*$  there exist  $l : X \rightarrow \mathbb{N}$  and a gauge  $\gamma$  such that for any  $x \in X$ , any  $n \geq l(x)$  and any cell  $J$  such that  $(J, x)$  is a  $\gamma$ -fine pixel,

$$|f_n(J, x) - f(J, x)| \leq e(x).$$

A series  $\sum_n f_n$  of pixel functions is **cu-convergent** if the sequence of partial sums  $(\sum_{k=0}^n f_k)$  is cu-convergent.

**Remark 3.5.** If  $\sum_n f_n$  is cu-convergent then for any  $e : X \rightarrow \mathbb{R}_+^*$ , there exist  $l : X \rightarrow \mathbb{N}$  and a gauge  $\gamma$  such that

$$|\sum_{k=n}^{n'} f_k(J, x)| \leq e(x)$$

for all  $x \in X$ , any nonempty cell  $J$  such that  $(J, x)$  is a  $\gamma$ -fine pixel and any  $n' \geq n \geq l(x)$ .

**Remark 3.6.** Note that if all the  $f_n$  and  $f$  are point-functions (i.e. independent of the cell  $J$  of a pixel  $(J, x)$ ), then the cu-convergence of  $(f_n)$  is equivalent to pointwise convergence.

**Remark 3.7.** If  $(f_n)$  is a sequence of point-dimension function, then the sequence  $(f_n)$  cu-converges to a point-dimension function  $f$  if and only if for any map  $e : X \rightarrow \mathbb{R}_+^*$ , there is  $l : X \rightarrow \mathbb{N}$ , and  $L : X \rightarrow \mathcal{F}(T)$  with countable image such that for any  $x \in X$ , and any  $n \geq l(x)$  and any  $N \supseteq L(x)$ ,

$$|(f_n)_N(x_N) - f_N(x_N)| \leq e(x).$$

**Theorem 3.8. (Series integration theorem)** Let  $h$  be a projectively integrable positive pixel function. Let  $(f_n)$  be a sequence of positive pixel functions in  $\mathcal{H}(X, h)$ , such that  $\sum_n f_n$  is cu-convergent to a function  $f$ , and such that  $\sum_n \int f_n h < \infty$ . Then :

1.  $(s_n h)$  is uniformly integrable, where  $s_n := \sum_{k=0}^n f_k$ .
2.  $f \in \mathcal{H}(X, h)$ .
3.  $\int fh = \sum_n \int f_n h$ .

*Proof.* Note that 2. and 3. follow directly from 1., linearity of the integral, and the uniform integrability property (Proposition 2.57, item 6).

To prove 1., we adapt the arguments of [11, p.51]. Let  $\varepsilon > 0$ , and fix for any  $n \in \mathbb{N}$  a  $(s_n h, \varepsilon/2^{n+3})$ -gauge  $\gamma_n$ . Fix also  $n_0 \in \mathbb{N}$  such that  $\sum_{n \geq n_0} \int f_n h < \varepsilon/4$ .

Since  $h$  is projectively integrable, there is a positive pixel function  $g$  such that  $gh$  is integrable with  $\int gh > 0$ , a strictly positive point function  $m : X \rightarrow \mathbb{R}_+^*$ , and a gauge  $\gamma' = (L', \delta')$  such that for any  $\gamma'$ -fine pixel  $(J, x)$ ,  $g(J, x) \geq m(x)$ . Since we can divide  $g$  by  $2 \int gh$ , we can suppose with no loss of generality that  $\int gh = \frac{1}{2}$ . We fix a  $(\frac{1}{2}, gh)$ -gauge  $\gamma_g = (L_g, \delta_g)$ , so that for any  $\gamma_g$ -resolution  $R, S(gh, R) \leq 1$ .

By hypothesis,  $\sum_n f_n$  is cu-convergent. Thus, there is map  $l : X \rightarrow \mathbb{N}$ , and a gauge  $\gamma_c = (L_c, \delta_c)$  such that for any  $x \in X$ , and  $j' \geq j \geq l(x)$  and cell  $J$  such that  $(J, x)$  is a  $\gamma_c$ -fine pixel,

$$\sum_{k=j}^{j'} f_k(J, x) < \varepsilon m(x)/4.$$

We set  $n_x := \max(n_0, l(x))$  for any  $x \in X$ . Define now the following gauge  $\gamma := (L, \delta)$ , where

$$\begin{aligned} L : x &\mapsto L'(x) \wedge L_g(x) \wedge L_c(x) \wedge L_0(x) \wedge \cdots \wedge L_{n_x}(x) \\ \delta_N : x &\mapsto \delta'_N(x) \wedge (\delta_g)_N(x) \wedge (\delta_c)_N(x) \wedge (\delta_0)_N(x) \wedge \cdots \wedge (\delta_{n_x})_N(x). \end{aligned}$$

Remark that  $\gamma \leq \gamma' \wedge \gamma_g \wedge \gamma_c \wedge \gamma_0 \wedge \cdots \wedge \gamma_{n_0}$ . Let  $R$  be a  $\gamma$ -resolution of  $X$  and let  $n \in \mathbb{N}$ . We want to prove that  $|S(s_n h, R) - \int s_n h| < \varepsilon$ . Note first that we can suppose  $n > n_0$ , since if  $n \leq n_0$ , then  $R \leq \gamma \leq \gamma_n$  and thus,  $|S(s_n h, R) - \int s_n h| < \varepsilon/2^{n+3} < \varepsilon$ . Consider then  $R_1$  as the subset of  $R$  consisting of all pixels with point  $x$  such that  $n_x \geq n$ , and  $R_2$  the subset of  $R$  consisting of all pixels with point  $x$  such that  $n_x < n$ , and we denote  $E_1 = \cup D_{R_1}$ ,  $E_2 = \cup D_{R_2}$ . We see then that, by the Saks-Henstock lemma, which can be applied to  $R_1$  since  $R_1$  is a  $\gamma_n$ -resolution here :

$$|S(s_n h, R) - \int s_n h| \leq |S(s_n h, R_1) - \int_{E_1} s_n h| + |S(s_n h, R_2) - \int_{E_2} s_n h| \leq \varepsilon/2^{n+3} + T < \varepsilon/4 + T$$

where  $T := |S(s_n h, R_2) - \int_{E_2} s_n h|$ . We claim that  $T \leq \frac{3}{4}\varepsilon$ . This will give us the desired conclusion. To prove this, note that :

$$\begin{aligned} S(s_n h, R_2) - \int_{E_2} s_n h &= \sum_{(J,x) \in R_2, 0 \leq k \leq n} (f_k h)(J, x) - \int_J f_k h \\ &= \sum_{(J,x) \in R_2} \left( \sum_{0 \leq k \leq n_x} (f_k h)(J, x) - \int_J f_k h \right) + \sum_{n_x+1 \leq k \leq n} (f_k h)(J, x) - \int_J f_k h \end{aligned}$$

which implies that  $T \leq T_1 + T_2 + T_3$  where we set the following :

$$\begin{aligned} T_1 &:= \left| \sum_{(J,x) \in R_2} ((s_{n_x} h)(J, x) - \int_J s_{n_x} h) \right| \\ T_2 &:= \sum_{(J,x) \in R_2} \sum_{n_x+1 \leq k \leq n} (f_k h)(J, x) \\ T_3 &:= \sum_{(J,x) \in R_2} \sum_{n_x+1 \leq k \leq n} \int_J f_k h. \end{aligned}$$

We first estimate  $T_1$ . Define  $s := \max\{n_x : (J, x) \in R_2\}$ . Thus, we obtain the decomposition  $R_2 = \cup_{j=0}^s R_2^{(j)}$  where  $R_2^{(j)} := \{(J, x) \in R_2 : n_x = j\}$ . This gives us the estimate

$$T_1 \leq \sum_{j=0}^s \left| \sum_{(J,x) \in R_2^{(j)}} ((s_j h)(J, x) - \int_J s_j h) \right|.$$

But since  $R_2^{(j)}$  is  $\gamma_j$  fine, for any  $j \in \{0, \dots, s\}$ , Saks-Henstock Lemma implies that

$$\left| \sum_{(J,x) \in R_2^{(j)}} ((s_j h)(J, x) - \int_J s_j h) \right| \leq \varepsilon/2^{j+3}.$$

This gives then  $T_1 \leq \varepsilon \sum_j 2^{-j-3} = \varepsilon/4$ .

Let us now estimate  $T_2$ . Since  $R_2$  is  $\gamma$ -fine, it is in particular  $\gamma' \wedge \gamma_g \wedge \gamma_c$ -fine. Thus we obtain  $T_2 \leq \sum_{(J,x) \in R_2} \frac{\varepsilon}{4} m(x) h(J, x) \leq \sum_{(J,x) \in R_2} \frac{\varepsilon}{4} g h(J, x) = \frac{\varepsilon}{4} S(g h, R_2) \leq \frac{\varepsilon}{4}$ .

We finally estimate  $T_3$ . We see that  $T_3 \leq \sum_{(J,x) \in R_2} \sum_{k=n_0}^n \int_J f_k h = \sum_{k=n_0}^n \int_{E_2} f_k h$ . And this last sum is less than  $\sum_{k=n_0}^{\infty} \int f_k h \leq \frac{\varepsilon}{4}$ . Finally, we obtain  $T \leq \frac{3}{4}\varepsilon$ .  $\square$

As consequence of the series integration theorem, we obtain :

**Theorem 3.9. (Monotonous convergence theorem)** *Let  $h$  be a projectively integrable positive pixel function. Let  $(f_n)$  be an increasing sequence of positive pixel functions in  $\mathcal{H}(X, h)$  that cu-converges to a pixel function  $f$  and such that  $\sup_n \int f_n h < \infty$ . Then  $f \in \mathcal{H}(X, h)$ ,  $(f_n h)$  is uniformly integrable and*

$$\int f h = \lim_{n \rightarrow \infty} \int f_n h.$$

*Proof.* This is a direct application of the previous theorem, by considering the series  $\sum_n g_n$  where  $g_n := f_{n+1} - f_n$ .  $\square$

### 3.2 Measure associated to a positive pixel function

The goal of this section is the construction of a measure on  $X$  from a positive pixel function  $h : \mathcal{P}X \rightarrow \mathbb{R}_+$ .

**Definition 3.10.** Let  $h : \mathcal{P}X \rightarrow \mathbb{R}_+$  be a positive pixel function. A subset  $A$  of  $X$  is said :

1.  **$h$ -integrable** if  $\mathbf{1}_A \in \mathcal{H}(X, h)$ .
2.  **$h$ -measurable** if for any  $h$ -integrable set  $B$ ,  $A \cap B$  is  $h$ -integrable.
3.  **$h$ -null** if  $A$  is  $h$ -integrable and  $\int \mathbf{1}_A h = 0$ .

We denote  $\mathcal{S}_h$  the set of all  $h$ -integrable subsets, and  $\mathcal{M}_h$  the set of all  $h$ -measurable subsets.

**Remark 3.11.** A cell  $J$  is  $h$ -integrable if and only if  $\mathbf{1}_J h$  is integrable, but this is not equivalent to saying that  $h$  is integrable on  $J$ , which is equivalent to :  $\mathbf{1}_{\mathcal{P}J} h$  is integrable. We will see in Theorem 3.23 a sufficient condition on a cell  $J$  that makes it  $h$ -integrable, supposing that  $h$  is integrable on  $J$ .

**Proposition 3.12.** *Let  $h$  be a positive pixel function. Then*

1. *Any  $h$ -integrable subset is  $h$ -measurable.*
2. *If  $A \subseteq A'$  and  $A'$  is  $h$ -integrable and  $A$  is  $h$ -measurable, then  $A$  is  $h$ -integrable.*
3. *A set  $A$  is  $h$ -null if and only if  $V(\mathbf{1}_A h) = 0$ . Moreover, any subset of a  $h$ -null set is  $h$ -null.*
4. *If  $h$  is projectively integrable,  $\mathcal{S}_h$  is a  $\delta$ -ring on  $X$ , and  $\mathcal{M}_h$  is a  $\sigma$ -algebra on  $X$ .*

*Proof.* 1. Suppose that  $A$  is  $h$ -integrable and that  $B$  is a  $h$ -integrable set. We want to prove that  $A \cap B$  is  $h$ -integrable, or in other words that  $\mathbf{1}_{A \cap B} h \in \mathcal{H}(X)$ . But since  $\mathbf{1}_{A \cap B} h = (\mathbf{1}_A h) \wedge (\mathbf{1}_B h)$  the result follows directly from Proposition 2.75, item 8.

2. Since we have in that case  $A = A' \cap A$ , and since  $A'$  is  $h$ -integrable and  $A$  is  $h$ -measurable, it follows from the definition of  $h$ -measurable sets that  $A$  is  $h$ -integrable.

3. Let  $A \subseteq X$ . By 2.71 item 4,  $\mathbf{1}_A h \in \mathcal{H}(X)$  if and only if  $\mathbf{1}_A h$  is regular and has bounded variation, and in that case  $V(\mathbf{1}_A h) = 0$ . Thus, if  $A$  is  $h$ -null, then  $V(\mathbf{1}_A h) = 0$ . Inversely, if  $V(\mathbf{1}_A h) = 0$ , then by Proposition 2.75,  $\mathbf{1}_A h \in \mathcal{H}(X)$ , and  $\int \mathbf{1}_A h = 0$ .

If  $B \subseteq A$ , and  $A$  is  $h$ -null, then  $\mathbf{1}_B h \leq \mathbf{1}_A h$  and 2.71 item 3 implies that  $V(\mathbf{1}_B h) \leq V(\mathbf{1}_A h) = 0$ , which implies that  $V(\mathbf{1}_B h) = 0$ . Thus,  $B$  is also  $h$ -null.

4. Let us first check that  $\mathcal{S}_h$  is a  $\delta$ -ring on  $X$ . Observe first that  $\mathcal{S}_h$  is stable by finite intersection and relative complement and finite union since  $\mathbf{1}_{A \cap B} h = \mathbf{1}_A h \wedge \mathbf{1}_B h$  and  $\mathbf{1}_{A \setminus B} h = \mathbf{1}_A h - \mathbf{1}_{A \cap B} h$  and  $\mathbf{1}_{A \cup B} h = \mathbf{1}_A h + \mathbf{1}_B h - \mathbf{1}_{A \cap B} h$  for any  $A, B \in \mathcal{S}_h$ .

Let us now check stability under countable intersection. Take  $(A_n)$  be a sequence in  $\mathcal{S}_h$ . Since  $\mathcal{S}_h$  is stable under finite intersection  $\cap A_n = \cap B_n$  where  $(B_n)$  is the following decreasing sequence  $B_n := A_0 \cap \dots \cap A_n$  in  $\mathcal{S}_h$ . Define now the following sequence  $(f_n)$  of pixel functions (which are

actually point functions)  $f_n := \mathbf{1}_{B_0} - \mathbf{1}_{B_n}$ . This is an increasing sequence that cu-converges to  $f := \mathbf{1}_{B_0} - \mathbf{1}_{\cap B_n}$ . The monotonous convergence theorem can be applied here since  $\sup \int f_n h \leq \int \mathbf{1}_{B_0} h < \infty$  and implies that  $f$  is integrable, and therefore  $\cap B_n \in \mathcal{S}_h$ .

Let us now check that  $\mathcal{M}_h$  is a  $\sigma$ -algebra. It is stable by complement since if  $A \in \mathcal{M}_h$ , and  $B$  is a  $h$ -integrable set, we have  $A^c \cap B = B \setminus (A \cap B)$  and  $\mathcal{S}_h$  is stable by relative complement. It is stable by finite intersection and finite union since, if  $A, B \in \mathcal{M}_h$  and  $C$  is a  $h$ -integrable set, we have  $A \cap B \cap C = (A \cap C) \cap (B \cap C)$ , and  $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$  and  $\mathcal{S}_h$  is stable by finite intersection and finite union.

Now consider a sequence  $(A_n)$  in  $\mathcal{M}_h$ . The union  $\cup A_n$  is also equal to  $\cup B_n$  where  $B_n := A_0 \cup \dots \cup A_n$ , and  $(B_n)$  is then an increasing sequence of elements in  $\mathcal{M}_h$  (since it is stable by finite union). Take now  $B$  a  $h$ -integrable set. Define now  $f_n := \mathbf{1}_{B_n \cap B}$ . Clearly,  $(f_n)$  is an increasing sequence of positive (point) functions that cpu-converges to  $\mathbf{1}_{(\cup B_n) \cap B}$ . Moreover,  $\sup_n \int f_n h \leq \int \mathbf{1}_B h < \infty$ . Thus the monotonous convergence theorem implies that  $\mathbf{1}_{(\cup B_n) \cap B}$  is  $h$ -integrable, and therefore  $\cup A_n \in \mathcal{M}_h$ .  $\square$

We arrive now at the first main result of this article :

**Definition-Theorem 3.13.** Let  $h : \mathcal{P}X \rightarrow \mathbb{R}_+$  be projectively integrable. Then :

$$\mu_h : \mathcal{M}_h \rightarrow \overline{\mathbb{R}}_+$$

$$A \mapsto \begin{cases} \int \mathbf{1}_A h, & \text{if } A \in \mathcal{S}_h \\ \infty & \text{otherwise} \end{cases}$$

is a complete measure on  $(X, \mathcal{M}_h)$ , called the **measure associated to  $h$** .

*Proof.* Let us check  $\sigma$ -additivity. Let  $(A_n)$  be a sequence of disjoint elements of  $\mathcal{M}_h$ . Consider the series of functions  $\sum_n \mathbf{1}_{A_n}$ . Suppose that  $A_n \in \mathcal{S}_h$  for all  $n$  and that  $\sum_n \int \mathbf{1}_{A_n} h < \infty$ . Then by the series integration theorem, since  $\sum_n \mathbf{1}_{A_n}$  cu-converges to  $\mathbf{1}_{\cup A_n}$ , the function  $\mathbf{1}_{\cup A_n}$  is  $h$ -integrable and  $\int \mathbf{1}_{\cup A_n} h = \sum_n \int \mathbf{1}_{A_n} h$ . This gives the equality  $\mu_h(\cup_n A_n) = \sum_n \mu_h(A_n)$ .

Suppose now that  $A_n \in \mathcal{S}_h$  for all  $n$ , and that  $\sum_n \int \mathbf{1}_{A_n} h = \infty$ . Then we must have  $\cup A_n$  outside  $\mathcal{S}_h$ . Indeed, if it is in  $\mathcal{S}_h$ , then by additivity, for any  $N \in \mathbb{N}$ ,  $\sum_{n=0}^N \int \mathbf{1}_{A_n} h = \int \mathbf{1}_{\cup_0^N A_n} h \leq \int \mathbf{1}_{\cup A_n} h < \infty$ . Thus we still have the equality  $\mu_h(\cup_n A_n) = \sum_n \mu_h(A_n)$ .

Finally the remaining case is when at least one of the  $A_n$ , say  $A_{n_0}$  (where  $n_0 \in \mathbb{N}$ ) is not in  $\mathcal{S}_h$ . In that case, it is clear that  $\sum_n \mu_h(A_n) = \infty$ . Moreover,  $\cup A_n$  is not in  $\mathcal{S}_h$  either, because if it were, since  $A_{n_0} \subseteq \cup A_n$  and  $A_{n_0}$  is  $h$ -measurable,  $A_{n_0}$  would be  $h$ -integrable by Proposition 3.12, 2. Thus, we also have the equality  $\mu_h(\cup_n A_n) = \sum_n \mu_h(A_n)$  in that case.

Completeness of  $\mu_h$  is a direct consequence of Proposition 3.12 item 3.  $\square$

**Lemma 3.14.** The product  $\sigma$ -algebra  $\mathcal{E}_X$  on  $X$  is generated by the set of all open cells of  $X$ .

*Proof.* Denoting  $\tau_t$  the topology of  $X_t$  for any  $t \in T$ , we know that  $\mathcal{C}_t \cap \tau_t$  is a basis of  $X_t$ . Since  $X_t$  is second-countable as a compact metrizable space,  $\mathcal{C}_t \cap \tau_t$  generates the Borel  $\sigma$ -algebra of  $X_t$ . Thus,  $\otimes_t \mathcal{C}_t \cap \tau_t$  generates the product  $\sigma$ -algebra  $\mathcal{E}_X$ . Since  $\otimes_t \mathcal{C}_t \cap \tau_t$  is precisely the set of all open cells of  $X$ , the result follows.  $\square$

**Definition 3.15.** A positive pixel function  $h$  is said  **$\sigma$ -cellular** if any open cell of  $X$  is a countable union of  $h$ -integrable elements of  $\mathcal{E}_X$ .

**Theorem 3.16.** Let  $h : \mathcal{P}X \rightarrow \mathbb{R}_+$  be positive, projectively integrable,  $\sigma$ -cellular pixel function. Then

1.  $\mathcal{M}_h$  contains  $\mathcal{E}_X$ , and the restriction of  $\mu_h$  on  $\mathcal{E}_X$  is  $\sigma$ -finite.
2. If  $T$  is countable, the restriction of  $\mu_h$  on  $\mathcal{E}_X$  is a Radon measure.

*Proof.* 1. Since  $\mathcal{S}_h \subseteq \mathcal{M}_h$  (by Proposition 3.12 1.), and any open cell is a countable union of  $h$ -integrable elements of  $\mathcal{E}_X$ , it follows from Proposition 3.12, item 3, that any open cell is in  $\mathcal{M}_h$ . By Lemma 3.14, open cells generate  $\mathcal{E}_X$ . Thus,  $\mathcal{E}_X$  is included in  $\mathcal{M}_h$ .

The fact that  $\mu_h$  is  $\sigma$ -finite on  $\mathcal{E}_X$  follows directly from the fact that  $X$  is an open cell and therefore a countable union of  $h$ -integrable elements of  $\mathcal{E}_X$ .

2. A countable product of second countable spaces is such that its Borel  $\sigma$ -algebra coincide with its product  $\sigma$ -algebra. Thus,  $\mathcal{E}_X = \mathcal{B}_X$ . Moreover, any Borel measure on a complete separable metric space is Radon. □

### 3.3 Lebesgue integrals as gauge integrals

Now that we can associate a measure  $\mu_h$  to any positive pixel function  $h$  that is projectively integrable and  $\sigma$ -cellular, we investigate the connection between the integrable functions for  $\mu_h$  in the Lebesgue sense, and the measurable functions whose module is  $h$ -integrable in the gauge integration sense.

**Assumption 3.17.** As before, we recall that  $(X_t)$  is fixed compact metrizable division family, and  $(X, \mathcal{C})$  is the associated product division space. We will only consider from now on the product  $\sigma$ -algebra  $\mathcal{E}_X$  on  $X$ . Thus, **measurable functions** on  $X$  are meant to be measurable in the  $\mathcal{E}_X$  sense.

Thus, if  $h$  is a given  $\sigma$ -cellular projectively integrable positive pixel function, we will consider the restriction of the measure  $\mu_h$  associated to  $h$  on  $\mathcal{E}_X$  (and still denote it  $\mu_h$ ). Note that by Theorem 3.16, this measure is  $\sigma$ -finite. We will denote, as it is usual,  $\mathcal{L}^1(X, \mu_h)$  the space of all complex valued measurable functions on  $X$  that are Lebesgue integrable with respect to  $\mu_h$ .

**Notation 3.18.** Given a positive pixel function  $h$ , the set of all measurable point functions  $f : X \rightarrow \mathbb{C}$  such that  $|f|$  is  $h$ -integrable is denoted  $\mathcal{L}^1(X, h)$ .

**Lemma 3.19.** *Suppose that  $h$  is a  $\sigma$ -cellular, projectively integrable, positive pixel function. Let  $f : X \rightarrow \mathbb{R}_+$  be a  $h$ -integrable positive function and  $A \in \mathcal{E}_X$ . Then  $f \mathbf{1}_A$  is  $h$ -integrable.*

*Proof.* Since  $h$  is  $\sigma$ -cellular, there exists a sequence  $(I_n)$  in  $\mathcal{S}_h \cap \mathcal{E}_X$  such that  $X = \cup_n I_n$ . Define  $A_n := I_0 \cup \dots \cup I_n$ . The increasing sequence  $(A_n)$  is in  $\mathcal{S}_h$  and  $\cup A_n = X$ . Define  $f_n := f \wedge n \mathbf{1}_{A_n \cap A}$  for all  $n$ . The increasing sequence  $(f_n)$  cu-converges to  $f \mathbf{1}_A$ . Since  $f$  is  $h$ -integrable and  $A_n \cap A$  is  $h$ -integrable, each  $f_n$  is  $h$ -integrable by Proposition 2.75, item 8. Moreover,  $\sup_n \int f_n h \leq \int f h$ . Thus, the monotonous convergence theorem (Theorem 3.9) implies that  $f \mathbf{1}_A$  is  $h$ -integrable. □

**Lemma 3.20.** *Suppose that  $h$  is a  $\sigma$ -cellular, projectively integrable, positive pixel function. Let  $f : X \rightarrow \mathbb{R}_+$  be a positive measurable function. Then  $f \in \mathcal{L}^1(X, \mu_h)$  if and only if  $f \in \mathcal{L}^1(X, h)$ . In that case*

$$\int f d\mu_h = \int f h.$$

*Proof.* By definition of  $\mu_h$ , this statement is obvious if  $f = \mathbf{1}_A$  for a measurable subset  $A$  of  $X$ . Suppose now that  $f$  is a simple function, of the form  $\sum_{k=1}^p a_k \mathbf{1}_{A_k}$  where the  $a_k$  are strictly positive numbers and the  $A_k$  are disjoint elements of  $\mathcal{E}_X$ . If  $f$  is in  $\mathcal{L}^1(X, \mu_h)$ , then each of the  $A_k$  have finite measure, and as a consequence, they are all  $h$ -integrable, and by linearity,  $f$  is also  $h$ -integrable. Inversely, if  $f$  is  $h$ -integrable, Lemma 3.19 implies that  $f \mathbf{1}_{A_k}$  is  $h$ -integrable for all  $k$ . But this function is simply  $a_k \mathbf{1}_{A_k}$ , which implies that  $A_k$  is  $h$ -integrable, and therefore with finite measure. By linearity,  $f \in \mathcal{L}^1(X, \mu_h)$  and the equality  $\int f h = \int f d\mu_h$  follows.

Consider now the general case. Since  $f$  is positive and measurable, there is an increasing sequence  $(e_n)$  of positive simple functions that converges pointwisely to  $f$ . By the monotonous convergence theorem for the measure  $\mu_h$ ,  $\int f d\mu_h = \lim \int e_n d\mu_h$ .

Suppose that  $f \in \mathcal{L}^1(X, \mu_h)$ . Then the  $e_n$  are in  $\mathcal{L}^1(X, \mu_h)$ , and by the preceding proven case,  $e_n$  is  $h$ -integrable and  $\int e_n d\mu_h = \int e_n h$ . Since  $(e_n)$  is cu-convergent to  $f$ , and  $\sup \int e_n h \leq \int f d\mu_h$ , the

monotonous convergence theorem for the gauge integral implies that  $f$  is  $h$ -integrable and  $\int fh = \lim \int e_n h = \int f d\mu_h$ .

Suppose now that  $f$  is  $h$ -integrable. Take  $A$  a  $h$ -integrable element of  $\mathcal{E}_X$ . By Lemma 3.19,  $f \mathbf{1}_A$  is  $h$ -integrable. The sequence of simple functions  $(e_n \mathbf{1}_A)$  is increasing, and converges pointwisely to  $f \mathbf{1}_A$ . Moreover, each  $e_n \mathbf{1}_A$  is  $h$ -integrable since it is of the form  $\sum_k a_k \mathbf{1}_{A \cap A_k}$  where the  $a_k$  are strictly positive constants, the  $A_k$  are elements of  $\mathcal{E}_X$ , and therefore the  $A \cap A_k$  are  $h$ -integrable. As a consequence  $e_n \mathbf{1}_A \in \mathcal{L}^1(X, \mu_h)$ . By the monotonous convergence theorem for  $\mu_h$ ,  $\int f \mathbf{1}_A d\mu_h = \lim \int e_n \mathbf{1}_A d\mu_h = \lim \int e_n \mathbf{1}_A h$  (as an equality in  $\overline{\mathbb{R}}_+$ ). Since  $\sup_n \int e_n \mathbf{1}_A h \leq \int f \mathbf{1}_A h < \infty$ , the monotonous convergence theorem for the gauge integral implies that  $\lim \int e_n \mathbf{1}_A h = \int f \mathbf{1}_A h$ , which is in  $\mathbb{R}_+$ . As a consequence,  $f \mathbf{1}_A$  is in  $\mathcal{L}^1(X, \mu_h)$ .

Consider now an increasing sequence  $(A_n)$  of  $h$ -integrable elements of  $\mathcal{E}_X$  such that  $\cup A_n = X$ , as in the beginning of the proof of Lemma 3.19. It is clear that  $f = \lim_n f \mathbf{1}_{A_n}$ , and therefore by monotonous convergence for  $\mu_h$ ,  $\int f d\mu_h = \lim_n \int f \mathbf{1}_{A_n} d\mu_h$ . We just proved that  $\int f \mathbf{1}_{A_n} d\mu_h$  is actually equal to  $\int f \mathbf{1}_{A_n} h$ . Moreover the monotonous convergence theorem for the gauge integral implies  $\lim_n \int f \mathbf{1}_{A_n} h = \int f h < \infty$ . Finally this yields  $f \in \mathcal{L}^1(X, \mu_h)$ . □

We can now formulate the second main result of this article :

**Theorem 3.21.** *Let  $h$  be a positive,  $\sigma$ -cellular, projectively integrable pixel function. Then*

1.  $\mathcal{L}^1(X, \mu_h) = \mathcal{L}^1(X, h)$ .
2. For all  $f \in \mathcal{L}^1(X, h)$ ,

$$\int f d\mu_h = \int f h.$$

*Proof.* 1. This follows directly from the previous lemma.

2. If  $f \in \mathcal{L}^1(X, h)$ , then it is in  $\mathcal{L}^1(X, \mu_h)$ . Since  $f = \Re(f)_+ - \Re(f)_- + i\Im(f)_+ - i\Im(f)_-$ , and the positive functions  $\Re(f)_\pm, \Im(f)_\pm$  are in  $\mathcal{L}^1(X, \mu_h)$  and therefore  $h$ -integrable by the previous lemma, it follows by linearity that  $f$  is  $h$ -integrable, and that  $\int f h = \int f d\mu_h$ . □

In order to check in practice that a given positive pixel function is  $\sigma$ -cellular, we give now a sufficient condition for a cell to be  $h$ -integrable :

**Definition 3.22.** Let  $h : \mathcal{P}X \rightarrow \mathbb{R}_+$  be a positive pixel function. A given cell  $J$  is said  **$h$ -regular** if  $h$  is integrable on  $J$  and for any  $\varepsilon > 0$ , there are elementary sets  $E, E'$  such that  $\overline{E} \subseteq J \subseteq (E')^\circ$  and there is a gauge  $\gamma$  such that for any  $\gamma$ -resolution  $R$  of  $E' \setminus E$ ,  $S(h, R) < \varepsilon$ .

**Theorem 3.23.** *Let  $h : \mathcal{P}X \rightarrow \mathbb{R}_+$  be a positive pixel function and  $J$  a  $h$ -regular cell. Then  $J$  is  $h$ -integrable and*

$$\int \mathbf{1}_J h = \int_J h.$$

*Proof.* Suppose that  $J_0$  is  $h$ -regular and fix  $\varepsilon > 0$ . Since  $J_0$  is  $h$ -regular,  $h$  is integrable on  $J_0$  and there is a  $(h_{J_0}, \varepsilon/2)$ -gauge  $\gamma_0$ . Moreover, there are elementary sets  $E, E'$  such that  $\overline{E} \subseteq J_0 \subseteq (E')^\circ$  and a gauge  $\gamma_1$  such that for any  $\gamma_1$ -resolution  $R$  of  $E' \setminus E$ ,  $S(h, R) < \varepsilon/2$ .

Let  $D_E$  and  $D_{E'}$  be two divisions of respectively  $E$  and  $E'$ , and let  $\gamma_E, \gamma_{E'}, \gamma_{J_0}$  be respectively  $D_E, D_{E'}, \{J_0\}$  gauges. Consider the gauge  $\gamma := \gamma_0 \wedge \gamma_1 \wedge \gamma_E \wedge \gamma_{E'} \wedge \gamma_{J_0}$  and take  $R$  a  $\gamma$ -resolution of  $X$ . We partition  $R$  the following way :  $R = R_1 \cup R_i \cup R_e \cup R_2$  where  $R_1$  is the subset of  $R$  of pixels  $(J, x)$  such that  $J \subseteq J_0$  and  $x \in J_0$ ,  $R_i$  is the subset of  $R$  of all pixels  $(J, x)$  such that  $J \subseteq J_0$  and  $x \notin J_0$  (but of course  $x \in \overline{J_0}$ ),  $R_e$  is the subset of  $R$  of all pixels  $(J, x)$  such that  $J \subseteq X \setminus J_0$  and  $x \in J_0$ , and  $R_2$  the subset of  $R$  of all pixels  $(J, x)$  such that  $J \subseteq X \setminus J_0$  and  $x \notin J_0$ .

Thus,  $S(\mathbf{1}_{J_0}h, R) = S(h, R_1) + S(h, R_e)$ ,  $S(h_{J_0}, R_1 \cup R_i) = S(h, R_1) + S(h, R_i)$ , and  $R_1 \cup R_i$  is a  $\gamma$ -resolution of  $J_0$ . This implies the estimate :

$$\begin{aligned} |S(\mathbf{1}_{J_0}h, R) - \int_{J_0} h| &\leq |S(h, R_1) + S(h, R_e) - (S(h, R_1) + S(h, R_i))| + |S(h_{J_0}, R_1 \cup R_i) - \int_{J_0} h| \\ &\leq S(h, R_e \cup R_i) + \varepsilon/2. \end{aligned}$$

Note that if  $(J, x) \in R_i$ , then either  $J \subseteq E$ , or  $J \subseteq J_0 \setminus E$ . But if  $J \subseteq E$ , then  $x \in \bar{E} \subseteq J_0$ , so we would get  $x \in J_0$  which is impossible. Thus,  $J \subseteq J_0 \setminus E$ . As a consequence  $R_i$  is a resolution in  $J_0 \setminus E$ .

Similarly, not that if  $(J, x) \in R_e$ , then either  $J \subseteq X \setminus E'$ , or  $J \subseteq E' \setminus J_0$ . But if  $J \subseteq X \setminus E'$ , then we would have  $x \in X \setminus E' = X \setminus (E')^\circ$  which is impossible since  $x \in J_0 \subseteq (E')^\circ$ . Thus we must have  $J \subseteq E' \setminus J_0$ , and thus  $R_e$  is a resolution in  $E' \setminus J_0$ .

Finally,  $R_e \cup R_i$  is a  $\gamma$ -resolution in  $E' \setminus E$  and as a consequence  $S(h, R_e \cup R_i) < \varepsilon/2$ . This yields the result.  $\square$

## 4 Examples

### 4.1 Lebesgue measure

In this section, we show how the Lebesgue measure can be obtained as a particular case of the construction given by Definition-Theorem 3.13. In all this section, we consider  $\bar{\mathbb{R}}^n$  as a division space, where the cells are of the form  $\prod_{i=1}^n J_i$  of where each  $J_i$  is an intervals of  $\bar{\mathbb{R}}$ .

We first fix the following conventions and notations :

**Definition 4.1.** A **bounded** subset of  $\bar{\mathbb{R}}^n$  is a bounded subset of  $\mathbb{R}^n$ .

**Notation 4.2.** We will denote  $\lambda_n$  the Lebesgue measure on the Borel subsets of  $\bar{\mathbb{R}}^n$ , defined as an extension of the standard Lebesgue measure on the borel subsets of  $\mathbb{R}^n$ , so that  $\lambda_n(\bar{\mathbb{R}}^n \setminus \mathbb{R}^n) = 0$ .

**Definition 4.3.** The  **$n$ -volume integrator** is the cell function defined by

$$v_n : J \mapsto \begin{cases} \prod_{i=1}^n l(J_i) & \text{if } J \text{ is bounded} \\ 0 & \text{otherwise} \end{cases}$$

where  $l(I)$  is the length of the bounded interval  $I$ .

**Remark 4.4.** Note that  $v_n$  is **not** additive. However, the restriction of  $v_n$  to all cells that are included in a given bounded elementary set, is additive.

**Remark 4.5.** All the definition and results of this section could be reformulated in an obvious way for the division space  $\bar{\mathbb{R}}^N$  where  $N$  is any finite set, instead of  $\bar{\mathbb{R}}^n$ . We work here with  $\bar{\mathbb{R}}^n$  just for notational simplifications. When we consider  $\bar{\mathbb{R}}^N$ , the  **$N$ -volume integrator** is by definition :

$$v_N : J \mapsto \begin{cases} \prod_{i \in N} l(J_i) & \text{if } J \text{ is bounded} \\ 0 & \text{otherwise} \end{cases}$$

**Definition 4.6.** Let  $f : \bar{\mathbb{R}}^n \rightarrow \mathbb{C}$  be a function, and  $E$  an elementary set of  $\bar{\mathbb{R}}^n$ . We say that  $f$  is **Henstock-Kurzweil (or HK) integrable** on  $E$ , if  $(f v_n)_E := (f v_n)|_{\varnothing E}$  is gauge integrable on  $E$  (i.e.  $(f v_n)_E \in \mathcal{H}(E)$ ).

**Remark 4.7.** The value of  $f$  on  $\bar{\mathbb{R}}^n \setminus \mathbb{R}^n$  has no importance in the following sense : if  $f$  and  $g$  coincide on  $\mathbb{R}^n$ , and if  $f$  is HK-integrable, then so is  $g$  and their integral are the same. This follows from the fact that  $v_n(J) = 0$  for unbounded cells of  $\bar{\mathbb{R}}^n$ . Thus, we do not need here to set to zero the value of  $f$  on  $\bar{\mathbb{R}}^n \setminus \mathbb{R}^n$ .

**Remark 4.8.** Since we are dealing here with a finite product of division spaces, any gauge  $\gamma$  is such that there is a finer gauge  $\gamma' \leq \gamma$  of the form  $\gamma' = (L_{max}, \delta)$ , where  $L_{max}$  is the constant function  $x \mapsto \{1, \dots, n\}$  (or  $x \mapsto N$  if we are working with  $\overline{\mathbb{R}}^N$ ). Thus, we could restrict with no loss of generality to gauges of the type  $(L_{max}, \delta)$  in the definitions of  $\mathcal{H}(\overline{\mathbb{R}}^n)$  (or  $\mathcal{H}(\overline{\mathbb{R}}^N)$ ). In other words, in the case of  $\overline{\mathbb{R}}^n$ , the function  $L$  in a given gauge  $(L, \delta)$  does not play any role.

**Remark 4.9.** Definition 4.6 contains as a particular case the classical Henstock–Kurzweil integrable functions on Euclidean space [11, p.83].

Indeed, the classical definition is the equivalent to the following formulation : a function  $f : \overline{\mathbb{R}}^n \rightarrow \mathbb{C}$  is (classically) HK-integrable to  $\alpha \in \mathbb{C}$  if it is zero on  $\overline{\mathbb{R}}^n \setminus \mathbb{R}^n$  and for any  $\varepsilon > 0$ , there is function  $\delta : \overline{\mathbb{R}}^n \rightarrow \mathbb{R}_+^*$  such that for any resolution  $R$  of  $\overline{\mathbb{R}}^n$ , such that all pairs  $(J, x) \in R$  satisfy  $J \subseteq I_{x, \delta(x)} := \prod_{i=1}^n I_{x_i, \delta(x)}$  (where  $I_{y,r} := ]y-r, y+r[$  if  $y \in \mathbb{R}$ ,  $I_{\infty, r} := ]1/r, \infty]$ ,  $I_{-\infty, r} := [-\infty, -1/r[$ ), we have  $|\sum_{(J,x) \in R} f(x) \lambda_n(J) - \alpha| < \varepsilon$ , with the convention  $0\infty = 0$ , and where  $\lambda_n$  is the Lebesgue measure. Note that in  $\sum_{(J,x) \in R} f(x) \lambda_n(J)$ , no nonzero term where  $J$  is unbounded or where  $x \notin \mathbb{R}^n$  appear. Thus, this sum is actually equal to  $\sum_{(J,x) \in R} g(x) v_n(J)$  where  $g$  is any function that coincide with  $f$  on  $\mathbb{R}^n$ .

Since  $I_{x, \delta(x)}$  is an open set containing  $x$ , for any  $x$ , there is a smallest natural number  $n_x$  such that  $B(x, 1/n_x) \subseteq I_{x, \delta(x)}$ , where  $B(x, r)$  is the open ball centered at  $x$ , of radius  $r$  for a given fixed compatible distance on  $\overline{\mathbb{R}}^n$ . Moreover, Lemma 2.47 shows that there exists a gauge  $\gamma$  such that any  $\gamma$ -fine pixel  $(J, x)$  will be such that  $J \subseteq B(x, 1/n_x) \subseteq I_{x, \delta(x)}$ . Thus,  $f$  is HK-integrable on  $\overline{\mathbb{R}}^n$  in the sense of Definition 4.6.

**Lemma 4.10.** 1. Any bounded cell of  $\overline{\mathbb{R}}^n$  is  $v_n$ -regular.

2. The  $n$ -volume integrator  $v_n : \mathcal{P}\overline{\mathbb{R}}^n \rightarrow \mathbb{R}_+$  is  $\sigma$ -cellular.

*Proof.* 1. Let  $J$  be a nonempty bounded cell of  $\overline{\mathbb{R}}^n$ . Note first that since  $(v_n)_J$  is additive, it is therefore integrable.

Let  $\varepsilon \in ]0, 1]$ . Since  $J$  is a nonempty bounded cell, its interior is of the form  $\prod_{i=1}^n ]a_i, b_i[$  and its closure of the form  $\prod_{i=1}^n [a_i, b_i]$ , where for all  $1 \leq i \leq n$ ,  $a_i \leq b_i$  and  $a_i, b_i \in \mathbb{R}$ . Consider now  $E' := \prod_i [a_i + \varepsilon, b_i - \varepsilon]$  and  $E := \prod_i [a_i - \varepsilon, b_i + \varepsilon]$ , where  $\varepsilon < \min_i (b_i - a_i)/2$ . Note that  $\overline{E} \subseteq J \subseteq (E')^\circ$ . Since  $(v_n)_{E' \setminus E}$  is additive, it is integrable. Moreover, by additivity,

$$\int_{E' \setminus E} v_n = \prod_{i=1}^n (b_i - a_i + 2\varepsilon) - \prod_{i=1}^n (b_i - a_i - 2\varepsilon) = \varepsilon P(\varepsilon) \leq \varepsilon \|P\|_{\infty, [0,1]},$$

where  $P$  is a polynomial of degree  $n-1$  with coefficients that only depend on  $(a_i)$  and  $(b_i)$ . Since  $v_n$  is additive on  $E' \setminus E$ , for any resolution  $R$  of  $E' \setminus E$ ,  $S(v_n, R) = \int_{E' \setminus E} v_n \leq \varepsilon \|P\|_{\infty, [0,1]}$ . Thus  $J$  is  $v_n$ -regular.

2. By item 1 and Theorem 3.23, any bounded cell  $J$  of  $\overline{\mathbb{R}}^n$  is  $v_n$ -integrable.

If  $J$  is an unbounded cell, write down  $J = J' \cup J''$  where  $J' = J \cap \mathbb{R}^n$ , and  $J'' = J \setminus \mathbb{R}^n$ . Since  $J'$  is a countable union of bounded cells, it is a countable union of  $v_n$ -integrable cells.

We now check that  $J''$  is a finite union of  $v_n$ -integrable cells. We can decompose  $J''$  into the union of all  $J_{i,\eta}$  where  $i \in \{1, \dots, n\}$  and  $\eta \in \{-1, 1\}$ , and  $J_{i,\eta} = J_1 \times \dots \times (J_i \cap \{\eta\infty\}) \times \dots \times J_n$ . Note that each  $J_{i,\eta}$  is a cell. Let us check that it is  $v_n$ -integrable. Take any resolution  $R$  of  $\overline{\mathbb{R}}^n$ . We have  $S(\mathbf{1}_{J_{i,\eta}} v_n, R) = \sum_{(I,x) \in R'} v_n(I)$  where  $R' \subseteq R$  is the subset of  $R$  made of all the pixels  $(I, x)$  such that  $x \in J_{i,\eta}$ . If  $(I, x) \in R'$ , then  $I$  is unbounded and therefore  $v_n(I) = 0$ . The sum is  $S(\mathbf{1}_{J_{i,\eta}} v_n, R)$  is therefore equal to zero. This shows that the  $J_{i,\eta}$  are all  $v_n$ -integrable.

In conclusion, we showed that any cell is a countable union of  $v_n$ -integrable cells, and in particular, that  $v_n$  is  $\sigma$ -cellular. □

**Lemma 4.11.** The  $n$ -volume integrator  $v_n$  is projectively integrable.

*Proof.* Consider the following strictly positive function on  $\overline{\mathbb{R}}^n$  :

$$f : x \mapsto \begin{cases} e^{-\|x\|^2} & \text{if } x \in \mathbb{R}^n \\ 1 & \text{otherwise} \end{cases}$$

where  $\|\cdot\|$  is the Euclidean norm on  $\mathbb{R}^n$ . We will apply Theorem 2.79 to prove that  $f v_n$  is integrable on  $\overline{\mathbb{R}}^n$ . Consider the sequence  $(J_k)_{k \geq 0}$  of cubes  $J_k := [-k, k]^n$ . Each  $J_k$  is a closed bounded cell, and therefore  $f|_{J_k}$  is continuous and bounded. Moreover,  $v_n$  is additive (and therefore integrable) on  $J_k$ , and Theorem 2.78 implies that  $(f v_n)_{J_k}$  is integrable. Note also that for any pixel  $(J, x)$ ,  $(f v_n)(J, x)$  is zero whenever  $x$  is not in  $\cup_k J_k = \mathbb{R}^n$  since in that case  $J$  is unbounded. Define now  $E_k := J_k \setminus J_{k-1}$  when  $k \geq 1$  and  $E_0 := J_0$ . We have  $J_k = \cup_{j \leq k} E_j$ , and by additivity

$$\int_{J_k} f v_n = \sum_{j=0}^k \int_{E_j} f v_n.$$

Since  $(f v_n)_{E_k} \leq e^{-n(k-1)^2} (v_n)_{E_k}$  and  $e^{-n(k-1)^2} (v_n)_{E_k}$  is integrable, we see that for any  $k \geq 1$ ,  $\int_{E_k} f v_n \leq e^{-n(k-1)^2} \int_{E_k} v_n = e^{-n(k-1)^2} ((2k)^n - (2k-2)^n)$ . Thus, the increasing sequence  $(\int_{J_k} f v_n)_{k \geq 0}$  is bounded in  $\mathbb{R}_+$ , and therefore converges. Let  $L$  be its limit. If  $\varepsilon > 0$ , then there is  $k_0$  such that  $|L - \int_{J_k} f v_n| < \varepsilon$  for all  $k \geq k_0$ . Suppose that  $E$  is an elementary set that contains  $J_{k_0}$  and included in one  $J_k$ . We have then  $k \geq k_0$ , and  $\int_{J_{k_0}} f v_n \leq \int_E f v_n \leq \int_{J_k} f v_n$ . Thus,  $|L - \int_E f v_n| < \varepsilon$ . We can therefore apply Theorem 2.79, and obtain the fact that  $f v_n$  is integrable on  $\overline{\mathbb{R}}^n$ , and  $\int f v_n = L > 0$ . Since  $f$  is a strictly positive point function, this implies that  $v_n$  is projectively integrable (see Definition 3.2).  $\square$

We can now conclude :

**Theorem 4.12.** Any Borel subset of  $\overline{\mathbb{R}}^n$  is  $v_n$ -measurable and the map

$$\begin{aligned} \mathcal{B}_{\overline{\mathbb{R}}^n} &\rightarrow \overline{\mathbb{R}}_+ \\ A &\mapsto \begin{cases} \int \mathbf{1}_A v_n, & \text{if } A \in \mathcal{G}_{v_n} \\ \infty & \text{otherwise} \end{cases} \end{aligned}$$

is a  $\sigma$ -finite Borel measure on  $\overline{\mathbb{R}}^n$ , that coincides with the Lebesgue measure  $\lambda_n$ . Moreover,  $\mathcal{L}^1(\overline{\mathbb{R}}^n, \lambda_n) = \mathcal{L}^1(\overline{\mathbb{R}}^n, v_n)$ , and for any  $f \in \mathcal{L}^1(\overline{\mathbb{R}}^n, v_n)$ ,

$$\int f d\lambda_n = \int f v_n.$$

*Proof.* By Lemmas 4.10 and 4.11, the hypothesis of Theorems 3.13, 3.16 and 3.21 are satisfied. This shows that we can define the measure  $\mu$  on  $\mathcal{B}_{\overline{\mathbb{R}}^n}$  associated to  $v_n$ .

If  $I := \prod_{i=1}^n ]a_i, b_i]$ , then  $\int \mathbf{1}_I v_n = v_n(I)$  since  $I$  is  $v_n$ -regular by Lemma 4.10, item 1. Thus,  $\mu(I) = \lambda_n(I)$ . This proves that the restrictions of  $\mu$  and  $\lambda_n$  coincide on  $\mathcal{B}_{\overline{\mathbb{R}}^n}$ .

Let  $R$  be resolution of  $\overline{\mathbb{R}}^n$ . Clearly,  $S(R, \mathbf{1}_{\overline{\mathbb{R}}^n \setminus \mathbb{R}^n} v_n) = 0$  since any pixel  $(J, x)$  in  $R$  such that  $x \in \overline{\mathbb{R}}^n \setminus \mathbb{R}^n$  is such that  $J$  is not bounded, and therefore such that  $v_n(J) = 0$ . This proves that  $\int \mathbf{1}_{\overline{\mathbb{R}}^n \setminus \mathbb{R}^n} v_n = 0$ , and therefore  $\mu(\overline{\mathbb{R}}^n \setminus \mathbb{R}^n) = \lambda_n(\overline{\mathbb{R}}^n \setminus \mathbb{R}^n) = 0$ .

As a consequence,  $\mu$  coincide with the Lebesgue measure on  $\mathcal{B}_{\overline{\mathbb{R}}^n}$ .  $\square$

## 4.2 Gaussian measures on Hilbert spaces

In this section we consider the product division space  $(\overline{\mathbb{R}}^{\mathbb{N}}, \mathcal{G})$  (the Hilbert cube) of the division family  $(\overline{\mathbb{R}})_{n \in \mathbb{N}}$ . Here,  $\mathcal{G}$  is the set of all cells of  $\overline{\mathbb{R}}^{\mathbb{N}}$ , that is to say the set of all products  $\prod_{n \in \mathbb{N}} J_n$  where the  $J_n$  are all intervals of  $\overline{\mathbb{R}}$  such that  $I_n = \overline{\mathbb{R}}$  for cofinitely almost all  $n \in \mathbb{N}$ .

**Notation 4.13.** The Gaussian measure on  $\mathbb{R}$  of mean  $a \in \mathbb{R}$  and variance  $\kappa > 0$  is the probability measure on  $\mathbb{R}$  with density with respect to the Lebesgue measure equal to :

$$x \mapsto \frac{1}{\sqrt{2\pi\kappa}} e^{-\frac{1}{2\kappa}(x-a)^2}.$$

It will be denoted  $\mathcal{N}(a, \kappa)$ .

**Assumption 4.14.** In all the following, we fix a pair of real sequences  $\kappa := (\kappa_n)$  and  $a := (a_n)$ , where  $\kappa_n > 0$  for all  $n$ .

**Notation 4.15.** On the separable Fréchet space  $\mathbb{R}^{\mathbb{N}}$  the **Gaussian measure** on  $\mathbb{R}^{\mathbb{N}}$  of mean  $a$  and variance  $\kappa$  is defined as the following product of probability measures :

$$\gamma_{a,\kappa} := \otimes_{n \in \mathbb{N}} \mathcal{N}(a_n, \kappa_n).$$

We extend this measure to a measure on the Borel sets of  $\overline{\mathbb{R}^{\mathbb{N}}}$  by setting the value zero to any borel set of  $\overline{\mathbb{R}^{\mathbb{N}}}$  included in  $\overline{\mathbb{R}^{\mathbb{N}}} \setminus \mathbb{R}^{\mathbb{N}}$ . We keep the notation  $\gamma_{a,\kappa}$  for this extension.

**Remark 4.16.** Let  $H$  be a real separable Hilbert space,  $a \in H$ , and  $K$  a positive invertible trace-class operator on  $H$ . Take a Hilbert basis  $(e_n)$  of  $H$  of eigenvectors of  $K$  such that the eigenvalue sequence  $(\kappa_n)$  associated to  $(e_n)$  is decreasing. Denote  $\psi : \ell_{\mathbb{R}}^2 \rightarrow H$  the isomorphism of Hilbert spaces associated to the basis  $(e_n)$ . Then the Gaussian measure on  $H$  of mean  $a$  and covariance operator  $K$  is the push-forward by  $\psi$  of the Gaussian measure  $\gamma_{\psi^{-1}(a),\kappa}$  (restricted on  $\ell_{\mathbb{R}}^2$ ). Thus, the study of Gaussian measures on Hilbert spaces can be reduced to the special case we considered here of Gaussian measures on  $\mathbb{R}^{\mathbb{N}}$ .

**Definition 4.17.** Let  $N \in \mathcal{F}(\mathbb{N})$  be nonempty. The  $N$ -**Gaussian function** is the function

$$g_N : \overline{\mathbb{R}^N} \rightarrow \mathbb{R}_+$$

$$x \mapsto \begin{cases} (\prod_{n \in N} 2\pi\kappa_n)^{-1/2} e^{-\sum_{n \in N} \frac{1}{2\kappa_n} (x_n - a_n)^2} & \text{if } x \in \mathbb{R}^N \\ 0 & \text{otherwise} \end{cases}$$

**Proposition 4.18.** Let  $N \in \mathcal{F}(\mathbb{N})$  be nonempty. The  $N$ -Gaussian function  $g_N$  is  $\nu_N$ -integrable and  $g_N \in \mathcal{L}^1(\overline{\mathbb{R}^N}, \nu_N)$ . Moreover,

$$\int_{\overline{\mathbb{R}^N}} g_N \nu_N = 1.$$

*Proof.* The result follows immediately from Theorem 4.12. □

Thanks to the previous proposition, we can define :

**Definition 4.19.** The **Gaussian integrator** is the following cell function  $G_{a,\kappa}$  :

$$G_{a,\kappa} : J \mapsto \begin{cases} \int_J g_N \nu_N & \text{if } J \neq \overline{\mathbb{R}^N} \\ 1 & \text{otherwise} \end{cases}$$

**Theorem 4.20.** The Gaussian integrator  $G_{a,\kappa}$  is additive (and therefore integrable).

*Proof.* Let  $J$  be a cell of  $\overline{\mathbb{R}^N}$ , and  $D$  a division of  $J$ . We want to prove that  $G_{a,\kappa}(J) = \sum_{J' \in D} G_{a,\kappa}(J')$ . If  $J' \in D$ , we have  $J' = \underline{J}' \times \overline{\mathbb{R}^{N \setminus N_{J}'}}$ . Denote  $N_D := \cup_{J' \in D} N_{J}'$ . Clearly,  $N_J \subseteq N_{J'} \subseteq N_D$ . Let  $B := N_D \setminus N_J$ . Thus, we have for any  $J' \in D$ ,  $J' = J'' \times \overline{\mathbb{R}^{N \setminus N_D}}$ , where  $J'' = \underline{J}' \times \overline{\mathbb{R}^{N_D \setminus N_{J}'}}$ , and  $J = J_D \times \overline{\mathbb{R}^{N \setminus N_D}}$ , where

$J_D := \underline{J} \times \overline{\mathbb{R}}^B$ . Since  $D$  is a partition of  $J$ , the  $J''$  form of a partition of  $J_D$ . Using Theorem 4.12, we now obtain by the Fubini theorem and additivity

$$\begin{aligned} G_{a,\kappa}(J) &= \left( \prod_{n \in N_D} 2\pi\kappa_n \right)^{-1/2} \int_{\underline{J} \times \overline{\mathbb{R}}^B} e^{-\sum_{n \in N_D} \frac{1}{2\kappa_n} (x_n - a_n)^2} \nu_{N_D} \\ &= \left( \prod_{n \in N_D} 2\pi\kappa_n \right)^{-1/2} \sum_{J' \in D} \int_{\underline{J}' \times \overline{\mathbb{R}}^{N_D \setminus N_{J'}}} e^{-\sum_{n \in N_D} \frac{1}{2\kappa_n} (x_n - a_n)^2} \nu_{N_D} \\ &= \sum_{J' \in D} \left( \prod_{n \in N_{J'}} 2\pi\kappa_n \right)^{-1/2} \int_{\underline{J}'} e^{-\sum_{n \in N_{J'}} \frac{1}{2\kappa_n} (x_n - a_n)^2} \nu_{N_{J'}} = \sum_{J' \in D} G_{a,\kappa}(J'). \end{aligned}$$

The function  $G_{a,\kappa}$  is therefore additive, and thus integrable.  $\square$

**Lemma 4.21.** Any cell  $J$  of  $\overline{\mathbb{R}}^{\mathbb{N}}$  is  $G_{a,\kappa}$ -regular.

*Proof.* Let  $\varepsilon > 0$ . For convenience, we suppose that  $\varepsilon \leq 1$ . If  $I$  is an interval of  $\overline{\mathbb{R}}$  with lower extremity equal to  $a \in \overline{\mathbb{R}}$  and higher extremity equal to  $b \in \overline{\mathbb{R}}$ , we define  $I_\varepsilon$  as the interval  $]a + \varepsilon, b - \varepsilon[$  and  $I'_\varepsilon$  as the interval  $[a - \varepsilon, b + \varepsilon]$ . Here we use the usual convention for the addition in  $\overline{\mathbb{R}}$ .

Let  $J$  be a cell of  $\overline{\mathbb{R}}^{\mathbb{N}}$ . Define  $J_\varepsilon$  as the open cell  $\prod_{n \in N_J} (J_n)_\varepsilon \oplus \overline{\mathbb{R}}^{\mathbb{N} \setminus N_J}$ , and  $J'_\varepsilon$  as the closed cell  $\prod_{n \in N_J} (J_n)'_\varepsilon \oplus \overline{\mathbb{R}}^{\mathbb{N} \setminus N_J}$ . Note that  $J_\varepsilon \subseteq J \subseteq J'_\varepsilon$ . By Theorem 4.20,  $(G_{a,\kappa})_{J'_\varepsilon \setminus J_\varepsilon}$  is integrable. Moreover, by additivity, and Fubini theorem

$$\begin{aligned} \int_{J'_\varepsilon \setminus J_\varepsilon} G_{a,\kappa} &= \int_{J'_\varepsilon} G_{a,\kappa} - \int_{J_\varepsilon} G_{a,\kappa} = G_{a,\kappa}(J'_\varepsilon) - G_{a,\kappa}(J_\varepsilon) \\ &= \int_{\prod_{n \in N_J} (J_n)'_\varepsilon} g_{N_J} \nu_{N_J} - \int_{\prod_{n \in N_J} (J_n)_\varepsilon} g_{N_J} \nu_{N_J} \\ &= \prod_{n \in N_J} \int_{(J_n)'_\varepsilon} g_n(x) dx - \prod_{n \in N_J} \int_{(J_n)_\varepsilon} g_n(x) dx \end{aligned}$$

where for all  $n \in N_J$ ,  $g_n : x \mapsto (2\pi\kappa_n)^{-1/2} e^{-\frac{1}{2\kappa_n} (x - a_n)^2}$ . We see that

$$\int_{I'_\varepsilon} g_n(x) dx = \int_{I_\varepsilon} g_n(x) dx + \int_{a-\varepsilon}^{a+\varepsilon} g_n(x) dx + \int_{b-\varepsilon}^{b+\varepsilon} g_n(x) dx.$$

Moreover,  $\int_{a-\varepsilon}^{a+\varepsilon} g_n(x) dx \leq 2(2\pi\kappa_n)^{-1/2} \varepsilon$ . Thus, denoting  $R_n := \int_{(J_n)'_\varepsilon} g_n dx - \int_{(J_n)_\varepsilon} g_n(x) dx$ , we see that  $|R_n| \leq 4(2\pi\kappa_n)^{-1/2} \varepsilon \leq C\varepsilon$  where  $C := \max_{n \in N_J} 4(2\pi\kappa_n)^{-1/2}$ . Let us denote  $A_n := \int_{(J_n)_\varepsilon} g_n(x) dx$ . Note that  $A_n \leq 1$ . We have, denoting  $N_J := \{n_1, \dots, n_p\}$ ,

$$\int_{J'_\varepsilon \setminus J_\varepsilon} G_{a,\kappa} \leq \prod_{n \in N_J} (A_n + C\varepsilon) - \prod_{n \in N_J} A_n = \sum_{k=0}^{p-1} \sigma_k(A_{n_1}, \dots, A_{n_p})(C\varepsilon)^{p-k} \leq \varepsilon \sum_{k=0}^{p-1} \sigma_k(1, \dots, 1) C^{p-k}$$

where the  $\sigma_k$  are the elementary symmetric polynomials. This shows finally that  $J$  is  $G_{a,\kappa}$ -regular.  $\square$

**Theorem 4.22.** The Gaussian integrator  $G_{a,\kappa}$  is  $\sigma$ -cellular.

*Proof.* By the previous lemma, any open cell is  $G_{a,\kappa}$ -regular, and therefore  $G_{a,\kappa}$  is  $\sigma$ -cellular.  $\square$

**Theorem 4.23.** Any Borel subset of  $\overline{\mathbb{R}}^{\mathbb{N}}$  is  $G_{a,\kappa}$ -measurable and the map

$$\begin{aligned} \mathcal{B}_{\overline{\mathbb{R}}^{\mathbb{N}}} &\rightarrow \overline{\mathbb{R}}_+ \\ A &\mapsto \begin{cases} \int \mathbb{1}_A G_{a,\kappa}, & \text{if } A \in \mathcal{G}_{G_{a,\kappa}} \\ \infty & \text{otherwise} \end{cases} \end{aligned}$$

coincide with the Gaussian measure  $\gamma_{a,\kappa}$  on  $\overline{\mathbb{R}^{\mathbb{N}}}$ . Moreover,  $\mathcal{L}^1(\overline{\mathbb{R}^{\mathbb{N}}}, G_{a,\kappa}) = \mathcal{L}^1(\overline{\mathbb{R}^{\mathbb{N}}}, \gamma_{a,\kappa})$ , and for any  $f \in \mathcal{L}^1(\overline{\mathbb{R}^{\mathbb{N}}}, G_{a,\kappa})$ ,

$$\int f d\gamma_{a,\kappa} = \int f G_{a,\kappa}.$$

*Proof.* By Theorems 4.22 and 4.20, the hypothesis of Theorems 3.13, 3.16 and 3.21 are satisfied. This shows that we can define the measure  $\mu$  on  $\mathcal{B}_{\overline{\mathbb{R}^{\mathbb{N}}}}$  associated to  $G_{a,\kappa}$ . To prove that  $\mu$  is equal to  $\gamma_{a,\kappa}$ , by [13, Theorem 21.10, p. 503], it is enough to check that these measures coincide on the semialgebra of cells of  $\overline{\mathbb{R}^{\mathbb{N}}}$ , since this semialgebra generates  $\mathcal{B}_{\overline{\mathbb{R}^{\mathbb{N}}}}$ . Since  $\overline{\mathbb{R}^{\mathbb{N}}} \setminus \mathbb{R}^{\mathbb{N}}$  is a null set for  $\gamma_{a,\kappa}$ , for any cell  $J$  of  $\overline{\mathbb{R}^{\mathbb{N}}}$ ,  $\gamma_{a,\kappa}(J) = \gamma_{a,\kappa}(J \cap \mathbb{R}^{\mathbb{N}}) = \prod_{n \in \mathbb{N}} \mathcal{N}(a_n, \kappa_n)(J_n \cap \mathbb{R})$ . By definition of  $\mu$ , and the fact that  $J$  is  $G_{a,\kappa}$ -regular, the last term of this equality is actually equal to  $\mu(J)$ .  $\square$

### 4.3 Wiener measure

In this section, we will consider the Wiener measure as defined in [12]. This measure is defined over the compact metrizable space

$$\mathcal{P} := \overline{\mathbb{R}}^{\mathbb{Q}_+^*}.$$

We use here  $\overline{\mathbb{R}}$  as a compactification of  $\mathbb{R}$ , while the one-point compactification is used in [12], but this modification does not change the Wiener measure construction.

We see in the section the space  $\mathcal{P}$  as the product division space of the compact metrizable division family  $(\overline{\mathbb{R}})_{\mathbb{Q}_+^*}$ .

**Notation 4.24.** For any  $t > 0$ , and  $x \in \mathbb{R}$ , we denote

$$p(t, x) := (4\pi t)^{-1/2} e^{-\frac{1}{4t}x^2}.$$

If  $N \in \mathcal{F}(\mathbb{Q}_+^*)$  is nonempty and  $k = \text{card } N$ , we denote for all  $x \in \mathbb{R}^k$ ,

$$p_N(x) := p(t_1, x_1)p(t_2 - t_1, x_2 - x_1) \cdots p(t_k - t_{k-1}, x_k - x_{k-1})$$

where  $N = \{t_1, \dots, t_k\}$ ,  $t_1 < \dots < t_k$ .

**Definition-Theorem 4.25.** The **Wiener measure** on  $\mathcal{P}$  is the unique probability measure  $W$  on the Borel subsets of  $\mathcal{P}$  such that for any cell  $J \neq \mathcal{P}$  of  $\mathcal{P}$ ,

$$W(J) = \int_{[J]} p_{N_J}(x) dx.$$

where  $N_J = \{t_1, \dots, t_k\}$ ,  $t_1 < \dots < t_k$ ,  $k := \text{card } N_J$ , and for any  $A \subseteq \overline{\mathbb{R}^{N_J}}$ ,  $[A] \subseteq \mathbb{R}^k$  is the image of  $A \cap \mathbb{R}^{N_J}$  under the identification  $\overline{\mathbb{R}^{N_J}} \rightarrow \overline{\mathbb{R}^k}$ ,  $\omega \mapsto (\omega(t_1), \dots, \omega(t_k))$ .

*Proof.* The existence claim follows from [12, Theorem 16.1 and following remarks, p. 223], and the uniqueness claim follows from [13, Theorem 21.10, p. 503] and the fact that the semialgebra of cells generates  $\mathcal{B}_{\mathcal{P}}$ .  $\square$

The goal of this section is to show that the Wiener measure is the measure of associated to a particular integrator in the sense of the gauge integration. The natural integrator is given by the previous Wiener measure characterization. We thus set :

**Definition 4.26.** The **Wiener integrator** is the restriction of the Wiener measure on cells of  $\mathcal{P}$ . We denote  $w$  this restriction.

**Lemma 4.27.** Any cell  $J$  of  $\mathcal{P}$  is  $w$ -regular.

*Proof.* Let  $\varepsilon > 0$ . We keep the same notation  $I_\varepsilon$  and  $I'_\varepsilon$  for any interval  $I$  of  $\overline{\mathbb{R}}$  as defined in the proof of Lemma 4.21.

Let  $J$  be a cell of  $\mathcal{P}$ . Define  $J_\varepsilon$  as the open cell  $\prod_{t \in N_J} (J_t)_\varepsilon \oplus \overline{\mathbb{R}}^{\mathbb{Q}_+^* \setminus N_J}$ , and  $J'_\varepsilon$  as the closed cell  $\prod_{t \in N_J} (J_t)'_\varepsilon \oplus \overline{\mathbb{R}}^{\mathbb{Q}_+^* \setminus N_J}$ . Note that  $J_\varepsilon \subseteq J \subseteq J'_\varepsilon$ . Since  $w$  is additive (as the restriction of the measure  $W$ ),  $w_{J'_\varepsilon \setminus J_\varepsilon}$  is additive and thus integrable. Moreover, by additivity, and Fubini theorem

$$\begin{aligned} \int_{J'_\varepsilon \setminus J_\varepsilon} w &= \int_{J'_\varepsilon} w - \int_{J_\varepsilon} w = W(J'_\varepsilon) - W(J_\varepsilon) \\ &= \int_{[\prod_{t \in N_J} (J_t)'_\varepsilon]} p_{N_J}(x) dx - \int_{[\prod_{t \in N_J} (J_t)_\varepsilon]} p_{N_J}(x) dx \\ &= \int_{[D]} p_{N_J}(x) dx = \sum_{\sigma \in S} \int_{[\prod_{i=1}^k A_{t_i, \varepsilon}^{(\sigma_i)}]} p_{N_J}(x) dx. \end{aligned}$$

where we define :

$$\begin{aligned} S &:= \{-1, 1\}^k \setminus \{(-1, \dots, -1)\} \\ D &:= \cup_{\sigma \in S} \prod_{i=1}^k A_{t_i, \varepsilon}^{(\sigma_i)} \\ A_{t, \varepsilon}^{(-1)} &= (J_t)_\varepsilon \text{ and } A_{t, \varepsilon}^{(1)} = (J_t)'_\varepsilon \setminus (J_t)_\varepsilon. \end{aligned}$$

With the convention  $x_0 = t_0 = 0$ , for any  $x \in \mathbb{R}^k$ , and  $i \in \{1, \dots, k\}$ ,

$$p(t_i - t_{i-1}, x_i - x_{i-1}) p(t_{i+1} - t_i, x_{i+1} - x_i) \leq C$$

where  $C := \max_{1 \leq i \leq k} (4\pi(t_i - t_{i-1})(t_{i+1} - t_i))^{-1/2}$ . For each  $\sigma \in S$ , let  $i_\sigma$  be the first index  $i$  such that  $\sigma_{i_\sigma} \neq -1$ . Using the convention  $p_\emptyset = 1$ , we get

$$\int_{J'_\varepsilon \setminus J_\varepsilon} w \leq 2\varepsilon C \sum_{\sigma \in S} \int_{\mathbb{R}^{k-1}} p_{\{1, \dots, i_\sigma-1\}}(x) p_{\{i_\sigma+1, \dots, k\}}(x) dx$$

which leads to the result. □

**Theorem 4.28.** *The Wiener integrator is additive, integrable and  $\sigma$ -cellular.*

*Proof.* The additivity (and therefore integrability) of  $w$  is a direct consequence of the  $\sigma$ -additivity of  $W$ . The fact that  $w$  is  $\sigma$ -cellular follows directly from Lemma 4.27. □

**Theorem 4.29.** *Any Borel subset of  $\mathcal{P}$  is  $w$ -measurable and the map*

$$\begin{aligned} \mathcal{B}_\mathcal{P} &\rightarrow \overline{\mathbb{R}}_+ \\ A &\mapsto \begin{cases} \int \mathbb{1}_A w, & \text{if } A \in \mathcal{S}_w \\ \infty & \text{otherwise} \end{cases} \end{aligned}$$

*coincide with the Wiener measure  $W$  on  $\mathcal{P}$ . Moreover,  $\mathcal{L}^1(\overline{\mathbb{R}}^{\mathbb{N}}, w) = \mathcal{L}^1(\overline{\mathbb{R}}^{\mathbb{N}}, W)$ , and for any integrable function  $f \in \mathcal{L}^1(\overline{\mathbb{R}}^{\mathbb{N}}, w)$ ,*

$$\int f dW = \int f w.$$

*Proof.* By Theorem 4.28, the hypothesis of Theorems 3.13, 3.16 and 3.21 are satisfied. This shows that we can define the measure  $\mu$  on  $\mathcal{B}_\mathcal{P}$  associated to  $w$ . To prove that  $\mu$  is equal to  $W$ , by Definition-Theorem 4.25, it is enough to check that these measures coincide on the semialgebra of cells of  $\mathcal{P}$ . By Theorem 3.23 and Lemma 4.27, the result follows immediately. □

## 4.4 Haar measure on the infinite torus

We consider here the infinite torus  $\mathbb{T}^{\mathbb{N}}$  as the product of the compact metrizable division family  $(\mathbb{T})_{n \in \mathbb{N}}$  (see Example 2.16, item 4). The infinite torus has a natural compact metrizable abelian group structure, and its normalized Haar measure is the product measure  $\mu = \otimes_n \mu_n$  where  $\mu_n = \nu$  is the probability Haar measure on the circle  $\mathbb{T}$ .

To prove that this measure can be constructed from infinite gauge integrals over  $\mathbb{T}^{\mathbb{N}}$ , we proceed as in the case of Gaussian measures.

**Notation 4.30.** We denote  $m$  the restriction of  $\mu$  on the cells of  $\mathbb{T}^{\mathbb{N}}$ .

**Lemma 4.31.** Any cell of  $\mathbb{T}^{\mathbb{N}}$  is  $m$ -regular.

*Proof.* Let  $\varepsilon > 0$ . If  $A$  is a cell of  $\mathbb{T}$  different from  $\mathbb{T}$ , it is of the form  $A_I := \{e^{i2\pi\theta} : \theta \in I\}$  where  $I$  is an interval of length strictly smaller than  $1/2$ . Using the notation of the proof of Lemma 4.21 we define  $A_\varepsilon = A_{I_\varepsilon}$  and  $A'_\varepsilon = A_{I'_{\varepsilon'}}$  where  $\varepsilon' = \min\{\varepsilon, (\frac{1}{2} - (b-a))/3\}$  (so that  $I'_{\varepsilon'}$  is of length strictly less than  $\frac{1}{2}$ ), where  $a = \inf I$  and  $b = \sup I$ .

Let  $J$  be a cell of  $\mathbb{T}^{\mathbb{N}}$ . Define  $J_\varepsilon$  as the open cell  $\prod_{n \in \mathbb{N}_J} (J_n)_\varepsilon \oplus \mathbb{T}^{\mathbb{N} \setminus \mathbb{N}_J}$ , and  $J'_\varepsilon$  as the closed cell  $\prod_{n \in \mathbb{N}_J} (J_n)'_\varepsilon \oplus \mathbb{T}^{\mathbb{N} \setminus \mathbb{N}_J}$ . Note that  $J_\varepsilon \subseteq J \subseteq J'_\varepsilon$ . Since  $m$  is additive,  $(m)_{J'_\varepsilon \setminus J_\varepsilon}$  is integrable. Following the same arguments of the proof of Lemma 4.21, we obtain

$$\int_{J'_\varepsilon \setminus J_\varepsilon} m \leq C\varepsilon,$$

where  $C$  is a constant that is independent of  $\varepsilon$ , and only depend on  $J$ . The result follows.  $\square$

**Theorem 4.32.** The cell function  $m$  is additive (and thus integrable), and  $\sigma$ -cellular.

*Proof.* As in the case of the Wiener integrator, the additivity (and therefore integrability) of  $m$  is a direct consequence of the  $\sigma$ -additivity of  $\mu$ . The fact that  $m$  is  $\sigma$ -cellular follows directly from the previous lemma.  $\square$

**Theorem 4.33.** Any Borel subset of  $\mathbb{T}^{\mathbb{N}}$  is  $m$ -measurable and the map

$$\begin{aligned} \mathcal{B}_{\mathbb{T}^{\mathbb{N}}} &\rightarrow \overline{\mathbb{R}}_+ \\ A &\mapsto \begin{cases} \int \mathbf{1}_A m, & \text{if } A \in \mathcal{S}_m \\ \infty & \text{otherwise} \end{cases} \end{aligned}$$

coincide with the Haar measure  $\mu$  of  $\mathbb{T}^{\mathbb{N}}$ . Moreover,  $\mathcal{L}^1(\mathbb{T}^{\mathbb{N}}, m) = \mathcal{L}^1(\mathbb{T}^{\mathbb{N}}, \mu)$ , and for any  $f \in \mathcal{L}^1(\mathbb{T}^{\mathbb{N}}, m)$ ,

$$\int f d\mu = \int f m.$$

*Proof.* By Theorem 4.32, the hypothesis of Theorems 3.13, 3.16 and 3.21 are satisfied. As in the case of the Wiener measure, to prove that the measure associated to  $m$  is equal to the Haar measure  $\mu$ , it is enough to check that these measures coincide on the semialgebra of cells of  $\mathcal{T}^{\mathbb{N}}$  since that semialgebra generates the Borel sets of  $\mathbb{T}^{\mathbb{N}}$ . This fact follows directly from Lemma 4.31 and Theorem 3.23.  $\square$

## 5 Appendix

### 5.1 Generalized Cousin's lemma

We fix here a compact metrizable division family  $(X_t)_{t \in T}$  and denote  $(X, \mathcal{C})$  the associated product division space. The goal of this appendix is to prove the following generalization of Cousin's lemma :

**Theorem 5.1.** For any gauge  $\gamma$  and elementary set  $E$ , there exists a  $\gamma$ -resolution of  $E$ , i.e.  $\mathcal{R}_E^\gamma \neq \emptyset$ .

The proof relies on a contradiction argument, and is based on an adaptation of the arguments of [5], which deals with the case of  $\overline{\mathbb{R}}^T$  and a different notion of gauge.

We first introduce the following definition :

**Definition 5.2.** A gauge  $\gamma$  on  $X$  is **nonresolvable** if there is no  $\gamma$ -resolution of  $X$ . A cell  $J$  of  $X$  is **non- $\gamma$ -resolvable** if there is no  $\gamma$ -resolution of  $J$ .

**Remark 5.3.** Note the following preliminary fact about nonresolvable gauges (besides that they do not exist, as we aim to show) : if  $\gamma = (L, \delta)$  is nonresolvable then the function  $L : X \mapsto \mathcal{F}(T)$  is always such that  $L(x) \neq \emptyset$  for all  $x \in X$ . Indeed, if there is a point  $x \in X$  such that  $L(x) = \emptyset$ , then  $\{(X, x)\}$  is a  $\gamma$ -resolution of  $X$ , since we have then  $N_x = \emptyset$  and  $\delta_\emptyset(x) = \{X_\emptyset\}$ .

**Notation 5.4.** In the following, if  $(t_n)_{n \in \mathbb{N}}$  is a fixed sequence of elements of  $T$ , we will denote  $N_n := \{t_0, \dots, t_{n-1}\}$  (for  $n \geq 1$ ),  $N_0 := \emptyset$  and  $T_n := T \setminus N_n$ .

To prove Theorem 5.1, we first start with the following lemma :

**Lemma 5.5.** Let  $(t_n)_{n \in \mathbb{N}}$  be a fixed sequence in  $T$ . We define the relation  $R$  on the set  $S$  of all pairs  $(n, \gamma)$  where  $n \in \mathbb{N}$  and  $\gamma$  is a nonresolvable regular gauge on  $X_{T_n}$  the following way :  $(n, \gamma)R(n', \gamma')$  if and only if  $n' = n + 1$ , and  $\gamma' = (L', \delta')$  is such that there is a point  $x \in X_{\{t_n\}}$  such that for all  $N \in \mathcal{F}(T_{n+1})$ ,

$$L' = L(x \oplus \cdot) \setminus \{t_n\}, \quad (5.1)$$

$$\delta'_N = \delta_{N \cup \{t_n\}}(x \oplus \cdot)|_N. \quad (5.2)$$

Then the relation  $R$  is left-total, or in other words, for any  $(n, \gamma) \in S$ , there is  $(n', \gamma') \in S$  such that  $(n, \gamma)R(n', \gamma')$ .

*Proof.* Remark first that the equations (5.1) and (5.2) make sense since  $y \mapsto L(x \oplus y) \setminus \{t_n\}$  is a map from  $X_{T_{n+1}}$  into  $\mathcal{F}(T_{n+1})$ , and since  $\delta_{N \cup \{t_n\}}$  is a map from  $X_{T_n}$  into  $\mathcal{D}_{X_{N \cup \{t_n\}}}$ , the map  $y \mapsto (\delta_{N \cup \{t_n\}}(x \oplus y))|_N$  is a map from  $X_{T_{n+1}}$  into  $\mathcal{D}_{X_N}$ .

Let  $(n, \gamma = (L, \delta)) \in S$ . We want to prove that the regular gauge  $\gamma' = (L', (\delta')_N)$  on  $X_{T_{n+1}}$  defined by the equations (5.1) and (5.2) is nonresolvable for at least one  $x \in X_{\{t_n\}}$ .

Fix an enumeration  $(D_k)_{k \in \mathbb{N}}$  of the countable set

$$\{\delta_{N \cup \{t_n\}}(z)|_{\{t_n\}} : N \in \mathcal{F}(T_{n+1}) \text{ and } z \in X_{T_n}\}.$$

Note that each  $D_k$  is a division of  $X_{\{t_n\}}$ . Define now  $D'_k := \Delta \wedge D_0 \wedge \dots \wedge D_k$  for all  $k \in \mathbb{N}$ , where  $\Delta$  is a fixed division of  $X_{\{t_n\}}$  that is different from  $\{X_{\{t_n\}}\}$  (such division exists since  $X_{\{t_n\}}$  is nontrivial, see Remark 2.19). Thus, for any  $k$ ,  $D'_{k+1} \leq D'_k \leq D_k$ , and for any  $J \leq D'_k$ ,  $N_J = \{t_n\}$ .

Since  $\gamma$  is nonresolvable, there is at least one cell  $J$  in  $D'_0$  such that  $J \oplus X_{T_{n+1}}$  is non- $\gamma$ -resolvable. Indeed, if it was not the case, we would obtain a  $\gamma$ -resolution of  $X_{T_n}$  by considering the union  $\cup_{J \in D'_0} R_J$  where  $R_J$  is a  $\gamma$ -resolution of  $J \oplus X_{T_{n+1}}$ . Take  $J_0$  to be one such cell in  $D'_0$  such that  $J \oplus X_{T_{n+1}}$  is non- $\gamma$ -resolvable. Since  $J_0 \in D'_0$ , we have in particular that  $J_0 \leq D'_0$ .

Suppose now that for a given  $k \in \mathbb{N}$ , we have defined  $J_k$  as a cell of  $X_{\{t_n\}}$  such that  $J_k \leq D'_k$  and  $J_k \oplus X_{T_{n+1}}$  is non- $\gamma$ -resolvable. For the same reason as before, there is at least one cell  $J \in D'_{k+1}$  such that  $(J_k \cap J) \oplus X_{T_{n+1}}$  is non- $\gamma$ -resolvable. We define  $J_{k+1}$  to be  $J_k \cap J$  for one such cell  $J$ . Thus, we see that  $J_{k+1} \leq D'_{k+1}$ .

By the principle of dependent choices (DC), it follows that there exists a decreasing sequence  $(J_k)$  of cells of  $X_{\{t_n\}}$  such that  $J_k \leq D'_k$  and  $J_k \oplus X_{T_{n+1}}$  is non- $\gamma$ -resolvable, for all  $k \in \mathbb{N}$ .

The set  $\cap_{k \in \mathbb{N}} \overline{J}_k$  is nonempty since  $(\overline{J}_k)$  is a decreasing sequence of nonempty closed subsets of the compact space  $X_{\{t_n\}}$ . Take now  $x \in \cap_{k \in \mathbb{N}} \overline{J}_k$ . We claim that  $\gamma'$  as defined by (5.1) and (5.2) is nonresolvable. Suppose for contradiction that there exists a  $\gamma'$ -resolution  $R$  of  $X_{T_{n+1}}$ . For each  $k$ , we define

$$R_k := \{(J_k \oplus J, x \oplus y) : (J, y) \in R\}.$$

Clearly,  $R_k$  is a resolution of  $J_k \oplus X_{T_{n+1}}$ . Since  $R$  is  $\gamma'$ -fine, for all  $p = (J, y) \in R$ , we have  $N_J \supseteq L'(y)$  and  $\underline{J} \leq \delta'_{N_J}(y)$ . Since  $N_{J_k \oplus J} = \{t_n\} \cup N_J$  we get  $N_{J_k \oplus J} \supseteq L(x \oplus y)$  for all  $k$  and  $p = (J, y) \in R$ . Moreover, for all  $k$  and  $p = (J, y) \in R$ ,

$$\underline{J_k \oplus J} = J_k \oplus \underline{J} \leq D'_k \otimes \delta'_{N_J}(y) = D'_k \otimes \delta_{N_J \cup \{t_n\}}(x \oplus y)|_{N_J}.$$

By definition of the enumeration  $(D_k)_{k \in \mathbb{N}}$ , for any  $p = (J, y) \in R$ , there is  $k_p \in \mathbb{N}$  such that

$$D_{k_p} = \delta_{N_J \cup \{t_n\}}(x \oplus y)|_{\{t_n\}}.$$

We set  $k_{\max} := \max\{k_p : p \in R\}$ . Thus,  $J_{k_{\max}} \leq D'_{k_{\max}} \leq D'_{k_p} \leq D_{k_p}$  for each  $p \in R$ . Observe then that for all  $p = (J, y) \in R$ ,

$$\underline{J_{\max} \oplus J} \leq D'_{k_{\max}} \otimes \delta_{N_J \cup \{t_n\}}(x \oplus y)|_{N_J} \leq \delta_{N_J \cup \{t_n\}}(x \oplus y) = \delta_{N_{J_{\max} \oplus J}}(x \oplus y).$$

This shows finally that  $R_{k_{\max}}$  is a  $\gamma$ -resolution of  $J_{k_{\max}} \oplus X_{T_{n+1}}$ , which is a contradiction.  $\square$

The main work was done in the lemma, we can now conclude with the proof of the generalized Cousin's lemma :

*Proof of Theorem 5.1.* First observe that we can restrict ourselves to the case of a regular gauge. Indeed, by Proposition 2.32, if  $\gamma$  is a gauge, then there is a regular gauge  $\gamma'$  such that  $\gamma' \leq \gamma$ .

We first prove that  $X$  itself admits a  $\gamma$ -resolution. Suppose then for contradiction that  $\gamma = (L, \delta)$  is a nonresolvable regular gauge on  $X$ . Since  $L$  has a countable image, the set  $L(X)$  is countable, and there is a sequence  $(F_n)_{n \in \mathbb{N}}$  in  $\mathcal{F}(T)$  such that  $L(X) = \{F_n : n \in \mathbb{N}\}$ . Since  $\cup_n F_n$  is a countable subset of  $T$ , there is an enumeration  $(t_n)_n$  of  $\cup_n F_n$ . Thus, using the previous notations, we see that for any  $x \in X$ , there is  $n \in \mathbb{N}$  such that  $L(x) \subseteq N_n$ .

By the previous lemma, and (DC), there exists a sequence  $(x_n)_{n \in \mathbb{N}}$  such that  $x_n \in X_{\{t_n\}}$  for all  $n \in \mathbb{N}$ , and a sequence  $(\gamma_n = (L_n, (\delta_n)_N))_{n \in \mathbb{N}}$  with  $\gamma_0 = \gamma$ ,  $\gamma_n$  being a nonresolvable regular gauge on  $X_{T_n}$ , such that for any  $n \in \mathbb{N}$ , and  $N \in \mathcal{F}(T_{n+1})$ ,

$$\begin{aligned} L_{n+1} &= L_n(x_n \oplus \cdot) \setminus \{t_n\}, \\ (\delta_{n+1})_N &= (\delta_n)_{N \cup \{t_n\}}(x_n \oplus \cdot)|_N. \end{aligned}$$

Let  $x \in X$  be such that  $x(t_n) := x_n$  for all  $n \in \mathbb{N}$ . Observe that for all  $n \in \mathbb{N}$ ,  $L_{n+1}(x_{T_{n+1}}) = L_n(x_{T_n}) \setminus \{t_n\}$ . By induction, we obtain then for all  $n \in \mathbb{N}$ ,  $L_n(x_{T_n}) = L(x) \setminus N_n$ . Since  $\gamma_n$  is nonresolvable,  $L_n(x_{T_n})$  is nonempty, as noted in Remark 5.3. Thus, for any  $n$ ,  $L(x) \setminus N_n$  is nonempty. But this is impossible since  $L(x)$  is included in one  $N_n$  for a  $n \in \mathbb{N}$ . This concludes the proof of the claim that  $X$  admits a  $\gamma$ -resolution.

We now prove that if  $E$  is an elementary set, then there is a  $\gamma$ -resolution of  $E$ . Since  $E$  is an elementary set, there is a division  $D$  such that  $E = \cup D$ . And since  $X \setminus E$  is also an elementary set there is a division  $D'$  such that  $X \setminus E = \cup D'$ . The union  $D \cup D'$  is a division of  $X$ . Consider then the gauge  $\gamma' := \gamma \wedge \gamma_{D \cup D'}$ . By what we just proved, there is a  $\gamma'$ -resolution  $R$  of  $X$ . It follows then from Proposition 2.38 that  $R_1 := \{(J, x) \in R : J \subseteq E\}$  is a  $\gamma$ -resolution of  $E$ .  $\square$

## 5.2 Moore–Smith double limit theorem

In this appendix, we give a self-contained proof of the classical Moore-Smith theorem [7] on double nets. This result is used in Lemma 2.57 and provide a sufficient condition under which the exchange of limits for a given net indexed by a product of directed sets is possible.

Let  $A$  and  $B$  two (upward) directed sets. The product  $A \times B$  is naturally a directed set with the following preorder :  $(\alpha, \beta) \leq (\alpha', \beta')$  if and only if  $\alpha \leq \alpha'$  and  $\beta \leq \beta'$ .

**Theorem 5.6. (Moore–Smith double limit theorem)** *Let  $(E, d)$  be a complete metric space and fix three nets  $(x_{\alpha, \beta})_{(\alpha, \beta) \in A \times B}$ ,  $(\xi_\alpha)_{\alpha \in A}$  and  $(\zeta_\beta)_{\beta \in B}$  in  $E$ , respectively indexed by  $A \times B$ ,  $A$  and  $B$ . Suppose that*

1. *The net  $(x_{\alpha, \beta})_{\beta \in B}$  converges to  $\xi_\alpha$  in  $E$ , uniformly in  $\alpha \in A$ .*

2. The net  $(x_{\alpha,\beta})_{\alpha \in A}$  converges to  $\zeta_\beta$  in  $E$ , for any  $\beta \in B$ .

Then :

1. The three nets  $(x_{\alpha,\beta})_{(\alpha,\beta) \in A \times B}$ ,  $(\xi_\alpha)_{\alpha \in A}$  and  $(\zeta_\beta)_{\beta \in B}$  converge.

2. Those three nets have the same limit :

$$\lim_{(\alpha,\beta) \in A \times B} x_{\alpha,\beta} = \lim_{\alpha \in A} \xi_\alpha = \lim_{\beta \in B} \zeta_\beta.$$

*Proof.* The first step is to check that  $(\xi_\alpha)$  converges. For this, fix  $\varepsilon > 0$ . By hypothesis 1., there is  $\beta_0 \in B$  such that for any  $\beta \geq \beta_0$  and any  $\alpha \in A$ ,  $d(x_{\alpha,\beta}, \xi_\alpha) < \varepsilon/3$ . It follows from hypothesis 2. that  $(x_{\alpha,\beta_0})_\alpha$  is a Cauchy net. Thus, there is  $\alpha_0 \in A$  such that for any  $\alpha, \alpha' \geq \alpha_0$ ,  $d(x_{\alpha,\beta_0}, x_{\alpha',\beta_0}) < \varepsilon/3$ .

As a consequence, for any  $\alpha, \alpha' \geq \alpha_0$ , we have

$$d(\xi_\alpha, \xi_{\alpha'}) \leq d(\xi_\alpha, x_{\alpha,\beta_0}) + d(x_{\alpha,\beta_0}, x_{\alpha',\beta_0}) + d(x_{\alpha',\beta_0}, \xi_{\alpha'}) < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

By completeness of  $E$ , this implies that  $(\xi_\alpha)$  converges in  $E$ . Let  $L$  be its limit.

The second step is to check that  $(x_{\alpha,\beta})_{(\alpha,\beta) \in A \times B}$  converges to  $L$ . Fix again  $\varepsilon > 0$ . Since there is  $\alpha_0 \in A$  such that for any  $\alpha \geq \alpha_0$ ,  $d(\xi_\alpha, L) < \varepsilon/2$ , and since, by hypothesis 1., there is  $\beta_0 \in B$  such that for any  $\alpha \in A$ , and  $\beta \geq \beta_0$ ,  $d(x_{\alpha,\beta}, \xi_\alpha) < \varepsilon/2$ , we see that for any  $\alpha \geq \alpha_0$  and  $\beta \geq \beta_0$ ,

$$d(x_{\alpha,\beta}, L) \leq d(x_{\alpha,\beta}, \xi_\alpha) + d(\xi_\alpha, L) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This shows then that  $(x_{\alpha,\beta})_{(\alpha,\beta)}$  converges to  $L$

The third and final step is to check that  $(\zeta_\beta)$  converges to  $L$  too. Fix once more  $\varepsilon > 0$ . Since we just proved that  $(x_{\alpha,\beta})_{(\alpha,\beta)}$  converges to  $L$ , we know then that there exist  $(\alpha_0, \beta_0) \in A \times B$  such that for any  $\alpha \geq \alpha_0$  and  $\beta \geq \beta_0$ ,  $d(x_{\alpha,\beta}, L) < \varepsilon/2$ . Let  $\beta \geq \beta_0$  be given. Hypothesis 2. implies that there is  $\alpha'_0 \in A$  such that for any  $\alpha \geq \alpha'_0$ ,  $d(x_{\alpha,\beta}, \zeta_\beta) < \varepsilon/2$ . Take  $\alpha$  such that  $\alpha \geq \alpha'_0$  and  $\alpha \geq \alpha_0$ . We obtain the following estimation :

$$d(\zeta_\beta, L) \leq d(\zeta_\beta, x_{\alpha,\beta}) + d(x_{\alpha,\beta}, L) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This yields the expected result. □

## References

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