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Abstract

Autonomous vehicles (AVs) allow new ways of regulating the traffic flow on road networks. Most of available results in this direction are based on microscopic approaches, where ODEs describe the evolution of regular cars and AVs. In this paper, we propose a multiscale approach, based on recently developed models for moving bottlenecks. Our main result is the proof of existence of solutions for time-varying bottleneck speed, which corresponds to open-loop controls with bounded variation.

Key Words: Conservation laws; PDE-ODE system; macroscopic traffic models; moving bottlenecks; autonomous vehicles; control problems.

AMS Subject Classifications: 90B20, 35L65.

1 Introduction

Autonomous Vehicles (briefly AVs) represent the most disruptive technology for traffic regulation [14, 24, 31, 32, 34]. The effect of AVs in terms of influencing bulk traffic has been studied *in-silico* [7, 13, 17, 29, 33], artificial environment [16] and also in experiments [28]. In particular, the results of [28] showed a potential decrease of up to 40% in fuel consumption by dampening of traffic waves. Despite such achievements, a complete macroscopic theory for control of bulk traffic via AVs is still missing. The need of a macroscopic theory is due to the curse of dimensionality preventing control design for microscopic models [12].

Our approach to bypass current limitations is based on the idea of using macroscopic models for the bulk traffic, consisting of partial differential equations (PDEs), paired with microscopic ones for the AVs, consisting of ordinary differential equations (ODEs). The history of modeling traffic via partial differential equations started with the celebrated Lighthill-Whitham-Richards (briefly LWR) model for traffic flow on a road [23, 26]. The authors assumed that the average speed v depends only on local density, thus obtaining a scalar

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conservation law. More recently, the LWR model was paired to an ODE via a moving flux constraint producing a multiscale model, see [8, 9, 11]. Moreover, other works used coupled ODE-PDE systems [2, 8, 18, 19, 30].

Here we extend the multiscale model of [9] by assuming that the AV desired speed, described by a control u, is time-dependent, as in the case of regulation by a centralized controller. The resulting system is obtained by modifying the ODE for the AV, which is given by the minimum between the desired speed u and the average speed v of local downstream traffic. The proposed approach has already been investigated numerically in [25], using Model Predictive Control and proving its effectiveness in reducing traffic nuisance.

We base our work on the many results available for the multiscale model of [9]. In particular, existence and well-posedness of solutions was investigated in [11, 21, 22] and the numerical aspect in [4, 8, 10]. Moreover, the multiscale model was extended to second order in [30], replacing the LWR by Aw-Rascle-Zhang model.

The complete multiscale model is given by the system (2), consisting of a scalar conservation law, a controlled ODE and a moving flux constraint (plus initial conditions). To couple the equations, we need to specify the capacity reduction function F_α for every AV desired speed. This gives rise to undercompressive shocks also depending on the desired speed u, see (3). The definition of solution needs to reflect this choice via a modified entropy condition, see point 3 of Definition 3.1.

Our main result is the proof of existence of solutions for time-varying speed profile of the AV with bounded variation. The latter corresponds for a centralized controller to an open loop control u with bounded variation in time. In order to achieve the result, we construct approximate solutions via wave-front tracking approximations. In particular, we construct approximation grids allowing all left and right densities of undercompressive shocks for the discretized desired speed.

As usual, the sequence of approximate solutions satisfy some compactness estimate, that allows to pass to the limit. In our case, we need to define a Glimm type functional accounting for the density total variation, a term for variation due to undercompressive shocks and the time-variation of the time-varying speed. We can prove that at each wave interaction such functional either decreases (by a quantity bounded away from zero) or remains constant, but in this case the number of waves does not increase. Moreover, a detailed analysis is necessary to prove the convergence of the AV speed. Once a limit of approximate solutions is obtained, one has to prove that it satisfies the various conditions in the definition of solution. For this we use various results from the above mentioned literature adapted to our case. We remark that these results can be extended to the case of a finite number of autonomous vehicles, provided their trajectories do not interact. This extension is straightforward since all the waves generated by the multiscale system propagate with finite speed.

The paper is organized as follows. Section 2 describes the coupled PDE-ODE model and introduces the basic notation. Section 3 deals with the Cauchy problem. More precisely, in Subsection 3.1 we describe the wave-front tracking algorithm, we construct the approximating grids, and we define the Glimm type functional; in Subsection 3.2 we study the changes in the Glimm type functional, due to wave interactions; finally in Subsection 3.3 we state and prove the main theorem. The paper ends with Appendix A, which contains various technical lemmas used in Subsection 3.3, and with Appendix B, which describes in details the solution to the Riemann problem.

2 Description of the control model

Consider a unidirectional road I, model by the real line $\mathbb{R}$, where the traffic is described by the LWR model [23, 26]

$$\partial_t \rho + \partial_x f(\rho) = 0, \tag{1}$$

where $\rho = \rho(t,x) \in [0,R]$ denotes the macroscopic traffic density at time $t \geq 0$ and at position $x \in \mathbb{R}$, and $f = f(\rho)$ is the flux, which depends only on ρ. The constant R denotes the maximum possible density of the road. As usual the flux f is given by $\rho v(\rho)$, where $v \in \mathbf{C^2}([0,R];[0,+\infty))$ is the average speed of cars. We assume that the flux satisfies the condition

(F) $f : \mathbf{C^2}([0,R];[0,+\infty)), \quad f(0) = f(R) = 0,$
f strictly concave: $-B \leq f''(\rho) \leq -\beta < 0$ for all $\rho \in [0,R]$, for some $\beta, B > 0$.
Moreover, we assume $v'(\rho) < 0$ for every $\rho \in]0,R[$.

Note that, by **(F)**, the speed v is a strictly decreasing function.

Assume that a vehicle, e.g. a police or autonomous vehicle (AV), whose position is described by the variable $y = y(t)$, aims at controlling the behavior of traffic, selecting its maximal speed. Hence the evolution of such a vehicle is described by the ordinary differential equation (ODE)

$$\dot{y}(t) = \omega(\rho(t,y(t));u) := \min\{u(t), v(\rho(t,y(t)_+))\},$$

where $u = u(t) \in [0,V]$, with $V := \max_{\rho\in[0,R]} v(\rho)$, is a control function, selecting the *desired speed* of the vehicle. Indeed, the vehicle can move at its desired speed as long as the downstream traffic moves faster, otherwise it has to adapt. The notations $\rho(t,x_-)$ and $\rho(t,x_+)$ stand respectively for the left and right trace, with respect to the variable x, of ρ at the point (t,x), i.e.

$$\rho(t,x_-) := \lim_{\xi\to x^-} \rho(t,\xi) \qquad \text{and} \qquad \rho(t,x_+) := \lim_{\xi\to x^+} \rho(t,\xi),$$

whose existence is ensured provided $\rho(t,\cdot)$ has finite total variation. In the sequel, the notation is used also for the trace with respect to the time variable t.

Following the model proposed in [9, 19], we consider the following system

$$\partial_t \rho(t,x) + \partial_x f(\rho(t,x)) = 0, \qquad t > 0,\ x \in \mathbb{R}, \tag{2a}$$

$$\dot{y}(t) = \min\{u(t), v(\rho(t,y(t)_+))\}, \qquad t > 0, \tag{2b}$$

$$f(\rho(t,y(t))) - \dot{y}(t)\rho(t,y(t)) \leq F_\alpha(\dot{y}(t)) := \max_{\rho\in[0,R]} \left(\alpha f(\rho/\alpha) - \rho\dot{y}(t)\right), \qquad t > 0, \tag{2c}$$

$$\rho(0,x) = \rho_0(x), \qquad x \in \mathbb{R}, \tag{2d}$$

$$y(0) = y_0. \tag{2e}$$

Above, ρ_0 and y_0 are the initial traffic density and AV position, while the function F_α in (2c), $\alpha \in]0,1[$, represents the road capacity reduction due to the presence of the AV, acting as a moving bottleneck which imposes a unilateral flux constraint at the AV position. To determine the function F_α, we consider reduced flux function

$$\begin{array}{rccc} f_\alpha : & [0,\alpha R] & \longrightarrow & \mathbb{R}^+ \\ & \rho & \longmapsto & \rho v(\rho/\alpha) = \alpha f(\rho/\alpha), \end{array}$$

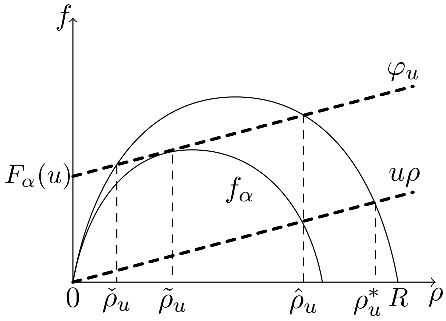


Figure 1: The definition of $\tilde{\rho}_u$, $\check{\rho}_u$, $\hat{\rho}_u$ and ρ_u^*.

which is a strictly concave function satisfying $f_\alpha(0) = f_\alpha(\alpha R) = 0$. For every $u \in [0, V]$, define the point $\tilde{\rho}_u$ as the unique solution to the equation $f'_\alpha(\rho) = u$. Introduce also, for every $u \in [0, V]$, the function

$$\begin{array}{rccc} \varphi_u : & [0, R] & \longrightarrow & \mathbb{R}^+ \\ & \rho & \longmapsto & f_\alpha(\tilde{\rho}_u) + u\,(\rho - \tilde{\rho}_u)\,. \end{array}$$

Hence, if $\dot{y}(t) = u$, the function F_α in (2c) is defined by

$$\begin{array}{rccc} F_\alpha : & [0, V] & \longrightarrow & \mathbb{R}^+ \\ & u & \longmapsto & \varphi_u(0) = f_\alpha(\tilde{\rho}_u) - u\tilde{\rho}_u. \end{array}$$

If $\dot{y}(t) = v(\rho(t, y(t)_+))$, then the inequality (2c) is trivially satisfied since the left-hand side is zero. Finally, the points $0 \leq \check{\rho}_u \leq \tilde{\rho}_u \leq \hat{\rho}_u \leq \rho_u^* \leq R$ are uniquely defined by

$$\check{\rho}_u = \min \mathcal{I}_u, \quad \hat{\rho}_u = \max \mathcal{I}_u, \quad \mathcal{I}_u = \{\rho \in [0, R] : \, f(\rho) = \varphi_u(\rho)\}\,, \tag{3}$$

and implicitly by

$$v(\rho_u^*) = u, \tag{4}$$

see [9] and Figure 1. It is straightforward to see that $\check{\rho}_V = \tilde{\rho}_V = \hat{\rho}_V = \rho_V^* = 0$.

Lemma 2.1 *Assume that the flux f satisfies* **(F)**. *Let $\check{\rho}_u \leq \hat{\rho}_u \leq \rho_u^*$ be defined by* (3) *and* (4). *Then the maps $\check{\rho}(u) : u \mapsto \check{\rho}_u$ and $\hat{\rho}(u) : u \mapsto \hat{\rho}_u$ are strictly decreasing on $[0, V]$ and satisfy*

$$-\infty < \check{\rho}'(u) < 0 \qquad -\infty < \hat{\rho}'(u) < 0 \tag{5}$$

for every $u \in [0, V[$. Moreover, the maps $\check{\rho}(u)$ and $\rho^*(u) : u \mapsto \rho_u^* = v^{-1}(u)$ are also Lipschitz continuous functions. Finally we have

$$\check{\rho}_u < \tilde{\rho}_u < \frac{\tilde{\rho}_u}{\alpha} < \hat{\rho}_u < \rho_u^* \tag{6}$$

for every $u \in [0, V[$.

Proof. First note that $f'\left(\frac{\tilde{\rho}_u}{\alpha}\right) = u$; thus differentiating this expression with respect to u we deduce that $f''\left(\frac{\tilde{\rho}_u}{\alpha}\right)\frac{\tilde{\rho}'_u}{\alpha} = 1$ and so $\tilde{\rho}'_u < 0$ for every $u \in [0, V]$. This proves that $\tilde{\rho}_u$ is strictly decreasing with respect to u.

Observe that $f_\alpha(\rho) < f(\rho)$ for every $\rho \in]0, \alpha R]$ and that the function $H : [0, R] \to \mathbb{R}$, defined by $H(\rho) = f(\rho) - \varphi_u(\rho)$, is strictly concave. By (3), $H(\check{\rho}_u) = H(\check{\rho}_u) = 0$. Moreover $H(\tilde{\rho}_u) = f(\tilde{\rho}_u) - f_\alpha(\tilde{\rho}_u) > 0$ for every $u \in [0, V[$. The concavity of H implies that

$$\check{\rho}_u < \tilde{\rho}_u < \hat{\rho}_u < \rho_u^* \tag{7}$$

for every $u \in [0, V[$. Besides, since the map $\rho \mapsto f(\rho) - u\rho$ is strictly concave for every fixed $u \in [0, V]$, with point of maximum at $\tilde{\rho}_u/\alpha$, by (3) we also deduce that $\check{\rho}_u < \tilde{\rho}_u/\alpha < \hat{\rho}_u$. Therefore (7) implies (6).

By (3), we have

$$f(\check{\rho}_u) - u\check{\rho}_u = f_\alpha(\tilde{\rho}_u) - u\tilde{\rho}_u \qquad \text{and} \qquad f(\hat{\rho}_u) - u\hat{\rho}_u = f_\alpha(\tilde{\rho}_u) - u\tilde{\rho}_u. \tag{8}$$

Differentiating equations (8) w.r.t. u, for every $u \in [0, V[$ we have

$$\check{\rho}'(u) = \frac{\check{\rho}_u - \tilde{\rho}_u}{f'(\check{\rho}_u) - u} = \frac{\check{\rho}_u - \tilde{\rho}_u}{f''(\check{\rho}_u - \theta(\check{\rho}_u - \tilde{\rho}_u/\alpha))(\check{\rho}_u - \tilde{\rho}_u/\alpha)} \geq \frac{1}{f''(\check{\rho}_u - \theta(\check{\rho}_u - \tilde{\rho}_u/\alpha))} \geq -\frac{1}{\beta}, \tag{9}$$

$$\hat{\rho}'(u) = \frac{\hat{\rho}_u - \tilde{\rho}_u}{f'(\hat{\rho}_u) - u} = \frac{\hat{\rho}_u - \tilde{\rho}_u}{f''(\hat{\rho}_u - \theta'(\hat{\rho}_u - \tilde{\rho}_u/\alpha))(\hat{\rho}_u - \tilde{\rho}_u/\alpha)} \leq \frac{1}{f''(\hat{\rho}_u - \theta'(\hat{\rho}_u - \tilde{\rho}_u/\alpha))} \leq -\frac{1}{B} \tag{10}$$

for some $\theta, \theta' \in [0, 1]$. Using (6), (10), **(F)**, and the fact that $u = f'(\tilde{\rho}_u/\alpha)$, we deduce that (5) holds and that both $\check{\rho}_u$ and $\hat{\rho}_u$ are strictly decreasing as functions of $u \in [0, V[$. Finally, by (9), we deduce that

$$-\frac{1}{\beta} \leq \check{\rho}'(u) < 0$$

for every $u \in [0, V[$, proving that the function $\check{\rho}(u)$ is Lipschitz continuous. By **(F)**, v is a continuously differentiable function and $v'(\rho) < 0$ for every $\rho \in]0, R[$. Using the Inverse Function Theorem, we deduce that v^{-1} is a continuously differentiable function and $\rho^*(u) = v^{-1}(u)$. This concludes the proof. $\square$

Lemma 2.2 *Let us consider the sequence ω_n, defined by*

$$\omega_0 = 0, \qquad \omega_{n+1} = \hat{\rho}^{-1}(\check{\rho}(\omega_n)) \qquad \forall n \in \mathbb{N}. \tag{11}$$

Then ω_n is a strictly increasing sequence such that

$$\lim_{n \to +\infty} \omega_n = V.$$

PROOF. By Lemma 2.1, the functions

$$\check{\rho} : [0, V] \to [0, \check{\rho}(0)], \qquad \hat{\rho} : [0, V] \to [0, \hat{\rho}(0)],$$

are strictly decreasing and so invertible. This implies that the sequence ω_n, defined in (11), is well defined and $\omega_n \in [0, V]$ for every $n \in \mathbb{N}$. Note also that (5) in Lemma 2.1 implies that the inverse functions of $\check{\rho}$ and $\hat{\rho}$ are strictly decreasing functions.

First we claim that $\omega_n < V$ for every $n \in \mathbb{N}$. We proceed by induction on n. When $n = 0$, clearly $\omega_0 = 0 < V$. Assume now that $\omega_{n-1} < V$ and so $\check{\rho}(\omega_{n-1}) > 0 = \hat{\rho}(V)$. Hence $\hat{\rho}^{-1}(\check{\rho}(\omega_{n-1})) < V$, i.e. $\omega_n < V$, proving the claim.

We prove now that ω_n is a strictly increasing sequence. Fix $n \in \mathbb{N}$. Since $\omega_n < V$, we deduce, by (6), that $\check{\rho}(\omega_n) < \hat{\rho}(\omega_n)$ and so $\hat{\rho}^{-1}(\check{\rho}(\omega_n)) > \omega_n$, i.e. $\omega_{n+1} > \omega_n$.

Since ω_n is an increasing sequence, then it has a limit L. Clearly $L \in [0, V]$. By (11), we deduce that

$$L = \hat{\rho}^{-1}(\check{\rho}(L)) \quad \Longleftrightarrow \quad \hat{\rho}(L) = \check{\rho}(L) \quad \Longleftrightarrow \quad L = V,$$

concluding the proof. □

Remark 1 If $u_1 < u_2$, then either

$$\check{\rho}(u_2) < \check{\rho}(u_1) < \hat{\rho}(u_2) < \hat{\rho}(u_1)$$

or

$$\check{\rho}(u_2) \le \hat{\rho}(u_2) < \check{\rho}(u_1) < \hat{\rho}(u_1).$$

3 The Cauchy problem

Let us consider an initial density $\rho_0 \in \mathbf{L^1}(\mathbb{R};[0,R])$ with finite total variation, an initial position of the AV $y_0 \in \mathbb{R}$, and an open-loop control $u \in \mathbf{L^1}(\mathbb{R}^+;[0,V])$ with finite total variation. Following [1, 5, 11], we define solutions to (2) as follows.

Definition 3.1 *The couple (ρ, y) provides a solution to (2) if the following conditions hold.*

1. $\rho \in \mathbf{C^0}\left(\mathbb{R}^+;\mathbf{L^1}(\mathbb{R};[0,R])\right)$ *and* $\mathrm{TV}(\rho(t)) < +\infty$ *for all* $t \in \mathbb{R}^+$*;*
2. $y \in \mathbf{W^{1,1}_{loc}}(\mathbb{R}^+;\mathbb{R})$*;*
3. *For every $\kappa \in \mathbb{R}$ and for all $\varphi \in \mathbf{C^1_c}(\mathbb{R}^2;\mathbb{R}^+)$ it holds*

$$\begin{aligned}\int_{\mathbb{R}^+}\int_{\mathbb{R}} \left(|\rho-\kappa|\partial_t\varphi + \mathrm{sgn}(\rho-\kappa)(f(\rho)-f(\kappa))\partial_x\varphi\right)dx\,dt + \int_{\mathbb{R}}|\rho_0-\kappa|\varphi(0,x)\,dx \\ + 2\int_{\mathbb{R}^+}\left(f(\kappa)-\dot{y}(t)\kappa - \min\{f(\kappa)-\dot{y}(t)\kappa, F_\alpha(\dot{y}(t))\}\right)\varphi(t,y(t))\,dt \ge 0\,;\end{aligned} \tag{12}$$

4. *For a.e.* $t > 0$, $f(\rho(t, y(t)_\pm)) - \dot{y}(t)\rho(t,y(t)_\pm) \le F_\alpha(\dot{y}(t))$*;*
5. *For a.e.* $t > 0$, $\dot{y}(t) = \min\{u(t), v(\rho(t,y(t)_+))\}$.

Remark 2 We observe that the definition introduced in [9], requiring only the entropy admissibility on $\mathbb{R}^+ \times\,]-\infty, y(t)[$ and $\mathbb{R}^+ \times\,]y(t), +\infty[$, is not sufficiently strong to single out a unique non-classical solution to Riemann problems. For example, the undercompressive shock between ρ^*_u and 0 traveling with speed u would be an admissible solution in the sense of [9, Definition 4.1] for the initial datum $\rho_0(x) = \tilde{\rho}_u$. Instead, condition (12), as well as its equivalent formulation used in [11], ensures that, if an undercompressive shock is present at $x = y(t)$, it satisfies point 4 in Definition 3.1 as an equality.

3.1 Wave-front tracking approximation

Since solutions to Riemann problems are known analytically (see details in Appendix B), we are able to construct piecewise constant approximations of solutions to (2) via the wave-front tracking algorithm, see [3, 15] for the general theory.

Definition 3.2 *Given $\varepsilon > 0$, we say that the maps ρ_ε, y_ε, and u_ε provide an ε-approximate wave-front tracking solution to (2) if the following conditions hold.*

1. *$\rho_\varepsilon \in \mathbf{C^0}\left(\mathbb{R}^+; \mathbf{L^1_{loc}}(\mathbb{R}; [0, R])\right)$ is piecewise constant, with discontinuities occurring along finitely many straight lines in the (t, x)-plane. Moreover jumps of ρ_ε can be shocks, rarefactions, or undercompressive shocks, and they are indexed by $\mathcal{J}(t) = \mathcal{S}(t) \cup \mathcal{R}(t) \cup \mathcal{U}(t)$. For simplicity, the jumps of ρ_ε are called **ρ-waves**. Moreover we call **classical waves** the ρ-waves described by the jump sets $\mathcal{S}$ and $\mathcal{R}$.*

2. *$u_\varepsilon \in \mathbf{L^1}(\mathbb{R}^+; \mathbb{R}^+)$ is a piecewise constant function with a finite number of discontinuities.*

3. *$y_\varepsilon \in \mathbf{W^{1,1}_{loc}}(\mathbb{R}^+; \mathbb{R})$ is a piecewise affine function. We refer to it as the **y-wave**.*

4. *Along each shock $x(t) = x_\alpha(t)$, $\alpha \in \mathcal{S}(t)$, we have, for a.e. $t > 0$,*
$$\rho_\varepsilon(t, x_\alpha(t)_-) < \rho_\varepsilon(t, x_\alpha(t)_+).$$
Moreover, for a.e. $t > 0$,
$$\left| \dot{x}_\alpha(t) - \frac{f(\rho_\varepsilon(t, x_\alpha(t)_-)) - f(\rho_\varepsilon(t, x_\alpha(t)_+))}{\rho_\varepsilon(t, x_\alpha(t)_-) - \rho_\varepsilon(t, x_\alpha(t)_+)} \right| \leq \varepsilon.$$

5. *Along each rarefaction front $x(t) = x_\alpha(t)$, $\alpha \in \mathcal{R}(t)$, we have, for a.e. $t > 0$,*
$$\rho_\varepsilon(t, x_\alpha(t)_+) < \rho_\varepsilon(t, x_\alpha(t)_-) \leq \rho_\varepsilon(t, x_\alpha(t)_+) + \varepsilon.$$
Moreover, for a.e. $t > 0$,
$$\dot{x}_\alpha(t) \in \left[f'(\rho_\varepsilon(t, x_\alpha(t)_-)), f'(\rho_\varepsilon(t, x_\alpha(t)_+)) \right].$$

6. *Along an undercompressive shock we have, for a.e. $t > 0$, $x_\alpha(t) = y_\varepsilon(t)$, $\alpha \in \mathcal{U}$, and*
$$\rho_\varepsilon(t, y_\varepsilon(t)_-) = \hat{\rho}_{u_\varepsilon(t)} > \check{\rho}_{u_\varepsilon(t)} = \rho_\varepsilon(t, y_\varepsilon(t)_+).$$
Moreover, for a.e. $t > 0$,
$$\dot{x}_\alpha(t) = \dot{y}_\varepsilon(t) = u_\varepsilon(t).$$

7. *The following estimates hold*
$$\|\rho_\varepsilon(0, \cdot) - \rho_0\|_{L^1(\mathbb{R})} < \varepsilon, \qquad \|u_\varepsilon - u\|_{L^1(0,+\infty)} < \varepsilon.$$

8. *For a.e. $t > 0$*
$$\dot{y}_\varepsilon(t) = \min\left\{u_\varepsilon(t),\, v\left(\rho_\varepsilon\left(t, y_\varepsilon(t)_+\right)\right)\right\}.$$

Remark 3 With relation to Definition 3.2, we recall here the various types of waves. With the terms *ρ-wave* and *y-wave* we denote respectively a discontinuity for ρ_ε and the curve $t \mapsto (t, y_\varepsilon(t))$. *Shocks*, *rarefactions* and *undercompressive shocks* are all ρ-waves. Moreover an *undercompressive shock* is also a y-wave. Finally, by *classical wave* we mean a ρ-wave, which is not a y-wave.

We describe here a possible algorithm for constructing a sequence of approximate wave-front tracking solutions. First of all, given $\nu \in \mathbb{N}$, let us define the following grids on the interval $[0, R]$ for density and on $[0, V]$ for velocity.

Step 1. Let $u_0^\nu = 0$ and set recursively

$$u_j^\nu > u_{j-1}^\nu \quad \text{such that} \quad \hat{\rho}_{u_j^\nu} = \check{\rho}_{u_{j-1}^\nu} \quad \text{for} \quad j \geq 1.$$

and stop the procedure at $j = J_\nu$ such that $V - u_{J_\nu+1}^\nu < 2^{-\nu}$, see Figure 2a. By Lemma 2.2, the iterative procedure is finite. We set $\rho_j^\nu = \check{\rho}_{u_{J_\nu - j}^\nu}$ for $j = 0, \ldots, J_\nu$.

Step 2. Divide the interval $[0, u_1^\nu]$ into 2^ν parts, define the grid points

$$u_{k,0}^\nu := k u_1^\nu 2^{-\nu}, \qquad k = 0, \ldots, 2^\nu - 1,$$

the corresponding $\rho_{J_\nu + k}^\nu = \hat{\rho}_{u_{k,0}^\nu} \in \,]\check{\rho}_0, \hat{\rho}_0[$ and consider the recursive sequence

$$u_{k,i}^\nu \in \left]u_i^\nu, u_{i+1}^\nu\right[\quad \text{such that} \quad \hat{\rho}_{u_{k,i}^\nu} = \check{\rho}_{u_{k,i-1}^\nu} \quad \text{for} \quad i = 1, \ldots, J_\nu.$$

Step 3. To complete the mesh, we divide the remaining sub-interval $[\hat{\rho}_0, R]$ in 2^ν parts, defining

$$\rho_\ell^\nu := \hat{\rho}_0 + \ell(R - \hat{\rho}_0) 2^{-\nu}, \qquad\qquad \ell = 1, \ldots, 2^\nu.$$

Step 4. Relabeling the points defined above, we obtain the grids $\mathcal{M}_\nu := \{\rho_i^\nu\}_{i=0}^{M_\nu} \subset [0, R]$ (with $\rho_0^\nu = 0$ and $\rho_{M_\nu}^\nu = R$) and $\mathcal{U}_\nu := \{u_i^\nu\}_{i=0}^{N_\nu} \subset [0, V]$ (with $u_0^\nu = 0$ and $u_{N_\nu}^\nu = V$), where the points are labeled in increasing order. We also set

$$\delta_\rho^\nu := \min_{i=1,\ldots,M_\nu} \left(\rho_i^\nu - \rho_{i-1}^\nu\right) \geq c_\nu 2^{-\nu}, \qquad \delta_u^\nu := \min_{i=1,\ldots,M_\nu} \left(u_i^\nu - u_{i-1}^\nu\right) \geq c_\nu 2^{-\nu},$$
$$\varepsilon_\rho^\nu := \max_{i=1,\ldots,M_\nu} \left(\rho_i^\nu - \rho_{i-1}^\nu\right) \leq C_\nu 2^{-\nu}, \qquad \varepsilon_u^\nu := \max_{i=1,\ldots,M_\nu} \left(u_i^\nu - u_{i-1}^\nu\right) \leq C_\nu 2^{-\nu},$$

for some $c_\nu, C_\nu > 0$ depending on α, f'' and J_ν.

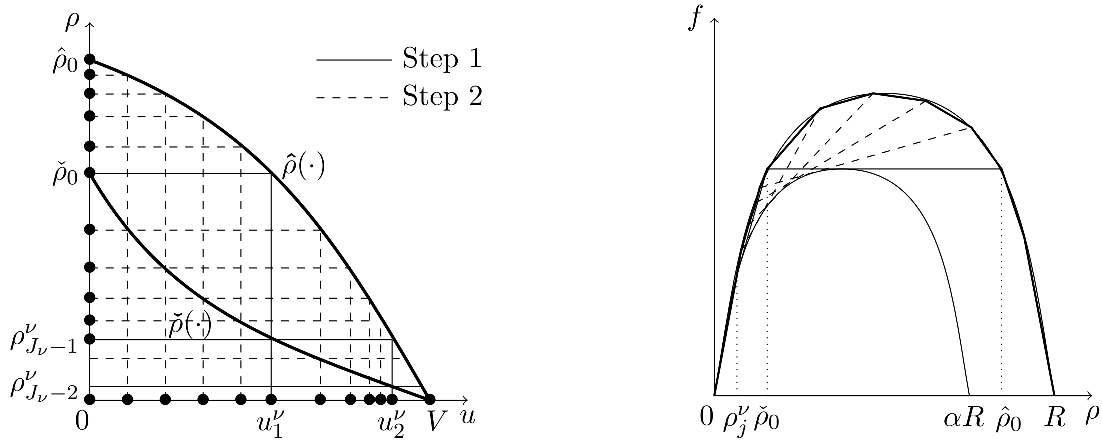
Also, let $f_\nu : [0, R] \to [0, +\infty[$ be the piecewise linear function such that $f_\nu(\rho_i^\nu) = f(\rho_i^\nu)$ for each $\rho_i^\nu \in \mathcal{M}_\nu$, see Figure 2b.

Given an initial traffic density $\rho_0 \in \mathbf{L^1}(\mathbb{R}; [0, R])$ with finite total variation, we consider a sequence of piece-wise constant functions $\rho_{0,\nu} : \mathbb{R} \to \mathcal{M}_\nu$ such that $\rho_{0,\nu}$ has a finite number of discontinuities and

$$\lim_{\nu \to +\infty} \|\rho_{0,\nu} - \rho_0\|_{\mathbf{L^1}(\mathbb{R})} = 0 \qquad \text{and} \qquad \mathrm{TV}(\rho_{0,\nu}) \leq \mathrm{TV}(\rho_0) \qquad \text{for all } \nu \in \mathbb{N}. \tag{13}$$

Besides, given $u \in \mathbf{L^1}(\mathbb{R}^+; [0, V])$ with finite total variation, we fix a sequence of piecewise constant functions $u_\nu : \mathbb{R}^+ \to \mathcal{U}_\nu$, such that u_ν has a finite number of discontinuities and

$$\lim_{\nu \to +\infty} \|u_\nu - u\|_{\mathbf{L^1}(\mathbb{R}^+)} = 0 \qquad \text{and} \qquad \mathrm{TV}(u_\nu) \leq \mathrm{TV}(u) \qquad \text{for all } \nu \in \mathbb{N}. \tag{14}$$



(a) Construction of the grids $\mathcal{M}_\nu$ and $\mathcal{U}_\nu$.

(b) Construction of f_ν.

Figure 2: Visual representation of the construction of the grids for ρ_ν and u_ν and of the construction of f_ν.

For every $\nu \in \mathbb{N} \setminus \{0\}$, we apply the following procedure. At time $t = 0$, we solve all the (classical) Riemann problems determined by a discontinuity of $\rho_{0,\nu}$ and the constrained Riemann problem at the AV initial position y_0 (see Appendix B), replacing the function f by f_ν. In this way, for small times $t > 0$ we obtain a piecewise constant function $\rho_\nu = \rho_\nu(t, x)$ with values in $\mathcal{M}_\nu$ and whose jump discontinuities satisfy the Rankine-Hugoniot condition

$$\dot{x}_\alpha(t) = \frac{f_\nu(\rho_\nu(t, x_\alpha(t)_-)) - f_\nu(\rho_\nu(t, x_\alpha(t)_+))}{\rho_\nu(t, x_\alpha(t)_-) - \rho_\nu(t, x_\alpha(t)_+)} = \frac{f(\rho_\nu(t, x_\alpha(t)_-)) - f(\rho_\nu(t, x_\alpha(t)_+))}{\rho_\nu(t, x_\alpha(t)_-) - \rho_\nu(t, x_\alpha(t)_+)},$$

thus providing a weak solution of (2a). In particular, every rarefaction wave is approximated by a rarefaction fan, formed by rarefaction shocks of strength less than ε_ρ^ν. We repeat the previous construction at every time $\bar{t}$ at which the following possibilities occur:

1. two classical waves (shock or rarefaction jumps) of ρ_ν interact together;

2. a classical discontinuity of ρ_ν interacts with y_ν;

3. $u_\nu(\bar{t}_-) \neq u_\nu(\bar{t}_+)$.

In this way we construct a piecewise constant function ρ_ν and a piecewise linear function y_ν.

Remark 4 By slightly modifying the wave speeds, we may assume that, at every positive time t, at most one of the previous interactions happens.

Remark 5 As usual, since rarefaction waves are generated only at time $t = 0$ or along the AV trajectory, we need to split rarefaction waves into rarefaction fans just at time $t = 0$ and possibly at the discontinuity points for u_ν.

Given a wave-front tracking approximate solution $(\rho_\varepsilon, y_\varepsilon, u_\varepsilon)$ to (2), we define the Glimm type functional

$$\Upsilon(t) = \Upsilon(\rho_\varepsilon(t, \cdot), u_\varepsilon) := \mathrm{TV}\left(\rho_\varepsilon(t, \cdot)\right) + 2R + \gamma(t) + \frac{6}{\beta}\mathrm{TV}\left(u_\varepsilon(\cdot); [t, +\infty[\right), \tag{15}$$

where γ is given by

$$\gamma(t) := \begin{cases} -2\left(\hat{\rho}_{u_\varepsilon(t)} - \check{\rho}_{u_\varepsilon(t)}\right) & \text{if } \rho_\varepsilon(t, y_\varepsilon(t)_-) = \hat{\rho}_{u_\varepsilon(t)},\ \rho_\varepsilon(t, y_\varepsilon(t)_+) = \check{\rho}_{u_\varepsilon(t)}, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that Υ is well defined for a.e. $t \geq 0$ and it changes only at discontinuity points of $\bar{u}_\varepsilon$ or at interaction times. Moreover, we observe that $\Upsilon(t) \geq \mathrm{TV}\left(\rho_\varepsilon(t, \cdot)\right) \geq 0$ and, with the above choices (13), (14), $\Upsilon(0) \leq \mathrm{TV}(\rho_0) + 2R + \frac{6}{\beta}\mathrm{TV}(u)$. The functional Υ will serve to provide an uniform estimate on the total variation of the approximate solutions ρ_ν constructed above.

3.2 Interaction estimates

This subsection we will show the following property of the functional Υ.

Proposition 3.1 *Let $\{\rho_\nu, y_\nu, u_\nu\}_{\nu\in\mathbb{N}}$ be the sequence of approximate solutions constructed in Section 3.1. For any $\nu \in \mathbb{N}$, at any wave interaction or jump in u_ν the map $t \mapsto \Upsilon(t) = \Upsilon(\rho_\nu(t, \cdot), u_\nu)$ either decreases by at least $\min\left\{\delta_\rho^\nu, 6\delta_u^\nu/\beta\right\}$, or it remains constant and the number of waves does not increases.*

Proof. We will detail the different types of interactions separately. To this end, we introduce the following notations:

- $\mathcal{F}_u$-*wave*: a wave denoting the AV trajectory with maximum speed u without discontinuity in ρ. The notation is to indicate a fictitious wave.
- $\mathcal{N}F_u$-*wave*: a wave denoting the AV trajectory with maximum speed u and discontinuity in ρ. The notation is to indicate a non fictitious wave. We can distinguish two cases:
  - UC_u-*wave*: a wave (ρ_l, ρ_r) denoting the AV trajectory with maximum speed u verifying $\rho_l = \hat{\rho}_u$ and $\rho_r = \check{\rho}_u$. The notation indicates an undercompressive shock.
  - C_u-*wave*: a wave (ρ_l, ρ_r) denoting the AV trajectory with speed u verifying $\rho_l < \rho_r$. The notation indicates a classical shock.

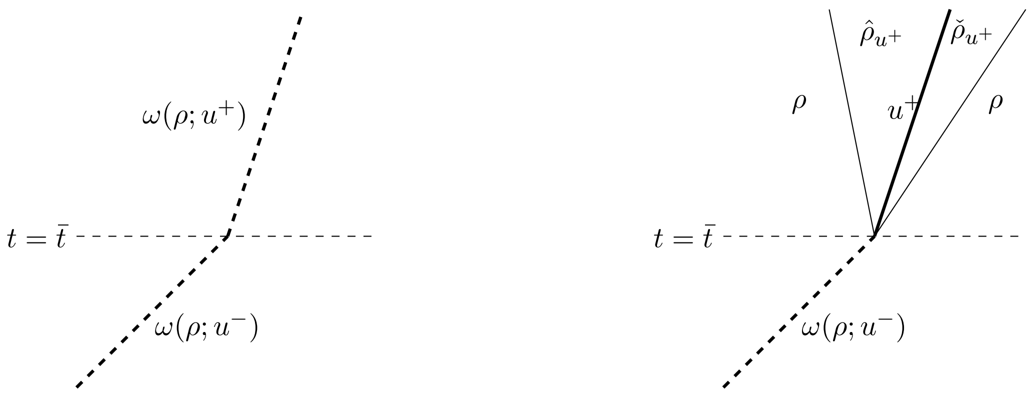
Let us consider an interaction occurring at time $t = \bar{t}$ away from the AV trajectory. In this case, either two shocks collide, or a shock and a rarefaction front interact. In both cases the number of waves diminishes. Moreover, $\mathrm{TV}(\rho_\nu(\bar{t}_+, \cdot)) \leq \mathrm{TV}(\rho_\nu(\bar{t}_-, \cdot))$, while the other terms in (15) remain constant and we conclude that $\Upsilon(\bar{t}_+) \leq \Upsilon(\bar{t}_-)$.

We then focus on events involving the AV trajectory. Interactions between classical waves and y_ν have been studied in [9]. Since the functional used there is equivalent to the one defined in (15) when the control u_ν does not jump, we can conclude as in [9, Lemma 2] that at any interactions of this type either Υ decreases of at least δ_ρ^ν, or remains constant and the number of waves does not increases.

Therefore, we focus here on the situations in which a jump in u occurs.

Lemma 3.1 *Assume that, at time $t = \bar{t}$, the control jumps from $u^- = u(\bar{t}_-)$ to $u^+ = u(\bar{t}_+)$ and that we have a $\mathcal{F}_{u^-}$-wave at $t = \bar{t}_-$. We have the following two cases.*

1. *At $\bar{t}_+$ we have a $\mathcal{F}_{u^+}$-wave and no wave is produced (see Figure 3a).*



(a) $\rho \notin [\check{\rho}_{u^-}, \hat{\rho}_{u^-}] \cup [\check{\rho}_{u^+}, \hat{\rho}_{u^+}]$ (b) $\rho \in [\check{\rho}_{u^+}, \hat{\rho}_{u^+}] \setminus [\check{\rho}_{u^-}, \hat{\rho}_{u^-}]$

Figure 3: Jump from u^- to u^+ at $t = \bar{t}$ with a $\mathcal{F}_{u^-}$-wave at $t = \bar{t}_-$.

2. At $\bar{t}_+$ we have a UC_{u^+}-wave and the number of waves increases (see Figure 3b).

In both cases, we have the estimate

$$\Delta\Upsilon(\bar{t}) = -\frac{6}{\beta}\left|u^+ - u^-\right| \le -\frac{6}{\beta}\delta_u^\nu < 0.$$

Proof. Before time $\bar{t}$, we have a $\mathcal{F}_{u^-}$-wave and we denote the density by ρ. Thus, the speed of the AV is $w(\rho; u^-)$ and $\rho \notin]\check{\rho}_{u^-}, \hat{\rho}_{u^-}[$. At $t = \bar{t}$, a jump in the control from u^- to u^+ occurs and two cases may happen:

1. $\rho \notin]\check{\rho}_{u^+}, \hat{\rho}_{u^+}[$: no new wave is produced (see Figure 3a). Thus, we have:

$$\begin{aligned}
&\Delta \mathrm{TV}(\rho(\bar{t}, \cdot)) = 0,\\
&\Delta\gamma(\bar{t}) = 0,\\
&\Delta \mathrm{TV}(u; [\bar{t}, +\infty[) = -\left|u^+ - u^-\right|.
\end{aligned}$$

   We conclude that $\Delta\Upsilon(\bar{t}) = -\frac{6}{\beta}\left|u^+ - u^-\right| \le -\frac{6}{\beta}\delta_u^\nu < 0$.

2. $\rho \in]\check{\rho}_{u^+}, \hat{\rho}_{u^+}[$: a UC_{u^+}-wave arises together with two shocks $(\rho, \hat{\rho}_{u^+})$ and $(\check{\rho}_{u^+}, \rho)$ (see Figure 3b). Thus, we have:

$$\begin{aligned}
&\Delta \mathrm{TV}(\rho(\bar{t}, \cdot)) = \left|\rho - \check{\rho}_{u^+}\right| + \left|\check{\rho}_{u^+} - \hat{\rho}_{u^+}\right| + \left|\hat{\rho}_{u^+} - \rho\right| = 2\left|\check{\rho}_{u^+} - \hat{\rho}_{u^+}\right|,\\
&\Delta\gamma(\bar{t}) = -2\left|\check{\rho}_{u^+} - \hat{\rho}_{u^+}\right|,\\
&\Delta \mathrm{TV}(u; [\bar{t}, \infty)) = -\left|u^+ - u^-\right|.
\end{aligned}$$

   We conclude that $\Delta\Upsilon(\bar{t}) = -\frac{6}{\beta}\left|u^+ - u^-\right| \le -\frac{6}{\beta}\delta_u^\nu < 0$.

This concludes the proof. □

Lemma 3.2 *Assume that, at time $t = \bar{t}$, the control jumps from $u^- = u(\bar{t}_-)$ to $u^+ = u(\bar{t}_+)$ and that we have a C_{u^-}-wave (ρ_l, ρ_r) at $t = \bar{t}_-$. Then, at $t = \bar{t}_+$, we have a $\mathcal{F}_{u^+}$-wave and no wave is produced (see Figure 4). Moreover*

$$\Delta\Upsilon(\bar{t}) = -\frac{6}{\beta}\left|u^+ - u^-\right| \le -\frac{6}{\beta}\delta_u^\nu < 0.$$

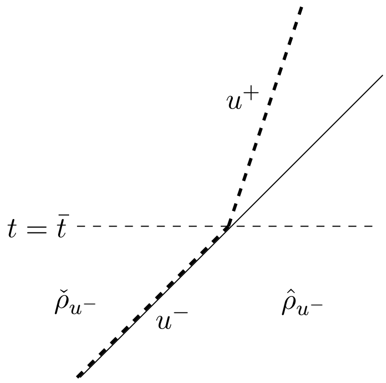


Figure 4: Jump from u^- to u^+ ($u^+ < u^-$) at $t = \bar{t}$ with a C_{u^-}-wave at $t = \bar{t}_-$: $\rho_l \leq \check{\rho}_{u^-}$ and $\rho_r \geq \hat{\rho}_{u^-}$.

PROOF. Since (ρ_l, ρ_r) is a C_{u^-}-wave, it holds $\rho_l \leq \check{\rho}_{u^-}$ and $\rho_r \geq \hat{\rho}_{u^-}$. Therefore, by Lemma 2.1, we have $\rho_l \leq \check{\rho}_{u^+}$ if $u^+ < u^-$ and $\rho_r \geq \hat{\rho}_{u^+}$ if $u^+ > u^-$. In both cases the classical shock (ρ_l, ρ_r) satisfies the constraint (2c) at the position $y_\nu(\bar{t})$ of the AV. Thus:

$$\begin{aligned}
&\Delta \mathrm{TV}(\rho(\bar{t}, \cdot)) = 0, \\
&\Delta \gamma(\bar{t}) = 0, \\
&\Delta \mathrm{TV}(u; [\bar{t}, +\infty[) = -\left|u^+ - u^-\right|.
\end{aligned}$$

Therefore

$$\Delta \Upsilon(\bar{t}) = -\frac{6}{\beta}\left|u^+ - u^-\right| \leq -\frac{6}{\beta}\delta_u^\nu < 0,$$

concluding the proof. □

Lemma 3.3 *Assume that, at time $t = \bar{t}$, the control jumps from u^- to u^+ and that we have a UC_{u^-}-wave at $t = \bar{t}_-$. The following two cases may happen.*

1. *The undercompressive shock is canceled. Then a rarefaction fan and a $\mathcal{F}_{u^+}$-wave arise (see Figure 5).*

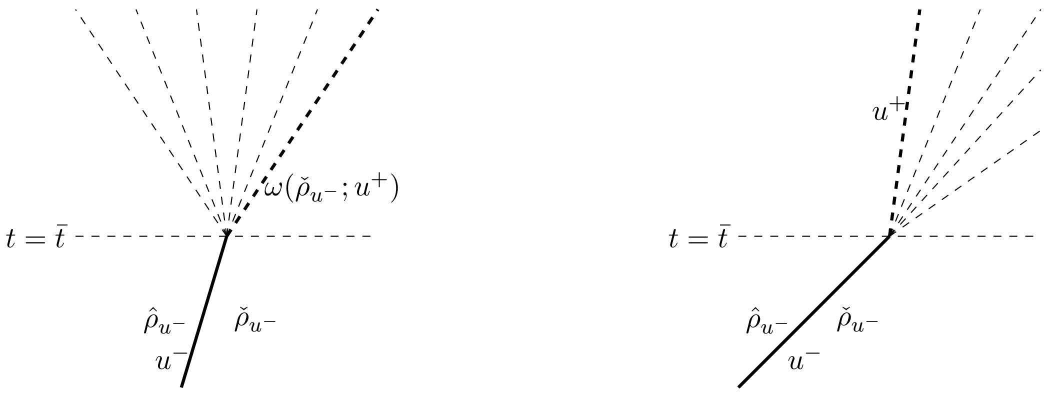
2. *At time $t = \bar{t}_+$ we have a UC_{u^+}-wave (see Figure 6). Then, the number of waves increases, since a rarefaction fan and a shock wave are produced.*

In both cases we have $\Delta \Upsilon(\bar{t}) \leq -\delta_\rho^\nu$.

PROOF. At $t = \bar{t}_-$, we have a UC_{u^-}-wave $(\hat{\rho}_{u^-}, \check{\rho}_{u^-})$ and the speed of the AV is u^-. At $t = \bar{t}$, a jump from u^- to u^+ occurs and two cases may happen.

1. The undercompressive shock disappears and so the Riemann problem at $t = \bar{t}$ is solved by a rarefaction fan. Hence we have

$$\begin{aligned}
&\Delta \mathrm{TV}(\rho(\bar{t}, \cdot)) = 0, \\
&\Delta \gamma(\bar{t}) = 2\left(\hat{\rho}_{u^-} - \check{\rho}_{u^-}\right), \\
&\Delta \mathrm{TV}(u; [\bar{t}, +\infty[) = -\left|u^+ - u^-\right|,
\end{aligned}$$



(a) $u^- < u^+$ (b) $u^- > u^+$

Figure 5: Jump in u at $t = \bar{t}$ with a UC_{u^-}-wave at $t = \bar{t}_-$ which gets canceled at $t = \bar{t}_+$.

which gives

$$\Delta\Upsilon(\bar{t}) = 2\left(\hat{\rho}_{u^-} - \check{\rho}_{u^-}\right) - \frac{6}{\beta}\left|u^+ - u^-\right|. \tag{16}$$

We distinguish two cases.

(a) $u^- < u^+$ and $\hat{\rho}_{u^+} \leq \check{\rho}_{u^-}$ (see Figure 5a). In this situation we have $f'(\check{\rho}_{u^-}) \leq u^+$ and $\check{\rho}_{u^+} \leq \hat{\rho}_{u^+}, \check{\rho}_{u^-} < \hat{\rho}_{u^-}$. By Taylor formula and **(F)**, there exists $\xi \in \left]\check{\rho}_{u^-}, \hat{\rho}_{u^-}\right[$ such that

$$\begin{aligned} u^+ - u^- &\geq f'(\check{\rho}_{u^-}) - \frac{f(\hat{\rho}_{u^-}) - f(\check{\rho}_{u^-})}{\hat{\rho}_{u^-} - \check{\rho}_{u^-}} \\ &= -\frac{f''(\xi)}{2}(\hat{\rho}_{u^-} - \check{\rho}_{u^-}) \\ &\geq \frac{\beta}{2}(\hat{\rho}_{u^-} - \check{\rho}_{u^-}). \end{aligned} \tag{17}$$

From (16) and (17) we deduce

$$\Delta\Upsilon(\bar{t}) \leq -\left(\hat{\rho}_{u^-} - \check{\rho}_{u^-}\right) \leq -\delta_\rho^\nu.$$

(b) $u^+ < u^-$ and $\hat{\rho}_{u^-} \leq \check{\rho}_{u^+}$ (see Figure 5b). In this situation we have $f'(\hat{\rho}_{u^-}) \geq u^+$ and $\check{\rho}_{u^-} \leq \hat{\rho}_{u-}, \check{\rho}_{u^+} < \hat{\rho}_{u^+}$. If $u^- = V$ (and thus $\check{\rho}_{u^-} = \hat{\rho}_{u-}$), we have

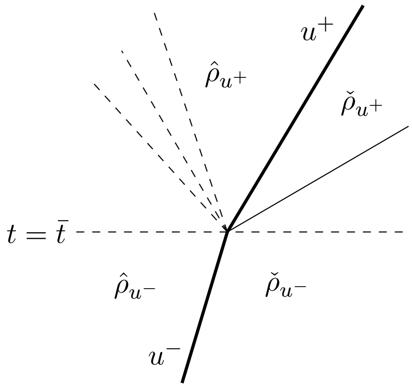
$$\Delta\Upsilon(\bar{t}) \leq -\frac{6}{\beta}\left|u^+ - u^-\right| \leq -\frac{6}{\beta}\delta_u^\nu.$$

Otherwise, by Taylor formula and **(F)**, there exists $\xi \in \left]\check{\rho}_{u^-}, \hat{\rho}_{u^-}\right[$ such that

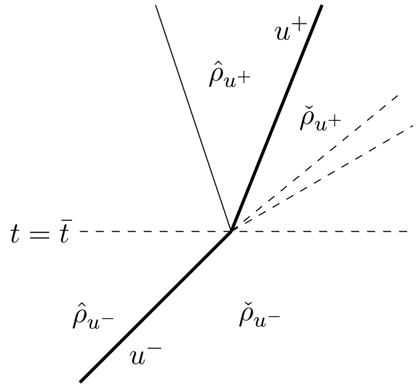
$$\begin{aligned} u^- - u^+ &\geq \frac{f(\check{\rho}_{u^-}) - f(\hat{\rho}_{u^-})}{\check{\rho}_{u^-} - \hat{\rho}_{u^-}} - f'(\hat{\rho}_{u^-}) \\ &= -\frac{f''(\xi)}{2}(\hat{\rho}_{u^-} - \check{\rho}_{u^-}) \\ &\geq \frac{\beta}{2}(\hat{\rho}_{u^-} - \check{\rho}_{u^-}). \end{aligned} \tag{18}$$

From (16) and (18),

$$\Delta\Upsilon(\bar{t}) \leq -\left(\hat{\rho}_{u^-} - \check{\rho}_{u^-}\right) \leq -\delta_\rho^\nu.$$



(a) $u^- < u^+$



(b) $u^- > u^+$

Figure 6: Jump in u at $t=\bar t$ with a UC_{u^-}-wave at $t=\bar t_-$ and a UC_{u^+}-wave at $t=\bar t_+$.

2. A new undercompressive shock arises at $t=\bar t_+$. Again, we distinguish two cases.

   (a) If $u^- < u^+$ and $\hat\rho_{u^+} > \check\rho_{u^-}$, then we have $\check\rho_{u^+} < \check\rho_{u^-} < \hat\rho_{u^+} < \hat\rho_{u^-}$ (see Remark 1). In this case, a UC_{u^+}-wave is created together with a rarefaction fan $(\hat\rho_{u^-}, \hat\rho_{u^+})$ and a shock wave $(\check\rho_{u^+}, \check\rho_{u^-})$ (see Figure 6a). Thus, we have

$$\begin{aligned}
&\Delta \mathrm{TV}(\rho(\bar t,\cdot)) = 2\left(\check\rho_{u^-} - \check\rho_{u^+}\right),\\
&\Delta\gamma(\bar t) = -2\left(\hat\rho_{u^+} - \check\rho_{u^+}\right) + 2\left(\hat\rho_{u^-} - \check\rho_{u^-}\right),\\
&\Delta \mathrm{TV}(u, [\bar t, +\infty[) = -\left|u^+ - u^-\right|,
\end{aligned}$$

   and we conclude that

$$\Delta\Upsilon(\bar t) = 2\left(\hat\rho_{u-} - \hat\rho_{u^+}\right) - \frac{6}{\beta}\left|u^+ - u^-\right|. \tag{19}$$

   Moreover, by Taylor's theorem and **(F)**,

$$\begin{aligned}
u^+ - u^- &\geq \frac{f(\hat\rho_{u^+}) - f(\check\rho_{u^-})}{\hat\rho_{u^+} - \check\rho_{u^-}} - \frac{f(\hat\rho_{u^-}) - f(\check\rho_{u^-})}{\hat\rho_{u^-} - \check\rho_{u^-}}\\
&= \left(\frac{f'(\xi)}{\xi - \check\rho_{u^-}} - \frac{f(\xi) - f(\check\rho_{u^-})}{(\xi - \check\rho_{u^-})^2}\right)\left(\hat\rho_{u^+} - \hat\rho_{u^-}\right)
\end{aligned}$$

   for some $\xi \in \,]\hat\rho_{u^+}, \hat\rho_{u^-}[$. Combining Taylor's theorem with **(F)**, we conclude that there exist $\tilde\xi \in \,]\xi, \hat\rho_{u^-}[$ such that

$$u^+ - u^- \geq -\frac{f''(\tilde\xi)}{2}\left(\hat\rho_{u^-} - \hat\rho_{u^+}\right) \geq \frac{\beta}{2}\left(\hat\rho_{u^-} - \hat\rho_{u^+}\right). \tag{20}$$

   From (19) and (20),

$$\Delta\Upsilon(\bar t) \leq -\left(\hat\rho_{u-} - \hat\rho_{u^+}\right) \leq -\delta_\rho^\nu.$$

   (b) If $u^+ < u^-$ and $\hat\rho_{u^-} > \check\rho_{u^+}$, then we have $\check\rho_{u^-} < \check\rho_{u^+} < \hat\rho_{u^-} < \hat\rho_{u^+}$ (see Remark 1). In this case, a UC_{u^+}-wave is created together with a shock wave $(\hat\rho_{u^-}, \hat\rho_{u^+})$ and a rarefaction fan $(\check{\rho}_{u^+}, \check{\rho}_{u^-})$ (see Figure 6b). Thus, we have

$$\begin{aligned}
\Delta \mathrm{TV}(\rho(\bar{t}, \cdot)) &= 2\left(\hat{\rho}_{u^+} - \hat{\rho}_{u^-}\right) \\
\Delta \gamma(\bar{t}) &= -2\left(\hat{\rho}_{u^+} - \check{\rho}_{u^+}\right) + 2\left(\hat{\rho}_{u^-} - \check{\rho}_{u^-}\right), \\
\Delta \mathrm{TV}(u; [\bar{t}, +\infty[) &= -\left|u^+ - u^-\right|,
\end{aligned}$$

and we conclude that

$$\Delta \Upsilon(\bar{t}) = 2\left(\check{\rho}_{u^+} - \check{\rho}_{u^-}\right) - \frac{6}{\beta}\left|u^+ - u^-\right|. \tag{21}$$

Moreover, by Taylor's theorem and **(F)**,

$$\begin{aligned}
u^- - u^+ &\geq \frac{f(\check{\rho}_{u^-}) - f(\hat{\rho}_{u^+})}{\check{\rho}_{u^-} - \hat{\rho}_{u^+}} - \frac{f(\check{\rho}_{u^+}) - f(\hat{\rho}_{u^+})}{\check{\rho}_{u^+} - \hat{\rho}_{u^+}} \\
&= \left(\frac{f'(\xi)}{\xi - \hat{\rho}_{u^+}} - \frac{f(\xi) - f(\hat{\rho}_{u^+})}{(\xi - \hat{\rho}_{u^+})^2}\right)\left(\check{\rho}_{u^-} - \check{\rho}_{u^+}\right)
\end{aligned}$$

for some $\xi \in \,]\check{\rho}_{u^-}, \check{\rho}_{u^+}[$. Combining again Taylor's theorem with **(F)**, we conclude that there exists $\tilde{\xi} \in \,]\check{\rho}_{u^-}, \xi[$ such that

$$u^- - u^+ \geq -\frac{f''(\tilde{\xi})}{2}\left(\check{\rho}_{u^+} - \check{\rho}_{u^-}\right) \geq \frac{\beta}{2}\left(\check{\rho}_{u^+} - \check{\rho}_{u^-}\right). \tag{22}$$

From (21) and (22) we obtain

$$\Delta \Upsilon(\bar{t}) \leq -\left(\check{\rho}_{u^+} - \check{\rho}_{u^-}\right) \leq -\delta_\rho^\nu.$$

The proof is so finished. □

This concludes the proof of Proposition 3.1. □

Proposition 3.1 ensures that the number of wave fronts in ρ_ν is finite and the wave-front tracking procedure can be prolonged for every positive time. Moreover, the total variation of ρ_ν is uniformly bounded in time:

Corollary 3.1 *Let $\{\rho_\nu, y_\nu, u_\nu\}_{\nu \in \mathbb{N}}$ be the sequence of approximate solutions constructed in Section 3.1. Then for any $\nu \in \mathbb{N}$, $t > 0$, it holds*

$$\mathrm{TV}\left(\rho_\nu(t, \cdot)\right) \leq \Upsilon(t) \leq \Upsilon(0) \leq \mathrm{TV}(\rho_0) + 2R + \mathrm{TV}(u).$$

3.3 Existence of a solution

The following is the main result of the paper.

Theorem 3.1 *Let the initial conditions $\rho_0 \in \mathbf{L^1}(\mathbb{R}; [0, R])$ with finite total variation, $y_0 \in \mathbb{R}$, and the open-loop control $u \in \mathbf{L^1}(\mathbb{R}^+; [0, V])$ with finite total variation. There exists a solution (ρ, y) to (2) in the sense of Definition 3.1.*

First, we prove that a limit of the sequence of approximate solutions $\{\rho_\nu, y_\nu, u_\nu\}_{\nu \in \mathbb{N}}$ to (2) constructed in Section 3.1 exists.

Lemma 3.4 *Let $\{\rho_\nu, y_\nu, u_\nu\}_{\nu\in\mathbb{N}}$ be the sequence of approximate solutions to* (2) *constructed in Section 3.1. Then, up to a subsequence, we have*

$$\rho_\nu \to \rho, \qquad in\ \mathbf{L^1_{loc}}(\mathbb{R}^+ \times \mathbb{R}; [0, R]), \tag{23a}$$

$$y_\nu \to y, \qquad in\ \mathbf{L^\infty_{loc}}(\mathbb{R}^+; \mathbb{R}), \tag{23b}$$

$$\dot{y}_\nu \to \dot{y}, \qquad in\ \mathbf{L^1_{loc}}(\mathbb{R}^+; \mathbb{R}), \tag{23c}$$

for some $\rho \in \mathbf{C^0}\left(\mathbb{R}^+; \mathbf{L^1}(\mathbb{R}; [0, R])\right)$ with $\mathrm{TV}\,(\rho(t)) < +\infty$ *for a.e. $t \in \mathbb{R}^+$ and $y \in \mathbf{W^{1,1}_{loc}}(\mathbb{R}^+; \mathbb{R})$, which is a Lipschitz continuous function with Lipschitz constant V.*

Proof. Fix $T > 0$. By Corollary 3.1, we know that $\mathrm{TV}(\rho_\nu(t, \cdot))$ is uniformly bounded for a.e. $t \in [0, T]$. This, together with the finite wave speed propagation, implies that

$$\int_{\mathbb{R}} |\rho_\nu(t, x) - \rho_\nu(s, x)|\, dx \le L\,|t - s| \qquad \text{for all } t, s \in [0, T],$$

for some L depending on f and on the total variation bound, but not on ν. Helly's Theorem, see [3, Theorem 2.4], implies the existence of $\rho \in \mathbf{C^0}\left([0, T]; \mathbf{L^1}(\mathbb{R}; [0, R])\right)$ such that $\mathrm{TV}\,(\rho(t)) < +\infty$ for a.e. $t \in [0, T]$ and a subsequence of $\{\rho_\nu\}_\nu$, which for simplicity we denote again by $\{\rho_\nu\}_\nu$. This implies that (23a) holds.

By construction (see point 8 of Definition 3.2), we deduce that

$$0 \le \dot{y}_\nu(t) \le V \tag{24}$$

for a.e. $t \in [0, T]$ and $\nu \in \mathbb{N} \setminus \{0\}$. Hence Ascoli Theorem [27, Theorem 7.25] implies that, there exists a function $y \in \mathbf{C^0}([0, T]; \mathbb{R})$ and a subsequence of $\{y_\nu\}_\nu$, which for simplicity we denote again by $\{y_\nu\}_\nu$, such that y_ν converges to y uniformly in $\mathbf{C^0}([0, T]; \mathbb{R})$. By the arbitrariness of T, the function y can be defined on $\mathbb{R}^+$ and (23b) holds. Moreover y is a Lipschitz continuous function with V as a Lipschitz constant.

To prove (23c), we aim to estimate the total variation of $\dot{y}_\nu$ on $[0, T]$. Since (24) holds, then it is sufficient to estimate the positive variation of $\dot{y}_\nu$, denoted with the symbol TV^+. Observe that $\dot{y}_\nu$ can jump only at interactions with waves coming from the right (see the interactions' estimates in [9, Section 4.2]) or at jumps in the control u_ν. In particular, $\dot{y}_\nu$ is non-decreasing at interactions with rarefaction fronts, which can be originated at $t = 0$, or at upward jumps in u_ν (see Figure 5a). We have the following two possibilities.

I. **A fan of rarefaction fronts interacts with the AV at most once.** If y_ν interacts over the time interval $[t_1, t_2]$ with rarefaction fronts, all originated from the point $(0, x_0)$, and u_ν is constant in $[t_1, t_2]$, then there exists a constant $C > 0$ such that

$$\mathrm{TV}^+(\dot{y}_\nu; [t_1, t_2]) \le |v(\rho_\nu(t_1, y_\nu(t_1)_-)) - v(\rho_\nu(t_2, y_\nu(t_2)_+))| \le C\,|\rho_0(x_{0-}) - \rho_0(x_{0+})|\,. \tag{25}$$

If y_ν interacts over $[t_1, t_2]$ with rarefaction fronts, all originated from $(\bar{t}, y_\nu(\bar{t}))$ with $\bar{t} > 0$, and u_ν is constant over $[t_1, t_2]$ (such rarefaction wave created at $\bar{t} > 0$ is described in Lemma 3.3; see also Figure 5 and Figure 6), then either a $\mathcal{F}_{u_\nu(\bar{t}_+)}$-wave arises (see Figure 5) or a $UC_{u_\nu(\bar{t}_+)}$-wave arises (see Figure 6). In the former case, the rarefaction wave connects $\hat{\rho}_{u_\nu(\bar{t}_-)}$ to $\check{\rho}_{u_\nu(\bar{t}_-)}$ and so, using (17) and (18), there exists a positive constant $C > 0$ such that

$$\mathrm{TV}^+(\dot{y}_\nu; [t_1, t_2]) \leq \left|v\left(\check{\rho}_{u_\nu(\bar{t}_-)}\right) - v\left(\hat{\rho}_{u_\nu(\bar{t}_-)}\right)\right| \leq C\left|u_\nu(\bar{t}_+) - u_\nu(\bar{t}_-)\right|. \tag{26}$$

In the latter case, when a $UC_{u_\nu(\bar{t}_+)}$-wave arises (see Figure 6), the rarefaction wave connects either $\hat{\rho}_{u_\nu(\bar{t}_-)}$ to $\hat{\rho}_{u_\nu(\bar{t}_+)}$ or $\check{\rho}_{u_\nu(\bar{t}_-)}$ to $\check{\rho}_{u_\nu(\bar{t}_+)}$. Using (20) and (22), we deduce that there exists a constant $C > 0$ such that

$$\mathrm{TV}^+(\dot{y}_\nu; [t_1, t_2]) \leq \left|v(\rho_\nu(t_1, y_\nu(t_1))) - v(\rho_\nu(t_2, y_\nu(t_2)))\right| \leq C\left|u_\nu(\bar{t}_+) - u_\nu(\bar{t}_-)\right|. \tag{27}$$

II. **The AV interacts with a fan of rarefaction shocks** (ρ_l, ρ_r) **modifying** n_ν **times its speed.** Define, for every $i \in \{1, \cdots, n_\nu\}$, the times t_{2i-1} and t_{2i}, respectively the

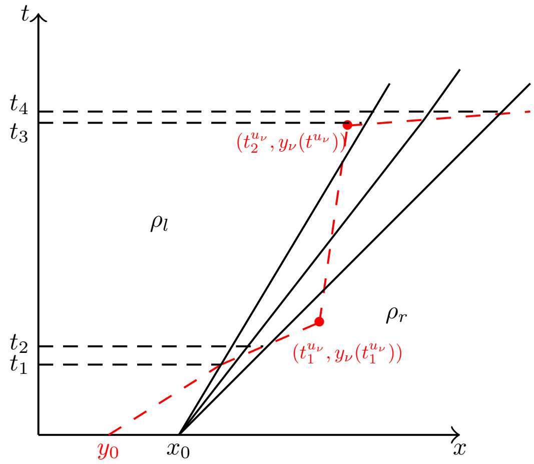


Figure 7: The AV interacts with a fan of rarefaction shocks (ρ_l, ρ_r) modifying its speed over the time intervals $[t_1, t_2]$ and $[t_3, t_4]$. In this situation $n_\nu = 2$ and, for every $i \in \{1, 2\}$, $u_\nu(t_i^{n_\nu}{}_-) \neq u_\nu(t_i^{n_\nu}{}_+)$.

times at which the AV enters from the left and exits from the right the rarefaction fan. Moreover, for every $i \in \{1, \cdots, n_\nu\}$, define by $t_{2i-1}^{u_\nu}$ and $t_{2i}^{u_\nu}$ the point of discontinuity for u_ν in such a way

$$t_0 < t_1 < t_2 < t_1^{u_\nu} < t_2^{u_\nu} < \cdots < t_{2n_\nu-3} < t_{2n_\nu-2} < t_{2n_\nu-3}^{u_\nu} < t_{2n_\nu-2}^{u_\nu} < t_{2n_\nu-1} < t_{2n_\nu},$$

where t_0 is the time at which the rarefaction fan is originated. For simplicity we denote with ρ_l and ρ_r respectively the left and the right states of the rarefaction fan; see Figure 7. Note that $v(\rho_l) < v(\rho_r)$.

Since, $\dot{y}_\nu$ can increase only at interactions with waves coming from the right or at jumps in u_ν, we have

$$\begin{aligned}\mathrm{TV}^+(\dot{y}_\nu; [t_1, t_{2n_\nu}]) &\leq \sum_{i=1}^{n_\nu} |v(\rho_l) - v(\rho_r)| + \sum_{i=1}^{2n_\nu-2} |u_\nu(t_i^{u_\nu}{}_-) - u_\nu(t_i^{u_\nu}{}_+)| \\ &= n_\nu\left[v(\rho_r) - v(\rho_l)\right] + \sum_{i=1}^{2n_\nu-2} |u_\nu(t_i^{u_\nu}{}_-) - u_\nu(t_i^{u_\nu}{}_+)|.\end{aligned}$$

The speed of AV is modified in the time interval $[t_{2i-1}, t_{2i}]$ for every $i \in \{1, \cdots, n_\nu\}$; in particular, we deduce that $v(\rho_l) \le v(\rho_r) \le u_\nu(t^{u_\nu}_{2i-1_-})$ for every $i \in \{1, \cdots, n_\nu - 1\}$. Moreover, since the speed of AV is constant over the time interval $\left]t^{u_\nu}_{2i-1}, t^{u_\nu}_{2i}\right[$ for every $i \in \{1, \cdots, n_\nu - 1\}$, we have $u_\nu(t^{u_\nu}_{2i-1_+}) \le v(\rho_l) \le v(\rho_r)$. Therefore

$$(n_\nu - 1)\,[v(\rho_r) - v(\rho_l)] \le \sum_{i=1}^{n_\nu - 1} \left| u_\nu(t^{u_\nu}_{2i-1_-}) - u_\nu(t^{u_\nu}_{2i-1_+}) \right|,$$

and so

$$\mathrm{TV}^+(\dot{y}_\nu; [t_1, t_{2n_\nu}]) \le [v(\rho_r) - v(\rho_l)] + 2 \sum_{i=1}^{u_\nu - 1} \left| u_\nu(t^{u_\nu}_{2i-1_-}) - u_\nu(t^{n_\nu}_{2i-1_+}) \right|. \tag{28}$$

We have

$$\begin{aligned} \mathrm{TV}\left(\dot{y}_\nu; [0,T]\right) &\le 2\mathrm{TV}^+\left(\dot{y}_\nu; [0,T]\right) + \|\dot{y}_\nu\|_\infty, \\ &\le 2 \sum_{i=0}^{N_\nu} \mathrm{TV}^+(\dot{y}_\nu;]t^{u_\nu}_i, t^{u_\nu}_{i+1}[) + 2 \sum_{i=1}^{N_\nu} |u_\nu(t^{u_\nu}_{i}{}_+) - u_\nu(t^{u_\nu}_{i}{}_-)| + \|\dot{y}_\nu\|_\infty. \end{aligned} \tag{29}$$

Above, $(t^{u_\nu}_i)_{i=1,\cdots,N_\nu}$ are the N_ν discontinuous points of u_ν such that, for every $i = 1, \cdots, N_\nu$, $t^{u_\nu}_i < T$. Moreover, by convention $t^{u_\nu}_0 = 0$ $t^{u_\nu}_{N_\nu+1} = T$. Combining (25), (26), (27), and (28), we deduce

$$\sum_{i=0}^{N_\nu} \mathrm{TV}^+(\dot{y}_\nu;]t^{u_\nu}_i, t^{u_\nu}_{i+1}[) \le C(TV(\rho_0) + TV(u)) + 2TV(u). \tag{30}$$

From (29) and (30), there exists a constant $C > 0$ such that

$$\mathrm{TV}\left(\dot{y}_\nu; [0,T]\right) \le 2C\mathrm{TV}(\rho_0) + (2C + 6)\mathrm{TV}(u) + V,$$

proving (23c), concluding the proof. □

Proof of Theorem 3.1. Consider a sequence of approximate solutions $\{\rho_\nu, y_\nu, u_\nu\}_{\nu \in \mathbb{N}}$ to (2) constructed in Section 3.1. By Lemma 3.4, there exist $\rho \in \mathbf{C^0}\left(\mathbb{R}^+; \mathbf{L^1}(\mathbb{R}; [0,R])\right)$ such that $\mathrm{TV}\left(\rho(t)\right) < +\infty$ for a.e. t and $y \in \mathbf{W^{1,1}_{loc}}(\mathbb{R}^+; \mathbb{R})$ such that, up to a subsequence, (23a), (23b), and (23c) hold. We prove that the couple (ρ, y) provides a solution to (2) according to Definition 3.1. Clearly the points 1 and 2 of Definition 3.1 hold.

Since ρ_ν is a weak entropy solution of (2a)–(2c)–(2d) in the sense of Definition 3.1, points 3 and 4, then, for every $\kappa \in \mathbb{R}$ and for all $\varphi \in \mathbf{C^1_c}(\mathbb{R}^2; \mathbb{R}^+)$, it holds

$$\begin{aligned} &\int_{\mathbb{R}^+} \int_{\mathbb{R}} \left(|\rho_\nu - \kappa| \partial_t \varphi + \mathrm{sgn}(\rho_\nu - \kappa)(f(\rho_\nu) - f(\kappa)) \partial_x \varphi\right) dx\, dt + \int_{\mathbb{R}} |\rho_{0,\nu} - \kappa| \varphi(0,x)\, dx \\ &\qquad + 2 \int_{\mathbb{R}^+} \left(f(\kappa) - \dot{y}_\nu(t)\kappa - \min\{f(\kappa) - \dot{y}_\nu(t)\kappa, F_\alpha(\dot{y}_\nu(t))\}\right) \varphi(t, y(t))\, dt \ge 0. \end{aligned} \tag{31}$$

Using (23a) and (23c) and passing to the limit in (31) as $\nu \to +\infty$, we conclude that ρ satisfies (12), hence point 3 of Definition 3.1 holds.

We deal now with the point 4 of Definition 3.1. Fix $T > 0$ and consider the sets

$$D_l := \{(t,x) \in [0,T] \times \mathbb{R} : \, x < y(t)\}\,, \quad D_r := \{(t,x) \in [0,T] \times \mathbb{R} : \, x > y(t)\}\,,$$
$$D_l^\nu := \{(t,x) \in [0,T] \times \mathbb{R} : \, x < y_\nu(t)\}\,, \quad D_r^\nu := \{(t,x) \in [0,T] \times \mathbb{R} : \, x > y_\nu(t)\}\,.$$

Fix $\psi \in \mathbf{C_c^1}(]0,T[\times\mathbb{R};\mathbb{R}^+)$. By [6, Theorem 2.2], since ρ_ν and ρ are weak solutions of (1), we deduce that

$$\int_{D_l^\nu} (\rho_\nu \partial_t \psi + f(\rho_\nu)\partial_x \psi)\, dt\, dx = \int_0^T \left[f(\rho_\nu(t, y_\nu(t)_-)) - \rho_\nu(t, y_\nu(t)_-)\dot{y}_\nu(t)\right]\psi(t, y_\nu(t))dt \tag{32}$$

and

$$\int_{D_l} (\rho \partial_t \psi + f(\rho)\partial_x \psi)\, dt\, dx = \int_0^T \left[f(\rho(t, y(t)_-)) - \rho(t, y(t)_-)\dot{y}(t)\right]\psi(t, y(t))dt \tag{33}$$

hold. The construction of (ρ_ν, y_ν), (32) and the fact that $\psi \geq 0$ imply

$$\int_{D_l^\nu} (\rho_\nu \partial_t \psi + f(\rho_\nu)\partial_x \psi)\, dt\, dx \leq \int_0^T F_\alpha(\dot{y}_\nu(t))\psi(t, y_\nu(t))dt. \tag{34}$$

Lemma 3.4 and the Dominated Convergence Theorem imply

$$\lim_{\nu\to\infty} \int_{D_l^\nu} (\rho_\nu \partial_t \psi + f(\rho_\nu)\partial_x \psi)\, dt\, dx = \int_{D_l} (\rho \partial_t \psi + f(\rho)\partial_x \psi)\, dt\, dx, \tag{35}$$

and

$$\lim_{\nu\to\infty} \int_0^T F_\alpha(\dot{y}_\nu(t))\psi(t, y_\nu(t))dt = \int_0^T F_\alpha(\dot{y}(t))\psi(t, y(t))dt. \tag{36}$$

Therefore, using (33), (34), (35), (36), we get

$$\int_0^T \left[f(\rho(t, y(t)_-)) - \rho(t, y(t)_-)\dot{y}(t)\right]\psi(t, y(t))dt \leq \int_0^T F_\alpha(\dot{y}(t))\psi(t, y(t))dt.$$

The same holds for the right traces. By the arbitrariness of ψ, we deduce that

$$f(\rho(t, y(t)_\pm)) - \rho(t, y(t)_\pm)\dot{y}(t) \leq F_\alpha(\dot{y}(t))$$

for a.e $t \in]0,T]$. Thus the couple (ρ, y) satisfies point 4 of Definition 3.1.

It remains to prove that the couple (ρ, y) satisfies point 5 of Definition 3.1. From Lemma 3.4, (14) and the construction of (ρ_ν, y_ν, u_ν), there exists a null set $\mathcal{N}$ such that, for every $\bar{t} \in \mathbb{R}_+^* \setminus \mathcal{N}$,

- $\lim_{\nu\to\infty} \dot{y}_\nu(\bar{t}) = \dot{y}(\bar{t})$,
- $\dot{y}_\nu(\bar{t}) = \min\{u_\nu(\bar{t}), v(\rho_\nu(\bar{t}, y_\nu(\bar{t})_+))\}$,
- y is continuously differentiable at $\bar{t}$,
- $\lim_{\nu\to\infty} \rho_\nu(\bar{t}, x) = \rho(\bar{t}, x)$ for a.e. $x \in \mathbb{R}$,

- $\lim_{\nu\to\infty} u_\nu(\bar{t}\,) = u(\bar{t}\,)$,
- $u(\bar{t}_-) = u(\bar{t}_+) =: \bar{u}$.

We have to prove that, if $\bar{t} \in \mathbb{R}^+ \setminus \mathcal{N}$, then

$$\lim_{\nu\to\infty} \min\left\{u_\nu(\bar{t}\,), v\left(\rho_\nu(\bar{t}, y_\nu(\bar{t})_+)\right)\right\} = \min\left\{u(\bar{t}\,), v\left(\rho(\bar{t}, y(\bar{t})_+)\right)\right\}. \tag{37}$$

To this aim, it is sufficient to prove that $\lim_{\nu\to\infty} v\left(\rho_\nu(\bar{t}, y_\nu(\bar{t})_+)\right) = v\left(\rho(\bar{t}, y(\bar{t})_+)\right)$. The proof follows closely the one given in [21, Section 3.3] for a constant control speed u.

Define $\rho_\pm = \lim_{x\to y(\bar{t})\pm} \rho(\bar{t}, x)$, which exist since $\mathrm{TV}(\rho(\bar{t}, \cdot); \mathbb{R})$ is finite. Various cases can occur.

1. Case: $\rho_-, \rho_+ \in [\rho^*_{\bar{u}}, R]$. From Lemma A.2, see also the entropy condition (12), the only possible case is $\rho^*_{\bar{u}} \le \rho_- \le \rho_+$.

   (a) If $\rho_+ = \rho_-$, then using Lemma A.1 and Lemma A.3, we have that for every $\epsilon > 0$ there exists $\bar{\nu} \in \mathbb{N}$ such that

   $$v(\min\{\rho_+ + 2\epsilon, R\}) \le \min\{u_\nu(\bar{t}), v(\rho_\nu(\bar{t}, y_\nu(\bar{t})_+))\} := \dot{y}_\nu(\bar{t}) \le \min\{u_\nu(\bar{t}), v(\rho_+ - 2\epsilon)\} \tag{38}$$

   for every $\nu \ge \bar{\nu}$. Since $\bar{t} \in \mathbb{R}^*_+ \setminus \mathcal{N}$, by passing to the limit in (38) as $\nu \to \infty$, we deduce (37), for the arbitrariness of ϵ.

   (b) If $\rho_+ \ne \rho_-$ and $y(\bar{t}) \le y_\nu(\bar{t})$ up to a subsequence; from Lemma A.1 we have that for every $\epsilon > 0$ there exists $\bar{\nu} \in \mathbb{N}$ such that

   $$v(\min\{\rho_+ + \epsilon, R\}) \le \min\{u_\nu(\bar{t}), v(\rho_\nu(t, y_\nu(\bar{t})_+))\} := \dot{y}_\nu(\bar{t}) \le v(\rho_+ - \epsilon) \tag{39}$$

   for every $\nu \ge \bar{\nu}$. Since $\bar{t} \in \mathbb{R}^*_+ \setminus \mathcal{N}$, the equality (37) holds by passing to the limit in (39) as $\nu \to \infty$.

   (c) If $\rho_+ \ne \rho_-$ and $y_\nu(\bar{t}) < y(\bar{t})$ up to a subsequence; in this case, from Lemma A.1 and Lemma A.3, we have that for every $\epsilon > 0$ there exists $\bar{\nu} \in \mathbb{N}$ such that $\rho_\nu(\bar{t}, y_\nu(\bar{t})_+) \in]\rho_- - 2\epsilon, \rho_+ + 2\epsilon[$ for every $\nu \ge \bar{\nu}$.
   For a.e $t > \bar{t}$,

   $$y_\nu(t) - y_\nu(\bar{t}) = \int_{\bar{t}}^{t} \dot{y}_\nu(s)ds \tag{40}$$

   and, by Lemma 3.4,

   $$\lim_{\nu\to\infty} y_\nu(t) = y(t).$$

   By Lemma A.4, there exists $c > 0$ and a sequence $t_\nu \searrow \bar{t}$ such that $\rho_\nu(s, y_\nu(s)_+) \in]\rho_+ - \epsilon, \rho_+ + \epsilon[$ for every $s \in [t_\nu, \bar{t} + c]$, hence $\dot{y}_\nu(s) \in]v(\rho_+ + \epsilon), v(\rho_+ - \epsilon)[$ for every $s \in [t_\nu, \bar{t} + c]$. By passing to the limit in (40), we have for a.e $t \in]\bar{t}, \bar{t} + c]$

   $$\frac{y(t) - y(\bar{t})}{t - \bar{t}} \in [v(\rho_+ + \epsilon), v(\rho_+ - \epsilon)].$$

   Using that y is continuously differentiable at time $\bar{t}$ and the arbitrariness of ϵ, we conclude that (37) holds.

2. Case: $\rho_-, \rho_+ \in [0, \rho^*_{\bar{u}}]$. Since $\bar{t} \in \mathbb{R}^*_+ \setminus \mathcal{N}$,

$$\dot{y}(\bar{t}) = \lim_{\nu\to\infty} y_\nu(\bar{t}) = \lim_{\nu\to\infty} \min\{u_\nu(\bar{t}), v(\rho_\nu(\bar{t}, y_\nu(\bar{t})_+)\}.$$

From Lemma A.5, for every $\epsilon > 0$ there exists $\bar{\nu} \in \mathbb{N}$ such that

$$v(\rho^*_{\bar{u}} + \epsilon) \leq \min\{u_\nu(\bar{t}), v(\rho_\nu(\bar{t}, y_\nu(\bar{t})_+)\} \leq u_\nu(\bar{t})$$

for $\nu \geq \bar{\nu}$. Since $\rho_+ \in [0, \rho^*_{\bar{u}}]$, we know that

$$\dot{y}(\bar{t}) = u(\bar{t}) = \min\{u(\bar{t}), v(\rho(\bar{t}, y(\bar{t})_+))\}.$$

3. Case: $\rho_+ < \rho^*_{\bar{u}} < \rho_-$. This case cannot occur by Lemma A.6.

4. Case: $\rho_- < \rho^*_{\bar{u}} < \rho_+$.

   (a) If $y(\bar{t}) \leq y_\nu(\bar{t})$ up to a subsequence, by Lemma A.1 for every $\epsilon > 0$ there exists $\bar{\nu}$ such that we have

   $$v(\min\{\rho_+ + \epsilon, R\}) \leq \min(u_\nu(\bar{t}), v(\rho_\nu(t, y_\nu(\bar{t})_+))) := \dot{y}_\nu(\bar{t}) \leq v(\rho_+ - \epsilon) \qquad (41)$$

   for $\nu \geq \bar{\nu}$. Since $\bar{t} \in \mathbb{R}^*_+ \setminus \mathcal{N}$, the equality (37) holds by passing to the limit in (41) as $\nu \to \infty$.

   (b) If $y_\nu(\bar{t}) < y(\bar{t})$ up to a subsequence, using Lemma A.7 and reasoning as in item 1c we get the conclusion.

This concludes the proof. □

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Appendix A Technical lemmas

Here we state some technical lemmas used in the proof of Theorem 3.1. In the following, we denote by ρ and y respectively the limit functions of wave front tracking approximate solutions ρ_ν and y_ν; see Lemma 3.4. Moreover, if $\bar{t} > 0$, then we define $\rho_- := \rho\,(\bar{t}, y(\bar{t})_-)$ and $\rho_+ := \rho\,(\bar{t}, y(\bar{t})_+)$.

Lemma A.1 [21, Lemma 4]. *Let $\bar t \in \mathbb{R}^+ \setminus \mathcal{N}$ and $\epsilon > 0$. Assume that $\rho_-, \rho_+ \in [0, R]$. There exist $0 < \tilde\delta < \delta$ such that*

$$\rho(\bar t, x) \in \begin{cases} \left]\max\{\rho_- - \epsilon/2, 0\}, \min\{\rho_- + \epsilon/2, R\}\right[, & x \in \left]y(\bar t) - \delta, y(\bar t)\right[, \\ \left]\max\{\rho_+ - \epsilon/2, 0\}, \min\{\rho_+ + \epsilon/2, R\}\right[, & x \in \left]y(\bar t), y(\bar t) + \delta\right[, \end{cases}$$

and, for $\nu \in \mathbb{N}^*$ sufficiently large,

$$\rho_\nu(\bar t, x) \in \begin{cases} \left]\max\{\rho_- - \epsilon, 0\}, \min\{\rho_- + \epsilon, R\}\right[, & x \in \left]\min\{y(\bar t), y_\nu(\bar t)\} - \tilde\delta, \min\{y(\bar t), y_\nu(\bar t)\}\right[, \\ \left]\max\{\rho_+ - \epsilon, 0\}, \min\{\rho_+ + \epsilon, R\}\right[, & x \in \left]\max\{y(\bar t), y_\nu(\bar t)\}, \max\{y(\bar t), y_\nu(\bar t)\} + \tilde\delta\right[; \end{cases}$$

see Figure 8.

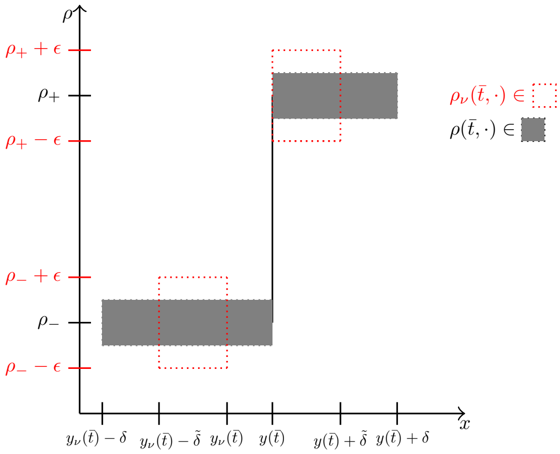

Figure 8: Illustration of Lemma A.1 in the case $\rho_- < \rho_+$ and $y_\nu(\bar t) < y(\bar t)$. The approximate density $\rho_\nu(\bar t, \cdot)$ in the space interval $]y_\nu(\bar t) - \tilde\delta, y_\nu(\bar t)[\cup]y(\bar t), y(\bar t) + \tilde\delta[$ belongs to the area surrounded by the red dotted lines, while $\rho(\bar t, \cdot)$ in the interval $]y(\bar t) - \delta, y(\bar t) + \delta[$ belongs to the grey shaded zone.

Lemma A.2 *Let $\bar t \in \mathbb{R}^+ \setminus \mathcal{N}$. If $\rho_-, \rho_+ \in [\rho^*_{\bar u}, R]$, then $\rho_- \leq \rho_+$.*

The proof is identical to that of [21, Lemma 5] remarking that, for every $\epsilon > 0$, there exists $\bar\nu \in \mathbb{N}$ such that $\rho^*_{u_\nu(\bar t)} < \rho^*_{\bar u} + \epsilon \leq \rho_+ + \epsilon$ for every $\nu \geq \bar\nu$.

Lemma A.3 *Let $\bar t \in \mathbb{R}^+ \setminus \mathcal{N}$ and $\epsilon > 0$. Assume that $\rho^*_{\bar u} \leq \rho_- \leq \rho_+$. Then there exists $\bar\nu \in \mathbb{N}$ such that*

$$\rho_\nu(\bar t, x) \in \left]\rho_- - 2\epsilon, \min(\rho_+ + 2\epsilon, R)\right[,$$

for every $\nu \geq \bar\nu$ and for every $x \in \left]\min\{y(\bar t), y_\nu(\bar t)\}, \max\{y(\bar t), y_\nu(\bar t)\}\right[$; see Figure 9.

The proof is identical to that of [21, Lemma 6] remarking that, for every $\epsilon > 0$, there exists $\bar\nu \in \mathbb{N}$ such that $\hat\rho_{u_\nu(\bar t)} < \hat\rho_{\bar u} + \epsilon \leq \rho^*_{\bar u} - 2\epsilon$ for every $\nu \geq \bar\nu$.

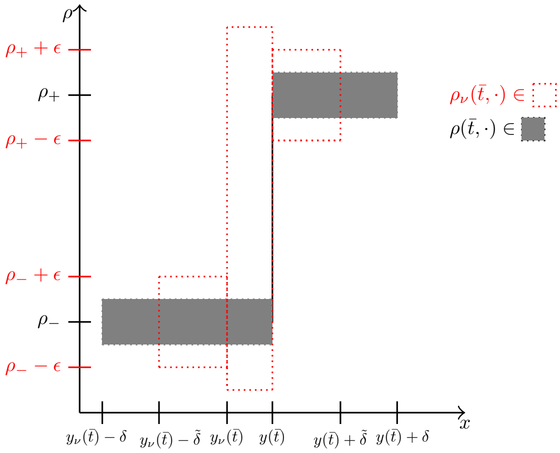

Figure 9: Illustration of Lemma A.3 in the case $\rho^*_{\bar u} \le \rho_- < \rho_+$ and $y_\nu(\bar t) < y(\bar t)$. The approximate density $\rho_\nu(\bar t, \cdot)$ in the space interval $]y_\nu(\bar t) - \tilde\delta, y(\bar t) + \tilde\delta[$ belongs to the area surrounded by the red dotted lines, while $\rho(\bar t, \cdot)$ in the interval $]y(\bar t) - \delta, y(\bar t) + \delta[$ belongs to the grey shaded zone.

Lemma A.4 [21, Lemma 7]. *Let $\bar t \in \mathbb{R}^+ \setminus \mathcal{N}$ and $\epsilon > 0$. Assume that $\rho^*_{\bar u} \le \rho_- < \rho_+$ and $y_\nu(\bar t) < y(\bar t)$ (up to a subsequence) for every $\nu \in \mathbb{N}$. Then there exist a domain $\mathcal{T}_0$ and, for every $\nu \in \mathbb{N}$, a piecewise constant function $\xi_\nu(\cdot)$ and a time $t_f^{\xi_\nu} > \bar t$ such that $(t, \xi_\nu(t)) \in \mathcal{T}_0$ for every $t \in [\bar t, t_f^{\xi_\nu}[$, and $\rho_\nu(t, x+) \in]\rho_+ - \epsilon, \rho_+ + \epsilon[$ for every $(t, x) \in ([\bar t, +\infty[\times]\xi_\nu(t), +\infty[) \cap \mathcal{T}_0$. Moreover, there exist $c > 0$ independent of ν and $t_\nu \in [\bar t, \bar t + c[$ such that $y_\nu(t_\nu) = \xi_\nu(t_\nu)$ and $\lim_{\nu \to \infty} t_\nu = \bar t$; see Figure 10.*

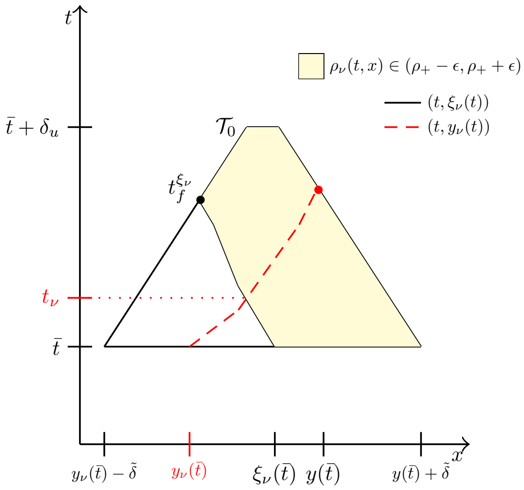

Figure 10: The situation of Lemma A.4 in the case $\rho^*_{\bar u} \le \rho_- < \rho_+ \le R$ with $y_\nu(\bar t) < y(\bar t)$.

Lemma A.5 [21, Lemma 8]. *Let $\bar t \in \mathbb{R}^+ \setminus \mathcal{N}$ and $\epsilon > 0$. Assume that $\rho_-, \rho_+ \in [0, \rho^*_{\bar u}]$. Then there exists $\delta > 0$ and $\bar\nu \in \mathbb{N}$ such that*

$$\rho_\nu(\bar t, x) \in]0, \rho^*_{\bar u} + 2\epsilon[$$

for every $\nu \geq \bar{\nu}$ *and* $x \in]\min\{y_\nu(\bar{t}), y(\bar{t})\} - \delta, \max\{y_\nu(\bar{t}), y(\bar{t})\} + \delta[$.

Lemma A.6 [21, Lemma 9]. *If* $\min\{\rho_+, \rho_-\} < \rho^*_{\bar{u}} < \max\{\rho_+, \rho_-\}$, *then it holds* $\rho_- < \rho^*_{\bar{u}} < \rho_+$.

Lemma A.7 [21, Lemma 10]. *Let* $\bar{t} \in \mathbb{R}^+ \setminus \mathcal{N}$ *and* $\epsilon > 0$. *Assume that* $\rho_- < \rho^*_{\bar{u}} < \rho_+$ *and* $y_\nu(\bar{t}) < y(\bar{t})$ *(up to a subsequence) for every* $\nu \in \mathbb{N}$. *Then there exist a domain* $\mathcal{T}_1$ *and, for every* $\nu \in \mathbb{N}$, *a piecewise constant function* $\xi^1_\nu(\cdot)$ *and a time* $t_f^{\xi^1_\nu} > \bar{t}$ *such that* $(t, \xi^1_\nu(t)) \in \mathcal{T}_1$ *for every* $t \in [\bar{t}, t_f^{\xi^1_\nu}[$, *and* $\rho_\nu(t, x+) \in]\rho_+ - \epsilon, \rho_+ + \epsilon[$ *for every* $(t, x) \in \big([\bar{t}, +\infty[\times]\xi^1_\nu(t), +\infty[\big) \cap \mathcal{T}_1$. *Moreover, there exist* $c > 0$ *independent of* ν *and* $t_\nu \in [\bar{t}, \bar{t} + c[$ *such that* $y_\nu(t_\nu) = \xi^1_\nu(t_\nu)$ *and* $\lim_{\nu \to \infty} t_\nu = \bar{t}$.

Appendix B The Riemann problem at the vehicle location

We recall here the definition of the solution to the Riemann problem, i.e. problem (2) with initial data

$$y_0 = 0 \qquad \text{and} \qquad \rho_0(x) = \begin{cases} \rho_l & \text{if } x < 0, \\ \rho_r & \text{if } x > 0. \end{cases} \tag{42}$$

Denote by $\mathcal{R}$ the standard (i.e., without the constraint (2c)) Riemann solver for (2a)-(42), i.e., the (right continuous) map $(t, x) \mapsto \mathcal{R}(\rho_L, \rho_R)(x/t)$ given by the standard weak entropy solution, see for instance [20]. Moreover, given $u \in [0, V]$, let $\check{\rho}_u$ and $\hat{\rho}_u$, with $\check{\rho}_u \leq \hat{\rho}_u$, be the points defined in (3).
Following [9, Definition 3.1], the constrained Riemann solver is defined as follows:

Definition B.1 *The constrained Riemann solver* $\mathcal{R}^u : [0, R]^2 \to \mathbf{L^1_{loc}}(\mathbb{R}; [0, R])$ *is defined as follows.*

1. *If* $f(\mathcal{R}(\rho_l, \rho_r)(u)) > F_\alpha + u\,\mathcal{R}(\rho_l, \rho_r)(u)$, *then*

$$\mathcal{R}^u(\rho_l, \rho_r)(x/t) = \begin{cases} \mathcal{R}(\rho_l, \hat{\rho}_u)(x/t) & \text{if } x < ut, \\ \mathcal{R}(\check{\rho}_u, \rho_r)(x/t) & \text{if } x \geq ut, \end{cases} \qquad \text{and} \quad y(t) = ut.$$

2. *If* $f(\mathcal{R}(\rho_l, \rho_r)(u)) \leq F_\alpha + u\,\mathcal{R}(\rho_L, \rho_R)(u)$, *then*

$$\mathcal{R}^u(\rho_l, \rho_r) = \mathcal{R}(\rho_l, \rho_r) \quad \text{and} \quad y(t) = \omega(\rho_r; u)\, t.$$

In the next subsections we detail the structure of solutions in some cases.

B.1 Case of a *shock wave* in ρ: $\rho_l < \rho_r$.

We suppose that $0 \leq \rho_l < \rho_r \leq R$. Denote by σ the speed of the shock wave, i.e.

$$\sigma = \frac{f(\rho_l) - f(\rho_r)}{\rho_l - \rho_r}.$$

First assume $\sigma < u$. The following possibilities hold.

1. $f(\rho_r) > \varphi_u(\rho_r)$. This hypothesis implies that $\check{\rho}_u < \rho_r < \hat{\rho}_u$, while $\sigma < u$ implies that the vehicle at y enters in the region with density ρ_r. Moreover $\sigma < u$ and $\rho_l < \rho_r$ imply that $\check{\rho}_u < \rho_l < \rho_r < \hat{\rho}_u$. Therefore the solution is given by

$$\rho(t,x) = \begin{cases} \rho_l & \text{if } x < \frac{f(\rho_l)-f(\hat{\rho}_u)}{\rho_l-\hat{\rho}_u}t \\ \hat{\rho}_u & \text{if } \frac{f(\rho_l)-f(\hat{\rho}_u)}{\rho_l-\hat{\rho}_u}t < x < ut \\ \check{\rho}_u & \text{if } ut < x < \frac{f(\rho_r)-f(\check{\rho}_u)}{\rho_r-\check{\rho}_u}t \\ \rho_r & \text{if } x > \frac{f(\rho_r)-f(\check{\rho}_u)}{\rho_r-\check{\rho}_u}t \end{cases}$$

and

$$y(t) = y(0) + ut;$$

see Figure 11. Finally note that $u \le v(\rho_l) \le v(\rho_r)$, so that the constraint in (42) for u is satisfied.

2. $u\rho_r \le f(\rho_r) \le \varphi_u(\rho_r)$. This hypothesis implies that either $0 \le \rho_r \le \check{\rho}_u$ or $\hat{\rho}_u < \rho_r < \rho_u^*$, while $\sigma < u$ implies that the vehicle at y enters in the region with density ρ_r. If $0 \le \rho_r \le \check{\rho}_u$, then the assumptions $\rho_l < \rho_r$ and $\sigma < u$ give a contradiction. Thus we deduce that $\hat{\rho}_u < \rho_r < \rho_u^*$. In this case the solution is given by

$$\rho(t,x) = \begin{cases} \rho_l & \text{if } x < \frac{f(\rho_l)-f(\rho_r)}{\rho_l-\rho_r}t \\ \rho_r & \text{if } x > \frac{f(\rho_l)-f(\rho_r)}{\rho_l-\rho_r}t \end{cases}$$

and

$$y(t) = y(0) + ut;$$

see Figure 12. Finally note that $u \le v(\rho_r)$, so that the constraint in (42) for u is satisfied.

3. $f(\rho_r) < u\rho_r$. This hypothesis implies that $\rho_r > \rho_u^*$, while $\sigma < u$ implies that the vehicle at y enters in the region with density ρ_r. In this case the solution is given by

$$\rho(t,x) = \begin{cases} \rho_l & \text{if } x < \frac{f(\rho_l)-f(\rho_r)}{\rho_l-\rho_r}t \\ \rho_r & \text{if } x > \frac{f(\rho_l)-f(\rho_r)}{\rho_l-\rho_r}t \end{cases}$$

and

$$y(t) = y(0) + v(\rho_r)t,$$

see Figure 13. Note that $u > v(\rho_l)$, so that the constraint in (42) is satisfied. The effective control is $u_e = v(\rho_r)$.

Now suppose $\sigma > u$. The following possibilities hold.

1. $f(\rho_l) > \varphi_u(\rho_l)$. This hypothesis implies that $\check{\rho}_u < \rho_l < \hat{\rho}_u$, while $\sigma > u$ implies that the vehicle at y enters in the region with density ρ_l. Moreover $\sigma > u$ and $\rho_l < \rho_r$ imply

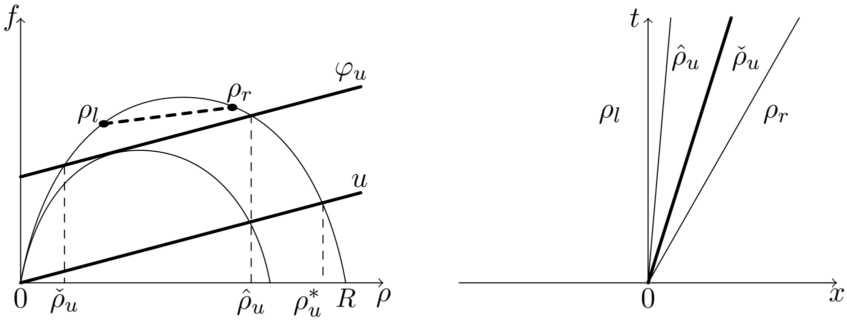

Figure 11: The Riemann problem. The situation of a shock with speed less than u and $f(\rho_r) > \varphi_u(\rho_r)$. In this case $y(t) = y(0) + ut$, while the solution for ρ is composed by two classical shocks and one non classical shock.

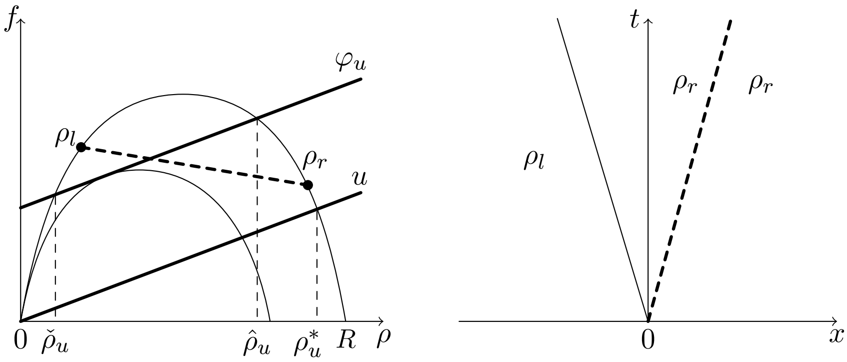

Figure 12: The Riemann problem. The situation of a shock with speed less than u and $u\rho_r \le f(\rho_r) \le \varphi_u(\rho_r)$. In this case $y(t) = y(0) + ut$, while the solution for ρ is composed by the classical shock connecting ρ_l to ρ_r.

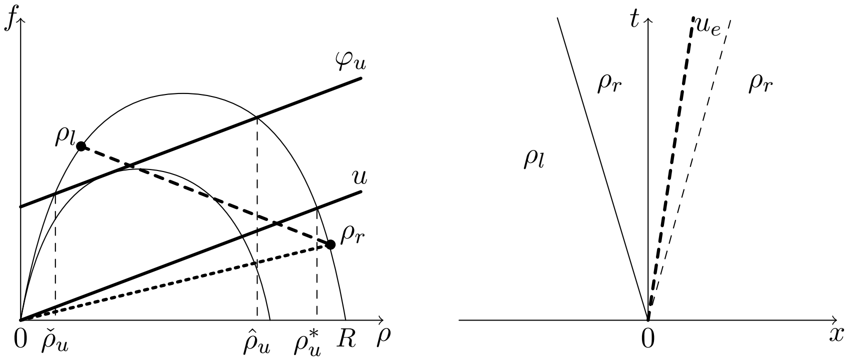

Figure 13: The Riemann problem. The situation of a shock with speed less than u and $f(\rho_r) < u\rho_r$. In this case $y(t) = y(0) + u_e t$ with $u_e = v(\rho_r)$, while the solution for ρ is composed by the classical shock connecting ρ_l to ρ_r.

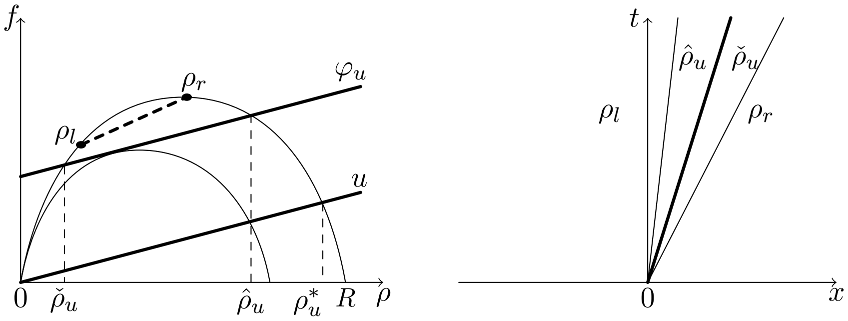

Figure 14: The Riemann problem. The situation of a shock with speed bigger than u and $f(\rho_l) > \varphi_u(\rho_l)$. In this case $y(t) = y(0) + ut$, while the solution for ρ is composed by two classical shocks and one non classical shock.

that $\check{\rho}_u < \rho_l < \rho_r < \hat{\rho}_u$. In this case the solution is given by

$$\rho(t,x) = \begin{cases} \rho_l & \text{if } x < \frac{f(\rho_l)-f(\hat{\rho}_u)}{\rho_l-\hat{\rho}_u}t \\ \hat{\rho}_u & \text{if } \frac{f(\rho_l)-f(\hat{\rho}_u)}{\rho_l-\hat{\rho}_u}t < x < ut \\ \check{\rho}_u & \text{if } ut < x < \frac{f(\rho_r)-f(\check{\rho}_u)}{\rho_r-\check{\rho}_u}t \\ \rho_r & \text{if } x > \frac{f(\rho_r)-f(\check{\rho}_u)}{\rho_r-\check{\rho}_u}t \end{cases}$$

and

$$y(t) = y(0) + ut;$$

see Figure 14. Finally note that $u \le v(\rho_l)$, so that the constraint in (42) for u is satisfied.

2. $u\rho_l \le f(\rho_l) \le \varphi_u(\rho_l)$. This hypothesis implies that either $0 \le \rho_l \le \check{\rho}_u$ or $\hat{\rho}_u \le \rho_l \le \rho_u^*$, while $\sigma > u$ implies that the vehicle at y enters in the region with density ρ_l. If $\hat{\rho}_u \le \rho_l \le \rho_u^*$, then the assumptions $\rho_l < \rho_r$ and $\sigma > u$ produce a contradiction. Thus we deduce that $0 \le \rho_l \le \check{\rho}_u$. In this case the solution is given by

$$\rho(t,x) = \begin{cases} \rho_l & \text{if } x < \frac{f(\rho_l)-f(\rho_r)}{\rho_l-\rho_r}t \\ \rho_r & \text{if } x > \frac{f(\rho_l)-f(\rho_r)}{\rho_l-\rho_r}t \end{cases}$$

and

$$y(t) = y(0) + ut;$$

see Figure 15. Finally note that $u \le v(\rho_l)$, so that the constraint in (42) for u is satisfied.

3. $f(\rho_l) < u\rho_l$. This hypothesis implies that $\rho_l > \rho_u^*$, while $\sigma > u$ implies that the vehicle at y enters in the region with density ρ_l. The fact that $\rho_l < \rho_r$ is in contradiction with $\sigma > u$, so that this case does not happen.

Finally suppose $\sigma = u$. The following possibilities hold.

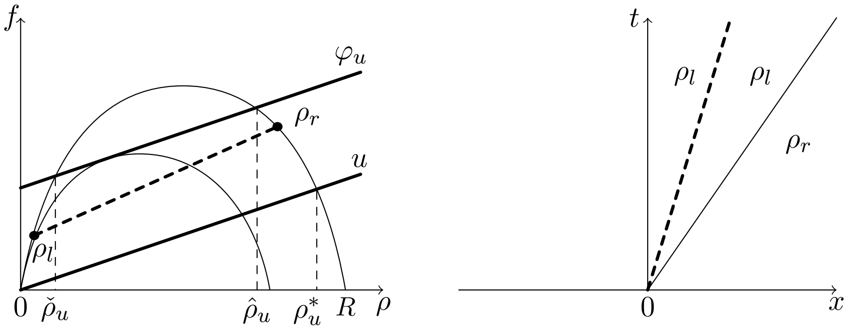

Figure 15: The Riemann problem. The situation of a shock with speed greater than u and $u\rho_l \le f(\rho_l) \le \varphi_u(\rho_l)$. In this case $y(t) = y(0) + ut$, while the solution for ρ is composed by the classical shock connecting ρ_l to ρ_r.

1. $f(\rho_r) > \varphi_u(\rho_r)$. This hypothesis implies that $\check{\rho}_u < \rho_r < \hat{\rho}_u$, while $\sigma = u$ implies that the vehicle at y has density ρ_r in front. Moreover $\sigma = u$ and $\rho_l < \rho_r$ imply that $\check{\rho}_u < \rho_l < \rho_r < \hat{\rho}_u$. In this case the solution is given by

$$\rho(t,x) = \begin{cases} \rho_l & \text{if } x < \frac{f(\rho_l)-f(\hat{\rho}_u)}{\rho_l-\hat{\rho}_u}t \\ \hat{\rho}_u & \text{if } \frac{f(\rho_l)-f(\hat{\rho}_u)}{\rho_l-\hat{\rho}_u}t < x < ut \\ \check{\rho}_u & \text{if } ut < x < \frac{f(\rho_r)-f(\check{\rho}_u)}{\rho_r-\check{\rho}_u}t \\ \rho_r & \text{if } x > \frac{f(\rho_r)-f(\check{\rho}_u)}{\rho_r-\check{\rho}_u}t \end{cases}$$

and

$$y(t) = y(0) + ut.$$

Finally note that $u \le v(\rho_r)$, so that the constraint in (42) for u is satisfied.

2. $u\rho_r \le f(\rho_r) \le \varphi_u(\rho_r)$. This hypothesis implies that either $0 \le \rho_r \le \check{\rho}_u$ or $\hat{\rho}_u \le \rho_r \le \rho_u^*$, while $\sigma = u$ implies that in front of the vehicle at y there is density ρ_r. If $0 \le \rho_r \le \check{\rho}_u$, then the assumptions $\rho_l < \rho_r$ and $\sigma = u$ produce a contradiction. Thus we deduce that $\hat{\rho}_u \le \rho_r \le \rho_u^*$. In this case the solution is given by

$$\rho(t,x) = \begin{cases} \rho_l & \text{if } x < \frac{f(\rho_l)-f(\rho_r)}{\rho_l-\rho_r}t \\ \rho_r & \text{if } x > \frac{f(\rho_l)-f(\rho_r)}{\rho_l-\rho_r}t \end{cases}$$

and

$$y(t) = y(0) + ut.$$

Finally note that $u \le v(\rho_r)$, so that the constraint in (42) for u is satisfied.

3. $f(\rho_r) < u\rho_r$. This hypothesis implies that $\rho_r > \rho_u$, while $\sigma = u$ implies that in the front of the vehicle at y there is density ρ_r. The fact that $\rho_l < \rho_r$ is in contradiction with $\sigma = u$, so that this case does not happen.

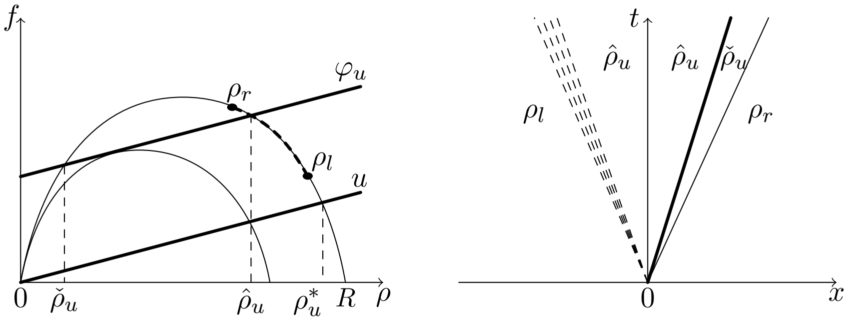

Figure 16: The Riemann problem. The situation of a rarefaction with speed less than u and $f(\rho_r) > \varphi_u(\rho_r)$. In this case $y(t) = y(0) + ut$, while the solution for ρ is composed by two classical waves and one non classical shock.

B.2 Case of a *rarefaction wave* in ρ: $\rho_r < \rho_l$.

Suppose that $0 \le \rho_r < \rho_l \le R$. Denote with σ_l and σ_r respectively the characteristic speeds of ρ_l and ρ_r, i.e.

$$\sigma_l = f'(\rho_l) \qquad \text{and} \qquad \sigma_r = f'(\rho_r).$$

Note that $\sigma_l < \sigma_r$.

First assume that $\sigma_r < u$, so that the density in front of the vehicle at y is ρ_r. The following possibilities hold.

1. $f(\rho_r) > \varphi_u(\rho_r)$. This hypothesis implies that $\check\rho_u < \rho_r < \hat\rho_u$. If $\rho_l \le \hat\rho_u$, then the solution for ρ is given by

$$\rho(t,x) = \begin{cases} \rho_l & \text{if } x < \frac{f(\rho_l)-f(\hat\rho_u)}{\rho_l-\hat\rho_u}t \\ \hat\rho_u & \text{if } \frac{f(\rho_l)-f(\hat\rho_u)}{\rho_l-\hat\rho_u}t < x < ut \\ \check\rho_u & \text{if } ut < x < \frac{f(\rho_r)-f(\check\rho_u)}{\rho_r-\check\rho_u}t \\ \rho_r & \text{if } x > \frac{f(\rho_r)-f(\check\rho_u)}{\rho_r-\check\rho_u}t, \end{cases}$$

while if $\rho_l > \hat\rho_u$, then the solution for ρ is given by

$$\rho(t,x) = \begin{cases} \rho_l & \text{if } x < f'(\rho_l)t \\ (f')^{-1}\left(\frac{x}{t}\right) & \text{if } f'(\rho_l)t < x < f'(\hat\rho_u)\,t \\ \hat\rho_u & \text{if } f'(\hat\rho_u)\,t < x < ut \\ \check\rho_u & \text{if } ut < x < \frac{f(\rho_r)-f(\check\rho_u)}{\rho_r-\check\rho_u}t \\ \rho_r & \text{if } x > \frac{f(\rho_r)-f(\check\rho_u)}{\rho_r-\check\rho_u}t. \end{cases}$$

The solution for y is given by

$$y(t) = ut;$$

see Figure 16. Finally note that $u \le v(\check\rho_u)$, so that the constraint in (42) for u is satisfied.

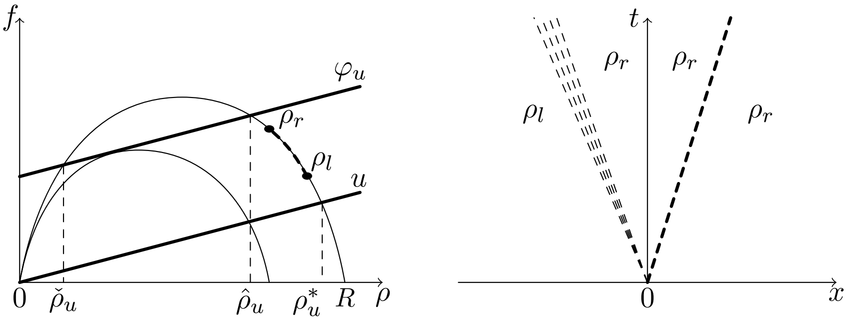

Figure 17: The Riemann problem. The situation of a rarefaction with speed less than u and $u\rho_r \le f(\rho_r) \le \varphi_u(\rho_r)$. In this case $y(t) = ut$, while the solution for ρ is composed by a classical rarefaction wave.

2. $u\rho_r \le f(\rho_r) \le \varphi_u(\rho_r)$. This hypothesis implies that either $0 \le \rho_r \le \check\rho_u$ or $\hat\rho_u \le \rho_r \le \rho_u^*$ If $0 \le \rho_r \le \check\rho_u$, then the assumption $\sigma_r < u$ gives a contradiction. Thus we deduce that $\hat\rho_u \le \rho_r \le \rho_u^*$. In this case the solution is given by

$$\rho(t,x) = \begin{cases} \rho_l & \text{if } x < f'(\rho_l)\,t \\ (f')^{-1}\left(\frac{x}{t}\right) & \text{if } f'(\rho_l)\,t < x < f'(\rho_r)\,t \\ \rho_r & \text{if } x > f'(\rho_r)\,t \end{cases}$$

and

$$y(t) = ut;$$

see Figure 17. Finally note that $u \le v(\rho_r)$, so that the constraint in (42) for u is satisfied.

3. $f(\rho_r) < u\rho_r$. This hypothesis implies that $\rho_r > \rho_u^*$. In this case the solution is given by

$$\rho(t,x) = \begin{cases} \rho_l & \text{if } x < f'(\rho_l)\,t \\ (f')^{-1}\left(\frac{x}{t}\right) & \text{if } f'(\rho_l)\,t < x < f'(\rho_r)\,t \\ \rho_r & \text{if } x > f'(\rho_r)\,t \end{cases}$$

and

$$y(t) = u_e t,$$

where $u_e = v(\rho_r) < u$; see Figure 18. Note that $u > v(\rho_l)$, so that the constraint in (42) for u is not satisfied. Thus the effective control is $u_e = v(\rho_r)$.

Assume now that $u < \sigma_l$, so that the density in front of the vehicle at y is ρ_l. The following possibilities hold.

1. $f(\rho_l) > \varphi_u(\rho_l)$. This hypothesis implies that $\check\rho_u < \rho_l < \hat\rho_u$. If $\rho_r \ge \check\rho_u$, then the solution for ρ is given by

$$\rho(t,x) = \begin{cases} \rho_l & \text{if } x < \frac{f(\rho_l)-f(\hat\rho_u)}{\rho_l-\hat\rho_u}t \\ \hat\rho_u & \text{if } \frac{f(\rho_l)-f(\hat\rho_u)}{\rho_l-\hat\rho_u}t < x < ut \\ \check\rho_u & \text{if } ut < x < \frac{f(\rho_r)-f(\check\rho_u)}{\rho_r-\check\rho_u}t \\ \rho_r & \text{if } x > \frac{f(\rho_r)-f(\check\rho_u)}{\rho_r-\check\rho_u}t, \end{cases}$$

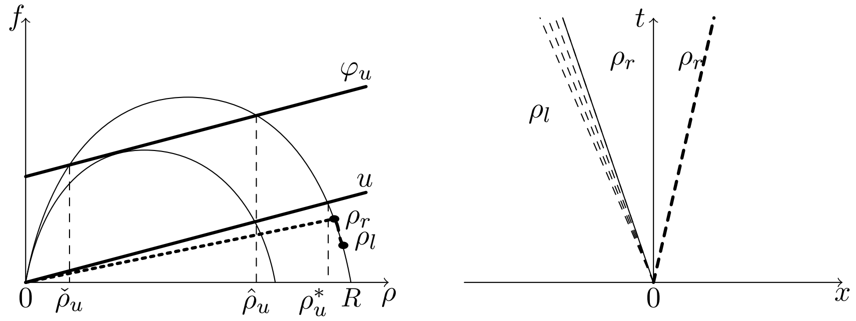

Figure 18: The Riemann problem. The situation of a rarefaction with speed less than u and $f(\rho_r) < u\rho_r$. In this case $y(t) = u_e t$, while the solution for ρ is composed by a classical rarefaction wave.

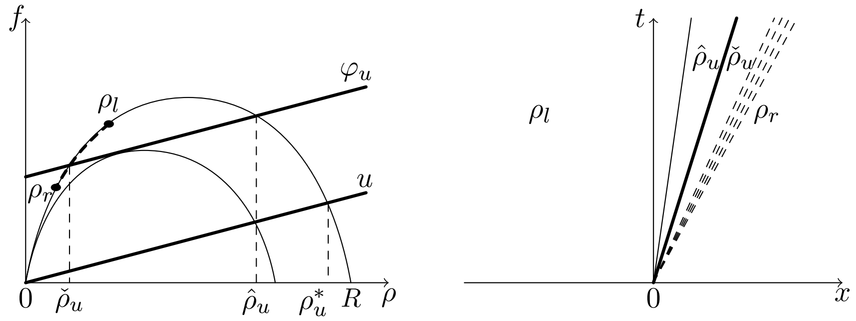

Figure 19: The Riemann problem. The situation of a rarefaction with speed greater than u and $f(\rho_l) > \varphi_u(\rho_l)$. In this case $y(t) = y(0) + ut$, while the solution for ρ is composed by two classical waves and one non classical shock.

while if $\rho_r < \check{\rho}_u$, then the solution for ρ is given by

$$\rho(t,x) = \begin{cases} \rho_l & \text{if } x < \frac{f(\rho_l)-f(\hat{\rho}_u)}{\rho_l-\hat{\rho}_u}\, t \\ \hat{\rho}_u & \text{if } \frac{f(\rho_l)-f(\hat{\rho}_u)}{\rho_l-\hat{\rho}_u}\, t < x < ut \\ \check{\rho}_u & \text{if } ut < x < f'(\check{\rho}_u)\, t \\ (f')^{-1}\left(\frac{x}{t}\right) & \text{if } f'(\check{\rho}_u)\, t < x < f'(\rho_r)\, t \\ \rho_r & \text{if } x > f'(\rho_r)\, t. \end{cases}$$

The solution for y is given by

$$y(t) = ut;$$

see Figure 19. Finally note that $u \le v(\check{\rho}_u)$, so that the constraint in (42) for u is satisfied.

2. $f(\rho_l) \le \varphi_u(\rho_l)$. Since $u < \sigma_l < \sigma_r$, this hypothesis implies that $\rho_r < \rho_l \le \check{\rho}_u$. In this case the solution is given by

$$\rho(t,x) = \begin{cases} \rho_l & \text{if } x < f'(\rho_l)\, t \\ (f')^{-1}\left(\frac{x}{t}\right) & \text{if } f'(\rho_l)\, t < x < f'(\rho_r)\, t \\ \rho_r & \text{if } x > f'(\rho_r)\, t \end{cases}$$

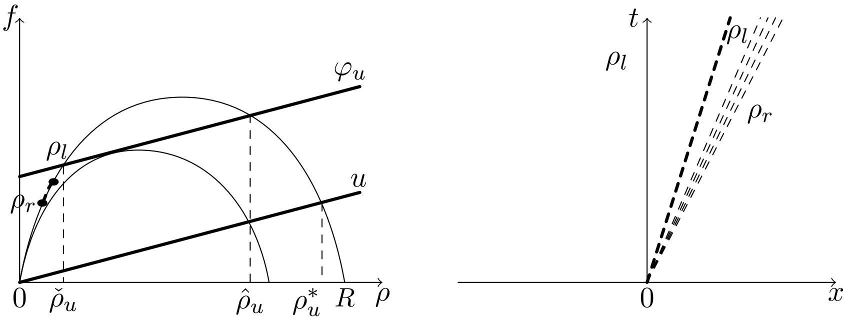

Figure 20: The Riemann problem. The situation of a rarefaction with speed greater than u and $f(\rho_l) \leq \varphi_u(\rho_l)$. In this case $y(t) = ut$, while the solution for ρ is composed by a classical rarefaction wave.

and

$$y(t) = ut;$$

see Figure 20. Finally note that $u \leq v(\rho_l)$, so that the constraint in (42) for u is satisfied.

Assume finally that $\sigma_l \leq u \leq \sigma_r$, so that the density in front of the vehicle at y is $\tilde{\rho} = (f')^{-1}(u)$. Note that $\check{\rho}_u < \tilde{\rho} < \hat{\rho}_u$. The following possibilities hold.

1. $\rho_l \leq \hat{\rho}_u$ and $\rho_r \geq \check{\rho}_u$. The solution for ρ is given by

$$\rho(t,x) = \begin{cases} \rho_l & \text{if } x < \frac{f(\rho_l)-f(\hat{\rho}_u)}{\rho_l-\hat{\rho}_u} t \\ \hat{\rho}_u & \text{if } \frac{f(\rho_l)-f(\hat{\rho}_u)}{\rho_l-\hat{\rho}_u} t < x < ut \\ \check{\rho}_u & \text{if } ut < x < \frac{f(\rho_r)-f(\check{\rho}_u)}{\rho_r-\check{\rho}_u} t \\ \rho_r & \text{if } x > \frac{f(\rho_r)-f(\check{\rho}_u)}{\rho_r-\check{\rho}_u} t, \end{cases}$$

while the solution for y is

$$y(t) = ut.$$

2. $\rho_l > \hat{\rho}_u$ and $\rho_r \geq \check{\rho}_u$. The solution for ρ is given by

$$\rho(t,x) = \begin{cases} \rho_l & \text{if } x < f'(\rho_l)\, t \\ (f')^{-1}\left(\frac{x}{t}\right) & \text{if } f'(\rho_l)\, t < x < f'(\hat{\rho}_u)\, t \\ \hat{\rho}_u & \text{if } f'(\hat{\rho}_u)\, t < x < ut \\ \check{\rho}_u & \text{if } ut < x < \frac{f(\rho_r)-f(\check{\rho}_u)}{\rho_r-\check{\rho}_u} t \\ \rho_r & \text{if } x > \frac{f(\rho_r)-f(\check{\rho}_u)}{\rho_r-\check{\rho}_u} t, \end{cases}$$

while the solution for y is

$$y(t) = ut.$$

3. $\rho_l \leq \hat{\rho}_u$ and $\rho_r < \check{\rho}_u$. The solution for ρ is given by

$$\rho(t,x) = \begin{cases} \rho_l & \text{if } x < \frac{f(\rho_l)-f(\hat{\rho}_u)}{\rho_l-\hat{\rho}_u} t \\ \hat{\rho}_u & \text{if } \frac{f(\rho_l)-f(\hat{\rho}_u)}{\rho_l-\hat{\rho}_u} t < x < ut \\ \check{\rho}_u & \text{if } ut < x < f'\left(\check{\rho}_u\right) t \\ \left(f'\right)^{-1}\left(\frac{x}{t}\right) & \text{if } f'\left(\check{\rho}_u\right) t < x < f'\left(\rho_r\right) t \\ \rho_r & \text{if } x > f'\left(\rho_r\right) t, \end{cases}$$

while the solution for y is

$$y(t) = ut.$$

4. $\rho_l > \hat{\rho}_u$ and $\rho_r < \check{\rho}_u$. The solution for ρ is given by

$$\rho(t,x) = \begin{cases} \rho_l & \text{if } x < f'\left(\rho_l\right) t \\ \left(f'\right)^{-1}\left(\frac{x}{t}\right) & \text{if } f'\left(\rho_l\right) t < x < f'\left(\hat{\rho}_u\right) t \\ \hat{\rho}_u & \text{if } f'\left(\hat{\rho}_u\right) t < x < ut \\ \check{\rho}_u & \text{if } ut < x < f'\left(\check{\rho}_u\right) t \\ \left(f'\right)^{-1}\left(\frac{x}{t}\right) & \text{if } f'\left(\check{\rho}_u\right) t < x < f'\left(\rho_r\right) t \\ \rho_r & \text{if } x > f'\left(\rho_r\right) t, \end{cases}$$

while the solution for y is

$$y(t) = ut.$$

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