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Frédéric Bayart, Fernando Costa Júnior, Dimitris Papathanasiou

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Frédéric Bayart, Fernando Costa Júnior, Dimitris Papathanasiou. Baire theorem and hypercyclic algebras. 2019. hal-02308466v1

**HAL Id: hal-02308466**

**<https://hal.science/hal-02308466v1>**

Preprint submitted on 9 Oct 2019 (v1), last revised 16 Sep 2020 (v2)

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# Baire theorem and hypercyclic algebras

Frédéric Bayart

Fernando Costa Júnior

Dimitris Papathanasiou

Author address:

*E-mail address:* Frederic.Bayart@uca.fr

*E-mail address:* Fernando.Vieira\_Costa\_Junior@uca.fr,

*E-mail address:* dpapath@bgsu.edu

LABORATOIRE DE MATHÉMATIQUES BLAISE PASCAL UMR 6620 CNRS,  
UNIVERSITÉ CLERMONT AUVERGNE, CAMPUS UNIVERSITAIRE DES CÉZEAUX, 3  
PLACE VASARELY, 63178 AUBIÈRE CEDEX, FRANCE.



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## Abstract

The question of whether a hypercyclic operator  $T$  acting on a Fréchet algebra  $X$  admits or not an algebra of hypercyclic vectors (but 0) have been explored in the recent literature. Analogous questions arise for stronger properties like frequent hypercyclicity, common hypercyclicity and disjoint hypercyclicity. In this paper we run over several different topics on this trend.

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Received by the editor October 11, 2019.

1991 *Mathematics Subject Classification.* 47A16.

*Key words and phrases.* hypercyclic operators, weighted shifts, convolution operators, hypercyclic algebras.

The first and the second author were partially supported by the grant ANR-17-CE40-0021 of the French National Research Agency ANR (project Front).



## CHAPTER 1

# Introduction

Among the many problems in linear dynamics, understanding the structure of the set of hypercyclic vectors is a major one. Let us introduce the relevant definitions. Let  $(X, T)$  be a linear dynamical system, namely  $X$  is a topological vector space and  $T$  is a bounded linear operator on  $X$ . A vector  $x \in X$  with dense orbit under  $T$  is called a hypercyclic vector, and we denote by  $HC(T)$  the set of hypercyclic vectors for  $T$ :

$$HC(T) = \{x \in X : \{x, Tx, T^2x, \dots\} \text{ is dense in } X\}.$$

This set  $HC(T)$  possesses interesting properties. When  $X$  is a Baire space, its nonemptiness implies its residuality, preventing it from being a non-trivial proper linear subspace of  $X$ . However, it is well known that, whenever  $HC(T)$  is nonempty, then  $HC(T) \cup \{0\}$  contains a dense linear manifold (see [20]). In many cases (not always)  $HC(T) \cup \{0\}$  contains a closed, infinite dimensional linear subspace (see [10, 27, 28, 26]). These properties reflect that we are working in a linear space.

Suppose now that  $X$  has a richer structure: it is an  $F$ -algebra, namely a metrizable and complete topological algebra. It is natural to ask whether  $HC(T) \cup \{0\}$  also contains a non-trivial subalgebra of  $X$ . Such an algebra will be called a *hypercyclic algebra*. The pioneering work in that direction has been done independently by Shkarin in [28] and by Bayart and Matheron in [8]: they showed that the derivation operator  $D : f \mapsto f'$ , acting on the Fréchet algebra  $H(\mathbb{C})$  of entire functions endowed with the pointwise multiplication, supports a hypercyclic algebra. However, this is not the case for all hypercyclic operators acting on an  $F$ -algebra: for instance, as pointed out in [5], the translation operators, acting on  $H(\mathbb{C})$ , do not support a hypercyclic algebra. Recent papers (see e.g. [6, 14, 15, 16, 22, 23]) give other examples of operators admitting a hypercyclic algebra.

Our aim, in this paper, is to shed new light on this problem and to study how it interacts with popular problems arising in linear dynamics. We are particularly interested in four questions.

### 1.1. Existence of hypercyclic algebras

All examples in the literature of operators supporting a hypercyclic algebra are generalizations of  $D$ . There are several ways to extend it. You may see  $D$  as a special convolution operator acting on  $H(\mathbb{C})$ . By [25], such an operator may be written  $\phi(D)$ , where  $\phi$  is an entire function with exponential type; if  $\phi$  is not constant, then  $\phi(D)$  is hypercyclic. When  $|\phi(0)| < 1$ , the existence of hypercyclic algebras is well-understood since [6]: such an algebra does exist if and only if  $\phi$  is not a multiple of an exponential function. When  $|\phi(0)| = 1$ , sufficient conditions are given in [6] or in [17] but almost nothing, except a very specific example, is known

when  $|\phi(0)| > 1$ . We partly fill this gap by proving the existence of a hypercyclic algebra when  $\phi$  goes to zero along some half-line.

**THEOREM 1.1.** *Let  $\phi$  be a nonconstant entire function with exponential type, not a multiple of an exponential function. Assume that  $|\phi(0)| > 1$  and that there exists some  $w \in \mathbb{C}$  such that  $|\phi(tw)| \rightarrow 0$  as  $t \rightarrow +\infty$ . Then  $\phi(D)$  supports a hypercyclic algebra.*

In particular, we shall see that if  $\phi(z) = P(z)e^z$  for some non-constant polynomial  $P$ , then  $\phi(D)$  supports a hypercyclic algebra.

Another way to generalize  $D$  is to see it as a weighted backward shift acting on  $H(\mathbb{C})$  considered as a sequence space. This was explored in [22]. The general context is that of a Fréchet sequence algebra  $X$ . Precisely we assume that  $X$  is a subspace of the space  $\omega = \mathbb{C}^{\mathbb{N}_0}$  of all complex sequences, whose topology is induced by a non-decreasing sequence of seminorms  $(\|\cdot\|_q)_{q \geq 1}$  and that  $X$  is endowed with a product  $\cdot$  such that, for all  $x, y \in X$ , all  $q \geq 1$ ,

$$\|x \cdot y\|_q \leq \|x\|_q \times \|y\|_q.$$

There are two natural products on a Fréchet sequence space: the coordinatewise product and the convolution or Cauchy product. It is clear that  $\ell_p$  and  $c_0$  are Fréchet sequence algebras for the coordinatewise product, and that  $\ell_1$  is also a Fréchet sequence algebra for the convolution product. Endowing  $H(\mathbb{C})$  with

$$\left\| \sum_{n \geq 0} a_n z^n \right\|_q = \sum_{n \geq 0} |a_n| q^n$$

and  $\omega$  with

$$\|(x_n)\|_q = \sum_{n=0}^q |x_n|,$$

we also obtain that  $H(\mathbb{C})$  and  $\omega$  are Fréchet sequence algebras for both products (on  $H(\mathbb{C})$ , the Cauchy product of  $f$  and  $g$  is nothing else but the product of the two functions  $f$  and  $g$ ). Another interesting source of examples for us will be the sequence spaces  $X = \{(x_n) \in \omega : \gamma_n x_n \rightarrow 0\}$  endowed with  $\|x\| = \sup_n \gamma_n |x_n|$ , where  $(\gamma_n) \in \mathbb{R}_+^{\mathbb{N}_0}$ . Provided  $\gamma_n \geq 1$  for all  $n$ ,  $X$  is a Fréchet sequence algebra for the coordinatewise product.

Given a sequence of nonzero complex numbers  $w = (w_n)_{n \in \mathbb{N}}$ , the (unilateral) weighted backward shift  $B_w$  with weight  $w$  is defined by

$$B_w(x_0, x_1, \dots) = (w_1 x_1, w_2 x_2, \dots).$$

The weight  $w$  will be called *admissible* (for  $X$ ) if  $B_w$  is a bounded operator on  $X$ . It is known that, provided the canonical basis  $(e_n)$  is a Schauder basis of  $X$ ,  $B_w$  is hypercyclic if and only if there exists a sequence  $(n_k)$  such that for all  $l \in \mathbb{N}$ ,  $((w_{l+1} \cdots w_{n_k+l})^{-1} e_{n_k+l})$  goes to zero.

Let us first assume that  $X$  is a Fréchet algebra under the coordinatewise product. Under a supplementary technical condition on  $X$ , a sufficient condition on  $w$  is given in [22] so that  $B_w$  supports a hypercyclic algebra. It turns out that we shall give a very natural characterization of this property when  $X$  admits a continuous norm. We recall that a Fréchet space  $(X, (\|\cdot\|_q))$  admits a continuous norm if there exists a norm  $\|\cdot\| : X \rightarrow \mathbb{R}$  that is continuous for the topology of  $X$ , namely there

exists  $q \in \mathbb{N}$  and  $C > 0$  with  $\|x\| \leq C\|x\|_q$  for all  $x \in X$ . In particular, for any  $q$  large enough,  $\|\cdot\|_q$  itself is a norm.

**THEOREM 1.2.** *Let  $X$  be a Fréchet sequence algebra for the coordinatewise product and with a continuous norm. Assume that  $(e_n)$  is a Schauder basis for  $X$ . Let also  $B_w$  be a bounded weighted shift on  $X$ . The following assumptions are equivalent.*

- (i)  $B_w$  supports a dense and not finitely generated hypercyclic algebra.
- (ii) There exists a sequence of integers  $(n_k)$  such that for all  $\gamma > 0$ , for all  $l \in \mathbb{N}$ ,  $((w_{l+1} \cdots w_{n_k+l})^{-\gamma} e_{n_k+l})$  tends to zero.

In particular, this theorem implies that on  $\ell_p$  or  $c_0$ , any hypercyclic weighted shift supports a hypercyclic algebra.

When  $X$  is a Fréchet algebra for the Cauchy product, it is shown in [22] that, under additional technical assumptions on  $X$ ,  $B_w$  supports a hypercyclic algebra as soon as it is mixing, namely as soon as  $(w_1 \cdots w_n)^{-1} e_n$  tends to zero. We shall improve that theorem by showing that any hypercyclic backward shift on a Fréchet sequence algebra for the Cauchy product supports a hypercyclic algebra. We will only require a supplementary assumption on  $X$  (to be regular) which is less strong than the assumption required in [22].

**THEOREM 1.3.** *Let  $X$  be a regular Fréchet sequence algebra for the Cauchy product and let  $B_w$  be a bounded weighted shift on  $X$ . The following assertions are equivalent.*

- (i)  $B_w$  is hypercyclic.
- (ii)  $B_w$  supports a dense and not finitely generated hypercyclic algebra.

In particular, if we compare this statement with Theorem 1.2, we see that, for the convolution product, we do not need extra assumptions on the weight, which was not the case for the coordinatewise product. Even if the proofs of Theorems 1.2, 1.3 share some similarities (the latter one being much more difficult), they also have strong differences, the main one being that, under the coordinatewise product, any power of  $x \in \omega$  keeps the same support, which is far from being the case if we work with the Cauchy product.

We also point out that this detailed study of the existence of hypercyclic algebras for weighted shifts has interesting applications. For instance, working with a bilateral shift, it will allow us to exhibit an invertible operator on a Banach algebra supporting a hypercyclic algebra and such that its inverse does not (see Example 4.12).

## 1.2. Common hypercyclic algebras

If  $(T_\lambda)_{\lambda \in \Lambda}$  is a family of hypercyclic operators acting on the same  $F$ -space  $X$ , it is natural to ask whether it admits a common hypercyclic vector, namely if  $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$  is non-empty. This is trivial if  $\Lambda$  is countable since each  $HC(T_\lambda)$  is a residual set. However, this becomes already an issue for the first natural example which comes in mind, that is the multiples of the unweighted backward shift  $B$ . It was proved by Abakumov and Gordon in [1] that indeed  $\bigcap_{\lambda > 1} HC(\lambda B) \neq \emptyset$  where  $B$  is seen acting on any  $\ell_p$ -space or on  $c_0$ . Further examples were given in subsequent papers (for instance in [21], [7])

Suppose now that we have a family  $(T_\lambda)_{\lambda \in \Lambda}$  of operators acting on the same  $F$ -algebra, each one supporting a hypercyclic algebra. It is natural to ask whether  $\bigcap_{\lambda \in \Lambda} HC(T_\lambda) \cup \{0\}$  contains a nontrivial algebra, which will be called a *common hypercyclic algebra*. Even when  $\Lambda$  has two elements, this is unclear. In this paper, we concentrate ourselves on the case where  $(T_\lambda)$  is a family of weighted shifts acting on  $X$ .

Firstly, regarding Fréchet sequence algebras endowed with the coordinatewise product, we have been able to get a general statement covering all the cases (and even more!) known for the existence of a common hypercyclic vector. As a corollary of our work, we have the following statement.

**THEOREM 1.4.** *(i) Let  $X = \ell_p$ ,  $1 \leq p < +\infty$ , or  $X = c_0$  endowed with the coordinatewise product. Then the family  $(\lambda B)_{\lambda > 0}$  admits a common hypercyclic algebra.*

*(ii) Let  $X = \ell_p$ ,  $1 \leq p < +\infty$ , or  $X = c_0$  endowed with the coordinatewise product. For  $\lambda > 0$ , define  $w(\lambda) = (1 + \lambda/n)_{n \geq 1}$ . Then the family  $(B_{w(\lambda)})_{\lambda > 0}$  admits a common hypercyclic algebra.*

*(iii) Let  $X = H(\mathbb{C})$  endowed with the coordinatewise product. Then the family  $(\lambda D)_{\lambda > 0}$  admits a common hypercyclic algebra.*

The statement of (ii) is particularly interesting. Indeed, for this family of weight, the existence of a single common hypercyclic vector was only known for  $\lambda > 1/p$ .

Again, it is much more difficult to handle Fréchet sequence algebras endowed with the Cauchy product. Nevertheless, we will be able to give a general sufficient condition which implies the following result.

**THEOREM 1.5.** *(i) Let  $X = \ell_1$  endowed with the Cauchy product. Then  $\bigcap_{\lambda > 1} HC(\lambda B) \cup \{0\}$  contains a non-trivial algebra.*

*(ii) Let  $X = H(\mathbb{C})$  endowed with the Cauchy product. Then  $\bigcap_{\lambda > 0} HC(\lambda D) \cup \{0\}$  contains a non-trivial algebra.*

### 1.3. Frequently and upper frequently hypercyclic algebras

Another fruitful subject in linear dynamics is frequent and upper frequent hypercyclicity. We say that  $T$  is frequently hypercyclic (resp. upper frequently hypercyclic) if there exists a vector  $x \in X$  such that, for all  $U \subset X$  open and non-empty, the set  $\{n \in \mathbb{N} : T^n x \in U\}$  has positive lower density (resp. positive upper density). Again, linearity allows to give a nice criterion to prove that an operator is (upper) frequently hypercyclic and gives rise to nice examples. For instance, if  $X$  is a Fréchet sequence space and  $B_w$  is a bounded weighted shift acting on  $X$ , it is known that the unconditional convergence of  $\sum_{n \geq 1} (w_1 \cdots w_n)^{-1} e_n$  implies that  $B_w$  is frequently hypercyclic. Moreover, in some spaces (for instance, on  $\ell_p$ -spaces), this condition is even necessary for the upper frequent hypercyclicity of  $B_w$ .

Of course, it is natural to ask if a (upper) frequently hypercyclic operator defined on an  $F$ -algebra  $X$  admits a (upper) frequently hypercyclic algebra, namely an algebra consisting only, except 0, of (upper) frequently hypercyclic vectors. Falcó and Grosse-Erdmann have shown recently ([23]) that this is not always the case: for instance,  $\lambda B$ ,  $\lambda > 1$ , acting on any  $\ell_p$  space ( $1 \leq p < +\infty$ ) or on  $c_0$ , endowed with the coordinatewise product, does not admit a frequently hypercyclic algebra.

Nevertheless, this leaves open the possibility for  $\lambda B$ ,  $\lambda > 1$ , to admit an upper frequently hypercyclic algebra.

We shall give two general results implying that a weighted shift on a Fréchet sequence algebra admits an upper frequently hypercyclic algebra. The first one deals with Fréchet sequence algebras endowed with the coordinatewise product. In view of Theorem 1.2, the natural extension of the above result for the existence of an upper frequently hypercyclic vector is to ask now for the unconditional convergence of the series  $\sum_{n \geq 1} (w_1 \cdots w_n)^{-1/m} e_n$  for all  $m \geq 1$ . This is sufficient!

**THEOREM 1.6.** *Let  $X$  be a Fréchet sequence algebra for the coordinatewise product and with a continuous norm. Assume that  $(e_n)$  spans a dense subspace of  $X$ . Let also  $B_w$  be a bounded weighted shift on  $X$  such that, for all  $m \geq 1$ ,  $\sum_{n \geq 1} (w_1 \cdots w_n)^{-1/m} e_n$  converges unconditionally. Then  $B_w$  admits an upper frequently hypercyclic algebra.*

In particular, for  $\lambda > 1$ ,  $\lambda B$  admits on any  $\ell_p$ -space ( $1 \leq p < +\infty$ ) and on  $c_0$  an upper frequently hypercyclic algebra. This last result was independently obtained by Falcó and Grosse-Erdmann in [23] in a different context (they concentrate themselves on  $\lambda B$  but allow different notions of hypercyclicity) and with a completely different proof.

Regarding Fréchet sequence algebras endowed with the convolution product, we also have been able to get a general statement (see the forthcoming Theorem 6.7). Its main feature is that we will only need the unconditional convergence of  $\sum_{n \geq 1} (w_1 \cdots w_n)^{-1} e_n$  and a technical condition to ensure the existence of an upper frequently hypercyclic algebra. As a corollary, we can state the following.

- COROLLARY 1.7.** (i) *Let  $X = \ell_1$  endowed with the convolution product and let  $\lambda > 1$ . Then  $\lambda B$  admits an upper frequently hypercyclic algebra.*  
(ii) *Let  $X = H(\mathbb{C})$  endowed with the convolution product. Then  $D$  admits an upper frequently hypercyclic algebra.*

Coming back to our initial problem, we show that it is possible to exhibit a weighted shift supporting a frequently hypercyclic algebra. The place to do this will be  $c_0$  endowed with the coordinatewise product; of course, the weight sequence will be much more complicated than that of the Rolewicz operator.

**THEOREM 1.8.** *There exists a weight  $(w_n)$  such that  $B_w$ , acting on  $c_0$  endowed with the coordinatewise product, supports a frequently hypercyclic algebra.*

The proof of this theorem will need the construction of disjoint subsets of  $\mathbb{N}$  with positive lower density and with some other extra properties, which seems interesting by itself.

#### 1.4. Disjoint hypercyclic algebras

The last problem in linear dynamics that we will investigate in view of finding algebras is that of disjointness. This notion was introduced in linear dynamics in [11] and [12]; we shall limit ourselves to the case of two operators. We say that two operators  $T_1$  and  $T_2$  acting on the same  $F$ -space  $X$  are disjointly hypercyclic if there exists a vector  $x \in X$ , called a disjointly hypercyclic vector for  $T_1$  and  $T_2$ , such that  $(T_1^n x, T_2^n x)_{n \geq 0}$  is dense in  $X^2$ . Many examples or counterexamples of disjointly hypercyclic operators can be found in the literature. In particular, in [12]

and in [13], our favourite examples, weighted backward shifts, are studied in depth: in [12], it is characterized when two shifts  $B_1^{r_1}$  and  $B_2^{r_2}$  are disjointly hypercyclic on  $\ell_p$  when they are not raised to the same power (namely  $r_1 < r_2$ ), whereas in [13] such a characterization is given when  $r_1 = r_2 = 1$ . From the proofs given in [12] and in [13] we see that the two cases are very different, as it will become clearer later on in this paper.

Assume now that  $X$  is an algebra. We say that  $T_1$  and  $T_2$  support a disjoint hypercyclic algebra if there exists a nontrivial subalgebra  $A$  of  $X$  such that any non zero element of  $A$  is a disjointly hypercyclic vector for  $T_1$  and  $T_2$ . We will study whether two backward shifts acting on a Fréchet sequence algebra  $X$  support a disjoint hypercyclic algebra or not. When  $X$  is endowed with the coordinatewise product, mixing the arguments of the present paper with that of [12, 13], we will get a complete characterization of the existence of a disjoint hypercyclic algebra for  $B_1^{r_1}$  and  $B_2^{r_2}$  (to avoid cumbersome statements, we will focus on the cases  $r_1 = r_2 = 1$  and  $r_1 = 1 < r_2 = 2$ ). This will be also the case when  $X$  is endowed with the convolution product and  $r_1 = r_2 = 1$ .

The most difficult case will happen for the Cauchy product, with two backward shifts raised to different powers (again, the unstability of the supports when taking powers will cause troubles). We shall concentrate ourselves on the multiples of  $B$  and  $D$ . In [12], it is shown that  $\lambda B$  and  $\mu B^2$  are disjointly hypercyclic on  $\ell_1$  if and only if  $1 < \lambda < \mu$  and that, for all  $\lambda, \mu > 0$ ,  $\lambda D$  and  $\mu D^2$  are disjointly hypercyclic on  $H(\mathbb{C})$ . Under this assumption they even support a disjointly hypercyclic algebra.

**THEOREM 1.9.** *Let  $\lambda, \mu > 0$ .*

- a) *Assume that  $1 < \lambda < \mu$ . Then  $\lambda B$  and  $\mu B^2$  support a disjoint hypercyclic algebra on  $\ell_1$  (endowed with the convolution product).*
- b)  *$\lambda D$  and  $\mu D^2$  support a disjoint hypercyclic algebra on  $H(\mathbb{C})$ .*

### 1.5. Organization of the paper

Up to now, there were two ways to produce hypercyclic algebras: by a direct construction (this is the method devised in [28] and in [22]) or by using a Baire argument (this method was initiated in [8]). In this paper, we improve the latter. We first give in Section 2.1 a general result for the existence of a hypercyclic algebra, enhancing the main lemma proved in [8]. This general theorem will be suitable to our new examples of operators supporting a hypercyclic algebra. Next, for each problem, we adapt the Baire argument to produce either common hypercyclic algebras or upper frequently hypercyclic algebras or disjoint hypercyclic algebras. Since the set of frequently hypercyclic vectors is always meagre, Theorem 1.8 cannot be proved using such an argument; it follows from a careful construction both of the weight and of the algebra.

We finish in the last section by making some remarks and asking some questions. In particular, we give a negative answer to a question raised by Shkarin about the existence of a closed hypercyclic algebra for the derivation operator.

### 1.6. Notations

The symbol  $\mathbb{N}$  will stand for the set of positive integers, whereas  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . We shall denote by  $\mathcal{P}_f(A)$  the set of finite subsets of a given set  $A$ .

For  $x = \sum_{n=0}^{+\infty} x_n e_n \in \omega$ , the support of  $x$  is equal to  $\text{supp}(x) = \{n \in \mathbb{N}_0 : x_n \neq 0\}$ . The notation  $c_{00}$  will denote the set of sequences in  $\omega$  with finite support.

For  $u \in X^d$  and  $\alpha \in \mathbb{N}_0^d$ ,  $u^\alpha$  will mean  $u_1^{\alpha_1} \cdots u_d^{\alpha_d}$ . If  $z$  is any complex number and  $m \in \mathbb{N}$ ,  $z^{1/m}$  will denote any  $m$ th root of  $z$ .

When working on a Fréchet space  $(X, \|\cdot\|_p)$ , it is often convenient to endow  $X$  with an  $F$ -norm  $\|\cdot\|$  defining the topology of  $X$  (see [24, Section 2.1]). Such an  $F$ -norm can be defined by the formula

$$\|x\| = \sum_{p=1}^{+\infty} \frac{1}{2^p} \min(1, \|x\|_p).$$

In particular, an  $F$ -norm satisfies the triangle inequality and the inequality

$$(1.1) \quad \forall \lambda \in \mathbb{C}, \forall x \in X, \|\lambda x\| \leq (|\lambda| + 1)\|x\|,$$

a property which replaces the positive homogeneity of the norm.

We finally recall some results on unconditional convergence in Fréchet spaces (see for instance [24, Appendix A]). A series  $\sum_{n=0}^{+\infty} x_n$  in a Fréchet space  $X$  is called unconditionally convergent if for any bijection  $\pi : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ , the series  $\sum_{n=0}^{+\infty} x_{\pi(n)}$  is convergent. This amounts to saying that, for any  $\varepsilon > 0$ , there is some  $N \in \mathbb{N}$  such that, whenever  $\sup_n |\alpha_n| \leq 1$ , the series  $\sum_{n=0}^{+\infty} \alpha_n x_n$  converges and

$$\left\| \sum_{n=N}^{+\infty} \alpha_n x_n \right\| < \varepsilon.$$



## CHAPTER 2

### A general criterion

#### 2.1. A transitivity criterion to get hypercyclic algebras

We first give a general statement which may be thought as a Birkhoff transitivity theorem for hypercyclic algebras. This criterion will be the main ingredient for the results from section 3.1 and some from section 4. Different versions of this approach will give rise to new criteria that will appear in sections 5.1, 6.1 and 7.1.

**THEOREM 2.1.** *Let  $T$  be a continuous operator on a separable commutative  $F$ -algebra  $X$  and let  $d \geq 1$ . Assume that for any  $A \subset \mathbb{N}_0^d \setminus \{(0, \dots, 0)\}$  finite and non-empty, for any non-empty open subsets  $U_1, \dots, U_d, V$  of  $X$ , for any neighbourhood  $W$  of  $0$ , there exist  $u = (u_1, \dots, u_d) \in U_1 \times \dots \times U_d$ ,  $\beta \in A$  and  $N \geq 1$  such that  $T^N(u^\beta) \in V$  and  $T^N(u^\alpha) \in W$  for all  $\alpha \in A$ ,  $\alpha \neq \beta$ . Then the set of  $d$ -tuples that generate a hypercyclic algebra for  $T$  is residual in  $X^d$ . Moreover, if the assumptions are satisfied for all  $d \geq 1$ , then  $T$  admits a dense and not finitely generated hypercyclic algebra.*

**PROOF.** Let  $(V_k)$  be a basis of open neighbourhoods of  $X$ . For  $A \in \mathcal{P}_f(\mathbb{N}_0^d)$ ,  $A \neq \emptyset$ ,  $(0, \dots, 0) \notin A$ , for  $s, k \geq 1$ , for  $\beta \in A$ , define

$$E(A, \beta, s) = \left\{ \sum_{\alpha \in A} \hat{P}(\alpha) z^\alpha \in \mathbb{C}[z_1, \dots, z_d] : \hat{P}(\beta) = 1 \text{ and } \sup_{\alpha \in A} |\hat{P}(\alpha)| \leq s \right\}$$

$$\mathcal{A}(A, \beta, s, k) = \{ u \in X^d : \forall P \in E(A, \beta, s), \exists N \geq 1, T^N(P(u)) \in V_k \}.$$

The sets  $E(A, \beta, s)$  are compact subsets of  $\mathbb{C}[z_1, \dots, z_d]$ . By continuity of the maps  $(u, P) \mapsto T^N(P(u))$ , this implies that each set  $\mathcal{A}(A, \beta, s, k)$  is open. Let us show that, for all  $A, k$  and  $s$ ,  $\bigcup_{\beta \in A} \mathcal{A}(A, \beta, s, k)$  is dense in  $X^d$ . Indeed, pick  $U_1, \dots, U_d$  non-empty open subsets of  $X$ . Let  $V \subset V_k$  and  $W = B(0, \varepsilon)$  be a neighbourhood of  $0$  such that  $V + B(0, (s+1)\text{card}(A)\varepsilon) \subset V_k$ . The assumptions of the proposition give the existence of  $u = (u_1, \dots, u_d) \in U_1 \times \dots \times U_d$ ,  $\beta \in A$  and  $N \geq 1$ . We claim that  $u$  belongs to  $\mathcal{A}(A, \beta, s, k)$ . Indeed,

$$T^N P(u) = \sum_{\alpha \neq \beta} \hat{P}(\alpha) T^N(u^\alpha) + T^N(u^\beta) \in V + B(0, (s+1)\text{card}(A)\varepsilon) \subset V_k$$

(observe that we have used (1.1)). Hence,  $\bigcap_{A, s, k} \bigcup_{\beta \in A} \mathcal{A}(A, \beta, s, k)$  is a residual subset of  $X^d$ . Pick  $u \in \bigcap_{A, s, k} \bigcup_{\beta \in A} \mathcal{A}(A, \beta, s, k)$ .

We show that for all non-zero polynomials  $P \in \mathbb{C}[z_1, \dots, z_d]$  with  $P(0) = 0$ ,  $P(u)$  belongs to  $HC(T)$ . We set  $A = \{ \alpha : \hat{P}(\alpha) \neq 0 \}$  and we first prove that  $\bigcup_{\beta \in A} \text{Orb} \left( T, \frac{1}{\hat{P}(\beta)} P(u) \right)$  is dense. Let us fix some  $k$  and let us set  $s = \sup_{\alpha, \beta \in A} |\hat{P}(\alpha)| / |\hat{P}(\beta)|$ . Let  $\beta \in A$  be such that  $u \in \mathcal{A}(A, \beta, s, k)$ . Define  $Q =$

$P/\hat{P}(\beta)$ . Then  $Q$  belongs to  $E(A, \beta, s)$  so that there exists  $N \geq 1$  satisfying

$$T^N \left( \frac{1}{\hat{P}(\beta)} P(u) \right) = T^N(Q(u)) \in V_k.$$

By the Bourdon-Feldman theorem, we deduce that there is some  $\beta_0 \in A$  such that  $\text{Orb} \left( T, \frac{1}{\hat{P}(\beta_0)} P(u) \right)$  is dense in  $X$ . Since any non-zero multiple of a hypercyclic vector remains hypercyclic, we finally deduce that  $P(u)$  is a hypercyclic vector for  $T$ .

The modification to obtain dense and infinitely generated algebras is easy. For  $A \in \mathcal{P}_f(\mathbb{N}_0^d)$ ,  $A \neq \emptyset$ ,  $(0, \dots, 0) \notin A$ , we now let

$$\mathcal{A}(A, \beta, s, k) = \{u \in X^{\mathbb{N}} : \forall P \in E(A, \beta, s), \exists N \geq 1, T^N(P(u)) \in V_k\}$$

and we still consider the set  $\bigcap_{A, s, k} \bigcup_{\beta \in A} \mathcal{A}(A, \beta, s, k)$  where now the intersection runs over all non-empty and finite sets  $A \subset \mathbb{N}^d \setminus \{(0, \dots, 0)\}$  with  $d \geq 1$  arbitrary. This intersection is still residual in  $X^{\mathbb{N}}$ . We also know from [16] that the set of  $u$  in  $X^{\mathbb{N}}$  that induce a dense algebra in  $X$  is residual in  $X^{\mathbb{N}}$ . Hence we may pick  $u \in X^{\mathbb{N}}$  belonging to  $\bigcap_{A, s, k} \bigcup_{\beta} \mathcal{A}(A, \beta, s, k)$  and inducing a dense algebra in  $X$ . It is plain that for any non-zero polynomial  $P$  with  $P(0) = 0$ ,  $P(u)$  is hypercyclic for  $T$ .

It remains to show that the algebra generated by  $u$  is not finitely generated. Assume on the contrary that it is generated by a finite number of  $P_1(u), \dots, P_p(u)$ . In particular, it is generated by a finite number of  $u_1, \dots, u_q$ . Then there exists a polynomial  $Q \in \mathbb{C}[z_1, \dots, z_q]$  such that  $Q(0) = 0$  and  $u_{q+1} = Q(u_1, \dots, u_q)$ . Define  $P(z) = z_{q+1} - Q(z)$ . Then  $P$  is a non-zero polynomial with  $P(0) = 0$ . Nevertheless,  $P(u) = 0$ , which contradicts the fact that  $P(u)$  is a hypercyclic vector for  $T$ .  $\square$

REMARK 2.2. Theorem 2.1 remains true if the algebra is not commutative. This is clear if  $d = 1$ . For the remaining cases, we have to replace in the proof polynomials in  $d$  commutative variables by polynomials in  $d$  non-commutative variables. Details are left to the reader.

We point out that, in the previous theorem, the index  $\beta$  may depend on  $A$ ,  $U_1, \dots, U_d, V$  and  $W$ . We will never use this possibility: we will only need that  $\beta$  may depend on  $A$  and we will denote  $\beta = \beta_A$ . For this particular case, we could give an easier proof avoiding the use of the Bourdon-Feldman theorem (see for instance the proofs of Propositions 5.1 or 6.1).

Let us give a couple of corollaries. The first one comes from [8, Remark 5.28] and was the key lemma in [8], [15] or [6] to get hypercyclic algebras.

COROLLARY 2.3. *Let  $T$  be a continuous operator on a separable  $F$ -algebra  $X$ . Assume that, for any pair  $(U, V)$  of non-empty open sets in  $X$ , for any open neighbourhood  $W$  of zero, and for any positive integer  $m$ , one can find  $u \in U$  and an integer  $N$  such that  $T^N(u^n) \in W$  for all  $n < m$  and  $T^N(u^m) \in V$ . Then  $T$  admits a hypercyclic algebra.*

PROOF. It is straightforward to show that the assumptions of Theorem 2.1 with  $d = 1$  and  $\beta_A = \max A$  are satisfied.  $\square$

In this work, we will often use the inverse choice for  $\beta_A$ .

**COROLLARY 2.4.** *Let  $T$  be a continuous operator on a separable  $F$ -algebra  $X$ . Assume that, for any pair  $(U, V)$  of non-empty open sets in  $X$ , for any open neighbourhood  $W$  of zero, and for any positive integers  $m_0 < m_1$ , one can find  $u \in U$  and an integer  $N$  such that  $T^N(u^m) \in W$  for all  $m \in \{m_0 + 1, \dots, m_1\}$  and  $T^N(u^{m_0}) \in V$ . Then  $T$  admits a hypercyclic algebra.*

**PROOF.** This is now Theorem 2.1 with  $d = 1$  and  $\beta_A = \min A$ . □

### 2.2. Hypercyclic algebras not contained in a finitely generated algebra

The conclusion of Theorem 2.1 about infinitely generated hypercyclic algebras does not prevent the possibility for such an algebra to be contained in a finitely generated algebra. Furthermore, it is well known that every at least two generated algebra contains an infinitely generated subalgebra. For all of the examples of this paper we may avoid this scenario thanks to the following result.

**COROLLARY 2.5.** *Let  $X$  be a separable commutative  $F$ -algebra that contains a dense freely generated subalgebra. Let  $T$  be a continuous operator on  $X$  and let  $d \geq 1$ . Assume that for any  $A \subset \mathbb{N}_0^d \setminus \{(0, \dots, 0)\}$  finite and non-empty, for any non-empty open subsets  $U_1, \dots, U_d, V$  of  $X$ , for any neighbourhood  $W$  of 0, there exist  $u = (u_1, \dots, u_d) \in U_1 \times \dots \times U_d$ ,  $\beta \in A$  and  $N \geq 1$  such that  $T^N(u^\beta) \in V$  and  $T^N(u^\alpha) \in W$  for all  $\alpha \in A$ ,  $\alpha \neq \beta$ . Then  $T$  admits a hypercyclic algebra whose minimal number of generators is  $d$ . Moreover, if the assumptions are satisfied for all  $d \geq 1$ , then  $T$  admits a dense hypercyclic algebra not contained in a finitely generated one.*

**PROOF.** By Theorem 2.1 the set of  $d$ -tuples generating a hypercyclic algebra for  $T$  is residual in  $X^d$ . Observe also that the set of  $u$  in  $X^d$  that induce an algebra which is not contained in a subalgebra of  $X$  with less than  $d$  generators is residual in  $X^d$  (see [16]). If we pick an element in the intersection of those two sets, the hypercyclic algebra generated by it will have minimal number of generators equal to  $d$ .

For the second claim, we consider  $X^{\mathbb{N}}$  endowed with the product topology. By the assumption, for each  $N \in \mathbb{N}$ , the set

$$H_N = \{(u_n) \in X^{\mathbb{N}} : (u_1, \dots, u_N) \text{ generates a hypercyclic algebra for } T\}$$

is residual in  $X^{\mathbb{N}}$  and hence, by the Baire category theorem, the set  $H = \bigcap_{N=1}^{\infty} H_N$  is residual as well. The algebra generated by any  $u \in H$  is hypercyclic for  $T$ . Furthermore, the set of sequences of  $X^{\mathbb{N}}$  which generates a dense algebra not contained in a finitely generated one is also residual (see [16]). The conclusion follows by one more application of the Baire category theorem. □

Hence, we need to provide for our examples a dense and freely generated subalgebra. This is quite easy for a Fréchet sequence algebra endowed with the Cauchy product: provided  $\text{span}(e_n)$  is dense, it always contain a dense and freely generated subalgebra, namely the unital algebra generated by the sequence  $e_1$ . This covers the case of  $H(\mathbb{C})$  and of the disc algebra. This is slightly more difficult for a Fréchet sequence algebra endowed with the coordinatewise product.

**LEMMA 2.6.** *Let  $X$  be a Fréchet sequence algebra, endowed with the pointwise product, and for which the sequence  $(e_n)$  is a Schauder basis. Then  $X$  has a dense freely generated subalgebra.*

PROOF. Let  $(b_n) \subset (0, 1)$  be such that the series  $\sum_{n=0}^{\infty} b_n \|e_n\|_n$  converges. Consider the sequence of natural numbers  $(a_n)$  such that  $a_0 = 0$  and  $a_n = a_{n-1} + n$ ,  $n \in \mathbb{N}$  and define  $c_n := \min_{l \in [a_{n-1}, a_n)} b_l$ , if  $n \in [a_{n-1}, a_n)$ . The series  $\sum_{n=0}^{\infty} c_n e_n$  converges absolutely since  $c_n \leq b_n$ ,  $n \in \mathbb{N}$ .

Define a sequence  $(\lambda_n) \subset (0, 1)$  inductively as follows: choose  $\lambda_0 \in (0, 1)$  and, for  $n \in \mathbb{N}$ , take  $\lambda_n \in (0, 1) \setminus \{\lambda_0^{p_0} \dots \lambda_{n-1}^{p_{n-1}} : p_0, \dots, p_{n-1} \in \mathbb{Q}\}$ . Observe that if  $\lambda = (\lambda_0, \dots, \lambda_p)$ ,  $p \in \mathbb{N}_0$  and  $\alpha \neq \beta \in \mathbb{N}_0^p$ , then  $\lambda^\alpha \neq \lambda^\beta$ . Let now, for each  $n \in \mathbb{N}_0$ ,

$$g_n = e_n + \sum_{k=1}^{\infty} \lambda_n^{n+k} c_{n+k} e_{n+k},$$

where the convergence of the series is ensured by the convergence of  $\sum_{n=0}^{\infty} b_n \|e_n\|_n$  and the inequality  $\lambda_n^{n+k} c_{n+k} < b_{n+k}$ . We claim that the algebra generated by the  $g_n$ ,  $n \in \mathbb{N}$  is dense and free.

First, we show that the sequence  $\{g_n : n \in \mathbb{N}\}$  is algebraically independent. For that reason, let

$$(2.1) \quad \sum_{\alpha \in A} a_\alpha g^\alpha = 0,$$

where  $A = \{\alpha(1), \dots, \alpha(q)\} \subset \mathbb{N}_0^p$ ,  $g = (g_1, \dots, g_p) \in X^p$ , and  $p \in \mathbb{N}$ . If we consider the coordinate  $N = n + k$  in equation (2.1), we get the following equation which holds for all  $N$  sufficiently large.

$$(2.2) \quad \sum_{i=1}^q a_{\alpha(i)} (\lambda_1^N c_N)^{\alpha_1(i)} \dots (\lambda_p^N c_N)^{\alpha_p(i)} = 0.$$

Now we may choose  $N$  sufficiently big such that  $N, \dots, N + q - 1 \in [a_{m-1}, a_m)$  for some  $m$ , which means that  $c_N = \dots = c_{N+q-1} = b_m$ . Equation (2.2) then becomes

$$\sum_{i=1}^q a_{\alpha(i)} (\lambda_1^M b_m)^{\alpha_1(i)} \dots (\lambda_p^M b_m)^{\alpha_p(i)} = 0,$$

where  $M$  ranges over  $N, \dots, N + q - 1$ . Setting  $A$  the matrix

$$\begin{bmatrix} (\lambda_1^N b_m)^{\alpha_1(1)} \dots (\lambda_p^N b_m)^{\alpha_p(1)} & \dots & (\lambda_1^N b_m)^{\alpha_1(q)} \dots (\lambda_p^N b_m)^{\alpha_p(q)} \\ \vdots & \ddots & \vdots \\ (\lambda_1^{N+q-1} b_m)^{\alpha_1(1)} \dots (\lambda_p^{N+q-1} b_m)^{\alpha_p(1)} & \dots & (\lambda_1^{N+q-1} b_m)^{\alpha_1(q)} \dots (\lambda_p^{N+q-1} b_m)^{\alpha_p(q)} \end{bmatrix},$$

we find the matrix equality

$$A \begin{bmatrix} a_{\alpha(1)} \\ \vdots \\ a_{\alpha(q)} \end{bmatrix} = 0.$$

The determinant of the square matrix  $A$ , after making use of the Vandermonde identity, is

$$\prod_{i=1}^q \prod_{j=1}^p (b_m \lambda_j^N)^{\alpha_j(i)} \prod_{i>j} [(\lambda_1^{\alpha_1(i)} \dots \lambda_p^{\alpha_p(i)}) - (\lambda_1^{\alpha_1(j)} \dots \lambda_p^{\alpha_p(j)})] \neq 0.$$

Hence, we get that  $a_{\alpha(i)} = 0$  for all  $i = 1, \dots, q$ .

Next, we show that the algebra generated by  $\{g_n : n \in \mathbb{N}\}$  is dense in  $X$ . We will show that the elements  $e_n, n \in \mathbb{N}$ , are in the closure of this algebra. Let us fix  $n \in \mathbb{N}$  and observe that, for all  $p \in \mathbb{N}$ ,

$$g_n^p - e_n = \sum_{k=1}^{\infty} \lambda_n^{(n+k)p} c_{n+k}^p e_{n+k}.$$

Fix  $q \in \mathbb{N}$  and let  $\varepsilon > 0$ . There exists  $N \geq q$  such that, for all  $p \in \mathbb{N}$ ,

$$\left\| \sum_{k>N} \lambda_n^{(n+k)p} c_{n+k}^p e_{n+k} \right\|_q \leq \sum_{k>N} b_{n+k} \|e_{n+k}\|_q < \varepsilon.$$

Since furthermore

$$\sum_{k=1}^N \lambda_n^{(n+k)p} c_{n+k}^p e_{n+k} \xrightarrow{p \rightarrow \infty} 0,$$

we conclude that  $g_n^p \xrightarrow{p \rightarrow \infty} e_n$ . □



## Convolution operators with $|\phi(0)| > 1$

### 3.1. Operators with many eigenvectors

In this section we shall deduce Theorem 1.1 from a more general assertion on operators having many eigenvalues. As Theorem 2.1 does for Corollary 2.3, this generalized statement also includes [6, Theorem 2.1] as a particular case. Before stating and proving it, let us add some notation.

Given  $p, d \in \mathbb{N}$ , we denote each set  $\{1, \dots, p\}$  by  $I_p$ , each  $d$ -tuple  $(j_1, \dots, j_d) \in I_p^d$  by the multi-index  $\mathbf{j} \in I_p^d$  and each product  $a_{j_1} \cdots a_{j_m}$  by the symbol  $a_{\mathbf{j}}$ . We allow  $d = 0$  with the convention that, in this case,  $a_{\mathbf{j}} = 1$ .

**THEOREM 3.1.** *Let  $X$  be an  $F$ -algebra and let  $T \in \mathcal{L}(X)$ . Assume that there exist a function  $E : \mathbb{C} \rightarrow X$  and an entire function  $\phi : \mathbb{C} \rightarrow \mathbb{C}$  satisfying the following assumptions:*

- (a) for all  $\lambda \in \mathbb{C}$ ,  $TE(\lambda) = \phi(\lambda)E(\lambda)$ ;
- (b) for all  $\lambda, \mu \in \mathbb{C}$ ,  $E(\lambda)E(\mu) = E(\lambda + \mu)$ ;
- (c) for all  $\Lambda \subset \mathbb{C}$  with an accumulation point, the linear span of  $\{E(\lambda) : \lambda \in \Lambda\}$  is dense in  $X$ ;
- (d)  $\phi$  is not a multiple of an exponential function;
- (e) for all  $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ , there exist  $m \in I$  and  $a, b \in \mathbb{C}$  such that  $|\phi(mb)| > 1$  and, for all  $n \in I$  and  $d \in \{0, \dots, n\}$ , with  $(n, d) \neq (m, m)$ ,  $|\phi(db + (n - d)a)| < |\phi(mb)|^{d/m}$ .

Then  $T$  supports a hypercyclic algebra.

The proof of this result follows the lines of that of Theorem 2.1 in [6], replacing Corollary 2.3 by the more general Theorem 2.1. For the sake of completeness, we include the details.

**PROOF.** Let  $(U, V, W)$  be a triple of non-empty open sets in  $X$ , with  $0 \in W$ , and let  $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ . By the hypothesis there are  $m \in I$  and  $a, b \in \mathbb{C}$  satisfying (e). Define  $w_0 := mb$  and let  $\delta > 0$  be small enough and  $w_1, w_2 \in B(w_0, \delta)$  so that

- (i)  $|\phi| > 1$  on  $B(w_0, \delta)$ ;
- (ii)  $t \mapsto \log |\phi(tw_1 + (1-t)w_2)|$  is strictly convex (the existence of  $w_1, w_2 \in B(w_0, \delta)$  comes from [6, Lemma 2.2] and is a consequence of (d));
- (iii) for all  $n \in I$  and  $d \in \{0, \dots, n\}$ , with  $(n, d) \neq (m, m)$ , and for all  $\lambda_1, \dots, \lambda_d \in B(w_0, \delta)$  and  $\gamma_1, \dots, \gamma_{n-d} \in B(a, \delta)$ ,

$$(3.1) \quad \left| \phi \left( \frac{\lambda_1 + \cdots + \lambda_d}{m} + \gamma_1 + \cdots + \gamma_{n-d} \right) \right| < (|\phi(\lambda_1)| \times \cdots \times |\phi(\lambda_d)|)^{1/m}.$$

The last condition can be satisfied because

$$\frac{\lambda_1 + \cdots + \lambda_d}{m} + \gamma_1 + \cdots + \gamma_{n-d} = db + (n - d)a + z,$$

where the size of  $z$  can be controlled through  $\delta$ . Now, since  $B(a, \delta)$  and  $[w_1, w_2]$  have accumulation points, we can find  $p, q \in \mathbb{N}$ ,  $a_1, \dots, a_p, b_1, \dots, b_q \in \mathbb{C}$ ,  $\gamma_1, \dots, \gamma_p \in B(a, \delta)$  and  $\lambda_1, \dots, \lambda_q \in [w_1, w_2]$  with

$$\sum_{l=1}^p a_l E(\gamma_l) \in U \text{ and } \sum_{j=1}^q b_j E(\lambda_j) \in V.$$

For some big  $N \in \mathbb{N}$  (which will be determined later in the proof) and each  $j \in \{1, \dots, q\}$ , let  $c_j := c_j(N)$  be any complex number satisfying

$$c_j^m(N) = \frac{b_j}{\phi(\lambda_j)^N}$$

and define

$$u := u(N) = \sum_{l=1}^p a_l E(\gamma_l) + \sum_{j=1}^q c_j E\left(\frac{\lambda_j}{m}\right).$$

For the powers of  $u$  we have the formula

$$(3.2) \quad u^n = \sum_{d=0}^n \sum_{\substack{\mathbf{l} \in I_p^{n-d} \\ \mathbf{j} \in I_q^d}} \alpha(\mathbf{l}, \mathbf{j}, d, n) a_1 c_j E\left(\gamma_{l_1} + \dots + \gamma_{l_{n-d}} + \frac{\lambda_{j_1} + \dots + \lambda_{j_d}}{m}\right).$$

We claim that, if  $N$  is taken large enough,  $u = u(N)$  satisfies the conditions of the general criterion with  $d = 1$ , what will complete the proof.

That  $u \in U$  for large  $N$  is clear since, from (i),  $c_j(N) \rightarrow 0$  as  $N \rightarrow \infty$ . Applying  $T^N$  to  $u^n$  we see that we need to study the behaviour (as  $N$  grows) of

$$(3.3) \quad c_j(N) \left[ \phi\left(\gamma_{l_1} + \dots + \gamma_{l_{n-d}} + \frac{\lambda_{j_1} + \dots + \lambda_{j_d}}{m}\right) \right]^N.$$

For  $n \in I \setminus \{m\}$  we have that (3.3) goes to 0 as  $N$  grows by the inequality (3.1) and the definition of  $c_j$ ,  $j = 1, \dots, q$ . This way we get  $T^N(u^n) \in W$  for all  $n \in I \setminus \{m\}$  if  $N$  is large enough. Now let us consider the case  $n = m$ . We have

$$\begin{aligned} u^m &= \sum_{d=0}^{m-1} \sum_{\substack{\mathbf{l} \in I_p^{m-d} \\ \mathbf{j} \in I_q^d}} \alpha(\mathbf{l}, \mathbf{j}, d, m) a_1 c_j E\left(\gamma_{l_1} + \dots + \gamma_{l_{m-d}} + \frac{\lambda_{j_1} + \dots + \lambda_{j_d}}{m}\right) \\ &+ \sum_{\mathbf{j} \in I_q^m \setminus D_q} \alpha(\mathbf{j}, m) c_j E\left(\frac{\lambda_{j_1} + \dots + \lambda_{j_m}}{m}\right) \\ &+ \sum_{j=1}^q c_j^m E(\lambda_j) \\ &=: v_1 + v_2 + v_3, \end{aligned}$$

where

$$v_1 := \sum_{d=0}^{m-1} \sum_{\substack{\mathbf{l} \in I_p^{m-d} \\ \mathbf{j} \in I_q^d}} \alpha(\mathbf{l}, \mathbf{j}, d, m) a_1 c_j E\left(\gamma_{l_1} + \dots + \gamma_{l_{m-d}} + \frac{\lambda_{j_1} + \dots + \lambda_{j_d}}{m}\right),$$

$$v_2 := \sum_{\mathbf{j} \in I_q^m \setminus D_q} \alpha(\mathbf{j}, m) c_{\mathbf{j}} E\left(\frac{\lambda_{j_1} + \cdots + \lambda_{j_m}}{m}\right), \quad v_3 := \sum_{j=1}^q c_j^m E(\lambda_j)$$

and  $D_q$  is the diagonal of  $I_q^m$ , that is, the set of all  $m$ -tuples  $(j, \dots, j)$  with  $1 \leq j \leq q$ . Again we have  $T^N(v_1) \rightarrow 0$  as  $N \rightarrow \infty$  from (3.1). Furthermore, since  $t \in [0, 1] \mapsto \log |\phi(tw_1 + (1-t)w_2)|$  is strictly convex, we have

$$\left| \phi\left(\frac{\lambda_{j_1} + \cdots + \lambda_{j_m}}{m}\right) \right| < |\phi(\lambda_{j_1})|^{1/m} \cdots |\phi(\lambda_{j_m})|^{1/m}.$$

From this we conclude that

$$\begin{aligned} & |c_{\mathbf{j}}(N)| \cdot \left| \phi\left(\frac{\lambda_{j_1} + \cdots + \lambda_{j_m}}{m}\right) \right|^N \\ &= \left| \frac{b_{\mathbf{j}}^{1/m}}{|\phi(\lambda_{j_1})|^{N/m} \cdots |\phi(\lambda_{j_m})|^{N/m}} \right| \cdot \left| \phi\left(\frac{\lambda_{j_1} + \cdots + \lambda_{j_m}}{m}\right) \right|^N \\ &= |b_{\mathbf{j}}^{1/m}| \cdot \left| \frac{\phi\left(\frac{\lambda_{j_1} + \cdots + \lambda_{j_m}}{m}\right)}{|\phi(\lambda_{j_1})|^{1/m} \cdots |\phi(\lambda_{j_m})|^{1/m}} \right|^N \rightarrow 0 \text{ as } N \rightarrow \infty, \end{aligned}$$

what shows that  $T^N(v_2)$  also tends to 0 as  $N \rightarrow \infty$ . Finally, by the definition of  $c_j$ ,  $j = 1, \dots, q$ , we get

$$T^N v_3 = \sum_{j=1}^q b_j E(\lambda_j) \in V$$

for all  $N \in \mathbb{N}$ . This completes the proof.  $\square$

We now deduce a more readable corollary, when the entire function  $\phi$  is “well behaved” in some half-line of the complex plane (like in the Figure 1 for example).

**COROLLARY 3.2.** *Let  $X$  be an  $F$ -algebra and let  $T \in \mathcal{L}(X)$ . Assume that there exist a function  $E : \mathbb{C} \rightarrow X$  and an entire function  $\phi : \mathbb{C} \rightarrow \mathbb{C}$  satisfying the following assumptions:*

- (a) for all  $\lambda \in \mathbb{C}$ ,  $TE(\lambda) = \phi(\lambda)E(\lambda)$ ;
- (b) for all  $\lambda, \mu \in \mathbb{C}$ ,  $E(\lambda)E(\mu) = E(\lambda + \mu)$ ;
- (c) for all  $\Lambda \subset \mathbb{C}$  with an accumulation point, the linear span of  $\{E(\lambda) : \lambda \in \Lambda\}$  is dense in  $X$ ;
- (d)  $\phi$  is not a multiple of an exponential function;
- (e) there exist  $v \in \mathbb{C}$  and a real number  $p > 0$  such that  $|\phi(v)| > 1$  and  $|\phi(tv)| \leq 1$  for all  $t > p$ .

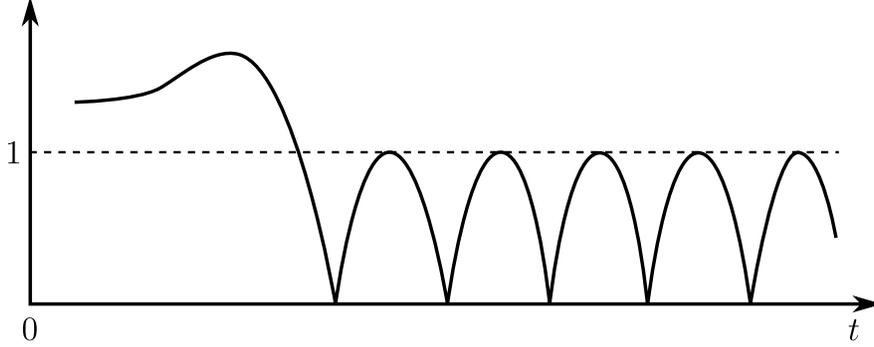
Then  $T$  supports a hypercyclic algebra.

For the proof of Corollary 3.2, we are going to need the following simple lemma.

**LEMMA 3.3.** *Let  $\phi$  be entire and  $\Lambda \subset \mathbb{R}$  have an accumulation point in it. If  $|\phi(t)| = 1$  for all  $t \in \Lambda$  then  $|\phi(t)| = 1$  for all  $t \in \mathbb{R}$ .*

**PROOF.** Since  $|\phi(t)| = 1$  for all  $t \in \Lambda \subset \mathbb{R}$ , we have  $\overline{\phi(\bar{t})} \cdot \phi(t) = \overline{\phi(t)} \cdot \phi(t) = 1$ , hence

$$\overline{\phi(\bar{t})} = \phi(t)^{-1}, \forall t \in \Lambda \subset \mathbb{R}.$$

FIGURE 1. Graph of  $t \mapsto |\phi(tv)|$ 

Since this is a holomorphic equality ( $t \mapsto \overline{\phi(\bar{t})}$  is entire), it extends to the whole complex plane. In particular it holds for all  $t \in \mathbb{R}$ , that is,

$$\overline{\phi(t)} = \overline{\phi(\bar{t})} = \phi(t)^{-1}, \forall t \in \mathbb{R},$$

thus

$$|\phi(t)| = 1, \forall t \in \mathbb{R},$$

as we wanted.  $\square$

**PROOF OF COROLLARY 3.2.** We may assume without loss of generality that  $v = 1$ . Let  $t_0 > 0$  be the smallest positive real number such that  $|\phi(t)| \leq 1$  for all  $t \geq t_0$ . We just need to prove that condition (e) of Theorem 3.1 is satisfied. So let  $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$  be arbitrary and set  $m = \min I$ . We can find  $t_1 < t_0$  near enough to  $t_0$  so that  $|\phi(t_1)| > 1$  and  $t_1 + \frac{t_1}{m} > t_0$ . Letting  $b := \frac{t_1}{m}$  we have  $|\phi(mb)| > 1$ . Now fix  $a_0 > t_0$  and take  $\epsilon \in (0, 1/2)$  such that  $a_0 - \epsilon > t_0$ . There exists  $a \in [a_0 - \epsilon, a_0 + \epsilon]$  such that  $|\phi(db + (n-d)a)| < 1$  for all  $n \in I$  and  $d \in \{0, \dots, n\}$  with  $(n, d) \neq (m, m)$ . In fact, if this is not the case then, for each  $a \in [a_0 - \epsilon, a_0 + \epsilon]$ , we can find a point  $t_a = d_a b + (n_a - d_a)a$ , with  $n_a \in I$ ,  $d_a \in \{0, \dots, n_a\}$  and  $(n_a, d_a) \neq (m, m)$ , such that  $|\phi(t_a)| \geq 1$ . Since  $m = \min(I)$ ,  $b = t_1/m$  and  $a > t_0$ , we get  $t_a > t_0$  so that  $|\phi(t_a)| \leq 1$ . This way, varying  $a \in [a_0 - \epsilon, a_0 + \epsilon]$  we find infinitely many points  $t_a$  within  $[t_0, \max I(a_0 + \epsilon)]$  in which  $|\phi|$  assumes the value 1. The set  $\Lambda$  composed by these points is infinite, closed and subset of the compact  $[t_0, \max(I)(a_0 + \epsilon)]$ , hence  $\Lambda$  has an accumulation point. By Lemma 3.3 we conclude that  $|\phi(t)| = 1$  for all  $t \in \mathbb{R}$ , which contradicts the fact that  $|\phi(t_1)| > 1$ . This completes the proof.  $\square$

### 3.2. Applications to convolution operators

We now observe that we may apply Corollary 3.2 to convolution operators  $\phi(D)$  with  $|\phi(0)| > 1$ , where  $E(\lambda)(z) = e^{\lambda z}$ . This yields immediately Theorem 1.1. We may also apply Corollary 3.2 to handle the case  $|\phi(0)| = 1$ .

**COROLLARY 3.4.** *Let  $P \in \mathbb{C}[z]$  be a non-constant polynomial and let  $\phi(z) = P(z)e^z$ . Then  $\phi(D)$  supports a hypercyclic algebra.*

**PROOF.** The case  $|P(0)| < 1$  is done in [6], the case  $|P(0)| > 1$  is settled by Theorem 1.1. It remains to consider the case  $|P(0)| = 1$ . Since  $|P(it)|$  tends to

$+\infty$  as  $t$  tends to  $+\infty$ , there exists  $t_0 \in \mathbb{R}$  such that  $|\phi(it_0)| > 1$ . By continuity of  $|\phi|$ , there exists  $v = |v|e^{i\theta}$  with  $\theta \in (\pi/2, 3\pi/2)$  such that  $|\phi(v)| > 1$ . Now, because  $v$  lies in the left half-plane,  $|\phi(tv)|$  tends to 0 as  $t$  tends to  $+\infty$ . We may conclude with Corollary 3.2.  $\square$

We finish this section by pointing out that Theorem 3.1 can also handle functions which do not satisfy the properties described above.

EXAMPLE 3.5. The convolution operator induced by  $\phi(z) = \frac{1}{2}e^z + e^{iz} - \frac{1}{4}$  supports a hypercyclic algebra (let us observe that  $|\phi(0)| > 1$  and that  $\phi$  does not tend to 0 along any ray). Indeed, for any  $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$  we choose  $m = \max(I)$  and take  $a = k(2\pi i)$  and  $b = k2\pi$  for some large integer  $k$ . Let  $n \in I$  and  $d \in \{0, \dots, n\}$  with  $(n, d) \neq (m, m)$ . Then

$$\phi(db + (n-d)a) = \frac{1}{2}e^{2dk\pi} + e^{-2(n-d)k\pi} - \frac{1}{4}.$$

In particular,

$$|\phi(mb)| = \frac{1}{2}e^{2mk\pi} + \frac{3}{4} > 1.$$

When  $d = 0$ ,

$$|\phi(na)| = \left| e^{-2nk\pi} + \frac{1}{4} \right| < 1.$$

Finally,

$$|\phi(db + (n-d)a)| \leq \frac{1}{2}e^{2dk\pi} + \frac{3}{4}$$

and we have, for all  $d = 1, \dots, m-1$ ,

$$\left( \frac{1}{2}e^{2dk\pi} + \frac{3}{4} \right)^m \leq \left( \frac{1}{2}e^{2mk\pi} + \frac{3}{4} \right)^d$$

if  $k$  is large enough.

REMARK 3.6. Combining the previous arguments with that of [6, Section 6], under the assumptions of Theorem 1.1,  $\phi(D)$  admits a dense hypercyclic algebra which is not contained in a finitely generated algebra.



## Weighted shifts on Fréchet sequence algebras

### 4.1. Fréchet sequence algebras with the coordinatewise product

We begin with the proof of Theorem 1.2. We first explain where the property of admitting a continuous norm comes into play.

LEMMA 4.1. *Let  $X$  be a Fréchet sequence algebra for the coordinatewise product and with a continuous norm. Then the sequence  $(e_n)$  is bounded below.*

PROOF. Let  $q \geq 1$  be such that  $\|\cdot\|_q$  is a norm on  $X$ . Then for all  $n \in \mathbb{N}_0$ ,

$$0 < \|e_n\|_q = \|e_n \cdot e_n\|_q \leq \|e_n\|_q^2$$

which shows that  $\|e_n\|_q \geq 1$ . □

We shall prove the following precised version of Theorem 1.2.

THEOREM 4.2. *Let  $X$  be a Fréchet sequence algebra for the coordinatewise product and with a continuous norm. Assume that  $(e_n)$  is a Schauder basis for  $X$ . Let also  $B_w$  be a bounded weighted shift on  $X$ . The following assumptions are equivalent.*

- (i)  $B_w$  supports a dense hypercyclic algebra not contained in a finitely generated algebra.
- (ii)  $B_w$  supports a hypercyclic algebra.
- (iii) For all  $m \geq 1$ , there exists  $x \in X$  such that  $x^m$  is a hypercyclic vector for  $B_w$ .
- (iv) For all  $m \geq 1$ , for all  $L \in \mathbb{N}$ , there exists a sequence of integers  $(n_k)$  such that, for all  $l = 0, \dots, L$ ,  $((w_{l+1} \cdots w_{n_k+l})^{-1/m} e_{n_k+l})$  tends to zero.
- (v) There exists a sequence of integers  $(n_k)$  such that, for all  $\gamma > 0$  and for all  $l \in \mathbb{N}$ ,  $((w_{l+1} \cdots w_{n_k+l})^{-\gamma} e_{n_k+l})$  tends to zero.

PROOF. The implications (i)  $\implies$  (ii) and (ii)  $\implies$  (iii) are trivial. The proof of (iii)  $\implies$  (iv) mimics that of the necessary condition for hypercyclicity. Let  $m \geq 1$  and  $x \in X$  be such that  $x^m \in HC(B_w)$ . Write  $x = \sum_{n=0}^{+\infty} x_n e_n$ . Since  $(e_n)$  is a Schauder basis, the sequence  $(x_n e_n)$  goes to zero. Moreover, there exists a sequence of integers  $(n_k)$  such that  $(B_w^{n_k}(x^m))_k$  goes to  $e_0 + \cdots + e_L$ . Since convergence in  $X$  implies coordinatewise convergence, for all  $l = 0, \dots, L$ ,  $(w_{l+1} \cdots w_{n_k+l} x_{n_k+l}^m)$  converges to 1. Hence the sequences  $((w_{l+1} \cdots w_{n_k+l})^{1/m} x_{n_k+l})$  are bounded below. Writing

$$(w_{l+1} \cdots w_{n_k+l})^{-1/m} e_{n_k+l} = \frac{1}{(w_{l+1} \cdots w_{n_k+l})^{1/m} x_{n_k+l}} \cdot x_{n_k+l} e_{n_k+l}$$

we get the result.

To prove that  $(iv) \implies (v)$ , observe that a diagonal argument ensure the existence of a sequence  $(n_k)$  such that, for all  $m \geq 1$  and all  $l \in \mathbb{N}$ , the sequence  $((w_{l+1} \cdots w_{n_k+l})^{-1/m} e_{n_k+l})$  tends to zero. Now we can conclude by observing that, since the sequence  $(e_n)$  is bounded below, if  $((w_{l+1} \cdots w_{n_k+l})^{-1/m} e_{n_k+l})$  tends to zero for some  $m$ , then  $(w_{l+1} \cdots w_{n_k+l})$  tends to  $+\infty$  and, in particular,  $((w_{l+1} \cdots w_{n_k+l})^{-\gamma} e_{n_k+l})$  tends to zero for all  $\gamma \geq 1/m$ .

It remains to prove the most difficult implication,  $(v) \implies (i)$ . We start by fixing a sequence of integers  $(n_k)$  such that for all  $\gamma > 0$  and for all  $l \in \mathbb{N}$ ,  $((w_{l+1} \cdots w_{n_k+l})^{-\gamma} e_{n_k+l})$  goes to zero. We intend to apply Theorem 2.1. Thus, let  $d \geq 1$  and  $A \subset \mathbb{N}_0^d \setminus \{(0, \dots, 0)\}$  be finite and non-empty. For  $\alpha \in A$  we define the linear form  $L_\alpha$  on  $\mathbb{R}^d$  by  $L_\alpha(\kappa) = \sum_{j=1}^d \alpha_j \kappa_j$ . Since  $L_\alpha$  and  $L_{\alpha'}$  coincide only on a hyperplane for  $\alpha \neq \alpha'$ , there exist  $\kappa \in (0, +\infty)^d$  and  $\beta = \beta_A \in A$  such that  $0 < L_\beta(\kappa) < L_\alpha(\kappa)$  for all  $\alpha \neq \beta$ ,  $\alpha \in A$ . Without loss of generality, we may assume that  $L_\beta(\kappa) = 1$ .

Let now  $U_1, \dots, U_d, V$  be non-empty open subsets of  $X$  and let  $W$  be a neighbourhood of zero. Let  $x_1, \dots, x_d$  belonging respectively to  $U_1, \dots, U_d$  with finite support and let  $y = \sum_{l=0}^p y_l e_l$  belonging to  $V$ . We set, for  $j = 1, \dots, d$ ,

$$u_j := u_j(n_k) = x_j + \sum_{l=0}^p \frac{y_l^{\kappa_j}}{(w_{l+1} \cdots w_{n_k+l})^{\kappa_j}} e_{n_k+l}.$$

Our assumption implies that, provided  $n_k$  is large enough,  $u_j$  belongs to  $U_j$  for all  $j = 1, \dots, d$ . Moreover, again if  $n_k$  is large enough (larger than the size of the support of each  $x_j$ ), for all  $\alpha \in A$ ,

$$B_w^{n_k}(u^\alpha) = \sum_{l=0}^p \frac{y_l^{L_\alpha(\kappa)}}{(w_{l+1} \cdots w_{n_k+l})^{L_\alpha(\kappa)-1}} e_l.$$

In particular, for  $\alpha = \beta$ ,  $B_w^{n_k}(u^\beta) = y \in V$  whereas, for  $\alpha \neq \beta$ , since  $L_\alpha(\kappa) - 1 > 0$  and since the sequences  $(w_{l+1} \cdots w_{n_k+l})$  tend to  $+\infty$ , we get  $B_w^{n_k}(u^\alpha) \in W$  provided  $n_k$  is large enough. Hence,  $B_w$  admits a dense and not finitely generated hypercyclic algebra.  $\square$

As recalled in the introduction, the hypercyclicity of  $B_w$  on  $X$  is equivalent to the existence of a sequence  $(n_k)$  such that, for all  $l \in \mathbb{N}$ ,  $((w_{l+1} \cdots w_{n_k+l})^{-1} e_{n_k+l})$  tends to zero. It is well known that this last condition is equivalent to the following one, which seems much weaker: there exists a sequence of integers  $(n_k)$  such that  $((w_1 \cdots w_{n_k})^{-1} e_{n_k})$  tends to zero. In view of this and of Theorem 4.2, it is tempting to conjecture that  $B_w$  supports a hypercyclic algebra if and only if there exists a sequence of integers  $(n_k)$  such that, for all  $\gamma > 0$ ,  $((w_1 \cdots w_{n_k})^{-\gamma} e_{n_k})$  tends to zero. Unfortunately, this is not the case, as the following example points out.

**EXAMPLE 4.3.** Let  $X = \{(x_n) \in \omega : |x_n| a_n \rightarrow 0\}$  where  $a_{2n} = 1$  and  $a_{2n+1} = 2^n$  endowed with  $\|x\| = \sup_n |x_n| a_n$  and let  $w$  be the weight such that  $w_1 \cdots w_{2n} = 2^{n-1}$  and  $w_1 \cdots w_{2n+1} = 2^{2n}$ . Then  $w$  is an admissible weight on  $X$ ,  $((w_1 \cdots w_{2n})^{-\gamma} e_{2n})$  tends to zero for all  $\gamma > 0$  but  $B_w$  does not admit a hypercyclic algebra.

**PROOF.** We first observe that, endowed with the coordinatewise product,  $X$  is a Fréchet sequence algebra (since  $a_n \geq 1$  for all  $n$ ). To prove that  $w$  is admissible, it suffices to observe that  $w_k \|e_{k-1}\| \leq 2 \|e_k\|$  for all  $k$ . The construction of  $w$  ensures

that  $w_{2n} = 2^{-(n-1)}$  and  $w_{2n+1} = 2^{n+1}$ . Hence the previous inequality is clearly satisfied if we separate the case  $k$  even and  $k$  odd. Moreover for all  $\gamma > 0$ ,

$$(w_1 \cdots w_{2n})^{-\gamma} \|e_{2n}\| = 2^{-\gamma(n-1)} \xrightarrow{n \rightarrow +\infty} 0.$$

To prove that  $B_w$  does not support a hypercyclic algebra, it suffices to observe that, for all  $n \geq 1$ ,  $(w_1 \cdots w_{2n+1})^{-1/2} \|e_{2n+1}\| = 1$ , which implies that condition (v) of Theorem 4.2 cannot be satisfied.  $\square$

Nevertheless, if we add an extra assumption on  $X$ , then we get the expected result.

**COROLLARY 4.4.** *Let  $X$  be a Fréchet sequence algebra for the coordinatewise product and with a continuous norm. Assume that  $(e_n)$  is a Schauder basis for  $X$ . Assume also that, for all admissible weights  $w$ , for all  $\gamma > 0$ ,  $w^\gamma$  is admissible. Let  $B_w$  be a bounded weighted shift on  $X$ . The following assumptions are equivalent.*

- (i)  $B_w$  supports a hypercyclic algebra.
- (ii) For all  $\gamma > 0$ , there exists a sequence  $(n_k)$  such that  $((w_1 \cdots w_{n_k})^{-\gamma} e_{n_k})$  tends to zero.

**PROOF.** We assume that (ii) is satisfied and we show that, for all  $\gamma > 0$  and for all  $L \in \mathbb{N}$ , there exists a sequence  $(m_k)$  such that  $((w_1 \cdots w_{m_k+l})^{-\gamma} e_{m_k+l})$  tends to zero. An application of Theorem 4.2 will then allow to conclude. It is easy to get this sequence  $m_k$ . Indeed, it is sufficient to set  $m_k = n_k - L$ , since in that case

$$(w_1 \cdots w_{m_k+l})^{-\gamma} e_{m_k+l} = (B_w^\gamma)^{L-l} ((w_1 \cdots w_{n_k})^{-\gamma} e_{n_k})$$

which goes to zero by continuity of  $B_w^\gamma$ .  $\square$

We may observe that our favorite sequence spaces (namely unweighted  $\ell_p$ -spaces or  $H(\mathbb{C})$ ) satisfy the assumptions of the last corollary. We may also observe that on unweighted  $\ell_p$ -spaces as well as on any Fréchet sequence algebra with a continuous norm such that  $(e_n)$  is bounded, the convergence of  $((w_1 \cdots w_{n_k})^{-\gamma} e_{n_k})$  to zero is equivalent to the convergence of  $(w_1 \cdots w_{n_k})$  to  $+\infty$ . Hence, we may formulate the following corollary.

**COROLLARY 4.5.** *Let  $X$  be a Fréchet sequence algebra for the coordinatewise product and with a continuous norm. Assume that  $(e_n)$  is a Schauder basis for  $X$  and that  $(e_n)$  is bounded. Assume also that, for all admissible weights  $w$ , for all  $\gamma > 0$ ,  $w^\gamma$  is admissible. Let  $B_w$  be a bounded weighted shift on  $X$ . The following assumptions are equivalent.*

- (i)  $B_w$  supports a hypercyclic algebra.
- (ii)  $B_w$  is hypercyclic.
- (iii) There exists a sequence  $(n_k)$  such that  $(w_1 \cdots w_{n_k})$  tends to  $+\infty$ .

**REMARK 4.6.** On  $H(\mathbb{C})$ , the sequence  $(z^n)$  is unbounded. Nevertheless, any hypercyclic weighted shift  $B_w$  on  $H(\mathbb{C})$  supports a hypercyclic algebra. Indeed, for a sequence of integers  $(n_k)$ ,

$$\begin{aligned} & (w_1 \cdots w_{n_k})^{-1} z^{n_k} \text{ tends to 0 in } H(\mathbb{C}) \\ \iff & \forall q \geq 1, (w_1 \cdots w_{n_k})^{-1} q^{n_k} \text{ tends to 0} \\ \iff & \forall q \geq 1, \forall \gamma > 0, (w_1 \cdots w_{n_k})^{-\gamma} q^{n_k} \text{ tends to 0} \\ \iff & \forall \gamma > 0, (w_1 \cdots w_{n_k})^{-\gamma} z^{n_k} \text{ tends to 0 in } H(\mathbb{C}). \end{aligned}$$

REMARK 4.7. Theorem 4.2 points out one difficulty when dealing with hypercyclic algebras: to admit a hypercyclic algebra is not a property preserved by similarity. Indeed, let  $X = \{(x_n) \in \omega : |x_n|2^n \rightarrow 0\}$  endowed with  $\|x\| = \sup_n |x_n|2^n$  and let  $w$  be the weight such that  $w_1 \cdots w_n = n \cdot 2^n$  for all  $n \geq 1$ . Then  $(w_1 \cdots w_n)^{-1}2^n$  goes to zero whereas  $(w_1 \cdots w_n)^{-1/2}2^n$  tends to  $+\infty$ , showing that  $B_w$  is hypercyclic but that no square vector  $x^2$  belongs to  $HC(B_w)$ .

Let now  $(\rho_n)$  be defined by  $\rho_1 = 1$  and  $\rho_n = n/(n-1)$  for  $n \geq 2$ . Then  $B_w$  acting on  $X$  is similar to  $B_\rho$  acting on  $c_0$ , the similarity being given by  $S : X \rightarrow c_0$ ,  $(x_n) \mapsto (2^n x_n)$ . But  $B_\rho$  admits a hypercyclic algebra, which is not the case of  $B_w$ . Of course, the problem is that  $S$  is not a morphism of algebra.

When  $X$  does not admit a continuous norm, one cannot apply Theorem 4.2. The space  $\omega$  is the prototypal example of a Fréchet space without a continuous norm (in fact, by a result of Bessaga and Pelczinski [18], a Fréchet space fails to admit a continuous norm if and only if it has a subspace isomorphic to  $\omega$ ) and we shall now concentrate on this space. On  $\omega$ , for all weight sequences  $w = (w_n)$ , the weighted shift  $B_w$  is bounded, hypercyclic and satisfies (iv). If the sequences  $(w_1 \cdots w_{n+l})$  converge to  $+\infty$  for all  $\ell \geq 0$ , then an easy modification of the proof of the previous theorem shows that  $B_w$  admits a hypercyclic algebra. On the other hand, if the sequences  $(w_1 \cdots w_{n+l})$  converge to 0 for all  $\ell \geq 0$ , then we may modify the previous proof using Corollary 2.3 instead of Corollary 2.4 to prove that  $B_w$  still admits a hypercyclic algebra. A completely different case is that of the unweighted shift  $B$ . It is a hypercyclic multiplicative operator on  $\omega$ . By [15, Theorem 16],  $B$  supports a hypercyclic algebra if and only if for each nonconstant polynomial  $P \in \mathbb{C}[X]$  with  $P(0) = 0$ , the map  $\tilde{P} : \omega \rightarrow \omega$ ,  $x \mapsto P(x)$  has dense range. This is clearly true.

We now show that every weighted shift on  $\omega$  admits a hypercyclic algebra showing that, coordinate by coordinate,  $B_w$  behaves like one of the three previous models.

THEOREM 4.8. *Every weighted shift  $B_w$  on  $\omega$  endowed with the coordinatewise product supports a hypercyclic algebra.*

PROOF. For  $V$  a non-empty open subset of  $\omega$ ,  $I \subset \mathbb{N}$  finite and non-empty and  $s > 0$ , let us define

$$E(I, s) = \left\{ P \in \mathbb{C}[z] : |\hat{P}(\min I)| \geq 1/s, |\hat{P}(\max I)| \geq 1/s, \right. \\ \left. |\hat{P}(n)| \leq s \text{ for all } n \in \mathbb{N}, \right. \\ \left. \hat{P}(n) = 0 \text{ when } n \notin I \right\}$$

$$\mathcal{A}(I, s, V) = \left\{ u \in \omega : \forall P \in E(I, s), \exists N \geq 1, T^N(P(u)) \in V \right\}.$$

As in the proof of Theorem 2.1, it is enough to prove that each set  $\mathcal{A}(I, s, V)$  is dense and open. The last property follows easily from the compactness of  $E(I, s)$ . Thus, let us fix  $I, s$  and  $V$  and let us prove that  $\mathcal{A}(I, s, V)$  is dense. We set  $m_0 = \min(I)$  and  $m_1 = \max(I)$ . Let  $U$  be a non-empty open subset of  $\omega$ . Let  $p \in \mathbb{N}_0$ ,  $u_0, \dots, u_p, v_0, \dots, v_p \in \mathbb{C}$  and  $\varepsilon > 0$  be such that, for all  $x, y \in \omega$ ,

$$|x_l - u_l| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } x \in U, \\ |y_l - v_l| < \varepsilon \text{ for all } l = 0, \dots, p \text{ implies } y \in V.$$

Let us first look at the sequence  $(w_1 \cdots w_n)$ . Three possibilities (which are not mutually exclusive) can occur:

- either  $(w_1 \cdots w_n)$  is bounded and bounded below;
- or it admits a subsequence going to zero;
- or it admits a subsequence going to  $+\infty$ .

Thus, we get the existence of a subsequence  $(w_1 \cdots w_{n_k})$  going to  $a_0 \in [0, +\infty]$ . We then do the same with  $(w_2 \cdots w_{n_{k+1}})$  and so on. By successive extractions, we get the existence of a sequence of integers  $(n_k)$  (we can assume that  $n_{k+1} - n_k > p$  for all  $k$  and that  $n_0 > p$ ) and of  $a_0, \dots, a_p \in [0, +\infty]$  such that, for all  $l = 0, \dots, p$ ,  $(w_{l+1} \cdots w_{n_k+l})$  tends to  $a_l$ . We set  $A_1 = \{l \in \{0, \dots, p\} : a_l = +\infty\}$ ,  $A_2 = \{l \in \{0, \dots, p\} : a_l = 0\}$  and  $A_3 = \{l \in \{0, \dots, p\} : a_l \in (0, +\infty)\}$ .

We fix now  $(\alpha(k))$ ,  $(\beta(k))$  two sequences of non-zero complex numbers and  $(z(k))$  a sequence in  $\mathbb{C}^{p+1}$  such that  $(\alpha(k), \beta(k), z(k))$  is dense in  $\mathbb{C}^{p+3}$ . We set

$$x = u + \sum_{k=0}^{+\infty} y(k)$$

where, for  $l = 0, \dots, p$ ,

$$y_{n_k+l}(k) = \begin{cases} \frac{v_l^{1/m_0}}{\alpha(k)^{1/m_0} (w_{l+1} \cdots w_{n_k+l})^{1/m_0}} & \text{provided } l \in A_1, \\ \frac{v_l^{1/m_1}}{\beta(k)^{1/m_1} (w_{l+1} \cdots w_{n_k+l})^{1/m_1}} & \text{provided } l \in A_2, \\ z_l(k) & \text{provided } l \in A_3 \end{cases}$$

and  $y_i(k) = 0$  if  $i \neq n_k, \dots, n_k + p$ .

We claim that  $x \in U \cap \mathcal{A}(I, s, V)$ . The definition of  $\varepsilon$  and  $p$  ensure that  $x \in U$ . Let  $P \in E(I, s)$ . There exists an increasing function  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  such that  $\alpha(\phi(k)) \rightarrow \hat{P}(m_0)$ ,  $\beta(\phi(k)) \rightarrow \hat{P}(m_1)$  and  $a_l P(z_l(\phi(k))) \rightarrow v_l$  for all  $l \in A_3$ . We claim that  $(B_w^{n_{\phi(k)}}(P(x)))$  belongs to  $V$  provided  $k$  is large enough. It suffices to prove that for  $l = 0, \dots, p$ , the  $l$ -th coordinate of  $B_w^{n_{\phi(k)}}(P(x))$  tends to  $v_l$ . Assume first that  $l \in A_1$ . This  $l$ -th coordinate is equal to

$$w_{l+1} \cdots w_{n_{\phi(k)}+l} P \left( \frac{v_l^{1/m_0}}{\alpha(\phi(k))^{1/m_0} (w_{l+1} \cdots w_{n_{\phi(k)}+l})^{1/m_0}} \right).$$

Now, since  $w_{l+1} \cdots w_{n_{\phi(k)}+l}$  tends to  $+\infty$ , and  $m_0 = \min(I)$ ,

$$w_{l+1} \cdots w_{n_{\phi(k)}+l} P \left( \frac{v_l^{1/m_0}}{\alpha(\phi(k))^{1/m_0} (w_{l+1} \cdots w_{n_{\phi(k)}+l})^{1/m_0}} \right) = \hat{P}(m_0) \frac{v_l}{\alpha(\phi(k))} + o(1)$$

and this tends to  $v_l$ . When  $l \in A_2$ , the proof is similar since now, because  $w_{l+1} \cdots w_{n_{\phi(k)}+l}$  tends to 0, and  $m_1 = \max(I)$ ,

$$w_{l+1} \cdots w_{n_{\phi(k)}+l} P \left( \frac{v_l^{1/m_1}}{\beta(\phi(k))^{1/m_1} (w_{l+1} \cdots w_{n_{\phi(k)}+l})^{1/m_1}} \right) = \hat{P}(m_1) \frac{v_l}{\beta(\phi(k))} + o(1)$$

and this also goes to  $v_l$ . Finally, when  $l \in A_3$ , the  $l$ -th coordinate of  $B_w^{n_{\phi(k)}}(P(x))$  is equal to  $w_{l+1} \cdots w_{n_{\phi(k)}+l} P(z_l(\phi(k)))$  which tends again to  $v_l$ .  $\square$

Theorem 4.8 has an analogue (with a completely different proof!) if we endow  $\omega$  with the Cauchy product: see [22, Corollary 3.9]. We also point out that the existence of a continuous norm is an important assumption in several problems in linear dynamics, for instance for the existence of a closed infinite dimensional subspace of hypercyclic vectors (see [26]).

#### 4.2. Bilateral shifts on Fréchet sequence algebras with the coordinatewise product

In this section, we investigate the case of bilateral shifts on a Fréchet sequence algebra  $X$  on  $\mathbb{Z}$ ; namely,  $X$  is a subset of  $\mathbb{C}^{\mathbb{Z}}$  endowed with the coordinatewise product under which it is an  $F$ -algebra. We intend to give an analogue of Theorem 4.2 for bilateral shifts on  $X$ . The statement and the methods are close to what happens for unilateral shifts. Since we do not want to give an exhaustive list of examples in this work, there is an extra interest for looking at bilateral shifts: a small subtlety appears in this case, since the condition that appears is not symmetric for the positive part of the weight and for the negative one. This will lead us to an interesting example of a hypercyclic operator  $T$  supporting a hypercyclic algebra such that  $T^{-1}$  does not.

**THEOREM 4.9.** *Let  $X$  be a Fréchet sequence algebra on  $\mathbb{Z}$  for the coordinatewise product, with a continuous norm. Assume that  $(e_n)_{n \in \mathbb{Z}}$  is a Schauder basis for  $X$ . Let also  $B_w$  be a bounded bilateral shift on  $X$  such that, for all  $\gamma \in (0, 1)$ ,  $B_w^\gamma$  is bounded. The following assertions are equivalent.*

- (i)  $B_w$  supports a hypercyclic algebra.
- (ii) For all  $m \geq 1$ , for all  $L \in \mathbb{N}$ , there exists a sequence of integers  $(n_k)$  such that, for all  $l = -L, \dots, L$ ,  $((w_{l+1} \cdots w_{n_k+l})^{-1/m} e_{n_k+l})$  and  $(w_l \cdots w_{-n_k+l+1} e_{-n_k+l})$  tend to zero.

**PROOF.** (ii)  $\implies$  (i). We intend to apply Corollary 2.4. Let  $1 \leq m_0 < m_1$ , let  $U, V$  be nonempty open subsets of  $X$  and let  $W$  be a neighbourhood of zero. Let  $x, y$  belonging to  $U$  and  $V$  respectively, with finite support contained in  $[-p, p]$ . Write  $y = \sum_{l=-p}^p y_l e_l$  and let  $(n_k)$  be the sequence given in (ii) for  $m = m_0$  and  $L = p$ . Define

$$u := u(n_k) = x + \sum_{l=-p}^p \frac{y_l^{1/m_0}}{(w_{l+1} \cdots w_{n_k+l})^{1/m_0}} e_{n_k+l}.$$

Provided  $k$  is large enough,  $u$  belongs to  $U$ . Moreover, for  $m \in \{m_0, \dots, m_1\}$ ,

$$B_w^{n_k}(u^m) = \sum_{l=-p}^p w_l \cdots w_{-n_k+l+1} x_l^m e_{-n_k+l} + \sum_{l=-p}^p \frac{y_l^{m/m_0}}{(w_{l+1} \cdots w_{n_k+l})^{\frac{m}{m_0}-1}} e_l.$$

For all values of  $m$ , it is clear that

$$\sum_{l=-p}^p w_l \cdots w_{-n_k+l+1} x_l^m e_{-n_k+l} \xrightarrow{k \rightarrow +\infty} 0.$$

Hence, for  $m = m_0$  and provided  $k$  is large enough,  $B_w^{n_k}(u^{m_0})$  belongs to  $V$ . Furthermore, if  $m > m_0$ , since each sequence  $(w_{l+1} \cdots w_{n_k+l})^{-1}$  goes to zero (recall that  $(e_n)$  is bounded below), then  $B_w^{n_k}(u^m)$  belongs to  $W$  for large values of  $k$ , showing that  $B_w$  admits a hypercyclic algebra.

(i)  $\implies$  (ii). The proof is slightly more difficult than for unilateral shifts. Fix  $m$  and  $L$  and let  $x \in X$  be such that  $x^m \in HC(B_w)$ . Let  $(s_k)$  be an increasing sequence of integers such that  $B_w^{s_k}(x^m)$  tends to  $e_{-L} + \dots + e_L$ . We fix some  $s \in \mathbb{N}$  (which can be taken equal to some  $s_{k_0}$ ) such that, for all  $l = -L, \dots, L$ , the  $l$ -th coordinate of  $B_w^s(x)$  is not equal to zero. We then consider  $y \in X$  defined by  $y_l = (w_{l+1} \dots w_{l+s})^{1/m} x_{l+s}$  (namely,  $y = B_w^{s/m}(x)$ ) and we set  $n_k = s_k - s$ . It is easy to check that  $B_w^{n_k}(y^m) = B_w^{s_k}(x^m)$ . Hence, it goes to  $e_{-L} + \dots + e_L$ . This implies that

- for all  $l = -L, \dots, L$ ,

$$w_l \dots w_{-n_k+l+1} y_l^m e_{-n_k+l} \text{ tends to } 0.$$

- for all  $l = -L, \dots, L$ ,

$$w_{l+1} \dots w_{n_k+l} y_{n_k+l}^m \text{ tends to } 1.$$

We conclude as in the unilateral case, using that  $y_l$  is never equal to zero for  $l = -L, \dots, L$ .  $\square$

We can then state corollaries similar to what happens in the unilateral case.

**COROLLARY 4.10.** *Let  $X$  be a Fréchet sequence algebra on  $\mathbb{Z}$  for the coordinatewise product and with a continuous norm. Assume that  $(e_n)_{n \in \mathbb{Z}}$  is a Schauder basis for  $X$ . Assume also that, for all admissible weights  $w$ , for all  $\gamma \in (0, 1)$ ,  $w^\gamma$  is admissible. Let  $B_w$  be a bounded bilateral weighted shift on  $X$ . The following assumptions are equivalent.*

- (i)  $B_w$  supports a hypercyclic algebra.
- (ii) For all  $\gamma > 0$ , there exists a sequence  $(n_k)$  such that  $((w_1 \dots w_{n_k})^{-\gamma} e_{n_k})$  tends to zero and  $(w_{-1} \dots w_{-n_k} e_{-n_k})$  tends to 0.

**COROLLARY 4.11.** *Let  $X$  be a Fréchet sequence algebra on  $\mathbb{Z}$  for the coordinatewise product and with a continuous norm. Assume that  $(e_n)$  is a Schauder basis for  $X$  and that  $(e_n)$  is bounded. Assume also that, for all admissible weights  $w$ , for all  $\gamma \in (0, 1)$ ,  $w^\gamma$  is admissible. Let  $B_w$  be a bounded bilateral weighted shift on  $X$ . The following assumptions are equivalent.*

- (i)  $B_w$  supports a hypercyclic algebra.
- (ii)  $B_w$  is hypercyclic.
- (iii) There exists a sequence  $(n_k)$  such that  $(w_1 \dots w_{n_k})$  and  $(w_{-1} \dots w_{-n_k})$  tend to  $+\infty$ .

On the contrary, the nonsymmetry of the conditions in (ii) of Theorem 4.9 proves to be useful to get the following example.

**EXAMPLE 4.12.** There exists an invertible operator  $T$  on a Banach algebra such that  $T$  supports a hypercyclic algebra and  $T^{-1}$  does not.

**PROOF.** Let

$$X = \left\{ x \in \mathbb{C}^{\mathbb{Z}} : x_n (|n| + 1) \xrightarrow{n \rightarrow \pm\infty} 0 \right\},$$

endowed with

$$\|x\| = \sup_n |x_n| (|n| + 1).$$

Equipped with the coordinatewise product,  $X$  is a Fréchet sequence algebra. Let  $w$  be the weight defined by  $w_0 = 1$ ,  $w_n = 2$  and  $w_{-n} = n^2/(n+1)^2$  for  $n > 0$ . For all  $\gamma > 0$ , the weighted shift  $B_{w^\gamma}$  is bounded on  $X$ . Moreover, it satisfies the

assumptions of Theorem 4.9 with  $(n_k)$  equal to the whole sequence of integers. In particular,  $w_{-1} \cdots w_{-n} \|e_{-n}\| = (n+1)^{-1}$  tends to zero.

It is plain that  $B_w$  is invertible and that its inverse is the forward shift  $F_\rho$ , defined by  $F_\rho(e_n) = \rho_{n+1}e_{n+1}$  with  $\rho_n = 1/w_n$ . Assume that  $F_\rho$  supports a hypercyclic algebra. Then we apply the symmetrized version of Theorem 4.9 adapted to forward shifts with  $m = 2$  to get the existence of a sequence  $(n_k)$  such that  $(\rho_{-1} \cdots \rho_{-n_k})^{-1/2} e_{-n_k}$  tends to zero. This is impossible since

$$\left\| (\rho_{-1} \cdots \rho_{-n_k})^{-1/2} e_{-n_k} \right\| \sim_{k \rightarrow +\infty} \frac{n_k}{n_k^{2 \times 1/2}} = 1.$$

□

### 4.3. Fréchet sequence algebras for the convolution product

This subsection is devoted to the proof of Theorem 1.3. We first have to give the meaning of a regular Fréchet sequence algebra. Let  $(X, (\|\cdot\|_q))$  be such a Fréchet sequence algebra for the Cauchy product. We will say that  $X$  is *regular* provided that it satisfies the following three properties:

- (a)  $X$  admits a continuous norm;
- (b)  $(e_n)$  is a Schauder basis of  $X$ ;
- (c) for any  $r \geq 1$ , there exists  $q \geq 1$  and  $C > 0$  such that, for all  $n, k \geq 0$ ,

$$\|e_n\|_r \cdot \|e_k\|_r \leq C \|e_{n+k}\|_q.$$

Let us make some comments on these assumptions. Conditions (a) and (b) are standard in this work. We shall use (a) by assuming that  $\|e_n\|_q > 0$  for all  $n \in \mathbb{N}_0$  and all  $q > 0$ . Regarding (c), it should be thought as a reverse inequality for the continuity of the product in  $X$ . Observe also that  $H(\mathbb{C})$  and  $\ell_1$  are clearly regular. However, this is not the case of all Fréchet sequence algebras for the Cauchy product. Pick for instance any sequence  $(a_n)$  of positive real numbers such that, for all  $n, p, q \in \mathbb{N}_0$ , with  $n = p+q$ ,  $a_n \leq a_p \cdot a_q$  and  $a_n^2/a_{2n} \rightarrow +\infty$  (the sequence  $a_n = 1/n!$  does the job). Then the Banach space  $X = \{x \in \omega : \|x\|_X = \sum_{n \geq 0} a_n |x_n| < +\infty\}$  is a Fréchet sequence algebra for the Cauchy product which does satisfy (c).

A consequence of (c) is the following technical lemma which will be crucial later.

**LEMMA 4.13.** *Let  $X$  be a regular Fréchet sequence algebra for the Cauchy product and let  $(w_n)$  be an admissible weight sequence on  $X$ . Then, for all  $M \geq 1$ , for all  $r \geq 1$ , and for all  $\rho \geq 0$ , there exist  $C > 0$  and  $q \geq r$  such that, for all  $n \geq M$ , for all  $u < v$  in  $\{n - M, \dots, n\}$ , for all  $k \in \{n - M + \rho, \dots, n + \rho\}^{v-u}$ ,*

$$(4.1) \quad \prod_{j=1}^{v-u} w_{k_j} \|e_u\|_r \leq C \|e_v\|_q.$$

Before to proceed with the proof, let us comment the statement of Lemma 4.13. The inequality (4.1) is nothing else than the continuity of  $B_w$  if we assume that  $k_j = u + j$  for  $j = 1, \dots, v - u$ . The regularity of  $X$  (and more precisely the third condition) will imply that we may slightly move the indices  $k_j$ .

**PROOF.** Let us fix  $M \geq 1$  and observe that  $v - u$  may only take the values  $1, \dots, M$ . Then, upon doing a finite induction and taking suprema, we need only

to prove that, for all  $r \geq 1$ , and for all  $\rho \geq 0$ , there exist  $C > 0$  and  $q \geq r$  such that, for all  $n \geq M$ , for all  $u \in \{n - M, \dots, n - 1\}$ , for all  $k \in \{n - M + \rho, \dots, n + \rho\}$ ,

$$(4.2) \quad w_k \|e_u\|_r \leq C \|e_{u+1}\|_q,$$

a property which should be thought as a strong version of the continuity of  $B_w$ . Assume first that  $u \geq k - 1$ . Then, writing  $e_u = e_{k-1} \cdot e_{u-(k-1)}$  and using the continuity of the product and of  $B_w$ , we get the existence of  $C > 0$  and  $q_1 \geq r$  such that

$$\begin{aligned} w_k \|e_u\|_r &\leq w_k \|e_{k-1}\|_r \cdot \|e_{u-(k-1)}\|_r \\ &\leq C_1 \|e_k\|_{q_1} \cdot \|e_{u-(k-1)}\|_{q_1}. \end{aligned}$$

We now use property (c) for  $r = q_1$  to deduce the existence of  $C_2 > 0$  and  $q_2 \geq q_1$  such that

$$w_k \|e_u\|_r \leq C_1 C_2 \|e_{u+1}\|_{q_2}.$$

Hence, (4.2) is proved for  $q = q_2$  and  $C = C_1 C_2$ . If we assume that  $u < k - 1$ , then the argument is completely similar by exchanging the place where we use the continuity of the product and property (c). Precisely,

$$\begin{aligned} w_k \|e_u\|_r &= w_k \frac{\|e_u\|_r \cdot \|e_{(k-1)-u}\|_r}{\|e_{(k-1)-u}\|_r} \\ &\leq C_1 w_k \frac{\|e_{k-1}\|_{q_1}}{\|e_{(k-1)-u}\|_r} \\ &\leq C_1 C_2 \frac{\|e_k\|_{q_2}}{\|e_{k-(u+1)}\|_r} \\ &\leq C_1 C_2 \frac{\|e_{k-(u+1)}\|_{q_2}}{\|e_{k-(u+1)}\|_r} \|e_{u+1}\|_{q_2}. \end{aligned}$$

Hence, (4.2) is proved for  $q = q_2$  and

$$C = \max \left\{ C_1 C_2 \frac{\|e_l\|_{q_2}}{\|e_l\|_r} : 1 \leq l \leq M + \rho - 1 \right\}.$$

□

Lemma 4.13 will be used through the following more particular form.

**COROLLARY 4.14.** *Let  $X$  be a regular Fréchet sequence algebra for the Cauchy product and let  $(w_n)$  be an admissible weight sequence on  $X$ . Then, for all  $m \geq 1$ , for all  $N \geq 1$ , for all  $r \geq 1$ , and for all  $\rho \geq 0$ , there exist  $C > 0$  and  $q \geq 1$  such that, for all  $n \geq mN$ , for all  $s \in \{1, \dots, N\}$ ,*

$$(w_{n-s+1+\rho})^{m-1} \cdots (w_{n-1+\rho})^{m-1} (w_{n+\rho})^{m-1} \|e_{n-ms+m\rho}\|_r \leq C \|e_{n-s+\rho}\|_q.$$

**PROOF.** We apply the previous lemma with  $M = mN$  to get  $q' \geq r$  and  $C' > 0$  such that

$$(w_{n-s+1+\rho})^{m-1} \cdots (w_{n-1+\rho})^{m-1} (w_{n+\rho})^{m-1} \|e_{n-ms}\|_r \leq C' \|e_{n-s}\|_{q'}.$$

Now, using property (c) and the continuity of the product on a Fréchet algebra, we get  $q \geq q' \geq r$  and  $C'' > 0$  with

$$\begin{aligned}
& (w_{n-s+1+\rho})^{m-1} \cdots (w_{n-1+\rho})^{m-1} (w_{n+\rho})^{m-1} \|e_{n-ms+m\rho}\|_r \\
& \leq (w_{n-s+1+\rho})^{m-1} \cdots (w_{n-1+\rho})^{m-1} (w_{n+\rho})^{m-1} \|e_{n-ms}\|_r \|e_\rho\|_r \|e_{(m-1)\rho}\|_r \\
& \leq C' \|e_{n-s}\|_{q'} \|e_\rho\|_{q'} \|e_{(m-1)\rho}\|_r \\
& \leq C' C'' \|e_{n-s+\rho}\|_q \|e_{(m-1)\rho}\|_r \\
& = C \|e_{n-s+\rho}\|_q,
\end{aligned}$$

where  $C = C' C'' \|e_{(m-1)\rho}\|_r$ .  $\square$

Before proceeding with the proof of Theorem 1.3, let us make some comments on the difference between the coordinatewise product and the convolution product. Let  $P(z) = \sum_{m \in I} \hat{P}(m) z^m$  be a nonzero polynomial,  $x, y$  with finite support. The work done in Section 2.1 shows that it is important for us to find  $u$  close to  $x$  and  $N$  such that  $B_w^N(P(u))$  is close to  $y$ . In both cases,  $u$  will be of the form  $u = x + z$ , where  $z$  has finite support and  $\min(\text{supp}(z)) \gg \max(\text{supp}(x))$ . For the coordinatewise product, each  $u^m$  has the same support as  $u$ . Moreover, since  $z$  has to be small,  $z^m$  becomes smaller as  $m$  increases. Hence, in  $P(u)$ , the most important term was  $u^{m_0}$ , where  $m_0 = \min(I)$  (we assume  $\hat{P}(m) \neq 0 \iff m \in I$ ) and it was natural to apply Corollary 2.4.

Regarding the convolution product, the support of  $u^m$  is moving to the right:  $\max(\text{supp}(u^{m+1})) \geq \max(\text{supp}(u^m))$ . By using a specific translation term (an idea coming from [28] and [23]), we will arrange the choice of  $N$  such that  $B_w^N(u^m) = 0$  when  $m < m_1 := \max(I)$  and  $B_w^N(u^{m_1})$  is close to  $y$ . This explains why we will rather use Corollary 2.3.

**PROOF OF THEOREM 1.3.** We start with a hypercyclic weighted shift  $B_w$  and prove that  $B_w$  supports a dense hypercyclic algebra which is not contained in a finitely generated algebra. Let  $d \geq 1$ ,  $A \in \mathcal{P}_f(\mathbb{N}_0^d) \setminus \{\emptyset\}$  and  $U_1, \dots, U_d, V, W \subset X$  be open and non-empty, with  $0 \in W$ . We choose  $\beta = \max A$  under the lexicographical order, say  $\beta = (m, \beta_2, \dots, \beta_d)$ . Upon interchanging the coordinates in  $\mathbb{N}^d$ , we may and will assume that  $m > 0$ . Let  $x_1, \dots, x_d$  belonging respectively to  $U_1, \dots, U_d$  and let  $y = \sum_{l=0}^p y_l e_l$  belonging to  $V$ . We can find  $r \geq 1$ ,  $\delta > 0$  and a ball  $B \subset W$  for the seminorm  $\|\cdot\|_r$  and with radius  $\delta$  such that  $y + B \subset V$  and  $x_i + B \subset U_i$ , for all  $i = 1, \dots, d$ .

Since  $\beta > \alpha$  for all  $\alpha \in A \setminus \{\beta\}$  under the lexicographical order, we may find integers  $s_1, \dots, s_d$  with  $s_i > 4p$  such that

$$(4.3) \quad (m - \alpha_1)s_1 + (\beta_2 - \alpha_2)s_2 + \cdots + (\beta_d - \alpha_d)s_d > 3p, \text{ for all } \alpha \in A \setminus \{\beta\}.$$

The procedure to do this is the following. First find  $s_d$  such that  $(\beta_d - \alpha_d)s_d > 3p$ , for all  $\alpha_d < \beta_d$ ,  $\alpha_d \in \pi_d(A)$ . Next, find  $s_{d-1}$  such that  $(\beta_{d-1} - \alpha_{d-1})s_{d-1} + (\beta_d - \alpha_d)s_d > 3p$ , for all  $\alpha_{d-1} < \beta_{d-1} \in \pi_{d-1}(A)$  and  $\alpha_d \in \pi_d(A)$ . Continuing inductively, after finitely many steps we define  $s_2$  such that  $(\beta_2 - \alpha_2)s_2 + \cdots + (\beta_d - \alpha_d)s_d > 3p$ , for all  $\alpha_2 < \beta_2$ ,  $\alpha_2 \in \pi_2(A)$ , and  $\alpha_i \in \pi_i(A)$ , for  $i = 3, \dots, d$ . Finally we chose  $s_1$  large enough so that  $(m - \alpha_1)s_1 + (\beta_2 - \alpha_2)s_2 + \cdots + (\beta_d - \alpha_d)s_d > 3p$ , for all  $\alpha_1 < m$ ,  $\alpha_1 \in \pi_1(A)$ ,  $\alpha_i \in \pi_i(A)$ , for  $i = 2, \dots, p$ . This way we get (4.3).

These  $s_i$  being fixed, we now choose positive real numbers  $\eta_2, \dots, \eta_d$  such that

$$(4.4) \quad \|\eta_i e_{s_i}\|_r < \delta, \text{ for all } i = 2, \dots, d.$$

We will distinguish two cases in order to apply Corollary 2.5. The most difficult one is  $m > 1$ , an assumption that we now make. We set  $\rho = \beta_2 s_2 + \dots + \beta_d s_d$  and we consider a sequence  $(J_k)$  going to  $+\infty$  such that, for all  $l = 0, \dots, p$ ,

$$(w_1 \cdots w_{mJ_k - 3p + l + \rho})^{-1} e_{mJ_k - 3p + l + \rho} \xrightarrow{k \rightarrow +\infty} 0.$$

Indeed, let  $(m_k)$  be a sequence of integers such that  $(w_1 \cdots w_{m_k})^{-1} e_{m_k}$  goes to zero and  $m_k \geq m + \rho$  for all  $k$ . Define  $J_k$  as the single integer such that  $m_k - m < mJ_k - 2p + \rho \leq m_k$ . Then, for all  $l = 0, \dots, p$ ,

$$(w_1 \cdots w_{mJ_k - 3p + \rho + l})^{-1} e_{mJ_k - 3p + \rho + l} = B_w^{m_k - mJ_k + 3p - \rho - l} ((w_1 \cdots w_{m_k})^{-1} e_{m_k})$$

which tends to zero by the continuity of  $B_w$  and because

$$0 \leq m_k - mJ_k + 3p - \rho - l \leq m + p.$$

We now proceed with the construction of the vectors  $u_1, \dots, u_d$  required to apply Corollary 2.5. We set, for  $k$  large enough,  $N = mJ_k - 3p + \rho$  and

$$\varepsilon = \max_{0 \leq l \leq p} \left( \frac{\|e_{J_k - 3p + l}\|_r}{w_1 \cdots w_{mJ_k - 3p + l + \rho}} \right)^{\frac{1}{2(m-1)}} \times \min \left( \frac{1}{\|e_{J_k}\|_r}, \frac{1}{(w_1 \cdots w_{mJ_k + \rho})^{1/m}} \right)^{\frac{1}{2}},$$

$$d_j = \frac{w_1 \cdots w_j y_j}{\eta_2^{\beta_2} \cdots \eta_d^{\beta_d} m \varepsilon^{m-1} w_1 \cdots w_{mJ_k - 3p + j + \rho}}.$$

We also define

$$u_1 = x_1 + \sum_{j=0}^p d_j e_{J_k - 3p + j} + \varepsilon e_{J_k},$$

$$u_i = x_i + \eta_i e_{s_i}, \text{ for } i = 2, \dots, d.$$

Let us postpone the proof of the following facts.

$$(4.5) \quad \varepsilon \|e_{J_k}\|_r \rightarrow 0 \text{ as } k \rightarrow +\infty,$$

$$(4.6) \quad |d_j| \cdot \|e_{J_k - 3p + j + \rho}\|_r \rightarrow 0 \text{ as } k \rightarrow +\infty, \text{ for all } j = 0, \dots, p,$$

$$(4.7) \quad \varepsilon^m w_1 \cdots w_{mJ_k + \rho} \rightarrow 0 \text{ as } k \rightarrow +\infty.$$

From (4.5) and (4.6) we get  $u_1 \in U_1$  if  $k$  is large enough and from (4.4) we get  $u_i \in U$  for  $i = 2, \dots, d$ . We claim that  $u^\alpha \in \ker B_w^N$  for all  $\alpha \in A$  with  $\alpha \neq \beta$ . In fact, for a given  $\alpha \in A \setminus \{\beta\}$ , say  $\alpha = (\alpha_1, \dots, \alpha_d)$ , we have

$$\max(\text{supp}(u^\alpha)) \leq \alpha_1 J_k + \alpha_2 s_2 + \dots + \alpha_d s_d,$$

so the claim follows by (4.3) since  $\alpha_1 \leq m$  and for  $k$  large enough,  $J_k \geq s_1$ . Finally, for the main power  $\beta$  we write

$$u^\beta = z + \sum_{j=0}^p \eta_2^{\beta_2} \cdots \eta_d^{\beta_d} m \varepsilon^{m-1} d_j e_{mJ_k - 3p + j + \rho} + \varepsilon^m \eta_2^{\beta_2} \cdots \eta_d^{\beta_d} e_{mJ_k + \rho},$$

where the maximum of the support of  $z$  is less than  $N = mJ_k - 3p + \rho$ . Indeed, a term in  $z$  can come

- either from a term in  $u_1^{\beta_1}$  with support in  $[0, mJ_k - 4p]$  so that the maximum of the support of this term is at most  $mJ_k - 4p + \rho < N$ ;
- or from a term in some  $u_i^{\beta_i}$ ,  $i = 2, \dots, d$ , with support in  $[0, (\beta_i - 1)s_i + p]$ . The maximum of the support of such a term is then at most  $mJ_k + \rho + p - s_i < N$  since  $s_i > 4p$ .

Hence, we get

$$B_w^N u^\beta = y + \frac{\varepsilon^m \eta_2^{\beta_2} \cdots \eta_d^{\beta_d} w_1 \cdots w_{mJ_k + \rho}}{w_1 \cdots w_{3p}} e_{3p},$$

which belongs to  $V$  by (4.7) if  $k$  is big enough. It remains now to show that properties (4.5), (4.6) and (4.7) hold true.

Let us first prove (4.5). By property (c) and an easy induction, there exist  $q \geq 1$  and  $C > 0$  (depending on  $r$  and  $m$ ) such that, for all  $k \geq 1$  and all  $l \in \{0, \dots, p\}$ ,

$$\begin{aligned} \frac{\|e_{J_k - 3p + l}\|_r^{\frac{1}{2(m-1)}} \cdot \|e_{J_k}\|_r}{\|e_{J_k}\|_r^{\frac{1}{2}}} &= \frac{(\|e_{J_k - 3p + l}\|_r \cdot \|e_{J_k}\|_r^{m-1} \cdot \|e_\rho\|_r)^{\frac{1}{2(m-1)}}}{\|e_\rho\|_r^{\frac{1}{2(m-1)}}} \\ &\leq C \|e_{mJ_k - 3p + l + \rho}\|_q^{\frac{1}{2(m-1)}}. \end{aligned}$$

Hence,

$$\varepsilon \|e_{J_k}\|_r \leq C \max_{0 \leq l \leq p} \left( \frac{\|e_{mJ_k - 3p + l + \rho}\|_q}{w_1 \cdots w_{mJ_k - 3p + l + \rho}} \right)^{\frac{1}{2(m-1)}}$$

and this goes to zero as  $k$  tends to  $+\infty$ .

Regarding (4.6), we first write

$$\begin{aligned} |d_j| \cdot \|e_{J_k - 3p + j + \rho}\|_r &\leq C \frac{\|e_{J_k - 3p + j + \rho}\|_r}{w_1 \cdots w_{mJ_k - 3p + j + \rho}} \times \min_{0 \leq l \leq p} \left( \frac{w_1 \cdots w_{mJ_k - 3p + l + \rho}}{\|e_{J_k - 3p + l}\|_r} \right)^{\frac{1}{2}} \\ &\quad \times \max \left( \|e_{J_k}\|_r, (w_1 \cdots w_{mJ_k + \rho})^{\frac{1}{m}} \right)^{\frac{m-1}{2}} \\ &\leq C \frac{\|e_\rho\|_r^{\frac{1}{2}} \cdot \|e_{J_k - 3p + j + \rho}\|_r^{\frac{1}{2}}}{(w_1 \cdots w_{mJ_k - 3p + j + \rho})^{\frac{1}{2}}} \\ &\quad \times \max \left( \|e_{J_k}\|_r, (w_1 \cdots w_{mJ_k + \rho})^{\frac{1}{m}} \right)^{\frac{m-1}{2}}, \end{aligned}$$

where the last line comes from the continuity of the product, more precisely from

$$\|e_{J_k - 3p + l + \rho}\|_r \leq \|e_{J_k - 3p + l}\|_r \cdot \|e_\rho\|_r.$$

Assume first that the maximum is attained for  $\|e_{J_k}\|_r$ . In that case, using (c) in a similar way we write

$$\frac{\|e_{J_k - 3p + j + \rho}\|_r}{w_1 \cdots w_{mJ_k - 3p + j + \rho}} \|e_{J_k}\|_r^{m-1} \leq C \frac{\|e_{mJ_k - 3p + j + \rho}\|_q}{w_1 \cdots w_{mJ_k - 3p + j + \rho}}$$

and the last parcel goes to zero. If the maximum is attained for  $(w_1 \cdots w_{mJ_k + \rho})^{\frac{1}{m}}$ , we now write

$$\begin{aligned} \frac{\|e_{J_k - 3p + j + \rho}\|_r (w_1 \cdots w_{mJ_k + \rho})^{\frac{m-1}{m}}}{w_1 \cdots w_{mJ_k - 3p + j + \rho}} &= \frac{(\|e_{J_k - 3p + j + \rho}\|_r^m (w_1 \cdots w_{mJ_k + \rho})^{m-1})^{\frac{1}{m}}}{w_1 \cdots w_{mJ_k - 3p + j + \rho}} \\ &\leq C_1 \frac{((w_1 \cdots w_{mJ_k + \rho})^{m-1} \|e_{mJ_k - m(3p-j) + m\rho}\|_q)^{\frac{1}{m}}}{w_1 \cdots w_{mJ_k - 3p + j + \rho}}. \end{aligned}$$

Now, using Corollary 4.14 for  $n = mJ_k$ ,  $N = 3p$  and  $s = 3p - j$ , we get the existence of  $C_2 > 0$  and  $q' \geq q$  (which does not depend on  $k$ ) such that

$$(w_1 \cdots w_{mJ_k + \rho})^{m-1} \|e_{mJ_k - m(3p-j) + m\rho}\|_q \leq C_2 (w_1 \cdots w_{mJ_k - 3p + j + \rho})^{m-1} \times \|e_{mJ_k - 3p + j + \rho}\|_{q'}$$

so that

$$\frac{\|e_{J_k - 3p + j + \rho}\|_r (w_1 \cdots w_{mJ_k + \rho})^{\frac{m-1}{m}}}{w_1 \cdots w_{mJ_k - 3p + j\rho}} \leq C_1 C_2^{\frac{1}{m}} \left( \frac{\|e_{mJ_k - 3p + j + \rho}\|_{q'}}{w_1 \cdots w_{mJ_k - 3p + j + \rho}} \right)^{\frac{1}{m}}$$

and this goes to zero.

Finally, let us prove (4.7). The proof is very similar. Indeed, for all  $l = 0, \dots, p$ ,

$$\begin{aligned} & \left( \frac{\|e_{J_k - 3p + l + \rho}\|_r}{w_1 \cdots w_{mJ_k - 3p + l + \rho}} \right)^{\frac{m}{2(m-1)}} (w_1 \cdots w_{mJ_k + \rho})^{\frac{1}{2}} \\ & \leq \frac{(\|e_{J_k - 3p + l + \rho}\|_r^m (w_1 \cdots w_{mJ_k + \rho})^{m-1})^{\frac{1}{2(m-1)}}}{(w_1 \cdots w_{mJ_k - 3p + l + \rho})^{\frac{m}{2(m-1)}}} \\ & \leq C_1 \frac{(\|e_{mJ_k - m(3p-l) + m\rho}\|_q (w_1 \cdots w_{mJ_k + \rho})^{m-1})^{\frac{1}{2(m-1)}}}{(w_1 \cdots w_{mJ_k - 3p + l + \rho})^{\frac{m}{2(m-1)}}} \\ & \leq C_1 C_2^{\frac{1}{2(m-1)}} \frac{(\|e_{mJ_k - 3p + l + \rho}\|_{q'} (w_1 \cdots w_{mJ_k - 3p + l + \rho})^{m-1})^{\frac{1}{2(m-1)}}}{(w_1 \cdots w_{mJ_k - 3p + l + \rho})^{\frac{m}{2(m-1)}}} \\ & \leq C_1 C_2^{\frac{1}{2(m-1)}} \left( \frac{\|e_{mJ_k - 3p + l + \rho}\|_{q'}}{w_1 \cdots w_{mJ_k - 3p + l + \rho}} \right)^{\frac{1}{2(m-1)}}, \end{aligned}$$

which achieves the proof of (4.7).

We now sketch briefly the proof when  $m = 1$ . We still set  $\rho = \beta_2 s_2 + \cdots + \beta_d s_d$  and we now consider a sequence  $(J_k)$  satisfying

$$(w_1 \cdots w_{J_k + l + \rho})^{-1} e_{J_k + l + \rho} \xrightarrow{k \rightarrow +\infty} 0$$

for all  $l = 0, \dots, p$ . We define

$$\begin{aligned} u_1 &= x_1 + \sum_{j=0}^p d_j e_{J_k + j + \rho}, \\ u_i &= x_i + \eta_i e_{s_i}, \text{ for } i = 2, \dots, d, \end{aligned}$$

where

$$d_j = \frac{y_j w_1 \cdots w_j}{\eta_2^{s_2} \cdots \eta_d^{s_d} w_1 \cdots w_{J_k + j + \rho}}.$$

Setting  $N = J_k + \rho$  for  $k$  sufficiently large, it is easy to check that  $u_i \in U_i$ ,  $i = 1, \dots, d$ ,  $B_w^N(u^\alpha) = 0$  if  $\alpha \in A \setminus \{\beta\}$ , and  $B_w^N(u^\beta) \in V$ , which concludes the proof.  $\square$



## Common hypercyclic algebras

### 5.1. How to get a common hypercyclic algebra

We begin this section with a result which mimics Theorem 2.1, now for the existence of a common hypercyclic algebra. We fix a topological space  $\Lambda$  which is a countable union of compact sets.

**PROPOSITION 5.1.** *Let  $\Lambda$  be a countable union of compact sets and let  $(T_\lambda)_{\lambda \in \Lambda}$  be a family of operators in  $\mathcal{L}(X)$  such that the map  $(\lambda, x) \mapsto T_\lambda(x)$  is continuous from  $\Lambda \times X$  into  $X$ . Assume that, for all compact sets  $K \subset \Lambda$ , for all  $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ , there exists  $m_0 \in I$  such that, for all  $U, V$  non-empty open subsets of  $X$ , for all neighborhood  $W$  of 0, one can find  $u \in U$  such that, for all  $\lambda \in K$ , there exists  $N \in \mathbb{N}$  satisfying*

$$\begin{cases} T_\lambda^N(u^m) \in W \text{ for } m \in I \setminus \{m_0\} \\ T_\lambda^N(u^{m_0}) \in V. \end{cases}$$

*Then the set of points generating a common hypercyclic algebra for  $(T_\lambda)_{\lambda \in \Lambda}$  is residual in  $X$ .*

**PROOF.** By the Baire Category theorem, it is enough to show the result assuming that  $\Lambda = K$  is a compact set. For  $V \subset X$  open and non-empty,  $W$  a neighborhood of 0,  $I \subset \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ , define

$$\mathcal{A}(I, V, W) = \left\{ u \in X : \forall \lambda \in \Lambda, \exists N \in \mathbb{N}, T_\lambda^N(u^m) \in W \text{ for } m \in I \setminus \{m_0(I)\} \text{ and } T_\lambda^N(u^{m_0(I)}) \in V \right\}$$

(here,  $m_0(I)$  is uniquely defined by  $I$  using the assumptions of the lemma). The assumption tells us that each set  $\mathcal{A}(I, V, W)$  is dense. The compactness of  $\Lambda$  together with the continuity of  $(\lambda, x) \mapsto T_\lambda(x)$  ensure that these sets are also open. Fix now  $(V_k)$  a basis of open sets of  $X$  and  $(W_l)$  a basis of open neighborhoods of 0 and let us consider  $u \in \bigcap_{I, k, l} \mathcal{A}(I, V_k, W_l)$ . Let  $P \in \mathbb{C}[z]$  be non-constant,  $P(z) = \sum_{m \in I} \hat{P}(m)z^m$  with  $\hat{P}(m) \neq 0$  for  $m \in I$  and  $P(0) = 0$ . As in the proof of Theorem 2.1, we may assume that  $\hat{P}(m_0(I)) = 1$ . Let  $V$  be any non-empty open subset of  $X$  and let  $k, l$  be such that  $V_k + \sum_{m \neq m_0(I)} |\hat{P}(m)|W_l \subset V$ . Since  $u \in \mathcal{A}(I, V_k, W_l)$ , for all  $\lambda \in \Lambda$ , there exists  $N \in \mathbb{N}$  such that

$$T_\lambda^N(P(u)) = T_\lambda^N(u^{m_0(I)}) + \sum_{m \neq m_0(I)} \hat{P}(m)T_\lambda^N(u^m) \in V.$$

Hence,  $P(u)$  belongs to  $\bigcap_{\lambda \in \Lambda} HC(T_\lambda)$  and the proof is complete.  $\square$

## 5.2. Common hypercyclic algebras for a family of backward shifts - coordinatewise product

We now specialize our study to backward shifts. Thus we fix  $X$  a Fréchet sequence algebra under the coordinatewise product with a continuous norm in which  $\text{span}(e_i)$  is dense. Let  $(w(\lambda))_{\lambda \in \Lambda}$  be a family of admissible weights. For simplicity, we will denote by  $T_\lambda$  the operator  $B_{w(\lambda)}$  and by  $S_{\lambda^r}$  the forward shift defined by  $S_{\lambda^r} e_i = \frac{1}{w_{i+1}^r(\lambda)} e_{i+1}$ .

Proposition 5.1 will be interpreted under the following form.

LEMMA 5.2. *Assume that the map  $(\lambda, x) \mapsto T_\lambda(x)$  is continuous. Assume that, for all  $m_0 \leq m_1 \in \mathbb{N}$ , for all  $v \in X$  with finite support, for all  $M > 0$ , for all  $\mathcal{O}$  neighborhood of 0, one can find*

- parameters  $\lambda_1, \dots, \lambda_q$ ,
- sets of parameters  $\Lambda_1, \dots, \Lambda_q \subset \Lambda$  with  $\Lambda \subset \bigcup_{i=1}^q \Lambda_i$  and  $\lambda_i \in \Lambda_i$  for all  $i$ ,
- integers  $N_1, \dots, N_q \in \mathbb{N}$  with  $N_1 > M$  and  $N_{i+1} - N_i > M$  for all  $i$ ,

such that

- (i)  $\sum_{i=1}^q S_{\lambda_i^{1/m_0}}^{N_i}(v) \in \mathcal{O}$ ;
- (ii)  $\sum_{j=i+1}^q T_\lambda^{N_i} \left( \left( S_{\lambda_j^{1/m_0}}^{N_j} v \right)^m \right) \in \mathcal{O}$  for all  $m \in [m_0, m_1]$ , all  $i \in \{1, \dots, q\}$ , all  $\lambda \in \Lambda_i$ ;
- (iii)  $T_\lambda^{N_i} \left( \left( S_{\lambda_i^{1/m_0}}^{N_i} v^{1/m_0} \right)^m \right) \in \mathcal{O}$  for all  $m \in (m_0, m_1]$ , all  $i \in \{1, \dots, q\}$ , all  $\lambda \in \Lambda_i$ ;
- (iv)  $T_\lambda^{N_i} \left( \left( S_{\lambda_i^{1/m_0}}^{N_i} v^{1/m_0} \right)^{m_0} \right) - v \in \mathcal{O}$  for all  $i \in \{1, \dots, q\}$ , all  $\lambda \in \Lambda_i$ .

Then the set of points generating a common hypercyclic algebra for  $(T_\lambda)_{\lambda \in \Lambda}$  is residual in  $X$ .

PROOF. We apply Proposition 5.1. Let  $I \subset \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ . We set  $m_0 = \min(I)$ ,  $m_1 = \max(I)$ . Let  $U, V, W$  be three non-empty open subsets of  $X$  with  $0 \in W$ . Let  $x \in U$ ,  $v \in V$  with finite support and let  $\mathcal{O}$  be a neighborhood of zero such that  $x + \mathcal{O} \subset U$ ,  $v + \mathcal{O} + \mathcal{O} \subset V$  and  $\mathcal{O} + \mathcal{O} \subset W$ .

Let  $M$  be bigger than any integer in the support of  $x$  and  $v$ . The assumptions of the lemma give us parameters  $\lambda_1, \dots, \lambda_q$ , sets of parameters  $\Lambda_1, \dots, \Lambda_q$  and integers  $N_1, \dots, N_q$ . We set

$$u = x + \sum_{i=1}^q S_{\lambda_i^{1/m_0}}^{N_i}(v^{1/m_0})$$

so that assumption (i) says that  $u$  belongs to  $U$ . Fix  $\lambda \in \Lambda$  and let  $i \in \{1, \dots, q\}$  be such that  $\lambda \in \Lambda_i$ . We intend to prove that  $T_\lambda^{N_i}(u^m) \in W$  for  $m \in I \setminus \{m_0\}$  and  $T_\lambda^{N_i}(u^{m_0}) \in V$ . Since we are working with the coordinatewise product and by the choice of  $M$ , we have

$$u^m = x^m + \sum_{i=1}^q \left( S_{\lambda_i^{1/m_0}}^{N_i}(v^{1/m_0}) \right)^m$$

and

$$T_\lambda^{N_i}(u^m) = T_\lambda^{N_i} \left( \left( S_{\lambda_i^{1/m_0}}^{N_i}(v^{1/m_0}) \right)^m \right) + \sum_{j=i+1}^q T_\lambda^{N_j} \left( \left( S_{\lambda_j^{1/m_0}}^{N_j}(v^{1/m_0}) \right)^m \right).$$

Assumptions (ii), (iii) and (iv) and the choice of  $v$  and  $\mathcal{O}$  allow us to conclude that  $T_\lambda^N(u^m) \in W$  for  $m \in I \setminus \{m_0\}$  and  $T_\lambda^N(u^{m_0}) \in V$ .  $\square$

We now give a more concrete application. We first need a definition.

**DEFINITION 5.3.** Let  $(x_{k,\alpha})_{k \geq 1, \alpha \in I}$  be a family of vectors of  $X$ . The series  $\sum_{k \geq 1} x_{k,\alpha}$ ,  $\alpha \in I$ , are said to be uniformly unconditionally convergent if, for any neighborhood  $\mathcal{O}$  of 0, there exists  $K \geq 1$  such that, for all  $\alpha \in I$ , for all sequences  $(\omega_k) \subset [-1, 1]^{\mathbb{N}}$ ,

$$\sum_{k \geq K} \omega_k x_{k,\alpha} \in \mathcal{O}.$$

Implicitly, in the previous definition, we assume that each series  $\sum_k \omega_k x_{k,\alpha}$  is convergent. Of course, if  $I$  is finite and each series  $\sum_k x_{k,\alpha}$  is unconditionally convergent, then they are uniformly unconditionally convergent.

**THEOREM 5.4.** Let  $\Lambda = [a, b]$  be a compact interval and let  $(w(\lambda))_{\lambda \in \Lambda}$  be a family of admissible weight sequences. Assume that

- (a)  $(e_i)$  is an unconditional basis of  $X$ ;
- (b) for each  $n$ , the function  $\log(w_n)$  is non-decreasing and Lipschitz on  $\Lambda$ , with Lipschitz constant less than or equal to  $L_n > 0$ .
- (c) for all  $p > 0$ , for all  $m_0 \geq 1$ , there exists an increasing sequence of integers  $(n_k)$  such that
  - the series  $\sum_{k \geq 1} \frac{1}{(w_{l+1}(a) \cdots w_{l+n_k}(a))^{1/m_0}} e_{l+n_k}$  are unconditionally convergent for all  $l = 0, \dots, p$ ;
  - the series  $\sum_{k \geq 1} \frac{1}{w_{l+1}(a) \cdots w_{l+n_k+i-n_i}(a)} e_{l+n_k+i-n_i}$ ,  $i \geq 1$ ,  $l = 0, \dots, p$  are uniformly unconditionally convergent;
  - for all  $k \geq 1$ ,  $w_{l+1}(a) \cdots w_{l+n_k}(a) \geq 1$ ;
  - $\sum_{i > 1} 1/C_{n_i} = +\infty$ , where  $C_n = \sum_{k=1}^n L_k$ .

Then the set of points generating a common hypercyclic algebra for  $(T_\lambda)_{\lambda \in \Lambda}$  is residual in  $X$ .

**PROOF.** Let us first prove that  $(\lambda, x) \mapsto T_\lambda(x)$  is continuous. We first observe that the family  $(T_\lambda)_{\lambda \in \Lambda}$  is pointwise bounded. Indeed, for all  $x \in X$ , since all functions  $(w_n)$  are non-decreasing, it follows from the unconditional convergence of  $\sum_{n \geq 1} w_n(b) x_n e_{n-1}$  that the family  $(T_\lambda(x))_{\lambda \in \Lambda}$  is bounded. Hence, by the Banach-Steinhaus theorem, the family  $(T_\lambda)_{\lambda \in \Lambda}$  is equicontinuous. Now let us fix some  $(\mu, y) \in \Lambda \times X$ . For all  $(\lambda, x) \in \Lambda \times X$ , we write

$$T_\lambda(x) - T_\mu(y) = T_\lambda(x - y) + T_\lambda(y) - T_\mu(y).$$

The equicontinuity of  $(T_\lambda)_{\lambda \in \Lambda}$  ensures that, fixing  $\varepsilon > 0$ , there exists  $\delta > 0$  such that, provided  $\|x - y\| < \delta$ , then  $\|T_\lambda(x - y)\| < \varepsilon$  for all  $\lambda \in \Lambda$ . On the other hand, we write

$$T_\lambda(y) - T_\mu(y) = \sum_{n \geq 1} (w_n(\lambda) - w_n(\mu)) y_n e_{n-1}.$$

Again, the unconditional convergence of  $\sum_{n \geq 1} w_n(b) y_n e_{n-1}$  implies that there exists  $N \geq 1$  such that, for all  $\lambda \in \Lambda$ ,

$$\left\| \sum_{n > N} (w_n(\lambda) - w_n(\mu)) y_n e_{n-1} \right\| < \varepsilon.$$

Finally, we observe that the continuity of each function  $w_n$  implies, provided  $|\lambda - \mu|$  is small enough,

$$\left\| \sum_{n=1}^N (w_n(\lambda) - w_n(\mu)) y_n e_{n-1} \right\| < \varepsilon.$$

We now show that the assumptions of Lemma 5.2 are satisfied. Let  $v = \sum_{l=0}^p v_l e_l \in X$  with finite support,  $m_0 \leq m_1$ ,  $M > 0$  and  $\mathcal{O}$  be a neighborhood of 0. By linearity (which is not destroyed by taking powers since we are using the coordinatewise product), it is enough to verify that the assumptions are satisfied for  $v = e_l$ . Let  $(n_k)$  be the sequence given by the assumptions of the theorem, and let  $K \geq 1$  be such that, for all  $i \geq 0$ , for all sequences  $(\omega_k) \subset [-1, 1]^{\mathbb{N}}$ ,

$$(5.1) \quad \sum_{k \geq K} \frac{\omega_k}{w_{l+1}(a) \cdots w_{l+n_{i+k}-n_i}(a)} e_{l+n_{i+k}-n_i} \in \mathcal{O}$$

$$(5.2) \quad \sum_{k \geq K} \frac{\omega_k}{(w_{l+1}(a) \cdots w_{l+n_k}(a))^{1/m_0}} e_{l+n_k} \in \mathcal{O}.$$

Without loss of generality, we may assume that  $K \geq M$ . We then set  $N_i = n_{K_i}$ . Since the sequence  $(C_n)$  is nondecreasing, we still have

$$\sum_{i \geq 1} \frac{1}{C_{N_i+p}} = +\infty.$$

Let  $\tau > 0$  be a small real number (a precised condition on  $\tau$  will be fixed later) and define  $\lambda_1 = a$ ,  $\lambda_{i+1} = \lambda_i + \frac{\tau}{C_{N_i+p}}$ . Let  $q$  be the first integer such that  $\lambda_{q+1} \geq b$  and define  $\Lambda_i = [\lambda_i, \lambda_{i+1}]$  for  $i = 1, \dots, q-1$ ,  $\Lambda_q = [\lambda_q, b]$ . Regarding (i) of Lemma 5.2, we write

$$\sum_{i=1}^q S_{\lambda_i}^{N_i} (e_l) = \sum_{i=1}^q \frac{1}{(w_{l+1}(\lambda_i) \cdots w_{l+N_i}(\lambda_i))^{1/m_0}} e_{l+N_i}.$$

Since each  $(w_n)$  is nondecreasing, using (5.2), we get that (i) is true.

Let now  $i \in \{1, \dots, q\}$ ,  $\lambda \in \Lambda_i$  and  $m \in [m_0, m_1]$ . Then

$$\sum_{j=i+1}^q T_{\lambda}^{N_i} \left( \left( S_{\lambda_j}^{N_j} (e_l) \right)^m \right) = \sum_{j=i+1}^q \frac{w_{l+N_j-N_i+1}(\lambda) \cdots w_{l+N_j}(\lambda)}{(w_{l+1}(\lambda_j) \cdots w_{l+N_j}(\lambda_j))^{m/m_0}} e_{l+N_j-N_i}$$

Now, since

$$(w_{l+1}(\lambda_j) \cdots w_{l+N_j}(\lambda_j))^{m/m_0} \geq w_{l+1}(\lambda_j) \cdots w_{l+N_j}(\lambda_j) \geq w_{l+1}(a) \cdots w_{l+N_j}(a)$$

and

$$w_{l+N_j-N_i+1}(\lambda) \cdots w_{l+N_j}(\lambda) \leq w_{l+N_j-N_i+1}(\lambda_j) \cdots w_{l+N_j}(\lambda_j)$$

we get

$$\sum_{j=i+1}^q T_{\lambda}^{N_i} \left( \left( S_{\lambda_j}^{N_j} (e_l) \right)^m \right) = \sum_{j=i+1}^q \frac{\alpha_{i,j}(\lambda)}{w_{l+1}(a) \cdots w_{l+N_j-N_i}(a)} e_{l+N_j-N_i}$$

where  $\alpha_{i,j}(\lambda) \in [0, 1]$ . Hence, by (5.1), (ii) of Lemma 5.10 is satisfied. Finally, we have for  $i \in \{1, \dots, q\}$  and  $\lambda \in \Lambda_i$ ,

$$T_\lambda^{N_i} \left( \left( S_{\lambda_i^{1/m_0}}^{N_i} e_l \right)^m \right) = \frac{w_{l+1}(\lambda) \cdots w_{l+N_i}(\lambda)}{(w_{l+1}(\lambda_i) \cdots w_{l+N_i}(\lambda_i))^{m/m_0}} e_l.$$

If  $m = m_0$ , we just write

$$\begin{aligned} \left| \frac{w_{l+1}(\lambda) \cdots w_{l+N_i}(\lambda)}{w_{l+1}(\lambda_i) \cdots w_{l+N_i}(\lambda_i)} - 1 \right| &\leq \left| \exp \left( \sum_{k=1}^{N_i} (\log w_{l+k}(\lambda) - \log w_{l+k}(\lambda_i)) \right) - 1 \right| \\ &\leq \left| \exp \left( \sum_{k=1}^{N_i} L_{l+k} |\lambda - \lambda_i| \right) - 1 \right| \\ &\leq |\exp(C_{N_i+p} |\lambda - \lambda_i|) - 1| \\ &\leq |\exp(\tau) - 1| \\ &< \min(\varepsilon, 1) \end{aligned}$$

provided  $\tau$  is small enough. Hence, (iv) is satisfied.

If  $m > m_0$ , the previous choice of  $\tau$  ensures that that

$$\left| \frac{w_{l+1}(\lambda) \cdots w_{l+N_i}(\lambda)}{w_{l+1}(\lambda_i) \cdots w_{l+N_i}(\lambda_i)} \right| \leq 2.$$

Hence, using again the monotonicity of each  $w_n$ , for all  $i \in \{1, \dots, q\}$  and all  $\lambda \in \Lambda_i$ ,

$$\frac{w_{l+1}(\lambda) \cdots w_{l+N_i}(\lambda)}{(w_{l+1}(\lambda_i) \cdots w_{l+N_i}(\lambda_i))^{m/m_0}} \leq \frac{2}{(w_{l+1}(a) \cdots w_{l+N_i}(a))^{1/m_0}}.$$

Using again (5.2) (with only one nonzero  $\omega_k$  now), we get that (iii) is satisfied, which closes the proof of Theorem 5.4.  $\square$

**REMARK 5.5.** If we are only interested in the existence of a common hypercyclic vector, we have a similar statement by verifying the assumptions of Theorem 5.4 only for  $m_0 = 1$ .

Before to give specific examples, let us explain the dependance between the behaviour of  $(C_n)$  and the choice of  $(n_k)$ . We work on  $X = \ell^1$  and we assume first that  $w_n(\lambda) = \lambda$  for all  $n \geq 1$ ,  $\lambda > 1$ . In that case,  $C_n \sim n$  and the sequence  $n_i$  cannot grow too fast in order to ensure the divergence of  $\sum_i 1/C_{n_i}$ . It is then natural to set  $n_i = Ni$  for some  $N$ , and this will be enough to ensure the uniform unconditional convergence of the series involved in Theorem 5.4, because the product  $w_1(\lambda) \cdots w_n(\lambda) = \lambda^n$  grows very fast.

Assume now that  $w_n(\lambda) = 1 + \lambda/n$ . In that case,  $L_n = 1/n$  and  $C_n$  behaves like  $\log n$ . This means that we may choose a sequence  $(n_i)$  going very fast to  $+\infty$ , like  $n_i = 2^i$ . This will be necessary to ensure uniform unconditional convergence, since now  $w_1(\lambda) \cdots w_n(\lambda) \sim n^\lambda$ , which grows much slowly, especially if we allow  $\lambda$  to be close to 0.

Let us proceed with the details. We first give a result which should be thought as a version for hypercyclic algebras of [7, Corollary 4.10].

**THEOREM 5.6.** *Let  $\Lambda \subset \mathbb{R}$  be an interval and let  $(w(\lambda))_{\lambda \in \Lambda}$  be a family of admissible weight sequences. Assume that*

(a)  $(e_i)$  is an unconditional basis of  $X$ ;

(b) all functions  $\log(w_n)$  are non-decreasing and are Lipschitz on compact sets with uniformly bounded Lipschitz constants;  
(c) for all  $m \geq 1$  and all  $\lambda \in \Lambda$ , the series  $\sum_n (w_1(\lambda) \cdots w_n(\lambda))^{-1/m} e_n$  converges.  
Then  $\bigcap_{\lambda \in \Lambda} HC(T_\lambda) \cup \{0\}$  contains a non-trivial algebra.

PROOF. We may assume that  $\Lambda = [a, b]$  is a compact interval. Let  $C > 0$  be such that all functions  $\log(w_n)$  are  $C$ -Lipschitz. We then set  $n_k = k$ ,  $k \geq 0$ , and observe that, for all  $i \geq 0$ ,

$$\sum_{k \geq 1} \frac{1}{(w_{l+1}(a) \cdots w_{l+n_{k+i}-n_i})^{1/m}} e_{l+n_{k+i}-n_i} = \sum_{k \geq 1} \frac{1}{(w_{l+1}(a) \cdots w_{l+k}(a))^{1/m}} e_{l+k},$$

which shows that the assumptions on unconditional uniform convergence are satisfied.  $\square$

COROLLARY 5.7. Let  $X$  be a Fréchet sequence algebra for the coordinatewise product with a continuous norm and such that  $(e_i)$  is an unconditional basis of  $X$ . Let  $w = (w_n)$  be an admissible weight sequence and define

$$\lambda_w = \inf \left\{ \lambda > 0 : \sum_n \lambda^{-n/m} (w_1 \cdots w_n)^{-1/m} e_n \text{ converges for all } m > 0 \right\}.$$

Then  $\bigcap_{\lambda > \lambda_w} HC(\lambda B_w) \cup \{0\}$  contains a non-trivial algebra.

This corollary includes the case of the families  $(\lambda B)_{\lambda > 1}$  on  $\ell_p$  or  $c_0$  and  $(\lambda D)_{\lambda > 0}$  on  $H(\mathbb{C})$ . For families  $(\lambda B_w)_{\lambda > 0}$  on  $\omega$  the same result holds with an even easier demonstration, although this space doesn't admit a continuous norm. We can also get examples of families which are not multiples of a single operator.

EXAMPLE 5.8. Let  $X = c_0$  and consider  $w_n(\lambda) = 1 + \frac{\lambda}{n}$ ,  $\lambda > 0$ . Then  $\bigcap_{\lambda > 0} HC(B_{w(\lambda)}) \cup \{0\}$  contains a non-trivial algebra.

To include the  $\ell_p$ -case, we need to change the choice of  $(n_k)$ .

EXAMPLE 5.9. Let  $X = \ell_p$ ,  $1 \leq p < \infty$  and consider  $w_n(\lambda) = 1 + \frac{\lambda}{n}$ ,  $\lambda > 0$ . Then  $\bigcap_{\lambda > 0} HC(B_{w(\lambda)}) \cup \{0\}$  contains a non-trivial algebra.

PROOF. Again, we need only to consider the case  $\Lambda = [a, b]$  with  $a > 0$ . We apply Theorem 5.4 with  $L_k = 1/k$  and  $n_k = 2^k$ . The uniform unconditional convergence of the involved series is ensured by the inequalities

$$w_1(a) \cdots w_n(a) \geq \exp(c \log n) = n^c \text{ for some } c > 0$$

and

$$\begin{aligned} & \left\| \sum_{k \geq K} \frac{1}{(w_{l+1}(a) \cdots w_{l+n_{k+i}-n_i})^{1/m}} e_{l+n_{k+i}-n_i} \right\| \\ & \leq \sum_{k \geq K} \frac{1}{(w_{l+1}(a) \cdots w_{l+2^{k+i}-2^i}(a))^{1/m}} \\ & \leq C_l \sum_{k \geq K} \frac{1}{2^{ic/m} (2^k - 1)^{c/m}} \\ & \leq C_l \sum_{k \geq K} \frac{1}{(2^k - 1)^{c/m}}. \end{aligned}$$

$\square$

### 5.3. Common hypercyclic algebras for a family of backward shifts - Cauchy product

In this subsection we are dealing with the question of the existence of common hypercyclic algebras, for families of weighted backward shifts when the underlying Fréchet sequence algebra is endowed with the Cauchy product. We will establish a general criterion for the existence of such algebras which encompasses the case of the multiples of the backward shift on  $\ell_1$  and of the multiples of  $D$  on  $H(\mathbb{C})$  as particular cases. Again, our main tool will be Proposition 5.1. Nevertheless, fixing  $[a, b] \subset \Lambda$ , we will not be able to devise a partition of each  $[a, b] \subset \Lambda$  as in the proof of Theorem 5.4. We will only succeed to do this for intervals  $[a, \kappa a]$ , where  $\kappa$  will be independent of  $a$ . To come back to the assumptions of Proposition 5.1, we will need an auxiliary lemma. For a family of admissible weighted sequences  $(w(\lambda))$  we will denote as usual by  $T_\lambda = B_{w(\lambda)}$  the corresponding weighted backward shift.

LEMMA 5.10. *Let  $X$  be a Fréchet sequence algebra under the Cauchy product in which  $\text{span}(e_i)$  is dense and let  $\Lambda \subset \mathbb{R}$  be an interval. Let  $(w(\lambda))_{\lambda \in \Lambda}$  be a family of admissible weights and assume that the map  $(\lambda, x) \mapsto T_\lambda x$  is continuous. Suppose that for all  $[a_0, b_0] \subset \Lambda$ , all  $m \geq 1$  and all non-empty open set  $V \subset X$ , there exists  $\kappa := \kappa(m, V, a_0, b_0) > 1$  such that, for all  $a \in [a_0, b_0]$ , for all  $\delta > 0$  and for all  $M_0 \in \mathbb{N}$ , one can find  $u \in X$  and  $M_1 \in \mathbb{N}$  satisfying*

- (i)  $u$  has finite support and  $\|u\| < \delta$ ;
- (ii) for each  $\lambda \in [a, \kappa a] \cap [a_0, b_0]$  there is  $N \leq M_1$  such that, for all  $x \in X$  with  $\text{supp}(x) \subset [0, M_0]$ ,  $(T_\lambda)^N(u+x)^n = 0$  for  $n \leq m-1$  and  $(T_\lambda)^N(u+x)^m \in V$ .

Then  $(T_\lambda)_{\lambda \in \Lambda}$  admits a common hypercyclic algebra.

PROOF. We are going to apply Proposition 5.1. So let  $K = [a_0, b_0] \subset \Lambda$ , let  $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ , set  $m = \max(I)$  and let  $U, V, W \subset X$  be non-empty open subsets, with  $0 \in W$ . We begin by considering  $\kappa$  given by the hypothesis. Let  $s$  be the first positive integer satisfying  $a_0 \kappa^s \geq b$  and define  $a_i = \kappa^i a_0$ , for  $i = 1, \dots, s-1$ , and  $a_s = b_0$ . We will use the assumptions of the lemma to construct a sequence of pairs  $(u(1), N_1), \dots, (u(s), N_s)$  satisfying, for all  $i = 1, \dots, s$ ,

- $u(i) \in U$ ;
- for all  $\lambda \in [a_0, a_i]$ , there is  $N \leq N_i$  such that  $(T_\lambda)^N(u(i)^n) \in W$ , for  $n < m$ , and  $(T_\lambda)^N(u(i)^m) \in V$ .

This construction being done, it is clear that  $u := u(s)$  is the desired point we are looking for in order to apply Proposition 5.1.

We fix  $u(0) \in U$  with finite support and we let  $M(0) = \max(\text{supp}(u(0)))$  and  $\delta(0) > 0$  such that  $B(u(0), \delta(0)) \subset U$ . For  $a = a_0$ ,  $\delta = \delta(0)$  and  $M_0 = M(0)$  in the lemma we find  $u \in X$  and  $M_1 =: N_1$  such that  $u$  has finite support,  $\|u\| < \delta(0)$  and, for each  $\lambda \in [a_0, a_1] \cap [a_0, b_0]$  there is  $N \leq N_1$  such that, for all  $x \in X$  with  $\text{supp}(x) \subset [0, M_0]$ ,  $(T_\lambda)^N(u+x)^n = 0$  for  $n \leq m-1$  and  $(T_\lambda)^N(u+x)^m \in V$ . In particular, for  $x = u(0)$  we have  $u(1) := u(0) + u \in U$  and, for all  $\lambda \in [a_0, a_1]$ , there is  $N \leq N_1$  such that  $(T_\lambda)^N(u(1)^n) = 0 \in W$ , for  $n < m$ , and  $(T_\lambda)^N(u(1)^m) \in V$ .

Assume that  $u(i)$  has been defined and let us define  $u(i+1)$ . Instead of applying directly the lemma we need to adjust things so that the corresponding interval is not just  $[a_i, a_{i+1}]$  but  $[a_0, a_{i+1}]$  in fact. By the inductive properties of  $u(i)$ , by the continuity of  $(\lambda, x) \mapsto T_\lambda x$  and of the product of  $X$  and by the compactness of  $[a_0, a_i]$ , we can find  $\delta(i) > 0$  such that

- $B(u(i), \delta(i)) \subset U$ ;
- for all  $y \in B(u(i), \delta(i))$  and all  $\lambda \in [a_0, a_i]$ , there is  $N \leq N_i$  such that  $(T_\lambda)^N(y^n) \in W$ , for  $n < m$ , and  $(T_\lambda)^N(y^m) \in V$ .

Now we apply the lemma with  $a = a_i$ ,  $M_0 = \max(\text{supp}(u(i)))$  and  $\delta = \delta(i)$  and we find  $u \in X$  and  $M_1 =: N_{i+1}$  such that  $u$  has finite support,  $\|u\| < \delta(i)$  and, for each  $\lambda \in [a_i, a_{i+1}] \cap [a_0, b_0]$  there is  $N \leq N_{i+1}$  such that, for all  $x \in X$  with  $\text{supp}(x) \subset [0, M_0]$ ,  $(T_\lambda)^N(u+x)^n = 0$  for  $n \leq m-1$  and  $(T_\lambda)^N(u+x)^m \in V$ . In particular, for  $x = u(i)$  we have  $u(i+1) := u(i) + u \in U$  and, for all  $\lambda \in [a_i, a_{i+1}]$ , there is  $N \leq N_{i+1}$  such that  $(T_\lambda)^N(u(i+1)^n) = 0 \in W$ , for  $n < m$ , and  $(T_\lambda)^N(u(i+1)^m) \in V$ . Since  $u(i+1) \in B(u(i), \delta(i))$ , we have that, for all  $\lambda \in [a_0, a_{i+1}]$ , there is  $N \leq N_{i+1}$  such that  $(T_\lambda)^N(u(i+1)^n) = 0 \in W$ , for  $n < m$ , and  $(T_\lambda)^N(u(i+1)^m) \in V$ .  $\square$

Now we are ready to prove the main theorem of this subsection.

**THEOREM 5.11.** *Let  $\Lambda \subset \mathbb{R}$  be an interval, let  $X$  be a regular Fréchet sequence algebra under the Cauchy product and let  $(w(\lambda))_{\lambda \in \Lambda}$  be a family of admissible weighted sequences such that all functions  $\log(w_n)$  are non-decreasing and Lipschitz on compact sets with uniformly bounded Lipschitz constants. Suppose that  $(e_n)$  is an unconditional basis of  $X$  and that*

(a) for all  $\lambda \in \Lambda$ ,

$$\sum_{n=1}^{\infty} \frac{1}{w_1(\lambda) \cdots w_n(\lambda)} e_n \in X;$$

(b) for all  $m \in \mathbb{N}$  and all  $[a_0, b_0] \subset \Lambda$  there exist  $c \in (0, 1)$  and  $\kappa_0 > 1$  such that

$$(5.3) \quad \lim_{\sigma \rightarrow \infty} \sum_{n=c\sigma}^{\sigma} \frac{[w_1(\kappa_0 a) \cdots w_{m\sigma}(\kappa_0 a)]^{\frac{m-1}{m}}}{w_1(a) \cdots w_{(m-1)\sigma+n}(a)} e_n = 0, \text{ for all } a \in [a_0, b_0].$$

Then  $(T_\lambda)_{\lambda \in \Lambda}$  admits a common hypercyclic algebra.

**PROOF.** First we observe that an application of the Banach-Steinhaus theorem as in the proof of Theorem 5.4 yields that the map  $(\lambda, x) \mapsto T_\lambda x$  is continuous. Given  $[a_0, b_0] \subset \Lambda$ ,  $m \geq 1$  and  $V \subset X$  open and non-empty, from condition (b) there exist  $c \in (0, 1) \cap \mathbb{Q}$  and  $\kappa_0 > 1$  such that (5.3) holds. Since the functions  $\lambda \mapsto w_n(\lambda)$  are nondecreasing and by unconditionality, (5.3) holds for all  $\kappa \in (1, \kappa_0)$ . Fix  $d \in (c, \frac{c+1}{2}) \cap \mathbb{Q}$ , take  $y = \sum_{j=0}^p y_j e_j \in V$  and find  $\rho \geq 1$  and  $\eta > 0$  so that  $y + B \subset V$ , where  $B$  is the ball for the seminorm  $\|\cdot\|_\rho$  centered at the origin and with radius  $3\eta$ . By unconditionality of the basis  $(e_n)$ , there exists  $\rho_0 \geq 1$  and  $C_{\rho, \rho_0} > 0$  such that, for all  $x = \sum_n x_n e_n$  and all  $\theta \in \ell_\infty$ ,

$$(5.4) \quad \left\| \sum_{n=0}^{+\infty} \theta_n x_n e_n \right\|_\rho \leq C_{\rho, \rho_0} \|\theta\|_\infty \left\| \sum_{n=0}^{+\infty} x_n e_n \right\|_{\rho_0}.$$

Take  $q > p$  big enough so that

$$(5.5) \quad \left\| \sum_{n \geq q} \frac{z_n}{w_1(a_0) \cdots w_n(a_0)} e_n \right\|_\rho < \eta,$$

for all  $z \in c_{00}$  with  $\|z\|_\infty \leq \|y\|_\infty (\max(1, w_1(b_0), \dots, w_p(b_0)))^p$  (this is possible by condition (a)). We also fix  $M > 0$  so that all the functions  $\log(w_n)$  are M-Lipschitz

on  $[a_0, b_0]$ . By continuity of  $\exp$  we can find  $\gamma > 0$  so that

$$(5.6) \quad |\exp(x) - 1| \leq \frac{\eta}{C_{\rho, \rho_0} \|y\|_{\rho_0}}, \text{ for all } x \in \mathbb{R} \text{ with } |x| \leq \gamma.$$

We then choose  $\tau > 0$  and  $\kappa \in (1, \kappa_0)$  such that

$$(5.7) \quad \tau < \frac{\gamma}{M}, \quad (\kappa - 1)b_0 \leq \tau \frac{d - c}{(m - 1 + d)q}.$$

Now let  $a \in [a_0, b_0]$ ,  $\delta > 0$  and  $M_0 \in \mathbb{N}$  be arbitrary. Take  $r > 0$  large enough such that  $cr > M_0$  and  $cr, dr \in \mathbb{N}$ . Later on we will request more conditions on the size of  $r$ . For  $j = cr, \dots, dr$ , let  $\lambda_j$  be defined inductively by  $\lambda_{cr} = a$  and  $\lambda_{j+1} = \lambda_j + \frac{\tau}{(m-1)qr+qj}$ . We notice that

$$\begin{aligned} \lambda_{dr} &= a + \sum_{j=cr}^{dr-1} \frac{\tau}{(m-1)qr+qj} \\ &\geq a + \tau \frac{d-c}{(m-1+d)q} \\ &\geq \kappa a. \end{aligned}$$

Define

$$u := u(r) = \sum_{j=cr}^{dr-1} \sum_{l=0}^p d_{j,l} e_{qj+l} + \varepsilon e_{qr},$$

where

$$\begin{aligned} d_{j,l} &:= \frac{y_l}{m\varepsilon^{m-1} w_{l+1}(\lambda_{j+1}) \cdots w_{(m-1)qr+qj+l}(\lambda_{j+1})}, \\ \varepsilon &= \left( \frac{1}{w_1(\kappa a) \cdots w_{mqr}(\kappa a)} \right)^{\frac{1}{m}}. \end{aligned}$$

Let also  $M_1 := mqr$ . We shall show that  $\|u\| < \delta$  if  $r$  is big enough. Let  $\sigma \geq 1$  and  $\sigma' \geq 1$ ,  $C_{\sigma, \sigma'}$  associated to  $\sigma$  as in (5.4) by the fact that  $(e_n)$  is unconditional. For the double sum we have

$$\left\| \sum_{j=cr}^{dr-1} \sum_{l=0}^p d_{j,l} e_{qj+l} \right\|_{\sigma} = \left\| \sum_{j=cr}^{dr-1} \sum_{l=0}^p \frac{y_l (w_1(\kappa a) \cdots w_{mqr}(\kappa a))^{\frac{m-1}{m}}}{m w_{l+1}(\lambda_{j+1}) \cdots w_{(m-1)qr+qj+l}(\lambda_{j+1})} e_{qj+l} \right\|_{\sigma}.$$

Now, we write the quotient of weights as

$$\frac{w_1(a) \cdots w_{(m-1)qr+qj+l}(a)}{w_{l+1}(\lambda_{j+1}) \cdots w_{(m-1)qr+qj+l}(\lambda_{j+1})} \times \frac{(w_1(\kappa a) \cdots w_{mqr}(\kappa a))^{\frac{m-1}{m}}}{w_1(a) \cdots w_{(m-1)qr+qj+l}(a)}.$$

Using unconditionality and the definition of  $C_{\sigma, \sigma'}$ , we get

$$\begin{aligned} & \left\| \sum_{j=cr}^{dr-1} \sum_{l=0}^p d_{j,l} e_{qj+l} \right\|_{\sigma} \\ & \leq C_{\sigma, \sigma'} \left\| \sum_{j=cr}^{dr-1} \sum_{l=0}^p \frac{y_l w_1(a) \cdots w_l(a)}{m} \frac{(w_1(\kappa a) \cdots w_{mqr}(\kappa a))^{\frac{m-1}{m}}}{w_1(a) \cdots w_{(m-1)qr+qj+l}(a)} e_{qj+l} \right\|_{\sigma'} \\ & \leq C_{\sigma, \sigma'} \left\| \sum_{n=cqr}^{qr} z_n \frac{(w_1(\kappa a) \cdots w_{mqr}(\kappa a))^{\frac{m-1}{m}}}{w_1(a) \cdots w_{(m-1)qr+n}(a)} e_n \right\|_{\sigma'}, \end{aligned}$$

for some eventually null sequence  $z = (z_n)$  satisfying

$$\|z\|_{\infty} \leq \|y\|_{\infty} (\max(1, w_1(b_0), \dots, w_p(b_0))^p) / m.$$

By assumption (b) and the unconditionality of  $(e_n)$ , we conclude that the sum  $\sum_{j=cr}^{dr-1} \sum_{l=0}^p d_{j,l} e_{qj+l}$  converges to 0 when  $r \rightarrow +\infty$ . For the term  $\varepsilon e_{qr}$  on the other hand, for all  $\sigma \geq 1$  and since  $X$  is regular, there exist  $\sigma' \geq \sigma$  and  $C_{\sigma, \sigma'} > 0$  such that

$$\begin{aligned} \|\varepsilon e_{qr}\|_{\sigma} &= \left\| \frac{1}{[w_1(\kappa a) \cdots w_{mqr}(\kappa a)]^{\frac{1}{m}}} e_{qr} \right\|_{\sigma} \\ &= \left[ \frac{1}{w_1(\kappa a) \cdots w_{mqr}(\kappa a)} \|e_{qr}\|_{\sigma}^m \right]^{\frac{1}{m}} \\ &\leq C_{\sigma, \sigma'} \left[ \frac{1}{w_1(\kappa a) \cdots w_{mqr}(\kappa a)} \|e_{mqr}\|_{\sigma'} \right]^{\frac{1}{m}} \\ &\leq C_{\sigma, \sigma'} \left\| \frac{1}{w_1(\kappa a) \cdots w_{mqr}(\kappa a)} e_{mqr} \right\|_{\sigma'}^{\frac{1}{m}}, \end{aligned}$$

and this converges to 0 as  $r \rightarrow +\infty$ . This shows that condition (i) of Lemma 5.10 is satisfied if  $r$  is big enough.

Now, taking  $\lambda \in [a, \kappa a]$ , there exists  $k \in \{cr, \dots, dr-1\}$  such that  $\lambda \in [\lambda_k, \lambda_{k+1}]$ . We choose  $N = (m-1)qr + qk$  and take  $x \in X$  with  $\text{supp}(x) \subset [0, M_0]$ . Clearly we have  $\max(\text{supp}((u+x)^n)) \leq (m-1)qr < N$  for all  $n = 1, \dots, m-1$ , which implies  $(T_{\lambda})^N(u+x)^n = 0$ . On the other hand,

$$(u+x)^m = x' + \sum_{j=cr}^{dr-1} \sum_{l=0}^p m \varepsilon^{m-1} d_{j,l} e_{(m-1)qr+qj+l} + \varepsilon^m e_{mqr},$$

with  $\text{supp}(x') \subset [0, (m-2)qr + 2(dqr+p)] \cup [0, (m-1)qr + M_0]$ . It follows that  $\max(\text{supp}(x')) < N$  if  $r$  is big enough. We also have that  $(m-1)qr + qj + p < N$

if  $j < k$ . Hence,

$$\begin{aligned}
& (T_\lambda)^N(u+x)^m \\
&= \sum_{l=0}^p m\varepsilon^{m-1} d_{k,l} w_{(m-1)qr+qk-N+l+1}(\lambda) \cdots w_{(m-1)qr+qk+l}(\lambda) e_{(m-1)qr+qk-N+l} \\
&\quad + \sum_{j=k+1}^{dr-1} \sum_{l=0}^p m\varepsilon^{m-1} d_{j,l} w_{(m-1)qr+qj-N+l+1}(\lambda) \cdots w_{(m-1)qr+qj+l}(\lambda) e_{(m-1)qr+qj-N+l} \\
&\quad + \varepsilon^m w_{mqr-N+1}(\lambda) \cdots w_{mqr}(\lambda) e_{mqr-N} \\
&=: P_1 + P_2 + P_3,
\end{aligned}$$

where, after substituting  $d_{j,l}$ ,  $N$  and  $\varepsilon$  by their values,

$$\begin{aligned}
P_1 &= \sum_{l=0}^p \frac{w_{l+1}(\lambda) \cdots w_{(m-1)qr+qk+l}(\lambda)}{w_{l+1}(\lambda_{k+1}) \cdots w_{(m-1)qr+qk+l}(\lambda_{k+1})} y_l e_l \\
P_2 &= \sum_{j=k+1}^{dr-1} \sum_{l=0}^p \frac{w_{(j-k)q+l+1}(\lambda) \cdots w_{(m-1)qr+qj+l}(\lambda)}{w_{l+1}(\lambda_{j+1}) \cdots w_{(m-1)qr+qj+l}(\lambda_{j+1})} y_l e_{(j-k)q+l} \\
P_3 &= \frac{w_{(r-k)q+1}(\lambda) \cdots w_{mqr}(\lambda)}{w_1(\kappa a) \cdots w_{mqr}(\kappa a)} e_{(r-k)q}.
\end{aligned}$$

From the definition of  $\lambda_{cr}, \dots, \lambda_{dr}$ , by the Lipschitz condition on the functions  $\log w_n$  and by (5.7), we get

$$\begin{aligned}
& \left| \sum_{j=1}^{(m-1)qr+qk} \log(w_{l+j}(\lambda)) - \log(w_{l+j}(\lambda_{k+1})) \right| \\
&\leq \sum_{j=1}^{(m-1)qr+qk} |\log(w_{l+j}(\lambda)) - \log(w_{l+j}(\lambda_{k+1}))| \\
&\leq \sum_{j=1}^{(m-1)qr+qk} M(\lambda_{k+1} - \lambda_k) \\
&= ((m-1)qr + qk) M \frac{\tau}{(m-1)qr + qk} \\
&< \gamma.
\end{aligned}$$

Hence, by (5.6) we have

$$\begin{aligned}
& \left| \frac{w_{l+1}(\lambda) \cdots w_{(m-1)qr+qk+l}(\lambda)}{w_{l+1}(\lambda_{k+1}) \cdots w_{(m-1)qr+qk+l}(\lambda_{k+1})} - 1 \right| \\
&= \left| \exp \left( \sum_{j=1}^{(m-1)qr+qk} \log(w_{l+j}(\lambda)) - \log(w_{l+j}(\lambda_{k+1})) \right) - 1 \right| \\
&\leq \frac{\eta}{C_{\rho, \rho_0} \|y\|_{\rho_0}}.
\end{aligned}$$

Writing

$$P_1 - y = \sum_{l=0}^p \theta_l y_l e_l \text{ with } |\theta_l| \leq \frac{\eta}{C_{\rho, \rho_0} \|y\|_{\rho_0}},$$

we get by (5.4) that  $\|P_1 - y\|_\rho < \eta$ . Furthermore, let us write

$$\begin{aligned} & \frac{w_{(j-k)q+l+1}(\lambda) \cdots w_{(m-1)qr+qj+l}(\lambda)}{w_{l+1}(\lambda_{j+1}) \cdots w_{(m-1)qr+qj+l}(\lambda_{j+1})} \\ &= \frac{w_1(\lambda_{j+1}) \cdots w_l(\lambda_{j+1})}{w_1(a_0) \cdots w_{(j-k)q+l}(a_0)} \prod_{s=1}^{(j-k)q+l} \frac{w_s(a_0)}{w_s(\lambda_{j+1})} \prod_{s=(j-k)q+l+1}^{(m-1)qr+qj+l} \frac{w_s(\lambda)}{w_s(\lambda_{j+1})}. \end{aligned}$$

Then we get

$$P_2 = \sum_{n \geq q} \frac{z_n}{w_1(a_0) \cdots w_n(a_0)} e_n$$

for some eventually null sequence  $(z_n)$  with

$$\|z\|_\infty \leq \|y\|_\infty (\max(1, w_1(b_0), \dots, w_p(b_0)))^p.$$

By (5.5) we conclude that  $\|P_2\| < \eta$ . Finally,

$$\begin{aligned} \frac{w_{(r-k)q+1}(\lambda) \cdots w_{mqr}(\lambda)}{w_1(\kappa a) \cdots w_{mqr}(\kappa a)} &\leq \frac{w_{(r-k)q+1}(\kappa a) \cdots w_{mqr}(\kappa a)}{w_1(\kappa a) \cdots w_{mqr}(\kappa a)} \\ &= \frac{1}{w_1(\kappa a) \cdots w_{(r-k)q}(\kappa a)} e^{(r-k)q} \\ &\leq \frac{1}{w_1(a_0) \cdots w_{(r-k)q}(a_0)} e^{(r-k)q}, \end{aligned}$$

hence  $\|P_3\|_\sigma < \eta$  if  $r$  is big enough. With this we conclude that

$$\|(T_\lambda)^N(u+x)^m - y\| < 3\eta,$$

which shows that  $(T_\lambda)^N(u+x)^m \in V$  and completes the proof.  $\square$

We now apply Theorem 5.11 to multiples of the backward shift and of the derivative operator.

**EXAMPLE 5.12.** On  $\ell_1$  endowed with the Cauchy product,  $\bigcap_{\lambda > 1} HC(\lambda B) \cup \{0\}$  contains a nontrivial algebra.

**PROOF.** We apply Theorem 5.11 with  $X = \ell_1$ ,  $\Lambda = (1, \infty)$  and  $w_n(\lambda) = \lambda$ . We have that the functions  $\log(w_n)$  are continuously first differentiable hence uniformly Lipschitz on compact subsets of  $\Lambda$ . Condition (a) trivially holds and, for (b), we have that

$$\left\| \sum_{n=c\sigma}^{\infty} \frac{[w_1(\kappa a) \cdots w_{m\sigma}(\kappa a)]^{\frac{m-1}{m}}}{w_1(a) \cdots w_{(m-1)\sigma+n}(a)} e_n \right\| = \frac{a}{a-1} \times \frac{\kappa^{(m-1)\sigma}}{a^{c\sigma}}$$

which, for  $m = 1$ , tends to 0 as  $\sigma$  goes to  $+\infty$  for any  $c \in (0, 1)$ ,  $\kappa > 1$  and  $a > 1$ , and for  $m > 1$  it goes to 0 for any  $c \in (0, 1)$  and  $1 < \kappa < a^{\frac{c}{m-1}}$ .  $\square$

**EXAMPLE 5.13.** On  $H(\mathbb{C})$  endowed with the Cauchy product,  $\bigcap_{\lambda > 0} HC(\lambda D) \cup \{0\}$  contains a non-trivial algebra.

**PROOF.** We apply Theorem 5.11 with  $X = H(\mathbb{C})$ ,  $\Lambda = (0, \infty)$  and  $w_n(\lambda) = \lambda n$ . Again the functions  $\log(w_n)$  are uniformly Lipschitz on compact subsets of  $\Lambda$  and

condition (a) is clearly satisfied. For (b) we fix  $m \geq 1$ ,  $a > 0$ ,  $\kappa > 1$  and  $r \geq 1$  and we have

$$\begin{aligned} & \left\| \sum_{n \geq c\sigma} \frac{(w_1(\kappa a) \cdots w_{m\sigma}(\kappa a))^{(m-1)/m}}{w_1(a) \cdots w_{(m-1)\sigma+n}(a)} e_n \right\|_r \\ &= \kappa^{(m-1)\sigma} \sum_{n \geq c\sigma} \frac{(m\sigma)!^{(m-1)/m}}{((m-1)\sigma+n)!} \left(\frac{r}{a}\right)^n \\ &= \left(\frac{\kappa a}{r}\right)^{(m-1)\sigma} (m\sigma)!^{(m-1)/m} \sum_{n \geq c\sigma} \frac{(r/a)^{n+(m-1)\sigma}}{(n+(m-1)\sigma)!} \\ &\leq C \left(\frac{\kappa a}{r}\right)^{(m-1)\sigma} (m\sigma)!^{(m-1)/m} \times \frac{(r/a)^{(m-1+c)\sigma}}{((m-1+c)\sigma)!}. \end{aligned}$$

Since for all  $\varepsilon > 0$ , Stirling's formula implies

$$\begin{aligned} (m\sigma)!^{(m-1)/m} &\leq C_\varepsilon \sigma^{(m-1+\varepsilon)\sigma} \\ ((m-1+c)\sigma)! &\geq C_\varepsilon \sigma^{(m-1+c-\varepsilon)\sigma}, \end{aligned}$$

choosing  $\varepsilon < 2c$ , it follows that for all  $c \in (0, 1)$ , all  $\kappa > 1$ , all  $a > 0$ , all  $m \geq 1$  and all  $r \geq 1$ , we have

$$\left(\frac{\kappa a}{r}\right)^{(m-1)\sigma} (m\sigma)!^{(m-1)/m} \times \frac{(r/a)^{(m-1+c)\sigma}}{((m-1+c)\sigma)!} \xrightarrow{\sigma \rightarrow +\infty} 0.$$

Hence, assumption (b) is satisfied.  $\square$

QUESTION 5.14. Let  $X = \ell_1$ ,  $\Lambda = (0, +\infty)$  and  $w_n(\lambda) = 1 + \lambda/n$ . In the previous section, we have shown that  $\bigcap_{\lambda \in \Lambda} HC(B_{w(\lambda)})$  is not empty. Does it contain (except 0) a nontrivial algebra?

We cannot apply Theorem 5.11 for this family of shifts since (b) is not satisfied. Indeed, using standard tools of calculus, it can be shown that, for all  $n \in \mathbb{N}$ , all  $a \in \Lambda$ , all  $c \in (0, 1)$  and all  $\kappa > 1$ , there exists  $C > 0$  such that, for all  $\sigma \geq 1$ ,

$$\left\| \sum_{n=c\sigma}^{\sigma} \frac{[w_1(\kappa_0 a) \cdots w_{m\sigma}(\kappa_0 a)]^{\frac{m-1}{m}}}{w_1(a) \cdots w_{(m-1)\sigma+n}(a)} e_n \right\|_1 \geq C \sigma^{a(\frac{\kappa(m-1)}{m}-1)+1}.$$

For  $a = m = 2$ , it is impossible to find  $\kappa > 1$  such that the right handside goes to 0.



## Frequently and upper frequently hypercyclic algebras

### 6.1. How to get upper frequently hypercyclic algebras

In [19], following the proof made in [9] that the set of upper frequently hypercyclic vectors is either empty or residual, the authors gave an analogue to Birkhoff's transitivity theorem for upper frequent hypercyclicity. We adapt it in order to get upper frequently hypercyclic algebras.

**PROPOSITION 6.1.** *Let  $T$  be a continuous operator on an  $F$ -algebra  $X$  satisfying the following condition: for each  $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ , there exists  $m_0 \in I$  such that, for each non-empty open subset  $V$  of  $X$  and each neighbourhood  $W$  of the origin, there is  $\delta > 0$  such that for each non-empty open subset  $U$  and each  $N_0 \in \mathbb{N}$ , there is  $u \in U$  and  $N \geq N_0$  satisfying*

$$\frac{1}{N+1} \text{card}\{p \leq N : T^p(u^m) \in W \text{ for } m \in I \setminus \{m_0\} \text{ and } T^p(u^{m_0}) \in V\} > \delta.$$

*Then  $T$  admits an upper frequently hypercyclic algebra.*

**PROOF.** Let  $(V_k)_k$  be a basis for the topology of  $X$  and let  $(W_j)_j$  be a basis of open neighbourhoods of the origin. By the assumption for each  $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ , there exists  $m_0 = m_0(I)$  such that, for each  $k, j$ , there is  $\delta_{k,j,I} > 0$  such that for each non-empty open subset  $U$  and each  $N_0 \in \mathbb{N}$ , there is  $u \in U$  and  $N \geq N_0$  satisfying

$$\frac{1}{N+1} \text{card}\{p \leq N : T^p(u^m) \in W_j \text{ for } m \in I \setminus \{m_0\} \text{ and } T^p(u^{m_0}) \in V_k\} > \delta_{k,j,I}.$$

We set

$$A = \bigcap_{k,j,I,N_0 \geq 1} \bigcup_{N \geq N_0} \left\{ u \in X : \frac{1}{N+1} \text{card}\{p \leq N : T^p(u^m) \in W_j \text{ for } m \in I \setminus \{m_0\} \text{ and } T^p(u^{m_0}) \in V_k\} > \delta_{k,j,I} \right\}$$

and show that  $A$  is residual.

For fixed  $k, j, I, N_0$  the set

$$\bigcup_{N \geq N_0} \left\{ u \in X : \frac{1}{N+1} \text{card}\{p \leq N : T^p(u^m) \in W_j \text{ for } m \in I \setminus \{m_0\} \text{ and } T^p(u^{m_0}) \in V_k\} > \delta_{k,j,I} \right\}$$

is clearly open, and the definition of  $\delta_{k,j,I}$  implies that it is also dense. By the Baire category theorem  $A$  is residual.

Next, we check that if  $u$  belongs to  $A$  and if  $P \in \mathbb{C}[z]$  is not constant, with  $P(0) = 0$ , then  $P(u) \in UFHC(T)$ . We write  $P(z) = \sum_{m \in I} \hat{P}(m)z^m$  for some  $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$  and  $\hat{P}(m) \neq 0$  for all  $m \in I$ . Since  $UFHC(T)$  is invariant under multiplication by a scalar we may assume that  $\hat{P}(m_0) = 1$ . Let  $V$  be a non-empty open set and find  $k, j$  such that  $V_k + (\text{card}(I) - 1)\|\hat{P}\|_\infty W_j \subset V$ . For each  $N_0 \in \mathbb{N}$  there is  $N \geq N_0$  such that

$$\frac{1}{N+1} \text{card} \{p \leq N : T^p(u^m) \in W_j \text{ for } m \in I \setminus \{m_0\} \text{ and } T^p(u^{m_0}) \in V_k\} > \delta_{k,j,I}.$$

But if  $T^p(u^m) \in W_j$  for  $m \in I \setminus \{m_0\}$  and  $T^p(u^{m_0}) \in V_k$ , then  $T^p(P(u)) \in V$ . Therefore,

$$\frac{1}{N+1} \text{card} \{p \leq N : T^p(P(u)) \in V\} > \delta_{k,j,I}$$

which yields that  $\overline{\text{dens}}(\{p \in \mathbb{N} : T^p(P(u)) \in V\}) > \delta_{k,j,I} > 0$ .  $\square$

We will apply this lemma either for  $m_0(I) = \min(I)$  or  $m_0 = \max(I)$ . The proposition gets the simpler forms:

**COROLLARY 6.2.** *Let  $X$  be an  $F$ -algebra. If for each nonempty subset  $V$  of  $X$ , for each neighbourhood  $W$  of the origin, for any positive integers  $m_0 < m_1$ , there is  $\delta > 0$  such that for each nonempty open subset  $U$  and each  $N_0 \in \mathbb{N}$ , there is  $u \in U$  and  $N \geq N_0$  satisfying*

$$\frac{1}{N+1} \text{card} \{p \leq N : T^p(u^m) \in W \text{ for } m \in \{m_0 + 1, \dots, m_1\} \text{ and } T^p(u^{m_0}) \in V\} > \delta$$

*then  $T$  admits an upper frequently hypercyclic algebra.*

**COROLLARY 6.3.** *Let  $X$  be an  $F$ -algebra. If for each nonempty subset  $V$  of  $X$ , for each neighbourhood  $W$  of the origin, for any positive integer  $m$ , there is  $\delta > 0$  such that for each nonempty open subset  $U$  and each  $N_0 \in \mathbb{N}$ , there is  $u \in U$  and  $N \geq N_0$  satisfying*

$$\frac{1}{N+1} \text{card} \{p \leq N : T^p(u^n) \in W \text{ for } n \in \{1, \dots, m-1\} \text{ and } T^p(u^m) \in V\} > \delta$$

*then  $T$  admits an upper frequently hypercyclic algebra.*

## 6.2. Existence of upper frequently hypercyclic algebras for weighted backward shifts - coordinatewise products

We intend to apply the previous method to backward shift operators and prove Theorem 1.6. The unconditional convergence of  $\sum_{n \geq 1} (w_1 \cdots w_n)^{-1} e_n$  will be used throughout the following lemma.

**LEMMA 6.4.** *Let  $(w_n)$  be a weight sequence such that  $\sum_{n \geq 1} (w_1 \cdots w_n)^{-1} e_n$  converges unconditionally. Then, for all  $\varepsilon > 0$ , for all  $p > 0$ , for all  $M > 0$ , there exists  $N \geq p$  such that, for each sequence of complex numbers  $(y(n, l))_{n \geq N, 0 \leq l \leq p}$  with  $|y(n, l)| \leq M$  for all  $n, l$ , then*

$$\left\| \sum_{n \geq N} \sum_{l=0}^p \frac{y(n, l)}{w_{l+1} \cdots w_{n+l}} e_{n+l} \right\| \leq \varepsilon.$$

PROOF. We first observe that the convergence of the series involved follows from the unconditional convergence of each series  $\sum_{n \geq 1} (w_{l+1} \cdots w_{n+l})^{-1} e_{n+l}$ . Setting  $z(n, l) = y(n, l)/M$  and using the triangle inequality, the (almost) homogeneity of the  $F$ -norm implies that

$$\left\| \sum_{n \geq N} \sum_{l=0}^p \frac{y(n, l)}{w_{l+1} \cdots w_{n+l}} e_{n+l} \right\| \leq (M+1) \sum_{l=0}^p \left\| \sum_{n \geq N} \frac{z(n, l)}{w_{l+1} \cdots w_{n+l}} e_{n+l} \right\|.$$

The existence of an  $N$  such that the last term is less than  $\varepsilon$  now follows directly from the unconditional convergence of the series  $\sum_n (w_{l+1} \cdots w_{n+l})^{-1} e_{n+l}$  (see the introduction).  $\square$

PROOF OF THEOREM 1.6. Let  $m_0 < m_1$  be two positive integers. Let  $V, W \subset X$  be open and non-empty with  $0 \in W$ . Let  $p \geq 0$ ,  $\varepsilon > 0$  and  $v = \sum_{l=0}^p v_l e_l$  be such that  $B(v, \varepsilon) \subset V$  and  $B(0, 2\varepsilon) \subset W$ . We also set  $M = \max(1, \|v\|_\infty)^{m_1/m_0}$ . Let  $N \in \mathbb{N}$  be given by Lemma 6.4 for these values of  $\varepsilon$ ,  $p$  and  $M$ . Without loss of generality, we may assume  $N > p$ . We set  $\delta = \frac{1}{2N}$ . Let  $U \subset X$  be open and non-empty and let  $x \in U$  with finite support. We also define, for  $k \geq 0$ ,

$$v(k) = \sum_{l=0}^p \frac{v_l^{1/m_0}}{(w_{l+1} \cdots w_{k+l})^{1/m_0}} e_{k+l}.$$

Let  $N_1$  be very large (precise conditions on it will be given later; for the moment we just assume that  $N_1$  is bigger than the maximum of the support of  $x$ ). We finally set

$$u = x + \sum_{k \geq N_1} v(Nk).$$

The unconditional convergence of the series  $\sum_k (w_{l+1} \cdots w_{k+l})^{-1/m_0} e_{k+l}$  ensures that  $u$  is well-defined and that  $\|u - x\| < \varepsilon$  provided  $N_1$  is large enough. Let now  $m \in \{m_0, \dots, m_1\}$  and  $j \geq N_1$ . Then, since  $x$  and the  $v(kN)$ ,  $k \geq N_1$ , have pairwise disjoint support,

$$\begin{aligned} B_w^{Nj} u^m &= \sum_{l=0}^p \frac{v_l^{m/m_0}}{(w_{l+1} \cdots w_{jN+l})^{\frac{m}{m_0} - 1}} e_l \\ &\quad + \sum_{k > j} \sum_{l=0}^p \frac{v_l^{m/m_0} w_{(k-j)N+l+1} \cdots w_{kN+l}}{(w_{l+1} \cdots w_{kN+l})^{\frac{m}{m_0}}} e_{(k-j)N+l} \\ &=: z(1, j, m) + z(2, j, m). \end{aligned}$$

If  $m = m_0$ , then  $z(1, j, m) = v$  whereas, if  $m \in \{m_0 + 1, \dots, m_1\}$ , then since  $|v_l|^{m/m_0} \leq M$  and since the sequences  $(w_{l+1} \cdots w_{l+n})_n$  go to  $+\infty$  (recall that  $X$  has a continuous norm), we may adjust  $N_1$  so that  $\|z(1, j, m)\| < \varepsilon$  for all  $j \geq N_1$ . On the other hand, we may write

$$z(2, j, m) = \sum_{n \geq N} \sum_{l=0}^p \frac{y(j, n, l, m)}{w_{l+1} \cdots w_{n+l}} e_{n+l}$$

where, for  $s = k - j \geq 1$ ,  $l = 0, \dots, p$ ,

$$y(j, sN, l, m) = \frac{v_l^{m/m_0}}{(w_{l+1} \cdots w_{(s+j)N})^{\frac{m}{m_0} - 1}}$$

and  $y(j, n, l, m) = 0$  if  $n$  is not a multiple of  $N$ . Again taking  $N_1$  large enough guarantees that  $|y(j, n, l, m)| \leq M$  for all  $j \geq N_1$ , all  $n \geq N$ , all  $l = 0, \dots, p$  and all  $m = m_0, \dots, m_1$ . By the choice of  $N$ , we get  $\|z(2, j, m)\| < \varepsilon$  for all  $j \geq N_1$ . Summarizing, we have proved that, for all  $j \geq N_1$ ,  $B_w^{Nj} u^{m_0} \in V$  and  $B_w^{Nj} u^m \in W$  for all  $m \in \{m_0 + 1, \dots, m_1\}$ . Hence we may apply Corollary 6.2 to prove that  $B_w$  supports an upper frequently hypercyclic algebra.  $\square$

The Theorem 1.6 can be applied to the following examples of Section 5.2:  $\lambda B$  on  $\ell_p$  for  $\lambda > 1$ ,  $\lambda D$  on  $H(\mathbb{C})$  for  $\lambda > 0$  or  $B_w$  on  $c_0$  with  $w_n = 1 + \lambda/n$ ,  $\lambda > 0$ . Regarding this last weight, on  $\ell_p$ ,  $B_w$  is upper frequently hypercyclic if and only if  $\lambda > 1/p$ . However we do not know the answer to the following question because of the divergence of  $\sum_n (w_1 \cdots w_n)^{-1/m} e_n$  for  $m > \lambda p$ .

QUESTION 6.5. Let  $X = \ell_p$  and  $w_n = 1 + \lambda/n$  for  $\lambda > 1/p$ . Does  $B_w$  supports an upper frequently hypercyclic algebra?

On  $\ell_p$ , it is known that  $B_w$  is (upper) frequently hypercyclic if and only if  $\sum_n (w_1 \cdots w_n)^{-p}$  is convergent (see [9]).

QUESTION 6.6. Let  $X = \ell_p$  endowed with the coordinatewise product and let  $w = (w_n)$  be an admissible weight sequence. Assume that  $B_w$  supports an upper frequently hypercyclic algebra. Does this imply that  $\sum_n (w_1 \cdots w_n)^{-\gamma}$  is convergent for all  $\gamma > 0$ ?

### 6.3. Existence of upper frequently hypercyclic algebras for weighted backward shifts - convolution product

We now study the existence of a upper frequently hypercyclic algebra for weighted backward shifts when the underlying Fréchet algebra is endowed with the convolution product. We shall give a general statement encompassing the case of the multiples of the backward shift and of the derivation operator.

THEOREM 6.7. *Let  $X$  be a regular Fréchet sequence algebra for the Cauchy product and let  $(w_n)$  be an admissible weight sequence. Assume that*

- (a)  $\sum_{n \geq 1} (w_1 \cdots w_n)^{-1} e_n$  converges unconditionally.
- (b) for all  $m \geq 2$ , there exists  $c \in (0, 1)$  such that

$$\lim_{\sigma \rightarrow +\infty} \sup_{z \in c_{00} \cap B_{\ell_\infty}} \left\| \sum_{n \geq c\sigma} \frac{z_n (w_1 \cdots w_{m\sigma})^{(m-1)/m}}{w_1 \cdots w_{(m-1)\sigma+n}} e_n \right\| = 0.$$

Then  $B_w$  admits an upper frequently hypercyclic algebra.

PROOF. We shall prove that the assumptions of Corollary 6.3 are satisfied. For  $m = 1$ , this follows from condition (a) which implies that  $T$  admits a dense set of (upper) frequently hypercyclic vectors. Thus, let us assume that  $m \geq 2$  and let  $c \in (0, 1)$  be given by (b). We also consider  $d \in (c, (1+c)/2) \subset (0, 1)$ . Let  $V$  be a non-empty open subset of  $X$ ,  $W$  a neighbourhood of 0,  $y = \sum_{l=0}^p y_l e_l \in V$  and  $\eta > 0$  such that  $B(0, 2\eta) + y \subset V$ . Let finally  $q > p$  be such that, for all  $z \in \ell_\infty$  with  $\|z\|_\infty \leq \|y\|_\infty (\max(1, w_1, \dots, w_p))^{p+1}$ ,

$$\left\| \sum_{n \geq q} \frac{z_n}{w_1 \cdots w_n} e_n \right\| < \eta.$$

We intend to prove that, for each non-empty open subset  $U$  and each  $N_0 \in \mathbb{N}$ , there is  $u \in U$  and  $N \geq N_0$  satisfying

$$\frac{1}{N+1} \text{card} \{s \leq N : B_w^s(u^n) \in W \text{ for } n < m \text{ and } B_w^s(u^m) \in V\} \geq \frac{d-c}{2((m-1)q+qd)}.$$

More precisely, we shall prove that, for all  $\sigma$  large enough, setting

$$E_\sigma = \{(m-1)q\sigma + qj : c\sigma \leq j < d\sigma\},$$

there exists  $u \in U$  such that, for all  $s \in E_\sigma$ ,  $B_w^s(u^n) = 0$  for  $n < m$  and  $B_w^s(u^m) \in V$ . Since

$$\lim_{\sigma \rightarrow +\infty} \frac{\text{card}(E_\sigma)}{\max(E_\sigma)} = \frac{d-c}{(m-1)q+qd},$$

we will get the claimed result.

We thus fix  $x \in U$  with finite support (we denote by  $p'$  the maximum of the support of  $x$ ) and let  $\sigma > 0$  be such that  $p' < c\sigma$ . Inspired by the proof of Theorem 1.3, we set

$$u = x + \sum_{j=c\sigma}^{d\sigma-1} \sum_{l=0}^p d_{j,l} e_{qj+l} + \varepsilon e_{q\sigma}$$

where

$$\varepsilon = \frac{1}{(w_1 \cdots w_{mq\sigma})^{1/m}}$$

$$d_{j,l} = \frac{y_l}{m\varepsilon^{m-1} w_{l+1} \cdots w_{(m-1)q\sigma+qj+l}}.$$

Let us first prove that, provided that  $\sigma$  is large enough,  $u$  belongs to  $U$ . Let  $r \geq 1$ . Since  $X$  is regular, there exists  $\rho \geq r$  and  $C > 0$  such that

$$\|\varepsilon e_{q\sigma}\|_r \leq \frac{C}{(w_1 \cdots w_{mq\sigma})^{1/m}} \|e_{mq\sigma}\|_\rho^{1/m} \xrightarrow{\sigma \rightarrow +\infty} 0.$$

Furthermore,

$$\left\| \sum_{j=c\sigma}^{d\sigma-1} \sum_{l=0}^p d_{j,l} e_{qj+l} \right\| = \left\| \sum_{j=c\sigma}^{d\sigma-1} \sum_{l=0}^p \frac{y_l w_1 \cdots w_l (w_1 \cdots w_{mq\sigma})^{(m-1)/m}}{m w_1 \cdots w_{(m-1)q\sigma+qj+l}} e_{qj+l} \right\|$$

$$= \left\| \sum_{n \geq cq\sigma} \frac{z_n (w_1 \cdots w_{mq\sigma})^{(m-1)/m}}{w_1 \cdots w_{(m-1)q\sigma+n}} e_n \right\|$$

for some eventually null sequence  $(z_n)$  such that

$$\|z\|_\infty \leq \|y\|_\infty (\max(1, w_1, \dots, w_p))^p / m.$$

Assumption (b) allows us to conclude that  $\sum_{j=c\sigma}^{d\sigma-1} \sum_{l=0}^p d_{j,l} e_{qj+l}$  tends to zero as  $\sigma$  goes to  $+\infty$ .

Observe now that, for  $n < m$ , the support of  $u^n$  is contained in  $[0, nq\sigma]$  so that, for  $s \in E_\sigma$ ,  $B_w^s(u^n) = 0$ . On the other hand,

$$u^m = z + m\varepsilon^{m-1} \sum_{j=c\sigma}^{d\sigma-1} \sum_{l=0}^p d_{j,l} e_{(m-1)q\sigma+qj+l} + \varepsilon^m e_{mq\sigma}$$

with  $\text{supp}(z) \subset [0, (m-2)q\sigma + 2qd\sigma] \cup [0, (m-1)q\sigma + p']$ . It is not difficult to see that, because  $d < (1+c)/2$ ,  $\max(\text{supp}(z)) \leq (m-1)q\sigma + qc\sigma$  for  $\sigma$  large enough. Thus, for any  $s = (m-1)q\sigma + qk \in E_\sigma$ ,

$$B_w^s(u^m) = y + \sum_{j=k+1}^{d\sigma-1} \sum_{l=0}^p \frac{y_l}{w_{l+1} \cdots w_{q(j-k)+l}} e_{q(j-k)+l} + \frac{1}{w_1 \cdots w_{q\sigma-qk}} e_{q\sigma-qk}.$$

We handle the second term of the right hand side of the equality by writing

$$\sum_{j=k+1}^{d\sigma-1} \sum_{l=0}^p \frac{y_l}{w_{l+1} \cdots w_{q(j-k)+l}} e_{q(j-k)+l} = \sum_{n \geq q} \frac{z_n}{w_1 \cdots w_n} e_n$$

for some sequence  $(z_n)$  such that  $\|z\|_\infty \leq \|y\|_\infty (\max(1, w_1, \dots, w_p))^p$ . By our choice of  $q$ , this has  $F$ -norm less than  $\eta$ . Finally,  $\|e_{q\sigma-qk}\|/(w_1 \cdots w_{q\sigma-qk})$  becomes also less than  $\eta$  provided  $\sigma$ , and thus  $q\sigma - qd\sigma$ , becomes large enough.  $\square$

Assumption (a) in Theorem 6.7 is what we need to get an (upper) frequently hypercyclic vector. Assumption (b) is also an assumption around unconditional convergence of the series  $\sum_{n \geq 1} (w_1 \cdots w_n)^{-1} e_n$ . This looks clearer by writing

$$\sum_{n \geq c\sigma} \frac{z_n (w_1 \cdots w_{m\sigma})^{(m-1)/m}}{w_1 \cdots w_{n+(m-1)\sigma}} e_n = \sum_{n \geq c\sigma} \frac{z_n (w_1 \cdots w_{m\sigma})^{(m-1)/m}}{w_{n+1} \cdots w_{n+(m-1)\sigma}} \times \frac{e_n}{w_1 \cdots w_n}.$$

Although it looks quite technical, it is satisfied by three natural examples (where we always endow the  $F$ -algebra with the Cauchy product).

EXAMPLE 6.8. Let  $X = \ell_1$  and  $w_n = \lambda > 1$  for all  $n \in \mathbb{N}$ . Then  $B_w$  supports an upper frequently hypercyclic algebra.

PROOF. The situation is very simple here because, for all  $n \geq 1$ ,

$$\frac{(w_1 \cdots w_{m\sigma})^{(m-1)/m}}{w_{n+1} \cdots w_{n+(m-1)\sigma}} = 1,$$

so that (b) is clearly satisfied.  $\square$

EXAMPLE 6.9. Let  $X = \ell_1$  and  $w_1 \cdots w_n = \exp(n^\alpha)$  for all  $n \in \mathbb{N}$  and some  $\alpha \in (0, 1)$ . Then  $B_w$  supports an upper frequently hypercyclic algebra.

PROOF. That (b) is satisfied follows from the classical asymptotic behavior

$$(6.1) \quad \sum_{n \geq N} \exp(-n^\alpha) \sim_{N \rightarrow +\infty} \frac{1}{\alpha} N^{1-\alpha} \exp(-N^\alpha).$$

Assuming (6.1) is true, we just write

$$\begin{aligned} & \left\| \sum_{n \geq c\sigma} \frac{z_n (w_1 \cdots w_{m\sigma})^{(m-1)/m}}{w_1 \cdots w_{(m-1)\sigma+n}} e_n \right\| \\ &= \exp\left(\frac{m-1}{m} m^\alpha \sigma^\alpha\right) \sum_{n \geq c\sigma} \exp(-((m-1)\sigma + n)^\alpha) \\ &\sim_{+\infty} C \sigma^{1-\alpha} \exp\left(\left(\frac{m-1}{m} m^\alpha - (m-1+c)^\alpha\right) \sigma^\alpha\right). \end{aligned}$$

Assumption (b) is satisfied for  $c$  close enough to 1, since in that case

$$(m-1+c)^\alpha > \frac{(m-1)m^\alpha}{m}.$$

For the sake of completeness, we just mention that (6.1) follows from the formula of integration by parts:

$$\int_N^{+\infty} \exp(-x^\alpha) dx = \frac{1}{\alpha} N^{1-\alpha} \exp(-N^\alpha) + \frac{1-\alpha}{\alpha} \int_N^{+\infty} x^{-\alpha} \exp(-x^\alpha) dx.$$

□

EXAMPLE 6.10. The derivation operator  $D$  supports an upper frequently hypercyclic algebra on  $H(\mathbb{C})$ .

PROOF. It is sufficient to verify (b) replacing  $\|\cdot\|$  by any seminorm  $\|\cdot\|_r$ . Now,

$$\left\| \sum_{n \geq c\sigma} \frac{z_n (w_1 \cdots w_{m\sigma})^{(m-1)/m}}{w_1 \cdots w_{(m-1)\sigma+n}} e_n \right\|_r = \sum_{n \geq c\sigma} \frac{(m\sigma)!^{(m-1)/m}}{((m-1)\sigma+n)!} r^n$$

and the convergence of this series has already been proved in section 5.3. □

Another natural operator that could admit an upper frequently hypercyclic algebra is the backward shift  $B_w$  with  $w_n = (1 + \frac{\lambda}{n})$ ,  $\lambda > 1$ , acting over  $\ell_1$  with the convolution product. Unfortunately, for this weight, the assumptions in Theorem 6.7 are not verified.

QUESTION 6.11. Let  $X = \ell_1$  endowed with the convolution product and  $w_n = (1 + \frac{\lambda}{n})$ ,  $\lambda > 1$ . Does  $B_w$  admit an upper frequently hypercyclic algebra?

We can ask a similar question for convolution operators  $\phi(D)$  on  $H(\mathbb{C})$ ,  $|\phi(0)| < 1$ , which are frequently hypercyclic and admit a hypercyclic algebra.

QUESTION 6.12. Let  $X = H(\mathbb{C})$  and let  $\phi : \mathbb{C} \rightarrow \mathbb{C}$  be a nonconstant entire function with exponential type, not a multiple of an exponential function, with  $|\phi(0)| < 1$ . Does  $\phi(D)$  supports an upper frequently hypercyclic algebra?

#### 6.4. Weighted shifts with a frequently hypercyclic algebra on $\omega$

Despite the result of Falcó and Grosse-Erdmann, it is not so difficult to exhibit operators supporting a frequently hypercyclic algebra if we work on the big space  $\omega$ .

THEOREM 6.13. *Let  $w = (w_n)_{n \geq 1}$  be a weight sequence such that  $(w_1 \cdots w_n)$  either tends to  $+\infty$  or to 0. Then  $B_w$ , acting on  $\omega$  endowed with the coordinatewise product, supports a frequently hypercyclic algebra.*

PROOF. We first assume that  $(w_1 \cdots w_n)$  tends to  $+\infty$  and observe that this clearly implies that, for all  $l \geq 0$ ,  $(w_{l+1} \cdots w_{n+l})$  tends to  $+\infty$ . Let  $(v(p), m(p))$  be a dense sequence in  $\omega \times \mathbb{N}$ , where each  $v(p)$  has finite support contained in  $[0, p]$ . We then write  $v(p) = \sum_{l=0}^p v_l(p) e_l$ . For  $(n, p) \in \mathbb{N}^2$ , we define

$$y(n, p) = \sum_{l=0}^p \frac{v_l(p)^{1/m(p)}}{(w_{l+1} \cdots w_{n+l})^{1/m(p)}} e_{n+l}.$$

By [8, Lemma 6.19] (see the forthcoming Lemma 6.14), there exists a sequence  $(A(p))$  of pairwise disjoint subsets of  $\mathbb{N}$ , with positive lower density, and such that

$|n - n'| \geq p + q + 1$  whenever  $n \neq n'$  and  $(n, n') \in A(p) \times A(q)$ . In particular, the vectors  $y(n, p)$  for  $p \in \mathbb{N}$  and  $n \in A(p)$  have disjoint support. Hence, we may define  $u = \sum_{p \in \mathbb{N}} \sum_{n \in A(p)} y(n, p)$  and we claim that  $u$  generates a frequently hypercyclic algebra.

Indeed, let  $P \in \mathbb{C}[z]$  be non-constant with  $P(0) = 0$ ,  $P(z) = \sum_{m=m_0}^{m_1} \hat{P}(m)z^m$ ,  $\hat{P}(m_0) \neq 0$ , and let  $V$  be a non-empty open subset of  $\omega$ . Let  $p \in \mathbb{N}$ ,  $\varepsilon > 0$  be such that  $m(p) = m_0$  and any vector  $x \in \omega$  satisfying  $|x_l - \hat{P}(m_0)v_l(p)| < \varepsilon$  for all  $l = 0, \dots, p$  belongs to  $V$ . Now, for  $l = 0, \dots, p$  and  $n \in A(p)$ ,

$$(B_w^n P(u))_l = \hat{P}(m_0)v_l(p) + \sum_{m=m_0+1}^{m_1} \frac{\hat{P}(m)v_l(p)^{\frac{m}{m_0}}}{(w_{l+1} \cdots w_{n+l})^{\frac{m}{m_0}-1}}.$$

Since  $(w_{l+1} \cdots w_{n+l})$  tends to  $+\infty$  for all  $l$ ,  $B_w^n P(u)$  belongs to  $V$  for all  $n$  in a cofinite subset of  $A(p)$ . Hence,  $P(u)$  is a frequently hypercyclic vector for  $B_w$ .

The proof is completely similar if we assume that  $(w_1 \cdots w_n)$  tends to 0. The only difference is that the dominant term is now given by the term of highest degree of  $P$ , namely we choose  $p$  such that  $m(p) = m_1$  and we write

$$(B_w^n P(u))_l = \hat{P}(m_1)v_l(p) + \sum_{m=m_0}^{m_1-1} \frac{\hat{P}(m)v_l(p)^{\frac{m}{m_1}}}{(w_{l+1} \cdots w_{n+l})^{\frac{m}{m_1}-1}}.$$

We will conclude because, for  $m < m_1$ ,  $(w_{l+1} \cdots w_{n+l})^{\frac{m}{m_1}-1}$  tends to  $+\infty$ . Details are left to the reader.  $\square$

The unweighted backward shift on  $\omega$  (still endowed with the coordinatewise product) supports a frequently hypercyclic algebra. Indeed, more generally, let  $T$  be a multiplicative operator on an  $F$ -algebra  $X$  with the property that for every non-zero polynomial  $P$  vanishing at the origin, the map

$$\tilde{P} : X \rightarrow X, x \mapsto P(x)$$

has dense range. Then if  $T$  is frequently hypercyclic, it supports a frequently hypercyclic algebra. The reason for that is the simple observation that if  $U$  is a non-empty open set of  $X$  and  $P$  a non-zero polynomial vanishing at the origin,

$$\{n \in \mathbb{N} : T^n(P(x)) \in U\} = \{n \in \mathbb{N} : T^n x \in \tilde{P}^{-1}(U)\}.$$

From the same observation we may conclude that the translation operators  $T_a : C^\infty(\mathbb{R}, \mathbb{C}) \rightarrow C^\infty(\mathbb{R}, \mathbb{C})$  for  $a \in \mathbb{R}, a \neq 0$ , admit a frequently hypercyclic algebra. The fact that  $C^\infty(\mathbb{R}, \mathbb{C})$  has the above mentioned property is proven in [15, Proposition 20].

### 6.5. A sequence of sets with positive lower density which are very far away from each other

The remaining part of this section is devoted to the proof of Theorem 1.8. The starting point to exhibit frequently hypercyclic vectors is the following lemma on the existence of subsets of  $\mathbb{N}$  with positive lower density which are sufficiently separated.

LEMMA 6.14 (Lemma 6.19 in [8]). *Let  $(a(p))$  be any sequence of positive real numbers. Then one can find a sequence  $(A(p))$  of pairwise disjoint subsets of  $\mathbb{N}$  such that*

- (i) each set  $A(p)$  has positive lower density;
- (ii)  $\min A(p) \geq a(p)$  and  $|n - n'| \geq a(p) + a(q)$  whenever  $n \neq n'$  and  $(n, n') \in A(p) \times A(q)$ .

To produce a frequently hypercyclic algebra for a weighted shift on  $c_0$ , we will need a refined version of this lemma where we add new conditions of separation.

**THEOREM 6.15.** *Let  $(a(p))$  be any sequence of positive real numbers. Then one can find a sequence  $(A(p))$  of pairwise disjoint subsets of  $\mathbb{N}$  such that*

- (i) each set  $A(p)$  has positive lower density;
- (ii)  $\min A(p) \geq a(p)$  and  $|n - n'| \geq a(p) + a(q)$  whenever  $n \neq n'$  and  $(n, n') \in A(p) \times A(q)$ .
- (iii) for all  $C > 0$ , there exists  $\kappa > 0$  such that, for all  $(n, n') \in A(p) \times A(q)$  with  $p \neq q$  and  $\max(n, n') \geq \kappa$ , then  $|n - n'| \geq C$ .

The proof of this theorem is rather long. The strategy is to construct a sequence of sets satisfying only (i) and (iii), and then to modify them to add (ii). We begin with two sets.

**LEMMA 6.16.** *Let  $E \subset \mathbb{N}$  be a set with positive lower density. There exist  $A, B \subset E$  disjoint, with positive lower density, and such that, for all  $C > 0$ , there exists  $\kappa > 0$  such that, for all  $n \in A$  and all  $n' \in B$  with  $\max(n, n') \geq \kappa$ , then  $|n - n'| \geq C$ .*

**PROOF.** We write  $E = \{n_j : j \in \mathbb{N}\}$  in an increasing order. We set, for  $k \geq 1$ ,  $u_k = k$ ,  $v_k = \lfloor \sqrt{k} \rfloor$ . We define sequences  $(M_k)$ ,  $(N_k)$ ,  $(P_k)$  and  $(Q_k)$  by setting  $M_1 = 1$  and, for  $k \geq 1$ ,

$$N_k = M_k + u_k, \quad P_k = N_k + v_k, \quad Q_k = P_k + u_k, \quad M_{k+1} = Q_k + v_k.$$

We then define

$$I = \bigcup_k [M_k, N_k), \quad J = \bigcup_k [P_k, Q_k),$$

$$A = \{n_j : j \in I\}, \quad B = \{n_j : j \in J\}.$$

The sets  $I$  and  $J$  have positive lower density. Indeed, for  $N \in \mathbb{N}$ , let  $k$  be such that  $N \in [M_k, M_{k+1})$ . Then

$$\frac{\text{card}(I \cap [1, \dots, N])}{N} \geq \frac{u_1 + \dots + u_{k-1}}{2(u_1 + \dots + u_k + v_1 + \dots + v_k)} \geq \frac{1}{4}$$

provided  $k$  is large enough. The same is true for  $J$ . Since  $E$  has positive lower density, this yields that  $A$  and  $B$  have positive lower density. Moreover, let  $C > 0$ . There exists  $k \geq 0$  such that  $v_{k-1} \geq C$ . We set  $\kappa = n_{M_k}$ . Let  $(n, n') \in A \times B$  with  $\max(n, n') \geq \kappa$ . Assume for instance that  $n \geq n_{M_k}$  and write  $n = n_j$ ,  $n' = n_{j'}$ . Then  $j \geq M_k$  and the construction of the sets  $I$  and  $J$  ensure that  $j'$  does not belong to  $[j - v_{k-1}, j + v_{k-1}]$ . Thus,  $|n - n'| \geq |j - j'| \geq C$ .  $\square$

It is not difficult to require that (ii) in Theorem 6.15 holds when we restrict ourselves to  $p = q$ .

**LEMMA 6.17.** *Let  $A \subset \mathbb{N}$  with positive lower density and  $a > 0$ . There exists  $B \subset A$  with positive lower density,  $\min(B) \geq a$  and  $|n - n'| \geq a$  for all  $n, n' \in B$ ,  $n \neq n'$ .*

PROOF. Write  $A = \{n_j : j \in \mathbb{N}\}$  in an increasing order and define  $B = \{n_{ka} : k \in \mathbb{N}\}$ .  $\square$

We then go inductively from two sets to a sequence of sets.

LEMMA 6.18. *There exists a sequence  $(A(p))$  of pairwise disjoint subsets of  $\mathbb{N}$  such that*

- (i) *each set  $A(p)$  has positive lower density;*
- (ii) *for all  $C > 0$ , there exists  $\kappa > 0$  such that, for all  $(n, n') \in A(p) \times A(q)$  with  $p \neq q$  and  $\max(n, n') \geq \kappa$ , then  $|n - n'| \geq C$ .*

PROOF. We shall construct by induction two sequences of sets  $(A(p))$  and  $(B(p))$  and a sequence of integers  $(\kappa_k)$  such that, at each step  $r$ ,

- (a) for all  $1 \leq p \leq r$ ,  $A(p)$  and  $B(p)$  are disjoint and have positive lower density.
- (b) for all  $1 \leq p < q \leq r$ ,  $A(q) \subset B(p)$  and  $B(q) \subset B(p)$ .
- (c) for all  $C > 0$ , there exists  $\kappa > 0$  such that, for all  $1 \leq p \leq q \leq r$ , for all  $n \in A(p)$  and  $n' \in B(q)$ ,  $\max(n, n') \geq \kappa \implies |n - n'| \geq C$ .
- (d) for all  $k \in \{1, \dots, r\}$ , for all  $1 \leq p \leq q \leq r$ , for all  $n \in A(p)$  and  $n' \in B(q)$ ,  $\max(n, n') \geq \kappa_k \implies |n' - n| \geq k$ .

It is straightforward to check that the resulting sequence  $(A(p))$  satisfies the conclusions of Lemma 6.18. Observe nevertheless that it is condition (d) together with the inclusion  $A(q) \subset B(p)$  for  $q > p$  which gives (ii) in this lemma (which is uniform with respect to  $p$  and  $q$ ). Condition (c) is only helpful for the induction hypothesis.

We initialize the construction by applying Lemma 6.16 to  $E = \mathbb{N}$ . We set  $A(1) = A$  and  $B(1) = B$  which satisfy (a), (b) and (c). In particular, applying (c) for  $C = 1$  we find some  $\kappa$  that we call  $\kappa_1$ .

Assume now that the construction has been done until step  $r$  and let us perform it for step  $r + 1$ . Let  $E$  be a subset of  $B(r)$  with positive lower density and  $|n - n'| \geq r + 1$  provided  $n \neq n'$  are in  $E$ . We apply Lemma 6.16 to this set  $E$  and we set  $A(r + 1) = A$  and  $B(r + 1) = B$ , so that (a) and (b) are clearly satisfied. Upon taking a maximum, (c) is also easily satisfied: indeed, the only case which is not settled by the induction hypothesis is  $p = q = r + 1$  (when  $p < r + 1$  and  $q = r + 1$ , use  $B(q) \subset B(r)$ ); this case is solved by the construction of  $A(r + 1)$  and  $B(r + 1)$ .

The proof of (d) is slightly more delicate. For  $k = 1, \dots, r$ , we have to verify that for  $1 \leq p \leq r + 1$ ,  $n \in A(p)$  and  $n' \in B(r + 1)$ ,  $\max(n, n') \geq \kappa_k \implies |n - n'| \geq k$ . When  $p \leq r$ , again this follows from  $B(r + 1) \subset B(r)$ . For  $p = r + 1$ , this follows from  $A(r + 1), B(r + 1) \subset E$  and the fact that distinct elements of  $E$  have distance greater than or equal to  $r + 1$ . Finally, applying (c) for  $C = k + 1$ , we define  $\kappa_{k+1}$ .  $\square$

We need now to ensure property (ii) in Theorem 6.15. This will be done again inductively, the main step being the following lemma.

LEMMA 6.19. *Let  $(A(p))$  be a sequence of pairwise disjoint subsets of  $\mathbb{N}$  with positive lower density and let  $(a(p))$  be a sequence of positive real numbers. There exists a sequence  $(B(p))$  of subsets of  $\mathbb{N}$  with positive lower density such that each  $B(p)$  is contained in  $A(p)$  and, for all  $n \in B(1)$ , for all  $n' \in B(p)$ ,  $p \geq 2$ ,  $|n - n'| \geq a(1) + a(p)$ .*

PROOF. Since  $A(1)$  has positive lower density, there exists  $N \in \mathbb{N}$  and  $\delta > 0$  such that, for all  $n \geq N$ ,

$$\frac{\text{card}(A(1) \cap [0, \dots, n])}{n+1} \geq \delta.$$

For  $p \geq 2$ , let  $B(p)$  be a subset of  $A(p)$  such that, for all  $n \in \mathbb{N}$ ,

$$\frac{\text{card}((B(p) + [-a(1) - a(p), a(1) + a(p)]) \cap [0, \dots, n])}{n+1} < \frac{\delta}{2^p}$$

but  $B(p)$  still has positive lower density. This is possible if, writing  $A(p) = \{n_j; j \in \mathbb{N}\}$ , we set  $B(p) = \{n_{ka}; k \geq 1\}$  for some sufficiently large  $a$ . We then define  $B(1) = A(1) \setminus \bigcup_{p \geq 2} (B(p) + [-a(1) - a(p), a(1) + a(p)])$ . Then, for all  $n \in B(1)$  and all  $n' \in B(p)$ ,  $p \geq 2$ , one clearly has  $|n - n'| \geq a(1) + a(p)$  whereas, for all  $n \geq N$ ,

$$\frac{\text{card}(B(1) \cap [0, \dots, n])}{n+1} \geq \delta - \sum_{p \geq 2} \frac{\delta}{2^p} \geq \frac{\delta}{2}$$

so that  $B(1)$  still has positive lower density.  $\square$

PROOF OF THEOREM 6.15. Applying Lemmas 6.18 and 6.17, we start from a sequence  $(A(p))$  of pairwise disjoint subsets of  $\mathbb{N}$ , satisfying properties (i) and (iii) of Theorem 6.15 and property (ii) when  $p = q$ . We construct by induction on  $r$  sets  $B(1), \dots, B(r)$ ,  $A_r(k)$  for  $k \geq r+1$  such that

- $B(k) \subset A(k)$  for all  $k \leq r$ ,  $A_r(k) \subset A(k)$  for all  $k \geq r+1$ ;
- $B(k)$  and  $A_r(k)$  have positive lower density;
- for all  $p \in \{1, \dots, r\}$ , for all  $q \geq p+1$ , for all  $n \in A(p)$ , for all  $n' \in B(q)$  if  $q \leq r$ , for all  $n' \in A_r(q)$  if  $q \geq r+1$ ,  $|n - n'| \geq a(p) + a(q)$ .

The sequence  $(B(p))$  that we get at the end will answer the problem. Now the construction is easily done by successive applications of Lemma 6.19 first with the sequence  $(A(p))_{p \geq 1}$ , then with the sequence  $(A_1(p))_{p \geq 2}$ , and so on.  $\square$

### 6.6. A weighted shift with a frequently hypercyclic algebra on $c_0$

Let us now define a weight  $(w_n)$  such that  $B_w$ , acting on  $c_0$  endowed with the coordinatewise product, supports a frequently hypercyclic algebra. We start with the sequence  $(A(p))_{p \geq 1}$  given by Theorem 6.15 for  $a_p = p$ . We then construct inductively a sequence of integers  $(M_k)$  such that, for all  $(n, n') \in A(p) \times A(q)$  with  $p \neq q$ ,  $\max(n, n') \geq M_{k+1} \implies |n - n'| \geq M_k$ . This follows directly from property (iii) of Theorem 6.15, applied successively with  $C = M_1 = 1$  to get  $M_2$ ,  $C = M_2$  to get  $M_3$ , and so on. We may also assume that the sequence  $(M_{k+1} - M_k)$  is non-decreasing.

We define the weight  $(w_n)_{n \geq 1}$  by the following inductive formulas:

- $w_n = 2$  for all  $n \leq M_2$ ;
- for all  $k \geq 2$ , for all  $n \in \{M_k + 1, \dots, M_{k+1}\}$ ,

$$w_n = (w_1 \cdots w_{M_k})^{\frac{1}{k(M_{k+1} - M_k)}}$$

so that, and this is the crucial point,

$$w_{M_{k+1}} \cdots w_{M_{k+1}} = (w_1 \cdots w_{M_k})^{\frac{1}{k}}.$$

Let us summarize the properties of the weight which will be useful later.

LEMMA 6.20. *The weight  $(w_n)$  satisfies the following properties:*

- for all  $n \geq 1$ ,  $w_n \geq 1$ ;
- $(w_n)$  is non-increasing;
- $(w_1 \cdots w_n)$  tends to  $+\infty$ ;
- for all  $\alpha > 0$ , for all  $l \geq 0$ ,  $\frac{w_{M_{k-1}+l+1} \cdots w_{M_{k+1}+l}}{(w_{l+1} \cdots w_{M_{k+1}+l})^\alpha} \xrightarrow{k \rightarrow +\infty} 0$ .

PROOF. The first property is clear. For the second one, it suffices to prove that if  $n \in \{M_k + 1, \dots, M_{k+1}\}$  and  $n' \in \{M_{k+1} + 1, \dots, M_{k+2}\}$  for some  $k$ , then  $w_{n'} \leq w_n$ . We now write

$$\begin{aligned} w_{n'} &= (w_1 \cdots w_{M_{k+1}})^{\frac{1}{(k+1)(M_{k+2}-M_{k+1})}} \\ &= (w_1 \cdots w_{M_k})^{\frac{1}{(k+1)(M_{k+2}-M_{k+1})}} (w_{M_{k+1}} \cdots w_{M_{k+1}})^{\frac{1}{(k+1)(M_{k+2}-M_{k+1})}} \\ &= (w_1 \cdots w_{M_k})^{\frac{1}{k(M_{k+2}-M_{k+1})}} \\ &\leq w_n. \end{aligned}$$

To prove that  $(w_1 \cdots w_n)$  tends to  $+\infty$ , we just observe that, for all  $k \geq 2$ ,

$$w_1 \cdots w_{M_k} = (w_1 \cdots w_{M_2}) \prod_{j=2}^{k-1} \left(1 + \frac{1}{j}\right),$$

and this goes to  $+\infty$  as  $k$  tends to  $+\infty$ . Finally, since  $(w_n)$  is bounded and bounded below, we need only to prove the last property for  $l = 0$ . Now we write

$$\begin{aligned} w_{M_{k-1}+1} \cdots w_{M_{k+1}} &= w_{M_{k-1}+1} \cdots w_{M_k} (w_1 \cdots w_{M_k})^{1/k} \\ &= (w_{M_{k-1}+1} \cdots w_{M_k})^{1+\frac{1}{k}} (w_1 \cdots w_{M_{k-1}})^{\frac{1}{k}} \\ &= (w_1 \cdots w_{M_{k-1}})^{\frac{1}{k-1} \left(1+\frac{1}{k}\right) + \frac{1}{k}} \\ &= (w_1 \cdots w_{M_{k-1}})^{\frac{2}{k-1}} \end{aligned}$$

so that

$$\frac{w_{M_{k-1}+1} \cdots w_{M_{k+1}}}{(w_1 \cdots w_{M_{k+1}})^\alpha} \leq \frac{1}{(w_1 \cdots w_{M_{k-1}})^{\alpha - \frac{2}{k-1}}}$$

which indeed tends to zero.  $\square$

We now prove that the operator  $B_w$  acting on  $c_0$  endowed with the coordinatewise product supports a frequently hypercyclic algebra. Let  $(v(p), m(p))$  be a sequence dense in  $c_0 \times \mathbb{N}$  such that each  $v(p)$  has finite support contained in  $[0, p]$ . We shall need a last technical lemma involving all the objects we constructed until now.

LEMMA 6.21. *There exists a sequence of integers  $(N(r))_{r \geq 1}$  satisfying the following properties:*

- (i) for all  $r \geq 1$ ,

$$\sup_{n \geq N(r), l=0, \dots, r} \left| \frac{v_l(r)}{(w_{l+1} \cdots w_{n+l})^{\frac{1}{m(r)+1}}} \right|^{\frac{1}{m(r)}} < \frac{1}{r}.$$

(ii) for all  $r \geq 2$ , for all  $s \in \{1, \dots, r-1\}$ , for all  $(j, j') \in A(r) \times A(s)$  with  $j \geq N(r)$ , for all  $l \in \{0, \dots, r\}$ , for all  $\alpha \geq \min\left(\frac{1}{m(r)}, \frac{1}{m(s)}\right)$ ,

$$\begin{aligned} j > j' &\implies \left| \frac{w_{l+(j-j')+1} \cdots w_{j+l} v_l(r)^\alpha}{(w_{l+1} \cdots w_{j+l})^\alpha} \right| < \frac{1}{r} \\ j' > j &\implies \left| \frac{w_{l+(j'-j)+1} \cdots w_{j'+l} v_l(s)^\alpha}{(w_{l+1} \cdots w_{j'+l})^\alpha} \right| < \frac{1}{r}. \end{aligned}$$

PROOF. Let  $r \geq 1$  be fixed. We first observe that it is easy to ensure (i), just by assuming that  $N(r)$  is large enough. Let us choose  $N(r)$  to ensure (ii). Upon taking a supremum, we may fix  $s$  and  $l$  and to simplify the notations, we will assume  $l = 0$ . Let  $\alpha_0 = \min\left(\frac{1}{m(r)}, \frac{1}{m(s)}\right)$  and  $C = \max(1, |v_0(r)|, |v_0(s)|)$ . We define three integers  $N_0$ ,  $k_0$  and  $k_1$  satisfying the following three conditions:

$$\begin{aligned} n \geq N_0 &\implies \frac{C^2}{w_1 \cdots w_n} < \frac{1}{r} \\ n \geq M_{k_0} &\implies \frac{C^2}{w_1 \cdots w_n} < \frac{1}{r} \\ k \geq k_1 &\implies \frac{w_{M_{k-1}+1} \cdots w_{M_{k+1}} C}{(w_1 \cdots w_{M_{k+1}})^{\alpha_0}} < \frac{1}{r}. \end{aligned}$$

We set  $N(r) = \max(N_0, M_{k_0+1}, M_{k_1})$ . Let  $(j, j') \in A(r) \times A(s)$  with  $j \geq N(r)$ . To fix the ideas, we assume that  $j > j'$ . If  $\alpha \geq 2$ , then

$$\left| \frac{w_{(j-j')+1} \cdots w_j v_0(r)^\alpha}{(w_1 \cdots w_j)^\alpha} \right| \leq \frac{w_{(j-j')+1} \cdots w_j}{w_1 \cdots w_j} \times \left| \frac{v_0(r)^2}{w_1 \cdots w_j} \right|^{\alpha/2} < 1 \times \left(\frac{1}{r}\right)^{\frac{\alpha}{2}} \leq \frac{1}{r}.$$

If  $\alpha \in [1, 2]$ , then

$$\left| \frac{w_{(j-j')+1} \cdots w_j v_0(r)^\alpha}{(w_1 \cdots w_j)^\alpha} \right| \leq \frac{C^2}{w_1 \cdots w_{j-j'}}.$$

Since  $j \geq N(r)$ ,  $j - j' \geq M_{k_0}$  so that the last term is less than  $1/r$ . For  $\alpha < 1$ , since  $j \geq N(r) \geq M_{k_1}$ , there exists a single integer  $k \geq k_1$  such that  $j \in [M_k, M_{k+1})$ . Then  $j - j' \geq M_{k-1}$  and

$$\begin{aligned} \left| \frac{w_{(j-j')+1} \cdots w_j v_0(r)^\alpha}{(w_1 \cdots w_j)^\alpha} \right| &\leq \frac{w_{M_{k-1}+1} \cdots w_j C}{(w_1 \cdots w_j)^{\alpha_0}} \\ &\leq \frac{w_{M_{k-1}+1} \cdots w_{M_{k+1}} C}{(w_1 \cdots w_{M_{k+1}})^{\alpha_0}} \\ &< \frac{1}{r}. \end{aligned}$$

□

PROOF OF THEOREM 1.8. We are now ready for the proof that  $B_w$  supports a frequently hypercyclic algebra. By Lemma 6.17, for each  $p \geq 1$ , let  $B(p)$  be a subset of  $A(p)$  with positive lower density such that  $\min(B(p)) \geq N(p)$  and  $|n - n'| \geq N(p)$  for all  $n \neq n' \in B(p)$ . We set

$$u(p) = \sum_{n \in B(p)} \sum_{l=0}^p \frac{1}{(w_{l+1} \cdots w_{n+l})^{1/m(p)}} v_l(p)^{1/m(p)} e_{n+l}.$$

Since  $(w_1 \cdots w_n)$  tends to  $+\infty$ ,  $u(p)$  belongs to  $c_0$ . Moreover, the choice of  $N(p)$  (here, (i) of Lemma 6.21) ensures that  $\|u(p)\| < 1/p$ . We also observe that the  $u(p)$  have pairwise disjoint support. Hence we may define  $u = \sum_{p \geq 1} u(p)$  which still belongs to  $c_0$ . We claim that the following property is true: for all  $p \geq 1$ , for all  $q \neq p$ , for all  $n \in B(p)$ ,

$$(6.2) \quad \left\| B_w^n u(p)^{m(p)} - v(p) \right\| < \frac{1}{p},$$

$$(6.3) \quad \forall m > m(p), \left\| B_w^n u(p)^m \right\| < \frac{1}{p},$$

$$(6.4) \quad \forall m \geq m(p), \left\| B_w^n u(q)^m \right\| < \frac{1}{p}.$$

Assume that these properties have been proved. Let  $P$  be a non-constant polynomial with  $P(0) = 0$  and write it  $P(z) = \sum_{m=m_0}^{m_1} \hat{P}(m)z^m$  with  $\hat{P}(m_0) \neq 0$ . We aim to prove that  $P(u)$  is a frequently hypercyclic vector for  $B_w$ . Without loss of generality, we can assume  $\hat{P}(m_0) = 1$ . Let  $V$  be a non-empty open subset of  $c_0$ . There exists  $p \geq 1$  such that  $B\left(v(p), \left(2 + \sum_{m=m_0+1}^{m_1} |\hat{P}(m)|\right)/p\right) \subset V$  and  $m(p) = m_0$ . Then, for all  $n \in B(p)$ ,

$$\begin{aligned} \left\| B_w^n P(u) - v(p) \right\| &\leq \left\| B_w^n u(p)^{m(p)} - v(p) \right\| + \left\| \sum_{q \neq p} B_w^n u(q)^{m(p)} \right\| \\ &\quad + \sum_{m=m_0+1}^{m_1} |\hat{P}(m)| \left\| \sum_{q \geq 1} B_w^n u(q)^m \right\| \\ &\leq \frac{2 + \sum_{m=m_0+1}^{m_1} |\hat{P}(m)|}{p}, \end{aligned}$$

where the last inequality follows from (6.2), (6.3), (6.4) and the fact that the  $B_w^n u(q)$  have pairwise disjoint support. Therefore, for all  $n$  in a set of positive lower density,  $B_w^n P(u)$  belongs to  $V$ , showing that  $P(u)$  is a frequently hypercyclic vector for  $B_w$ . Hence, it remains to prove (6.2), (6.3) and (6.4). We first observe that

$$B_w^n u(p)^{m(p)} - v(p) = \sum_{\substack{n' \in B(p) \\ n' \geq n}} \sum_{l=0}^p \frac{v_l(p)}{w_{l+1} \cdots w_{(n'-n)+l}} e_{(n'-n)+l}.$$

Since  $n' - n > N(p)$  for all  $n' > n$ ,  $n' \in B(p)$ , (6.2) follows from (i) in Lemma 6.21. Next, for  $m > m(p)$ , we may write  $B_w^n u(p)^m$  as

$$\sum_{\substack{n' \in B(p) \\ n' \geq n}} \sum_{l=0}^p \frac{v_l(p)^{\frac{m}{m(p)}}}{(w_{l+1} \cdots w_{(n'-n)+l})^{\frac{m}{m(p)}} (w_{(n'-n)+l+1} \cdots w_{n'+l})^{\frac{m}{m(p)} - 1}} e_{(n'-n)+l}.$$

There is an additional difficulty since now we may have  $n' = n$ . We overcome this difficulty by writing

$$\begin{aligned}
& \left| \frac{v_l(p)^{\frac{m}{m(p)}}}{(w_{l+1} \cdots w_{(n'-n)+l})^{\frac{m}{m(p)}} (w_{(n'-n)+l+1} \cdots w_{n'+l})^{\frac{m}{m(p)}-1}} \right| \\
& \leq \left| \frac{v_l(p)^{\frac{m}{m(p)}}}{(w_{l+1} \cdots w_{n'+l})^{\frac{m}{m(p)}-1}} \right| \\
& \leq \left| \frac{v_l(p)^{\frac{1}{m(p)}}}{(w_{l+1} \cdots w_{n'+l})^{\frac{1}{m(p)}-\frac{1}{m}}} \right|^m \\
& \leq \left| \frac{v_l(p)^{\frac{1}{m(p)}}}{(w_{l+1} \cdots w_{n'+l})^{\frac{1}{m(p)(m(p)+1)}}} \right|^m \\
& < \frac{1}{p}.
\end{aligned}$$

Finally, for  $m \geq m(p)$  and  $q \neq p$ , we write

$$B_w^n u(q)^m = \sum_{\substack{n' \in B(q) \\ n' > n}} \sum_{l=0}^q \frac{w_{l+(n'-n)+1} \cdots w_{n'+l}}{(w_{l+1} \cdots w_{n'+l})^{\frac{m}{m(q)}}} v_l(q)^{\frac{m}{m(q)}} e_{(n'-n)+l}.$$

For  $q > p$ , we apply (ii) of Lemma 6.21 with  $r = q$ ,  $s = p$ ,  $j = n'$ ,  $j' = n$  and  $\alpha = m/m(q)$ . For  $q < p$ , we apply (ii) of Lemma 6.21 with  $r = p$ ,  $s = q$ ,  $j = n$ ,  $j' = n'$  and  $\alpha = m/m(q)$ . In both cases, we immediately find that all the coefficients of  $B_w^n u(q)^m$  are smaller than  $1/p$ , yielding

$$\|B_w^n u(q)^m\| < \frac{1}{p}.$$

This closes the proof of Theorem 1.8.  $\square$

This technical construction leads to an example over the not so difficult space  $c_0$ , but the following question remains open.

**QUESTION 6.22.** Does there exist a weighted shift on  $\ell_p$  endowed with the pointwise product admitting a frequently hypercyclic algebra?

Of course, it would also be nice to get simpler examples! On the other hand, for sequence spaces endowed with the convolution product, we have neither positive nor negative examples. For instance, it would be very interesting to solve the following questions.

**QUESTION 6.23.** Does  $B$  on  $\omega$  endowed with the convolution product support a frequently hypercyclic algebra?

**QUESTION 6.24.** Does  $2B$  on  $\ell_1$  endowed with the convolution product support a frequently hypercyclic algebra?



## Disjoint hypercyclic algebras

### 7.1. How to get a disjoint hypercyclic algebra

We follow the strategy which has been proved to be powerful in our previous problems, by proving a transitivity criterion to show that two operators admit a disjoint hypercyclic algebra. We may observe the similarity with Theorem 2.1 and with [12, Proposition 2.3].

**PROPOSITION 7.1.** *Let  $T_1$  and  $T_2$  be two operators acting on the same  $F$ -algebra  $X$ . Assume that for each  $I \in \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ , there exists  $m_0 \in I$  such that, for all  $U, V_1$  and  $V_2$  nonempty open subsets of  $X$ , for all  $W$  neighbourhood of zero, there exist  $u \in U$  and  $N \in \mathbb{N}$  such that*

$$\begin{cases} T_1^N(u^n) \in W, T_2^N(u^n) \in W \text{ for all } n \in I, n \neq m_0 \\ T_1^N(u^{m_0}) \in V_1, T_2^N(u^{m_0}) \in V_2. \end{cases}$$

*Then  $T_1$  and  $T_2$  admit a disjoint hypercyclic algebra.*

**PROOF.** For  $V, V' \subset X$  open and nonempty, for  $W$  a neighbourhood of zero, for  $I \subset \mathcal{P}_f(\mathbb{N}) \setminus \{\emptyset\}$ , let us define

$$\mathcal{A}(I, V, V', W) = \left\{ u \in X : \begin{aligned} \exists N \in \mathbb{N}, \quad & T_1^N(u^n) \in W \text{ for all } n \in I \setminus \{m_0(I)\} \\ & T_2^N(u^n) \in W \text{ for all } n \in I \setminus \{m_0(I)\} \\ & T_1^N(u^{m_0(I)}) \in V, T_2^N(u^{m_0(I)}) \in V' \end{aligned} \right\}$$

Fixing  $(V_k)$  a basis of open subsets of  $X$  and  $(W_l)$  a basis of neighbourhoods of 0, we argue exactly like in the previous sections to prove that any vector in  $\bigcap_{I,k,k',l} \mathcal{A}(I, V_k, V_{k'}, W)$  generates a disjoint hypercyclic algebra for  $T_1$  and  $T_2$ .  $\square$

As before, for the Cauchy product we will apply this proposition for  $m_0 = \max I$  and for the coordinatewise product we will take  $m_0 = \min I$ . The corresponding corollaries are the following.

**COROLLARY 7.2.** *Let  $T_1$  and  $T_2$  be two operators acting on the same  $F$ -algebra  $X$ . Assume that for all  $m \geq 1$ , for all  $U, V_1$  and  $V_2$  nonempty open subsets of  $X$ , for all  $W$  neighbourhood of zero, there exist  $u \in U$  and  $N \in \mathbb{N}$  such that*

$$\begin{cases} T_1^N(u^n) \in W, T_2^N(u^n) \in W \text{ for all } n < m \\ T_1^N(u^m) \in V_1, T_2^N(u^m) \in V_2. \end{cases}$$

*Then  $T_1$  and  $T_2$  admit a disjoint hypercyclic algebra.*

**COROLLARY 7.3.** *Let  $T_1$  and  $T_2$  be two operators acting on the same  $F$ -algebra  $X$ . Assume that for all positive integers  $m_0 < m_1$ , for all  $U, V_1$  and  $V_2$  nonempty*

open subsets of  $X$ , for all  $W$  neighbourhood of zero, there exist  $u \in U$  and  $N \in \mathbb{N}$  such that

$$\begin{cases} T_1^N(u^n) \in W, T_2^N(u^n) \in W \text{ for all } n \in \{m_0 + 1, \dots, m_1\} \\ T_1^N(u^{m_0}) \in V_1, T_2^N(u^{m_0}) \in V_2. \end{cases}$$

Then  $T_1$  and  $T_2$  admit a disjoint hypercyclic algebra.

## 7.2. Disjoint hypercyclic algebras for backward shifts - coordinatewise product

We now show how these general results can be applied to get disjoint hypercyclic algebras for backward shifts operators. We first begin with Fréchet sequence algebras endowed with the coordinatewise product. We get complete characterizations either for the case where the shifts are raised to different powers as for the case where the powers are the same.

### 7.2.1. Backward shifts raised to different powers.

**THEOREM 7.4.** *Let  $X$  be a Fréchet sequence algebra for the coordinatewise product with a continuous norm and such that  $(e_n)$  is a Schauder basis. Let  $B_1 = B_{w^{(1)}}$  and  $B_2 = B_{w^{(2)}}$  be two bounded weighted shifts on  $X$ . The following assertions are equivalent.*

- (i)  $B_1$  and  $B_2^2$  support a disjoint hypercyclic algebra.
- (ii) For all  $p \in \mathbb{N}$ , for all  $m \geq 1$ , there exists a sequence of integers  $(n_k)$  such that, for all  $l = 0, \dots, p$ ,

$$\begin{aligned} (w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1/m} e_{n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \\ (w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{-1/m} e_{2n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \\ \frac{w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)}}{w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)}} e_{n_k+l} &\xrightarrow{k \rightarrow +\infty} 0. \end{aligned}$$

- (iii) There exists a sequence  $(n_k)$  such that, for all  $l \geq 0$ , for all  $\gamma > 0$ ,

$$\begin{aligned} (w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-\gamma} e_{n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \\ (w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{-\gamma} e_{2n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \\ \frac{w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)}}{w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)}} e_{n_k+l} &\xrightarrow{k \rightarrow +\infty} 0. \end{aligned}$$

**PROOF.** (i)  $\implies$  (ii). Let  $m \geq 1$  and let  $x \in X$ ,  $x = \sum_{n=0}^{+\infty} x_n e_n$ , be such that  $x^m$  is a disjointly hypercyclic vector for  $B_1$  and  $B_2^2$ . Let  $(n_k)$  be a sequence of integers such that  $(B_j^{j n_k}(x^m))_k$  goes to  $e_0 + \cdots + e_p$ ,  $j = 1, 2$ . Arguing as in Theorem 4.2 we get, for all  $l = 0, \dots, p$ ,

$$(7.1) \quad \begin{aligned} (w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1/m} e_{n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \\ (w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{-1/m} e_{2n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \\ w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)} x_{2n_k+l}^m &\xrightarrow{k \rightarrow +\infty} 1. \end{aligned}$$

Moreover, write

$$B_1^{n_k}(x^m) = z + \sum_{l=0}^p w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)} x_{2n_k+l}^m e_{n_k+l} + z',$$

with  $\text{supp}(z) \subset [0, n_k)$  and  $\text{supp}(z') \subset (n_k + p, +\infty)$ . Since convergence in  $X$  implies pointwise convergence, we get

$$w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)} x_{2n_k+l}^m e_{n_k+l} \xrightarrow{k \rightarrow +\infty} 0 \text{ for all } l = 0, \dots, p$$

which in turn, in view of (7.1), gives

$$\frac{w_{n_k+l+1}^{(1)} \cdots w_{n_k+l}^{(1)}}{w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)}} e_{n_k+l} \xrightarrow{k \rightarrow +\infty} 0 \text{ for all } l = 0, \dots, p.$$

(ii)  $\implies$  (iii). This follows from a diagonal argument, like in the proof of Theorem 4.2.

(iii)  $\implies$  (i). We intend to apply Corollary 7.3. Let  $m_0 < m_1$  be two positive integers, let  $U, V_1, V_2$  be nonempty open subsets of  $X$  and let  $W$  be a neighbourhood of 0. Let  $x \in U$  with finite support, let  $y_i = \sum_{l=0}^p y_{i,l} e_l$  belonging to  $V_i$ ,  $i = 1, 2$ . We set

$$u = x + \sum_{l=0}^p \frac{y_{1,l}^{1/m_0}}{(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{1/m_0}} e_{n_k+l} + \sum_{l=0}^p \frac{y_{2,l}^{1/m_0}}{(w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{1/m_0}} e_{2n_k+l},$$

which belongs to  $U$  for all large values of  $k$ . We recall that, since the sequence  $(e_n)$  is bounded below, for all  $l \geq 0$ , the sequences  $(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})$  and  $(w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})$  tend to  $+\infty$ . Let  $n \in \{m_0, \dots, m_1\}$  and let us compute  $B_2^{2n_k}(u^n)$ :

$$B_2^{2n_k}(u^n) = \sum_{l=0}^p \frac{y_{2,l}^{n/m_0}}{(w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{\frac{n}{m_0}-1}} e_l.$$

The assumptions tell us that  $B_2^{2n_k}(u^{m_0}) = y_2$  while  $B_2^{2n_k}(u^n)$  tends to 0 if  $n > m_0$ . On the other hand,

$$B_1^{n_k}(u^n) = \sum_{l=0}^p \frac{y_{1,l}^{n/m_0}}{(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{\frac{n}{m_0}-1}} e_l + \sum_{l=0}^p y_{2,l}^{n/m_0} \frac{w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)}}{(w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{\frac{n}{m_0}}} e_{n_k+l}.$$

Writing

$$\frac{w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)}}{(w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{\frac{n}{m_0}}} e_{n_k+l} = \frac{1}{(w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{\frac{n}{m_0}-1}} \times \frac{w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)}}{w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)}} e_{n_k+l},$$

we get that the second (finite) sum tends to zero for all  $n \geq m_0$ , whereas the first sum can be handled exactly as for  $B_2$ .  $\square$

**REMARK 7.5.** If we are just interested in disjointly hypercyclic operators, our proof can be easily modified to give the following extension of [12]: let  $X$  be a Fréchet sequence space such that  $(e_n)$  is a Schauder basis for  $X$ . Let  $B_1 = B_{w^{(1)}}$  and

$B_2 = B_{w^{(2)}}$  be two bounded weighted shifts on  $X$ . Then  $B_1$  and  $B_2^2$  are disjointly hypercyclic if and only if there exists a sequence  $(n_k)$  such that, for all  $l \geq 0$ ,

$$\begin{aligned} (w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1} e_{n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \\ (w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})^{-1} e_{2n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \\ \frac{w_{n_k+l+1}^{(1)} \cdots w_{n_k+l}^{(1)}}{w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)}} e_{n_k+l} &\xrightarrow{k \rightarrow +\infty} 0. \end{aligned}$$

We point out that we do not need that  $(e_n)$  is bounded below. This last assumption was only useful when  $n > m_0$ , where we used it to deduce that  $(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})$  and  $(w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)})$  tend to  $+\infty$ . For the existence of a disjointly hypercyclic vector, we always have  $n = m_0 = 1$ .

REMARK 7.6. In particular, if the sequence  $(e_n)$  is bounded, the conditions of Theorem 7.4 are equivalent to say that there exists a sequence  $(n_k)$  such that, for all  $l = 0, \dots, p$ ,

$$\begin{aligned} w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)} &\xrightarrow{k \rightarrow +\infty} +\infty, \\ w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)} &\xrightarrow{k \rightarrow +\infty} +\infty, \\ \frac{w_{n_k+l+1}^{(1)} \cdots w_{2n_k+l}^{(1)}}{w_{l+1}^{(2)} \cdots w_{2n_k+l}^{(2)}} e_{n_k+l} &\xrightarrow{k \rightarrow +\infty} 0. \end{aligned}$$

In turn, this amounts to saying that  $B_1$  and  $B_2^2$  are disjointly hypercyclic. Hence, all the examples given in [12] support a disjoint hypercyclic algebra.

It is plain that this result can be extended for a finite list  $B_{w^{(1)}}^{n_1}, \dots, B_{w^{(r)}}^{n_r}$  raised to different powers  $n_1, \dots, n_r$ . The statement of precise conditions are left to the reader.

### 7.2.2. Backward shifts raised to the same power.

THEOREM 7.7. *Let  $X$  be a Fréchet sequence algebra for the coordinatewise product with a continuous norm and such that  $(e_n)$  is a Schauder basis. Let  $B_1 = B_{w^{(1)}}$  and  $B_2 = B_{w^{(2)}}$  be two bounded weighted shifts on  $X$ . The following assertions are equivalent.*

- (i)  $B_1$  and  $B_2$  support a disjoint hypercyclic algebra.
- (ii) For all  $p \in \mathbb{N}$ , for all  $m \geq 1$ , there exists a sequence  $(n_k)$  such that

$$\begin{aligned} (w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1/m} e_{n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \text{ for all } l = 0, \dots, p, \\ \left\{ \left( \frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} \right)_{l \geq 0} : k \geq 0 \right\} &\text{ is dense in } \omega. \end{aligned}$$

- (iii) There exists a sequence  $(n_k)$  such that, for all  $\gamma > 0$ , for all  $l \geq 0$ ,

$$\begin{aligned} (w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-\gamma} e_{n_k+l} &\xrightarrow{k \rightarrow +\infty} 0 \\ \left\{ \left( \frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} \right)_{l \geq 0} : k \geq 0 \right\} &\text{ is dense in } \omega. \end{aligned}$$

PROOF. (i)  $\implies$  (ii). Let  $m \geq 1$  and let  $x \in X$ ,  $x = \sum_{n=0}^{+\infty} x_n e_n$ , be such that  $x^m$  is a disjointly hypercyclic vector for  $B_1$  and  $B_2$ . Let also  $(\lambda^{(k)})$  be a dense sequence in  $\omega$  of vectors with finite support. We may assume that  $\text{supp}(\lambda^{(k)}) \subset [0, k]$  and that, for any fixed  $k_0$ , there are infinitely many  $k$  such that  $\lambda^{(k)} = \lambda^{(k_0)}$ . We then choose an increasing sequence of integers  $(n_k)$  such that, for all  $k \geq 1$ ,

$$(7.2) \quad \left\| B_1^{n_k}(x^m) - \sum_{l=0}^k e_l \right\| < \frac{1}{k},$$

$$(7.3) \quad \left\| B_2^{n_k}(x^m) - \sum_{l=0}^k \lambda_l^{(k)} e_l \right\| < \frac{1}{k}.$$

As we have already observed, (7.2) implies that, for all  $l \geq 0$ ,

$$\begin{aligned} (w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1/m} e_{n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \\ w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)} x_{n_k+l}^m &\xrightarrow{k \rightarrow +\infty} 1. \end{aligned}$$

Let us now fix  $U$  a nonempty open subset of  $\omega$  and  $k_0 \in \mathbb{N}$  be such that  $\lambda^{(k_0)} \in U$ . Let  $(n_{\phi(k)})$  be a subsequence of  $(n_k)$  such that  $\lambda^{\phi(k)} = \lambda^{k_0}$  for all  $k$ . By (7.3), we know that

$$w_{l+1}^{(2)} \cdots w_{n_{\phi(k)}+l}^{(2)} x_{n_{\phi(k)}+l}^m \xrightarrow{k \rightarrow +\infty} \lambda_l^{(k_0)} \text{ for all } l \geq 0.$$

Combining the two last properties, we find that

$$\frac{w_{l+1}^{(2)} \cdots w_{n_{\phi(k)}+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_{\phi(k)}+l}^{(1)}} \xrightarrow{k \rightarrow +\infty} \lambda_l^{(k_0)} \text{ for all } l \geq 0.$$

Hence, for  $k$  large enough,

$$\left( \frac{w_{l+1}^{(2)} \cdots w_{n_{\phi(k)}+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_{\phi(k)}+l}^{(1)}} \right)_{l \geq 0} \in U.$$

(ii)  $\implies$  (iii). This follows from a diagonal argument.

(iii)  $\implies$  (i). Let  $m_0 < m_1$  be two positive integers, let  $U, V_1, V_2$  be nonempty open subsets of  $X$  and let  $W$  be a neighbourhood of 0. Let  $x \in U$  with finite support, let  $y_i = \sum_{l=0}^p y_{i,l} e_l$  belonging to  $V_i$ ,  $i = 1, 2$ . Without loss of generality, we may assume that  $y_{1,l} \neq 0$  for all  $l = 0, \dots, p$ . We then consider a sequence  $(n_k)$  of integers such that, for all  $l = 0, \dots, p$ ,

$$(7.4) \quad \begin{aligned} (w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1/m_0} e_{n_k+l} &\xrightarrow{k \rightarrow +\infty} 0, \\ \frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} &\xrightarrow{k \rightarrow +\infty} \frac{y_{2,l}}{y_{1,l}}. \end{aligned}$$

Observe that  $w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}$  goes to  $+\infty$  for all  $l \geq 0$ . We finally set

$$u = x + \sum_{l=0}^p \frac{y_{1,l}^{1/m_0}}{(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{1/m_0}} e_{n_k+l}.$$

Clearly,  $u$  belongs to  $U$  (provided  $k$  is large enough),  $B_1^{n_k}(u^{m_0}) = y$  and  $B_1^{n_k}(u^n)$  goes to 0 for all  $n > m_0$ . Now, let us write

$$B_2^{n_k}(u^n) = \sum_{l=0}^p y_{1,l}^{n/m_0} \frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{\frac{n}{m_0}}} e_l.$$

If  $n = m_0$ , then (7.4) tells us immediately that  $B_2^{n_k}(u^m)$  tends to  $y_2$ . For  $n > m_0$ ,

$$y_{1,l}^{\frac{n}{m_0}} \frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{\frac{n}{m_0}}} \sim_{k \rightarrow +\infty} \frac{y_{1,l}^{\frac{n}{m_0}-1} y_{2,l}}{(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{\frac{n}{m_0}-1}}.$$

Since  $(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})$  tends to  $+\infty$ , this implies that  $B_2^{n_k}(u^n)$  belongs to  $W$  for  $k$  large enough, so that we may apply Corollary 7.3 to prove that  $B_1$  and  $B_2$  support a disjoint hypercyclic algebra.  $\square$

REMARK 7.8. If we are just interested in disjointly hypercyclic operators, our proof can be easily modified to give the following result : let  $X$  be a Fréchet sequence space such that  $(e_n)$  is a Schauder basis for  $X$ . Let  $B_1 = B_{w^{(1)}}$  and  $B_2 = B_{w^{(2)}}$  be two bounded weighted shifts on  $X$ . Then  $B_1$  and  $B_2$  are disjointly hypercyclic if and only if there exists a sequence  $(n_k)$  such that  $((w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1} e_{n_k+l})$  tends to zero for all  $l \geq 0$  and the set

$$\left\{ \left( \frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} \right)_{l \geq 0} : k \geq 0 \right\}$$

is dense in  $\omega$ .

REMARK 7.9. In particular, if the sequence  $(e_n)$  is bounded below, the conditions of the above theorem are all equivalent to say that  $B_1$  and  $B_2$  are disjointly hypercyclic. Hence, all the examples given in [13] support a disjoint hypercyclic algebra.

### 7.3. Disjoint hypercyclic algebras for backward shifts - convolution product

**7.3.1. Backward shifts raised to different powers.** We now turn to the case of Fréchet sequence algebras endowed with the Cauchy product, and we first study the existence of a disjoint hypercyclic algebra for two backward shifts raised to different powers. This situation seems more complicated and we choose to present only two significant examples. The first one is the multiples of  $B$  and  $B^2$  on  $\ell_1$ . It is known by [12, Cor 4.2] that  $\lambda B$  and  $\mu B^2$  are disjointly hypercyclic if and only if  $1 < \lambda < \mu$ . We show that under this condition on  $\mu$ , they even support a disjoint hypercyclic algebra. The proof will be done in two steps. We first handle the case  $\lambda < \mu^{1/2}$ . The proof uses a shifting factor which is placed after the term to approximate (like in Theorem 1.3).

THEOREM 7.10. *Let  $1 < \lambda < \mu^{1/2}$ . Then  $\lambda B$  and  $\mu B^2$ , acting on  $\ell_1$  endowed with the Cauchy product, support a disjoint hypercyclic algebra.*

PROOF. We shall prove that the assumptions of Corollary 7.2 are satisfied. Let  $m \geq 1$ , let  $U, V_1, V_2$  be nonempty open sets and let  $W$  be a neighbourhood of 0. Let  $p \geq 1$ ,  $x \in U$ ,  $y_1 \in V_1$ ,  $y_2 \in V_2$  with  $\text{supp}(x), \text{supp}(y_1), \text{supp}(y_2) \subset [0, p]$  and let

us write  $y_i = \sum_{l=0}^p y_{i,l} e_l$ ,  $i = 1, 2$ . Let finally  $\delta > 0$  be such that  $B(y_i, 2\delta) \subset V_i$ . For  $\sigma$  a large integer, we set

$$\begin{aligned} N &= m\sigma - 3p \\ d_{1,l} &= \frac{y_{1,l}}{m\lambda^{N/m}\delta^{(m-1)/m}}, & \varepsilon_1 &= \frac{\delta^{1/m}}{\lambda^{N/m}} \\ d_{2,l} &= \frac{y_{2,l}}{m\mu^{N/m}\delta^{(m-1)/m}}, & \varepsilon_2 &= \frac{\delta^{1/m}}{\mu^{N/m}} \\ z_1 &= \sum_{l=0}^p d_{1,l} e_{\sigma-3p+l} + \varepsilon_1 e_\sigma \\ z_2 &= \sum_{l=0}^p d_{2,l} e_{2\sigma-6p+l} + \varepsilon_2 e_{2\sigma} \\ u &= x + z_1 + z_2. \end{aligned}$$

It is easy to check that, provided  $\sigma$ , hence  $N$ , is large enough, then  $u \in U$ . We shall prove that we also have

$$\begin{cases} (\lambda B)^N(u^m) \in V_1, (\lambda B)^N(u^n) \in W \text{ for } n < m, \\ (\mu B^2)^N(u^m) \in V_2, (\mu B^2)^N(u^n) \in W \text{ for } n < m. \end{cases}$$

The two last properties are easier to prove: there is no more difficulty to prove them than to prove that  $\mu B^2$  admits a hypercyclic algebra. Indeed, if  $n < m$ , then  $\text{supp}(u^n) \subset [0, 2n\sigma]$  with  $2n\sigma < 2N$  whence  $(\mu B^2)^N(u^n) = 0$ . On the other hand,

$$u^m = z + \sum_{l=0}^p \frac{y_{2,l}}{\mu^N} e_{2m\sigma-6p+l} + \frac{\delta}{\mu^N} e_{2m\sigma}$$

with  $\max(\text{supp}(z)) < 2N$ , hence

$$(\mu B^2)^N(u^m) = y_2 + \delta e_{6p} \in V_2.$$

The proofs of the first two properties are more difficult: the powers of  $z_2$  are mixed with that of  $x + z_1$ . Such a mixing term does not always disappear when you apply  $B^N$ . It is also not always smaller than  $\lambda^{-N}$ . But taking into account the strong condition  $\mu > \lambda^2$ , we will show that it keeps smaller than  $\lambda^{-N}$  if its support exceeds  $[0, N]$ . We start by writing

$$\begin{aligned} u^m &= (x + z_1)^m + \sum_{\substack{\alpha+\beta+\gamma=m \\ \gamma \geq 1}} \binom{m}{\alpha, \beta, \gamma} x^\alpha z_1^\beta z_2^\gamma \\ &= z + \sum_{l=0}^p \frac{y_{1,l}}{\lambda^N} e_{m\sigma-3p+l} + \frac{\delta}{\lambda^N} e_{m\sigma} + \sum_{\substack{\alpha+\beta+\gamma=m \\ \gamma \geq 1}} \binom{m}{\alpha, \beta, \gamma} x^\alpha z_1^\beta z_2^\gamma \end{aligned}$$

where

$$\binom{m}{\alpha, \beta, \gamma} = \frac{m!}{\alpha! \beta! \gamma!}$$

is the multinomial coefficient and  $\max(\text{supp}(z)) < N$ . For the first three terms, we proceed as before to observe that, provided  $\sigma$  is large enough,

$$(\lambda B)^N \left( z + \sum_{l=0}^p \frac{y_{1,l}}{\lambda^N} e_{m\sigma-3p+l} + \frac{\delta}{\lambda^N} e_{m\sigma} \right) = y_1 + \delta e_{3p}.$$

Let us now look at  $(\lambda B)^N(x^\alpha z_1^\beta z_2^\gamma)$  with  $\gamma \geq 1$  and  $\alpha + \beta + \gamma = m$ . Expanding the product and using the behaviour of the terms  $d_{i,j}$  and  $\varepsilon_i$ ,  $x^\alpha z_1^\beta z_2^\gamma$  is a sum of a finite number of terms  $c_j e_j$  (this finite number does not depend on  $\sigma$ ), with

$$j \leq p\alpha + \beta\sigma + 2\gamma\sigma$$

$$c_j \leq C_{\alpha,\beta,\gamma} \frac{1}{\lambda^{\beta N/m} \mu^{\gamma N/m}}$$

where the involved constant  $C_{\alpha,\beta,\gamma}$  does not depend on  $\sigma$  (of course, it also depends on  $x, y_1, y_2, \delta$  and  $m$ ). We then distinguish two cases. Assume first that  $\beta + 2\gamma < m$ . Then, provided  $\sigma$  is large enough,  $p\alpha + \beta\sigma + 2\gamma\sigma < N$ , hence  $(\lambda B)^N(x^\alpha z_1^\beta z_2^\gamma) = 0$ . On the contrary, if  $\beta + 2\gamma \geq m$ , then

$$\left\| (\lambda B)^N(x^\alpha z_1^\beta z_2^\gamma) \right\| \leq C_{\alpha,\beta,\gamma} \frac{\lambda^N}{\lambda^{\beta N/m} \mu^{\gamma N/m}}$$

and the right handside goes to 0 since  $\mu > \lambda^2$ , hence  $\mu^\gamma \lambda^\beta > \lambda^{1/m}$ . This shows that, provided  $\sigma$  is large enough,  $(\lambda B)^N(u^m) \in V_1$ . Finally, for  $n < m$ , it is easy to check that

$$(\lambda B)^N(u^n) = \sum_{\substack{\alpha+\beta+\gamma=n \\ \gamma \geq 1}} \binom{n}{\alpha, \beta, \gamma} (\lambda B)^N(x^\alpha z_1^\beta z_2^\gamma).$$

We use exactly the same arguments to prove that

- $(\lambda B)^N(x^\alpha z_1^\beta z_2^\gamma) = 0$  provided  $\beta + 2\gamma \leq m$ ;
- $(\lambda B)^N(x^\alpha z_1^\beta z_2^\gamma)$  tends to zero as  $\sigma$  tends to  $+\infty$  provided  $\beta + 2\gamma > m$ .

Hence,  $(\lambda B)^N(u^n)$  belongs to  $W$ , which finishes the proof of Theorem 7.10.  $\square$

We turn to the case  $\lambda \geq \mu^{1/2}$ . The proof follows a similar scheme, but we now put the shifting term before the approximating term.

**THEOREM 7.11.** *Let  $1 < \lambda < \mu \leq \lambda^2$ . Then  $\lambda B$  and  $\mu B^2$ , acting on  $\ell^1$  endowed with the Cauchy product, support a disjoint hypercyclic algebra.*

**PROOF.** Again we shall prove that the assumptions of Corollary 7.2 are satisfied. Thus, let  $m \geq 1$ , let  $U, V_1, V_2$  be nonempty open subsets of  $\ell_1$  and let  $W$  be a neighbourhood of 0. Let  $p \geq 1$ ,  $x \in U$ ,  $y_1 \in V_1$ ,  $y_2 \in V_2$  with  $\text{supp}(x), \text{supp}(y_1)$  and  $\text{supp}(y_2) \subset [0, p]$  and let us write  $y_i = \sum_{l=0}^p y_{i,l} e_l$ ,  $i = 1, 2$ . Let  $\sigma > 2p$  and let  $\varepsilon > 0$  be such that the ball  $B(x, 2\varepsilon)$  is contained in  $U$ . For  $N$  a large integer, we define

$$z_1 = \frac{1}{m\varepsilon^{m-1}\lambda^N} \sum_{l=0}^p y_{1,l} e_{N-(m-1)\sigma+l}$$

$$z_2 = \frac{1}{m\varepsilon^{m-1}\mu^N} \sum_{l=0}^p y_{2,l} e_{2N-(m-1)\sigma+l}$$

$$u = x + \varepsilon e_\sigma + z_1 + z_2.$$

The definition of  $\varepsilon$  ensures that, for  $N$  large enough,  $u$  belongs to  $U$ . Let us show that the other assumptions of Corollary 7.2 are satisfied. First, consider  $n < m$  and write

$$u^n = w_1 + w_2$$

where

$$w_1 = \sum_{\substack{\alpha+\beta+\gamma=n \\ \gamma \leq 1}} \binom{n}{\alpha, \beta, \gamma} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma$$

$$w_2 = \sum_{\substack{\alpha+\beta+\gamma=n \\ \gamma > 1}} \binom{n}{\alpha, \beta, \gamma} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma + \sum_{\substack{\alpha+\beta+\gamma+\delta=n \\ \delta \geq 1}} \binom{n}{\alpha, \beta, \gamma, \delta} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma z_2^\delta.$$

The support of  $w_1$  is contained in  $[0, N - (m-1)\sigma + p + (n-1)\sigma] \subset [0, N)$  since  $\sigma > p$  and  $n < m$ . Thus  $(\lambda B)^N w_1^n = 0$ . Moreover, it is easy to check that

$$\|w_2\| \leq C \max\left(\frac{1}{\lambda^{2N}}, \frac{1}{\mu^N}\right)$$

where the constant  $C$  depends on  $x, p, \varepsilon, y_1, y_2$  but not on  $N$ . Since  $1 < \lambda < \mu$ , the norm of  $(\lambda B)^N w_2$  can be adjusted to be arbitrarily small, so that  $(\lambda B)^N u^n$  belongs to  $W$ . In the same vein, write

$$u^n = w'_1 + w'_2$$

where

$$w'_1 = \sum_{\substack{\alpha+\beta+\gamma=n \\ \gamma \leq 2}} \binom{n}{\alpha, \beta, \gamma} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma + \sum_{\substack{\alpha+\beta+\delta=n \\ \delta=1}} \binom{n}{\alpha, \beta, \delta} x^\alpha (\varepsilon e_\sigma)^\beta z_2^\delta$$

$$w'_2 = \sum_{(\alpha, \beta, \gamma, \delta) \in E_n} \binom{n}{\alpha, \beta, \gamma, \delta} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma z_2^\delta$$

with

$$E_n = \left\{ (\alpha, \beta, \gamma, \delta) \in \mathbb{N}_0^4 : \alpha + \beta + \gamma + \delta = n, (\gamma \geq 3 \text{ and } \delta = 0) \right. \\ \left. \text{or } (\delta \geq 2 \text{ and } \gamma = 0) \text{ or } (\gamma \geq 1 \text{ and } \delta \geq 1) \right\}.$$

As before, the support of  $w'_1$  is contained in  $[0, 2N)$ , so that  $(\mu B^2)^N (w'_1) = 0$ . Since

$$\|w'_2\| \leq C \max\left(\frac{1}{\lambda^{3N}}, \frac{1}{\mu^{2N}}, \frac{1}{\lambda^N \mu^N}\right)$$

and  $\mu \leq \lambda^2 < \lambda^3$ , we get that  $(\mu B^2)^N (u^n)$  belongs to  $W$  provided that  $N$  is large enough. Let us now inspect  $u^m$ . We first write it

$$u^m = \tilde{w}_1 + \frac{1}{\lambda^N} \sum_{l=0}^p y_{1,l} e_{N+l} + \tilde{w}_2$$

where

$$\tilde{w}_1 = \sum_{\alpha+\beta+\gamma \in F_m} \binom{m}{\alpha, \beta, \gamma} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma$$

$$\tilde{w}_2 = \sum_{\substack{\alpha+\beta+\gamma=m \\ \gamma > 1}} \binom{m}{\alpha, \beta, \gamma} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma + \sum_{\substack{\alpha+\beta+\gamma+\delta=m \\ \delta \geq 1}} \binom{m}{\alpha, \beta, \gamma, \delta} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma z_2^\delta$$

with  $F_m = \{(\alpha, \beta, \gamma) \in \mathbb{N}_0^3 : \alpha + \beta + \gamma = m, \gamma \leq 1\} \setminus \{(0, m-1, 1)\}$ . The same proof shows that  $\text{supp}(\tilde{w}_1) \subset [0, N - (m-1)\sigma + p + (m-2)\sigma + p] \subset [0, N)$  and

that  $(\lambda B)^N(\tilde{w}_2)$  tends to 0 as  $N$  tends to  $+\infty$ . Hence,  $(\lambda B)^N(u^m) \in V_1$  for  $N$  large enough. We also write

$$\begin{aligned} u^m &= \tilde{w}'_1 + \frac{1}{\mu^N} \sum_{l=0}^p y_{2,l} e_{2N+l} + \tilde{w}'_2 \\ \tilde{w}'_1 &= \sum_{\substack{\alpha+\beta+\gamma=m \\ \gamma \leq 2}} \binom{m}{\alpha, \beta, \gamma} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma + \sum_{(\alpha, \beta, \delta) \in F'_m} \binom{m}{\alpha, \beta, \delta} x^\alpha (\varepsilon e_\sigma)^\beta z_2^\delta \\ w'_2 &= \sum_{(\alpha, \beta, \gamma, \delta) \in E_m} \binom{m}{\alpha, \beta, \gamma, \delta} x^\alpha (\varepsilon e_\sigma)^\beta z_1^\gamma z_2^\delta \\ F'_m &= \{(\alpha, \beta, \delta) \in F_m : \delta = 1\}. \end{aligned}$$

Looking at the support of  $\tilde{w}'_1$  and at the norm of  $\tilde{w}'_2$ , we show again that  $(\mu B^2)^N(u^m)$  belongs to  $V_2$  for  $N$  large enough, showing that the assumptions of Corollary 7.2 are satisfied.  $\square$

We end up this section by proving the existence of a disjoint hypercyclic algebra for the multiples of  $D$  and  $D^2$ . The proof follows the scheme of that of Theorem 7.10, but we need a more careful analysis.

**THEOREM 7.12.** *Let  $\lambda, \mu > 0$ . Then  $\lambda D$  and  $\mu D^2$  support a disjoint hypercyclic algebra on  $H(\mathbb{C})$  endowed with the convolution product.*

**PROOF.** The scheme of the proof is similar to that of Theorem 7.10; nevertheless, we will need to be more careful when we analyze  $(\lambda D)^N(x^\alpha z_1^\beta z_2^\gamma)$  for  $\beta + 2\gamma = m$ . We fix  $m \geq 1$ ,  $U, V_1, V_2$  and  $W$  as above. Let  $q \geq 0$  and  $x \in U$  with support in  $[0, q]$ . Let now  $p \geq 1$  be such that  $2p > qm$  and such that there exists  $y_i \in V_i$  with  $\text{supp}(y_i) \subset [0, p]$ ,  $i = 1, 2$ . We again write  $y_i = \sum_{l=0}^p y_{i,l} e_l$  and we consider  $r \geq 1$  and  $\delta > 0$  such that  $B_{\|\cdot\|_r}(y_i, 2\delta) \subset V_i$ . We then consider  $\omega > 1/m$  satisfying the following property: for all integers  $\alpha, \beta, \gamma \geq 0$  with  $\alpha + \beta + \gamma \leq m$  and  $\beta + 2\gamma > m$ , we have

$$(7.5) \quad \beta + 2\gamma(m - m(m-1)\omega) > m$$

(observe that when  $\omega = 1/m$ , then the left-hand side is equal to  $\beta + 2\gamma$ , which is greater than  $m$ , and that there is only a finite number of constraints to satisfy). For  $\sigma$  a large integer, we set

$$\begin{aligned} N &= m\sigma - 3p, \\ d_{1,l} &= \frac{l! y_{1,l}}{m \varepsilon_1^{m-1} \lambda^N (N+l)!}, & \varepsilon_1 &= \frac{((3p)!)^{1/m} \delta^{1/m}}{(3p)^{r/m} \lambda^{N/m} [(N+3p)!]^{1/m}}, \\ d_{2,l} &= \frac{l! y_{2,l}}{m \varepsilon_2^{m-1} \mu^N (2N+l)!}, & \varepsilon_2 &= \frac{\delta^{1/m}}{\mu^{N/m} [(2N+6p)!]^\omega}, \\ z_1 &= \sum_{l=0}^p d_{1,l} e_{\sigma-3p+l} + \varepsilon_1 e_\sigma, \\ z_2 &= \sum_{l=0}^p d_{2,l} e_{2\sigma-6p+l} + \varepsilon_2 e_{2\sigma}, \\ u &= x + z_1 + z_2. \end{aligned}$$

The first part of the proof of Theorem 7.10 carries on to our new context. First, we observe that, using Stirling's formula, we now have the crude estimates, for some  $C > 0$  and some  $\kappa > 0$ ,

$$\begin{aligned} |d_{1,l}| &\leq \frac{C\kappa^\sigma}{\sigma^\sigma} \\ |d_{2,l}| &\leq \frac{C\kappa^\sigma}{\sigma^{2(m-m(m-1)\omega)\sigma}} \\ \varepsilon_1 &\leq \frac{C\kappa^\sigma}{\sigma^\sigma} \\ \varepsilon_2 &\leq \frac{C\kappa^\sigma}{\sigma^{2m\omega\sigma}} \leq \frac{C\kappa^\sigma}{\sigma^{2(m-m(m-1)\omega)\sigma}} \end{aligned}$$

(the very last inequality is a consequence of  $\omega > 1/m$ ). Hence, we still have  $u \in U$  provided  $\sigma$  is large enough. Looking at the support of  $u^n$  shows that  $(\mu D^2)^N(u^n) = 0$  for all  $n < m$ , whereas the values of  $\varepsilon_1^m$  and  $m\varepsilon_1^{m-1}d_{1,l}$  ensure that

$$(\mu D^2)^N(u^m) = y_2 + \frac{\delta(6p)!}{((2N+6p)!)^{m\omega-1}} e_{6p}.$$

Since  $\omega > 1/m$ , we clearly have  $\|(\mu D^2)^N(u^m) - y_2\|_r < 2\delta$  provided  $N$  is large enough.

Regarding  $(\lambda D)^N$ , the support of each element and the definition of  $d_{1,l}$ ,  $\varepsilon_1$  ensure that

$$\begin{aligned} (\lambda D)^N(u^m) &= y_1 + \frac{\delta}{(3p)^r} e_{3p} + \sum_{\substack{\alpha+\beta+\gamma=m \\ \gamma \geq 1, \beta+2\gamma \geq m}} \binom{m}{\alpha, \beta, \gamma} (\lambda D)^N(x^\alpha y^\beta z^\gamma) \\ (\lambda D)^N(u^n) &= \sum_{\substack{\alpha+\beta+\gamma=n \\ \gamma \geq 1, \beta+2\gamma \geq m}} \binom{n}{\alpha, \beta, \gamma} (\lambda D)^N(x^\alpha y^\beta z^\gamma) \text{ for } n < m. \end{aligned}$$

Hence, it remains to show that  $(\lambda D)^N(x^\alpha y^\beta z^\gamma)$  tends to zero as  $\sigma$  tends to  $+\infty$  when  $\alpha + \beta + \gamma \leq m$ ,  $\gamma \geq 1$  and  $\beta + 2\gamma \geq m$ . We first assume that  $\beta + 2\gamma > m$ . The product  $x^\alpha y^\beta z^\gamma$  may be written as a finite sum (whose number of terms do not depend on  $\sigma$ )  $\sum_j c_j e_j$  where

$$\begin{aligned} j &\leq q\alpha + \beta\sigma + 2\gamma\sigma \\ |c_j| &\leq \frac{C_{\alpha, \beta, \gamma} \kappa_{\alpha, \beta, \gamma}^\sigma}{\sigma^{\beta\sigma + 2\gamma(m - (m-1)\omega)\sigma}}. \end{aligned}$$

Thus, using  $\|(\lambda D)^N e_j\|_r \leq \lambda^N j^r j^N$ , we get that for some different constants  $C_{\alpha, \beta, \gamma}$  and  $\kappa_{\alpha, \beta, \gamma}$ ,

$$\|(\lambda D)^N(x^\alpha y^\beta z^\gamma)\|_r \leq \frac{C_{\alpha, \beta, \gamma} \kappa_{\alpha, \beta, \gamma}^\sigma \sigma^{m\sigma}}{\sigma^{\beta\sigma + 2\gamma(m - (m-1)\omega)\sigma}}.$$

The choice of  $\omega$  (sufficiently close to  $1/m$ , see (7.5)) guarantees that this goes to 0 as  $\sigma$  tends to  $+\infty$ . Finally, assume that  $\beta + 2\gamma = m$  and define

$$w_2 = \sum_{l=0}^p d_{2,l} e_{2\sigma - 6p + l}$$

so that

$$\begin{aligned} x^\alpha z_1^\beta z_2^\gamma &= x^\alpha z_1^\beta (w_2 + \varepsilon_2 e_{2\sigma})^\gamma \\ &= \sum_{k=0}^{\gamma} \binom{\gamma}{k} \varepsilon_2^{\gamma-k} x^\alpha z_1^\beta w_2^k e_{2(\gamma-k)\sigma}. \end{aligned}$$

Assume first that  $k \neq 0$ . Then

$$\begin{aligned} \max(\text{supp}(x^\alpha z_1^\beta w_2^k e_{2(\gamma-k)\sigma})) &\leq q\alpha + \beta\sigma + 2\gamma\sigma - 5p \\ &\leq qm + m\sigma - 5p \\ &< N \end{aligned}$$

since  $qm < 2p$ . This implies that

$$(\lambda D)^N (x^\alpha z_1^\beta w_2^k e_{2(\gamma-k)\sigma}) = 0$$

and it only remains to handle the term  $\varepsilon_2^\gamma x^\alpha z_1^\beta e_{2\gamma\sigma}$ . As before, it is equal to a finite sum  $\sum_j c_j e_j$  with  $j \leq q\alpha + (\beta + 2\gamma)\sigma$  but now we can be slightly more precise on  $|c_j|$ :

$$|c_j| \leq \frac{C_{\alpha,\beta,\gamma} \kappa_{\alpha,\beta,\gamma}^\sigma}{\sigma^{\beta\sigma + 2\gamma m \omega \sigma}}.$$

Therefore,

$$\left\| (\lambda D)^N (x \varepsilon_2^\gamma x^\alpha z_1^\beta e_{2\gamma\sigma}) \right\|_r \leq \frac{C_{\alpha,\beta,\gamma} \kappa_{\alpha,\beta,\gamma}^\sigma \sigma^{m\sigma}}{\sigma^{\beta\sigma + 2\gamma m \omega \sigma}}.$$

Now, since  $\omega > 1/m$  and  $\beta + 2\gamma = m$ , this tends to 0 as  $\sigma$  goes to  $+\infty$ . This finishes the proof of Theorem 7.12.  $\square$

**7.3.2. Backward shifts raised to the same power.** We conclude this section by studying the existence of hypercyclic algebras for two backward shifts (raised to the same power). We have been able to get a general result saying that two disjointly hypercyclic backward shifts support a disjoint hypercyclic algebra.

**THEOREM 7.13.** *Let  $X$  be a regular Fréchet sequence algebra for the Cauchy product and let  $B_1 = B_{w^{(1)}}$ ,  $B_2 = B_{w^{(2)}}$  be two bounded weighted shifts on  $X$ . The following assertions are equivalent.*

- (i)  $B_1$  and  $B_2$  are disjointly hypercyclic.
- (ii)  $B_1$  and  $B_2$  support a disjoint hypercyclic algebra.
- (iii) There exists a sequence  $(n_k)$  such that  $((w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1} e_{n_k+l})$  tends to zero for all  $l \geq 0$  and the set

$$\left\{ \left( \frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} \right)_{l \geq 0} : k \geq 0 \right\}$$

is dense in  $\omega$ .

**PROOF.** The implication (ii)  $\implies$  (i) is trivial. Let us prove the other ones.

(iii)  $\implies$  (ii). Let  $m \geq 1$ , let  $U, V_1, V_2$  be nonempty open subsets of  $X$  and let  $W$  be a neighbourhood of 0. Let  $p \geq 1$ ,  $x \in U$ ,  $y_1 \in V_1$ ,  $y_2 \in V_2$  with  $\text{supp}(x), \text{supp}(y_1), \text{supp}(y_2) \subset [0, p]$  and let us write  $y_i = \sum_{l=0}^p y_{i,l} e_l$ . Without loss of generality, we may assume  $y_{1,l} \neq 0$  for all  $l = 0, \dots, p$ . Let finally  $B$  be a ball for

the seminorm  $\|\cdot\|_r$  with radius  $\delta$  such that  $B \subset W$ ,  $x + B \subset U$  and  $y_i + B \subset V_i$ ,  $i = 1, 2$ . We consider  $(n_k)$  a sequence of integers such that

$$(w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)})^{-1} e_{n_k+l} \text{ tends to } 0 \text{ for all } l = 0, \dots, p$$

$$\frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} \text{ tends to } \frac{y_{2,l}}{y_{1,l}} \text{ for all } l = 0, \dots, p$$

$$\frac{w_{l+1}^{(2)} \cdots w_{n_k+l}^{(2)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} \text{ tends to } 1 \text{ for all } l = 3p, \dots, 3p + (m-1).$$

We then consider for each  $k \geq 1$  the integer  $J_k$  defined by

$$n_k + 3p \leq mJ_k < n_k + 3p + m$$

and we set  $N_k = J_k - (mJ_k - n_k)$  so that  $3p \leq J_k - N_k < 3p + m$ . We finally define

$$\varepsilon = \max_{0 \leq l \leq p} \left( \frac{\|e_{N_k+l}\|_r}{w_1^{(1)} \cdots w_{n_k+l}^{(1)}} \right)^{\frac{1}{2(m-1)}} \times \min \left( \frac{1}{\|e_{J_k}\|_r}, \frac{1}{(w_1^{(1)} \cdots w_{mJ_k}^{(1)})^{1/m}} \right)^{\frac{1}{2}},$$

$$d_l = \frac{y_{1,l}}{m\varepsilon^{m-1} w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}},$$

$$u = x + \sum_{l=0}^p d_l e_{N_k+l} + \varepsilon e_{J_k}.$$

Let us assume for a while that the following facts are true:

$$(7.6) \quad \varepsilon \|e_{J_k}\|_r \rightarrow 0 \text{ as } k \rightarrow +\infty,$$

$$(7.7) \quad |d_l| \cdot \|e_{N_k+l}\|_r \rightarrow 0 \text{ as } k \rightarrow +\infty, \text{ for all } l = 0, \dots, p,$$

$$(7.8) \quad \varepsilon^m w_1^{(1)} \cdots w_{mJ_k}^{(1)} \rightarrow 0 \text{ as } k \rightarrow +\infty.$$

Then it is clear that  $u$  belongs to  $U$  for a large integer  $k$  and that  $B_i^{n_k}(u^n) = 0$  for  $i = 1, 2$  and  $n < m$ , since  $\max(\text{supp}(u^n)) \leq (m-1)J_k < n_k$ . We can also write

$$u^m = z + \sum_{l=0}^p \frac{y_{1,l}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} e_{N_k+(m-1)J_k+l} + \varepsilon^m e_{mJ_k},$$

with

$$\max(\text{supp}(z)) \leq (m-2)J_k + 2N_k + 2p < N_k + (m-1)J_k = n_k.$$

Hence,

$$B_i^{n_k}(u^m) = \sum_{l=0}^p \frac{w_{l+1}^{(i)} \cdots w_{n_k+l}^{(i)}}{w_{l+1}^{(1)} \cdots w_{n_k+l}^{(1)}} y_{1,l} e_l + \varepsilon^m w_{mJ_k-n_k+1}^{(i)} \cdots w_{mJ_k}^{(i)} e_{mJ_k-n_k}.$$

When  $i = 1$ , the above sum is equal to  $y_1$ ; when  $i = 2$ , it goes to  $y_2$ . Hence, we have to prove that, for  $i = 1, 2$ ,

$$\varepsilon^m w_{mJ_k-n_k+1}^{(i)} \cdots w_{mJ_k}^{(i)} e_{mJ_k-n_k} \xrightarrow{k \rightarrow +\infty} 0.$$

Since  $(mJ_k - n_k)$  is a bounded sequence, this amounts to saying that

$$\varepsilon^m w_{mJ_k-n_k+1}^{(i)} \cdots w_{mJ_k}^{(i)} \xrightarrow{k \rightarrow +\infty} 0.$$

For  $i = 1$ , this is (7.8). For  $i = 2$ , we just observe that  $mJ_k - n_k \in \{3p, \dots, 3p + m - 1\}$  and we write

$$\frac{w_{mJ_k - n_k + 1}^{(2)} \cdots w_{mJ_k}^{(2)}}{w_{mJ_k - n_k + 1}^{(1)} \cdots w_{mJ_k}^{(1)}} = \frac{w_{mJ_k - n_k + 1}^{(2)} \cdots w_{n_k + (mJ_k - n_k)}^{(2)}}{w_{mJ_k - n_k + 1}^{(1)} \cdots w_{n_k + (mJ_k - n_k)}^{(1)}} \xrightarrow{k \rightarrow +\infty} 1.$$

It remains to prove (7.6), (7.7), (7.8). The proofs are completely similar to the proofs of the corresponding properties in Theorem 1.3. For the sake of completeness, we just show how it works for (7.7). We first write

$$|d_l| \cdot \|e_{N_k + l}\|_r \leq C \left( \frac{\|e_{N_k + l}\|_r}{w_1^{(1)} \cdots w_{n_k + l}^{(1)}} \right)^{1/2} \times \max \left( \|e_{J_k}\|_r, (w_1^{(1)} \cdots w_{mJ_k}^{(1)})^{1/m} \right)^{(m-1)/2}.$$

If the maximum is attained for  $\|e_{J_k}\|_r$ , then using

$$\begin{aligned} \|e_{N_k + l}\|_r \cdot \|e_{J_k}\|_r^{m-1} &\leq C \|e_{(m-1)J_k + N_k + l}\|_q \\ &\leq C \|e_{n_k + l}\|_q, \end{aligned}$$

we get

$$|d_l| \cdot \|e_{N_k + l}\|_r \leq C \left( \frac{\|e_{n_k + l}\|_q}{w_1^{(1)} \cdots w_{n_k + l}^{(1)}} \right)^{1/2}$$

and this goes to 0. Otherwise, we apply Corollary 4.14 with  $n = mJ_k$  and  $s = mJ_k - n_k - l$  to get

$$\begin{aligned} (w_1^{(1)} \cdots w_{mJ_k}^{(1)})^{m-1} \|e_{N_k + l}\|_r^m &\leq C (w_1^{(1)} \cdots w_{mJ_k}^{(1)})^{m-1} \|e_{mN_k + ml}\|_q \\ &= C (w_1^{(1)} \cdots w_{mJ_k}^{(1)})^{m-1} \|e_{mJ_k - ms}\|_q \\ &\leq C' (w_1^{(1)} \cdots w_{mJ_k - s}^{(1)})^{m-1} \|e_{mJ_k - s}\|_{q'} \\ &= C' (w_1^{(1)} \cdots w_{n_k + l}^{(1)})^{m-1} \|e_{n_k + l}\|_{q'} \end{aligned}$$

whence

$$|d_j| \cdot \|e_{N_k + l}\|_r \leq C'' \left( \frac{\|e_{n_k + l}\|_{q'}}{w_1^{(1)} \cdots w_{n_k + l}^{(1)}} \right)^{1/m},$$

which again goes to 0.

(i)  $\implies$  (iii). The proof is exactly the same as the proof of (i)  $\implies$  (ii) in Theorem 7.7, restricted to the case  $m = 1$ .  $\square$

## Concluding remarks and open questions

### 8.1. Closed hypercyclic algebras

As pointed out in the introduction, provided  $T$  is hypercyclic,  $HC(T) \cup \{0\}$  always contains a dense subspace. When moreover  $T$  satisfies the hypercyclicity criterion, there is a necessary and sufficient condition to determine whether  $HC(T) \cup \{0\}$  contains an infinite-dimensional closed subspace (see for instance [8]). In our context, it is natural to ask whether, for some of our examples,  $HC(T) \cup \{0\}$  contains a closed non-trivial algebra (we will say that  $T$  supports a closed hypercyclic algebra).

The third author and K. Grosse-Erdmann have shown that it is the case if  $T$  is a translation operator acting on the space  $\mathcal{C}^\infty(\mathbb{R}, \mathbb{C})$ . The fact that  $T$  is an algebra homomorphism plays an important role here. We now give several negative results. The first one solves a question of [28].

**PROPOSITION 8.1.** *No convolution operators  $P(D)$  induced by a nonconstant polynomial  $P \in \mathbb{C}[z]$  admit a closed hypercyclic algebra.*

**PROOF.** We write  $P(z) = \sum_{s=0}^t \hat{P}(s)z^s$ , with  $\hat{P}(t) \neq 0$ , and let  $f \in HC(P(D))$ . We shall prove that the closed algebra generated by  $f$  contains a non-zero and non-hypercyclic vector. Write  $f(z) = a_0 + \sum_{n \geq p} a_n z^n$ , with  $a_p \neq 0$ . Without loss of generality, we may assume that  $a_p = 1$ . We shall construct by induction a sequence of complex numbers  $(b_k)$  such that

$$|b_k| \leq \left( \frac{|\hat{P}(0)| + 1}{|\hat{P}(t)|} \right)^{kp} \times \frac{1}{(ktp)!}$$

for all  $k$  and, setting  $P_k(z) = \sum_{l=1}^k b_l(z - a_0)^{lt}$ , then

$$|P(D)^{lp}(P_k \circ f)(0)| \geq (|\hat{P}(0)| + 1)^{lp}$$

for all  $1 \leq l \leq k$ . The conclusion follows easily. In fact,  $(P_k)$  converges uniformly on compact subsets of  $\mathbb{C}$  to some entire function  $g$ . From the uniformity of the convergence, we conclude that the function  $g \circ f$  satisfies

$$|P(D)^{lp}(g \circ f)(0)| \geq (|\hat{P}(0)| + 1)^{lp}$$

for all  $l \geq 1$ . Let us set  $h = g - g(0)$ . The function  $h \circ f$ , which is in the algebra generated by  $f$ , satisfies

$$\begin{aligned} |P(D)^{lp}(h \circ f)(0)| &\geq |P(D)^{lp}(g \circ f)(0)| - |P(D)^{lp}(g(0))| \\ &\geq (|\hat{P}(0)| + 1)^{lp} - |\hat{P}(0)|^{lp}|g(0)| \\ &\xrightarrow{l \rightarrow +\infty} +\infty. \end{aligned}$$

Hence,  $h \circ f$  is nonzero and it cannot be hypercyclic for  $P(D)^p$ . In particular, since  $HC(P(D)) \subset \bigcap_{n \geq 1} HC(P(D)^n)$  (see [2, Theorem 1]),  $h \circ f$  cannot be hypercyclic for  $P(D)$  as well.

For the proof we will use the formula

$$P(D)^q = \sum_{\mathbf{j} \in I_q} \binom{q}{\mathbf{j}} \hat{P}(0)^{j_0} \dots \hat{P}(t)^{j_t} D^{j_1+2j_2+\dots+tj_t},$$

where  $I_q = \{\mathbf{j} = (j_0, \dots, j_t) \in \mathbb{N}_0^{t+1} : j_0 + \dots + j_t = q\}$  and  $\binom{q}{\mathbf{j}}$  denote the multinomial coefficient

$$\binom{q}{j_0, \dots, j_t} = \frac{q!}{j_0! \dots j_t!}.$$

Let us set  $P_0(z) = 0$  and let us assume that the construction has been done until step  $k-1$ . Then

$$\begin{aligned} P_{k-1} \circ f + b(f - a_0)^{kt} &= P_{k-1} \circ f + b(z^p + a_{p+1}z^{p+1} + \dots)^{kt} \\ &= P_{k-1} \circ f + bz^{ktp} + \sum_{j \geq ktp+1} c_j z^j. \end{aligned}$$

Hence, for  $1 \leq l \leq k$ ,

$$\begin{aligned} P(D)^{lp}(P_{k-1} \circ f + b(f - a_0)^{kt})(0) &= P(D)^{lp}(P_{k-1} \circ f)(0) + P(D)^{lp}(b(f - a_0)^{kt})(0) \\ &= P(D)^{lp}(P_{k-1} \circ f)(0) + g_l(0), \end{aligned}$$

where

$$g_l(z) = P(D)^{lp} \left( bz^{ktp} + \sum_{j \geq ktp+1} c_j z^j \right).$$

If  $l \leq k-1$  then  $\deg P^{lp} \leq (k-1)tp < ktp$ , hence  $g_l(0) = 0$ . By the induction hypothesis it follows that

$$\begin{aligned} |P(D)^{lp}(P_{k-1} \circ f + b(f - a_0)^{kt})(0)| &= |P(D)^{lp}(P_{k-1} \circ f)(0) + g_l(0)| \\ &= |P(D)^{lp}(P_{k-1} \circ f)(0)| \geq (|\hat{P}(0)| + 1)^{lp} \end{aligned}$$

whatever the value of  $b$  is. On the other hand, if  $l = k$ , then

$$g_k(z) = b\hat{P}(t)^{kp}(ktp)!$$

$$+ \sum_{\mathbf{j} \in I_{kp} \setminus \{(0, \dots, 0, kp)\}} \binom{kp}{\mathbf{j}} \hat{P}(0)^{j_0} \dots \hat{P}(t)^{j_t} D^{j_1+2j_2+\dots+tj_t} \left( bz^{ktp} + \sum_{j \geq ktp+1} c_j z^j \right),$$

hence  $g_k(0) = b\hat{P}(t)^{kp}(ktp)!$ , that is,

$$\begin{aligned} |P(D)^{kp}(P_{k-1} \circ f + b(f - a_0)^{kt})(0)| &= |P(D)^{kp}(P_{k-1} \circ f)(0) + g_k(0)| \\ &= |P(D)^{kp}(P_{k-1} \circ f)(0) + b\hat{P}(t)^{kp}(ktp)!|, \end{aligned}$$

so we can find  $b$  satisfying

$$|b| \leq \left( \frac{|\hat{P}(0)| + 1}{|\hat{P}(t)|} \right)^{kp} \times \frac{1}{(ktp)!}$$

such that

$$|P(D)^{lp}(P_{k-1} \circ f + b(f - a_0)^{kt})(0)| \geq (|\hat{P}(0)| + 1)^{lp}.$$

The proof is now done by taking  $b_k = b$ .  $\square$

QUESTION 8.2. Does there exist an entire function  $\phi$  of exponential type such that  $\phi(D)$  supports a closed hypercyclic algebra?

PROPOSITION 8.3. *Let  $X = \ell_p$ ,  $X = c_0$  or  $X = \omega$ , endowed with the coordinatewise product. No backward shift on  $X$  supports a closed hypercyclic algebra.*

PROOF. We first consider the case  $X = c_0$ . Let  $x \in X$  be a non-zero sequence, and let  $D \subset \mathbb{C}$  be a compact disc centered at the origin and omitting at least one of the terms of  $x$ . For each  $n \in \mathbb{N}$ , consider the compact set  $K_n$  as in Figure 1 and  $f$  a holomorphic function defined on a neighbourhood of  $K_n$  and satisfying that  $f(z) = 0$  if  $z \in D$  and  $f(z) = 1$  if  $z \in K_n \setminus D$ .

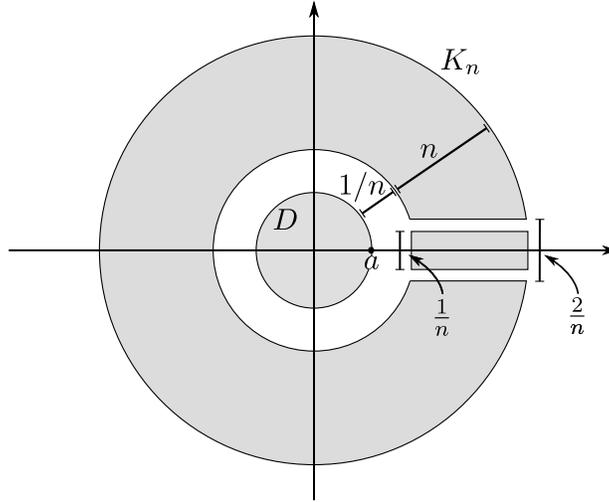


FIGURE 1

By Runge's approximation theorem we get a polynomial  $P_n$  such that  $\|P_n - f\|_{K_n} < \frac{1}{n}$ . We end up with a sequence of polynomials  $(P_n)$ , satisfying that  $P_n(z) \rightarrow 0$  uniformly on  $D$ , and  $P_n(z) \rightarrow 1$  pointwise on  $\mathbb{C} \setminus D$ . Redefining  $P_n$  by  $P_n - P_n(0)$ , we may also assume that  $P_n(0) = 0$ , for every  $n \in \mathbb{N}$ .

Since  $x \in c_0$ , it follows that eventually all the terms of  $x$  belong to  $D$  which yields that  $P_n(x) \rightarrow y = (y_k)$  in  $c_0$ , where  $y_k = 0$ , if  $x_k \in D$  and  $y_k = 1$  otherwise. We conclude, that  $y$  is a non-zero element in the closed algebra generated by  $x$  which is not hypercyclic for any weighted backward shift on  $c_0$ .

Let now  $X = \ell^p, p \geq 1$ . Consider an  $x \in X, x \neq 0$ , and  $D$  and  $(P_n)$  defined as above. Cauchy's formula ensures that  $P'_n \rightarrow 0$  uniformly on  $\frac{1}{2}D$ . Let

$$C = \sup \left\{ |P'_n(z)| : n \in \mathbb{N}, z \in \frac{1}{2}D \right\}.$$

Fix  $\varepsilon > 0$  and let  $k_0 \in \mathbb{N}$  be such that, for  $k > k_0$ ,  $x_k \in \frac{1}{2}D$  and

$$2^p C^p \sum_{k > k_0} |x_k|^p < \frac{\varepsilon}{2}.$$

Find  $N \in \mathbb{N}$  such that for all  $m, n \geq N$ ,

$$\sum_{k=1}^{k_0} |P_m(x_k) - P_n(x_k)|^p < \frac{\varepsilon}{2}.$$

We have

$$\begin{aligned} \|P_m(x) - P_n(x)\|_p^p &\leq \sum_{k=1}^{k_0} |P_m(x_k) - P_n(x_k)|^p + \sum_{k>k_0} (|P_m(x_k)| + |P_n(x_k)|)^p \\ &\leq \sum_{k=1}^{k_0} |P_m(x_k) - P_n(x_k)|^p + 2^p C^p \sum_{k>k_0} |x_k|^p < \varepsilon. \end{aligned}$$

That means that the sequence  $(P_n(x))$  is Cauchy in  $\ell_p$  and the conclusion follows exactly as in the previous case.

Finally, we consider the case  $X = \omega$ . Letting  $D = \{0\}$  and  $K_n$  be as above, and by using Runge's approximation theorem, we get a sequence of polynomials  $(Q_n)$  such that  $Q_n(0) = 0$ , for every  $n \in \mathbb{N}$  and  $Q_n(z) \rightarrow 1$  for each  $z \in \mathbb{C} \setminus \{0\}$ . If  $x \in \omega$  is a non-zero sequence, then  $Q_n(x) \rightarrow y = (y_k)$ , where  $y_k = 0$  if  $x_k = 0$ , and  $y_k = 1$  otherwise. It is immediate that  $y$  is a non-zero element in the closed algebra generated by  $x$  which fails to be hypercyclic for any weighted backward shift on  $\omega$ .  $\square$

QUESTION 8.4. Does there exist a weight  $(w_n)$  such that  $B_w$ , acting on  $\ell_1$  endowed with the Cauchy product, supports a closed hypercyclic algebra?

## 8.2. Hypercyclic algebras in the ideal of compact operators

Beyond the examples given in that paper, there are other examples where the existence of a hypercyclic algebra would be natural. One of them is given by hypercyclic operators acting on separable ideals of operators. For instance, assume that  $H$  is a separable Hilbert space and denote by  $X = \mathcal{K}(H)$  the (non-commutative) algebra of compact operators in  $H$ , endowed with the norm topology.

For  $T \in \mathcal{L}(H)$ , denote by  $L_T$  the operator of left multiplication by  $T$ , defined on  $\mathcal{K}(H)$ . It is known (see for instance [8, Chapter 8]) that if  $T$  satisfies the hypercyclicity criterion, then  $L_T$  is a hypercyclic operator on  $\mathcal{K}(H)$ . This latter space being an algebra, it is natural to study whether  $L_T$  supports a hypercyclic algebra. We do not know the answer to this question, but we point out that a positive answer would require different techniques. Indeed, Theorem 2.1 can never be applied to these operators.

PROPOSITION 8.5. *Let  $T \in \mathcal{L}(H)$ . Then  $L_T$ , acting on  $\mathcal{K}(H)$ , does not satisfy the assumptions of Theorem 2.1 even for  $d = 1$ .*

PROOF. We fix  $x \in H$ ,  $x^* \in H^*$  with  $x^*(x) = 1$ ,  $\|x\| = 1$ ,  $\|x^*\| = 1$ . Using the notations of Theorem 2.1, let  $A = \{1, 2\}$ ,

$$\begin{aligned} U &= V = \{u \in \mathcal{L}(H) : \|u - x^* \otimes x\| < 1/4\}, \\ W &= \{u \in \mathcal{L}(H) : \|u\| < 1/8\}. \end{aligned}$$

Assume first that  $\beta = 1$  and that there exist  $u \in U$ ,  $N \in \mathbb{N}$  with  $T^N u \in V$  and  $T^N u^2 \in W$ . Then we know that

$$\|T^N u^2(x) - x^*(u(x))x\| < \frac{\|u(x)\|}{4}$$

since  $T^N u \in V$ . Now,  $\|u(x) - x\| < 1/4$  so that  $\|u(x)\| < 5/4$  and  $|x^*(u(x))| > 3/4$ . Hence,

$$\|T^N u^2(x)\| > \frac{3}{4} - \frac{5}{16} > \frac{1}{8}.$$

This contradicts  $T^N u^2 \in W$ .

If we assume that  $\beta = 2$  and that there exist  $u \in U$ ,  $N \in \mathbb{N}$  with  $T^N u \in W$  and  $T^N u^2 \in V$ , then we get successively

$$\begin{aligned} \|T^N u^2(x) - x\| &< \frac{1}{4} \quad (\text{since } T^N u^2 \in V) \\ \|T^N u^2(x)\| &< \frac{1}{8} \|u(x)\| \quad (\text{since } T^N u \in W) \\ \|u(x) - x\| &< \frac{1}{4}. \end{aligned}$$

These three inequalities yield easily a contradiction.  $\square$

### 8.3. Further question and remark

As independently shown by Ansari in [3] and later-on by Bernal-González in [4], every separable Banach space supports a hypercyclic operator. In the context of algebras a natural question arises.

**QUESTION 8.6.** Is it true that every separable Banach algebra supports a hypercyclic operator admitting a hypercyclic algebra?

In all known results, the set of generators for hypercyclic algebras is either empty or residual. We observe below that this set can be non-empty and meager.

**REMARK 8.7.** For every pair  $(X, T)$  where  $X$  is a Banach space and  $T$  a hypercyclic operator with a non-hypercyclic vector, we may define a product on  $X$  turning it into a commutative Banach algebra and such that the set of generators for a hypercyclic algebra for  $T$  is non-empty and nowhere dense.

**PROOF.** Let  $x \in HC(T)$  and  $y$  a non-hypercyclic vector for  $T$  with  $\|y\| = 1$ . Consider  $f \in X^*$  with  $\|f\| = 1$  and such that  $f(x) = 0$  and  $f(y) = 1$ . We define the product

$$z \cdot w = f(z)f(w)y, \text{ with } z, w \in X$$

and observe that it turns  $X$  into a commutative Banach algebra. Now,  $x^2 = 0$  so  $A(x) = \text{span}(x)$ , and thus  $x$  is a generator for a hypercyclic algebra for  $T$ . Moreover, it is easy to check that the following holds,

$$\{x \in X : A(x) \setminus \{0\} \subset HC(T)\} = HC(T) \cap Ker(f).$$

Since  $Ker(f)$  is a proper, closed hyperplane of  $X$ , we conclude that the set of generators for a hypercyclic algebra for  $T$  is non-empty and nowhere dense.  $\square$



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