



HAL
open science

Variational solutions to an evolution model for MEMS with heterogeneous dielectric properties

Philippe Laurençot, Christoph Walker

► **To cite this version:**

Philippe Laurençot, Christoph Walker. Variational solutions to an evolution model for MEMS with heterogeneous dielectric properties. *Discrete and Continuous Dynamical Systems - Series S*, 2021, 14 (2), pp.677–694. hal-02308431

HAL Id: hal-02308431

<https://hal.science/hal-02308431>

Submitted on 8 Oct 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

VARIATIONAL SOLUTIONS TO AN EVOLUTION MODEL FOR MEMS WITH HETEROGENEOUS DIELECTRIC PROPERTIES

PHILIPPE LAURENÇOT AND CHRISTOPH WALKER

ABSTRACT. The existence of weak solutions to the obstacle problem for a nonlocal semilinear fourth-order parabolic equation is shown, using its underlying gradient flow structure. The model governs the dynamics of a microelectromechanical system with heterogeneous dielectric properties.

1. INTRODUCTION

The existence of variational solutions is shown for an evolution problem describing the space-time dynamics of a microelectromechanical system (MEMS) with heterogeneous dielectric properties. Specifically, a MEMS device such a switch is made of a thin rigid conducting plate above which a thin conducting elastic plate is suspended and clamped at its boundary. The shape of the undeformed elastic plate is identical to that of the rigid one. Holding the two plates at different electrostatic potentials generates a deformation of the top elastic plate to compensate the induced electrostatic force. It is by now well-known that a sufficiently large potential difference can lead to a *pull-in instability* or *touchdown*, a situation which corresponds to the top plate coming into contact with the bottom one and results in a short circuit due to the potential difference [2, 4, 11, 16, 17]. Clearly, such a phenomenon may alter the properties or the operating conditions of the MEMS device. However, it can be prevented, for instance, by covering the ground plate with an insulating layer of positive thickness [2, 4, 13, 14]. We consider this situation herein and thus assume that the bottom plate is covered by a non-deformable layer of positive thickness, possibly having heterogeneous dielectric properties characterized by a permittivity $\sigma_1 > 0$, which differs from the constant permittivity $\sigma_2 > 0$ of the surrounding medium. Touchdown may still occur, in the sense that the top plate may come into contact with the upper side of the insulating layer. However, such a situation does not generate a singularity, as the top plate cannot penetrate the layer. Assuming further that the physical state of the MEMS device is fully described by the deformation u of the top plate and the electrostatic potential ψ_u between the two plates, the dynamics of the MEMS is then governed by the competition between mechanical and electrostatic forces, and is given by a time relaxation towards critical points of the total energy, the latter including mechanical, contact, and electrostatic contributions.

To convert this rough description of the model into mathematical equations, we assume that there is no variation in one of the two horizontal directions and consider a two-dimensional MEMS device in which the rigid ground plate and the undeformed elastic plate have the same one-dimensional shape $D := (-L, L)$, $L > 0$, the former being located at height $z = -H - d$,

Date: October 8, 2019.

2010 Mathematics Subject Classification. 35K86, 74H20, 35Q74, 35M86, 35K25.

Key words and phrases. MEMS, gradient flow, transmission problem, obstacle problem, fourth-order equation.

Partially supported by the CNRS Projet International de Coopération Scientifique PICS07710.

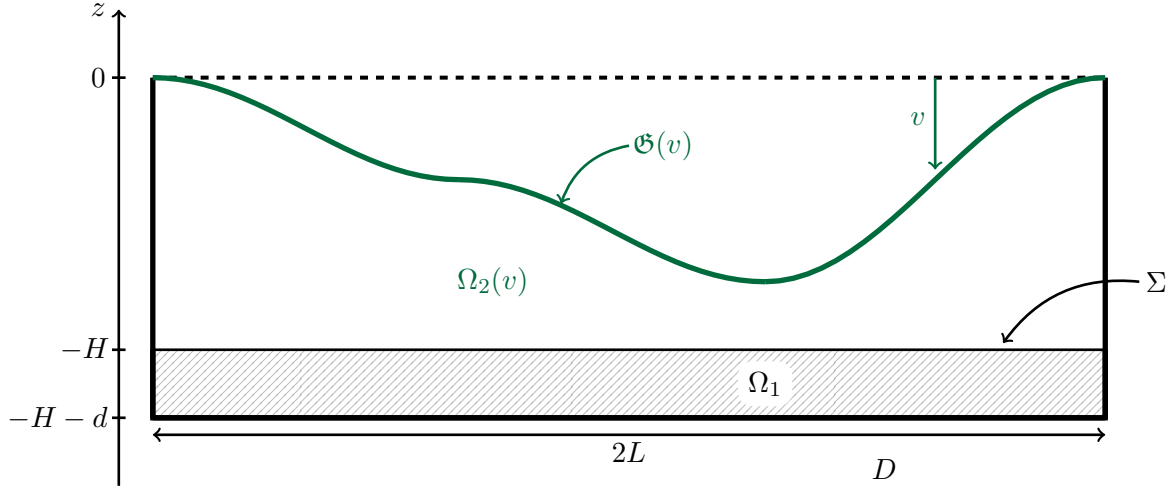


FIGURE 1. Geometry of $\Omega(u)$ for a state $u = v$ with empty coincidence set (green).

$H > 0$, $d > 0$, while the latter is clamped at its boundary $(\pm L, 0)$. The bottom plate is then $D \times \{-H - d\}$ and it is covered by an insulating layer

$$\Omega_1 := D \times (-H - d, -H)$$

of positive thickness d . The dielectric properties of Ω_1 are characterized by the permittivity $\sigma_1 > 0$ which may vary with the horizontal coordinate $x \in D$ and the height $z \in (-H - d, -H)$. As for the elastic top plate, we assume that the relevant physical framework is restricted to small deformations and, denoting the vertical deformation of the top plate by u , the top plate is given by

$$\mathfrak{G}(u) := \{(x, z) \in D \times \mathbb{R} : z = u(x)\}.$$

Recalling that the top plate cannot be deformed beyond the surface $\Sigma := D \times \{-H\}$ of the insulated layer Ω_1 , the deformation u is bounded from below by $-H$. The region $\Omega_2(u)$ between the top plate and the surface of the insulating layer is defined by

$$\Omega_2(u) := \{(x, z) \in D \times \mathbb{R} : -H < z < u(x)\}.$$

The dielectric permittivity is assumed to be a positive constant $\sigma_2 > 0$ in $\Omega_2(u)$ and differs in general from $\sigma_1(\cdot, -H)$, so that there is a jump discontinuity of the permittivity across the interface

$$\Sigma(u) := \{(x, -H) : x \in D \text{ and } u(x) > -H\},$$

separating Ω_1 and $\Omega_2(u)$. Observe that $\Omega_2(u)$ is an open subset of $D \times (-H, \infty)$, which is connected when the coincidence set

$$\mathcal{C}(u) := \{x \in D : u(x) = -H\}. \quad (1.1)$$

is empty, while it is disconnected and non-Lipschitz (if u is smooth enough) otherwise. The two situations are depicted in Figure 1 and Figure 2, respectively. Independent of whether or not $\mathcal{C}(u)$ is empty, the domain

$$\Omega(u) := \Omega_1 \cup \Sigma(u) \cup \Omega_2(u),$$

is Lipschitz (again if u is smooth enough).

$D \times \{-H - d\}$ and $h_u \equiv V$ along the elastic top plate $\mathfrak{G}(u)$ with positive potential value $V > 0$ (see assumption (2.3e) below).

The modeling assumption is then that the evolution of u is governed (at least formally) by the gradient flow associated with E , which reads

$$\partial_t u + \partial_u E(u) = 0, \quad t > 0, \quad u(0) = u_0, \quad (1.4)$$

supplemented with clamped boundary conditions; that is,

$$u(t) \in H_D^2(D) := \{v \in H^2(D) : v(\pm L) = \partial_x v(\pm L) = 0\}, \quad t \geq 0.$$

As already noted in [12, Section 5], where we studied the existence of minimizers of E , the interpretation of equation (1.4) needs some care for several reasons:

First, the contact energy involves a non-smooth convex function and it is rather the notion of subdifferential which is appropriate and requires a suitable functional setting. Specifically, we define

$$S_0 := \{v \in H_D^2(D) : v > -H \text{ in } D\}, \quad \bar{S}_0 := \{v \in H_D^2(D) : v \geq -H \text{ in } D\},$$

and recall that \bar{S}_0 is a closed convex set in $H_D^2(D)$. The “derivative” of E_c with respect to u is then given by the subdifferential $\partial \mathbb{I}_{\bar{S}_0}(u)$ of the indicator function $\mathbb{I}_{\bar{S}_0}$ of the set \bar{S}_0 . It is a subset of the dual space

$$H^{-2}(D) := (H_D^2(D))'$$

of $H_D^2(D)$ and, for $v \in \bar{S}_0$, it is given by:

$$\xi \in \partial \mathbb{I}_{\bar{S}_0}(v) \iff \langle \xi, w - v \rangle_{H_D^2} \leq 0, \quad w \in \bar{S}_0,$$

where $\langle \cdot, \cdot \rangle_{H_D^2}$ denotes the duality pairing between $H^{-2}(D)$ and $H_D^2(D)$.

Second, the electrostatic energy $E_e(u)$ depends on u not only through the integral over $\Omega(u)$ but also through the solution ψ_u to the transmission problem (1.3). Its differentiability is then a tricky and by no means obvious issue but can be handled with the help of shape optimization tools. It follows from the analysis performed in [12] that the functional E_e at $u \in \bar{S}_0$ has a directional derivative $g(u)$ given by

$$g(u)(x) := \begin{cases} \frac{\sigma_2}{2} (1 + (\partial_x u(x))^2) (\partial_z \psi_{u,2}(x, u(x)))^2, & x \in D \setminus \mathcal{C}(u), \\ \frac{\sigma_1(x, -H)^2}{2\sigma_2} (\partial_z \psi_{u,1}(x, -H))^2, & x \in \mathcal{C}(u), \end{cases} \quad (1.5)$$

where $\psi_{u,1} := \psi_u|_{\Omega_1}$ and $\psi_{u,2} := \psi_u|_{\Omega_2(u)}$. In fact, $g(u)$ corresponds to the electrostatic force acting on the elastic plate. It is worth emphasizing here that the derivation of this result owes much to the book by Henrot & Pierre [10] (and its french version [9]), which has been a constant source of inspiration in our studies of differentiability properties of the electrostatic energy involved in the modeling of MEMS. In fact, for $u \in S_0$, the coincidence set $\mathcal{C}(u)$ defined in (1.1) is empty and the functional E_e is actually Fréchet differentiable at u , a feature which is proved along the lines of [10, Sections 5.3.3-5.3.4], see [12, Proposition 4.2]. The formula (1.5) for $g(u)$ then reduces to its first line. The situation is strikingly different when the coincidence set $\mathcal{C}(u)$ is non-empty, which corresponds to $u \in \bar{S}_0 \setminus S_0$. In that situation, the trace of the solution ψ_u to (1.3) at a point $(x, -H)$, $x \in D$, is given either by the transmission condition (1.3b) (if $x \in D \setminus \mathcal{C}(u)$) or by the Dirichlet boundary condition (1.3c) (if $x \in \mathcal{C}(u)$) and both cases may alternate infinitely often while x ranges in D . Identifying the derivative of E_e at such a function u requires a rather delicate analysis, see [12, Corollary 4.3] and Lemma 3.3 below for a precise statement. Let us finally point

out that the functional $g(u)$ involves the traces of $\partial_z \psi_u$ on $\mathfrak{G}(u)$ and $\mathcal{C}(u) \times \{-H\}$, which are well-defined only if ψ_u is sufficiently regular. However, since $\Omega(u)$ is only a Lipschitz domain while $\Omega_2(u)$ might be even non-Lipschitz when $\mathcal{C}(u) \neq \emptyset$, the $H^2(\Omega_1)$ -regularity of $\psi_{u,1}$ and the $H^2(\Omega_2(u))$ -regularity of $\psi_{u,2}$ are not straightforward and a large part of the analysis performed in [12] is devoted to this regularity issue.

Since the computation of the derivative of the mechanical energy with respect to u is classical, collecting the outcome of the above discussion yields the following parabolic variational inequality for u :

$$\partial_t u + \beta \partial_x^4 u - (\tau + a \|\partial_x u\|_2^2) \partial_x^2 u + \partial \mathbb{I}_{\bar{S}_0}(u) \ni -g(u) \quad \text{in } (0, \infty) \times D, \quad (1.6a)$$

supplemented with the constraint

$$u(t) \in \bar{S}_0, \quad t \geq 0, \quad (1.6b)$$

and the initial condition

$$u(0) = u_0, \quad x \in D. \quad (1.6c)$$

Assuming $\beta > 0$, we note that (1.6a) is a fourth-order parabolic variational inequality and the main purpose of this paper is to investigate the existence of weak solutions to (1.6) for a suitable class of boundary data h_u occurring in (1.3c), see (2.3) below. In the following, we interpret $\partial_x^4 v$ for $v \in \bar{S}_0$ as an element of $H^{-2}(D)$ by virtue of

$$\langle \partial_x^4 v, \phi \rangle_{H_D^2} := \int_D \partial_x^2 v \partial_x^2 \phi \, dx, \quad \phi \in H_D^2(D).$$

A definition of a weak solution to (1.6) is then as follows.

Definition 1.1. *Let $\beta > 0$ and $u_0 \in \bar{S}_0$. A weak solution to (1.6) is a function*

$$u \in C([0, \infty), H^1(D)) \cap L_{\infty, \text{loc}}([0, \infty), H_D^2(D)) \cap H_{\text{loc}}^1([0, \infty), L_2(D))$$

satisfying (1.6b), (1.6c), the weak formulation

$$\begin{aligned} \int_D (u(t) - u_0) v \, dx &= - \int_0^t \int_D \{g(u(s)) - (\tau + a \|\partial_x u(s)\|_2^2) \partial_x^2 u(s)\} v \, dx ds \\ &\quad - \int_0^t \int_D \beta \partial_x^2 u(s) \partial_x^2 v \, dx ds - \int_0^t \langle \zeta(s), v \rangle_{H_D^2(D)} \, ds \end{aligned}$$

for any $t > 0$ and $v \in H_D^2(D)$, where

$$\zeta := -\partial_t u - \beta \partial_x^4 u + (\tau + a \|\partial_x u\|_2^2) \partial_x^2 u - g(u) \in L_{2, \text{loc}}([0, \infty), H^{-2}(D)) \quad (1.7)$$

with

$$\zeta(t) \in \partial \mathbb{I}_{\bar{S}_0}(u(t)) \quad \text{for a.a. } t \geq 0, \quad (1.8)$$

and the energy inequality

$$\frac{1}{2} \int_0^t \|\partial_t u(s)\|_2^2 \, ds + E(u(t)) \leq E(u_0), \quad t > 0. \quad (1.9)$$

The main result of this paper is then the following existence result.

Theorem 1.2. *Let $\beta > 0$ and $\tau, a \geq 0$. Suppose that the functions σ and h satisfy, respectively, (2.1) and (2.3) below. Then, given $u_0 \in \bar{S}_0$, there is a weak solution to (1.6) in the sense of Definition 1.1.*

Owing to the variational structure (1.4) of (1.6), the proof of Theorem 1.2 is performed with the help of a time implicit Euler scheme. Given a time step $\delta > 0$, we construct by induction a sequence $(u_n^\delta)_{n \geq 0}$ such that $u_0^\delta := u_0$ and, for $n \geq 0$, u_{n+1}^δ is a minimizer of the auxiliary functional

$$F_n^\delta(v) := \frac{1}{2\delta} \|v - u_n^\delta\|^2 + E(v), \quad v \in \bar{S}_0.$$

Since F_n^δ includes a negative contribution from the electrostatic energy E_e , we begin the proof by showing that F_n^δ is bounded below, provided $\delta \in (0, \delta_0)$ is sufficiently small, the smallness condition depending only on D , the parameters H, d, β, τ, a , the permittivity σ , and the function h defining the boundary data in (1.3c). The existence of a minimizer u_{n+1}^δ of F_n^δ on \bar{S}_0 then relies on the lower semicontinuity of the convex part $E_m + E_c$ of the energy and the properties of the electrostatic energy E_e established in [12]. As a consequence of u_{n+1}^δ being a minimizer of F_n^δ on \bar{S}_0 , we further derive a handful of estimates on $(u_n^\delta)_{n \geq 1}$, which allows us to show that the family $(u^\delta)_{\delta \in (0, \delta_0)}$ of piecewise constant functions in time defined by

$$u^\delta(t) := \sum_{n=0}^{\infty} u_n^\delta \mathbf{1}_{[n\delta, (n+1)\delta)}(t), \quad t \geq 0,$$

has the right compactness properties, so that its cluster points as $\delta \rightarrow 0$ are weak solutions to (1.6) in the sense of Definition 1.1. Here again, a key ingredient in the proof is the continuity of the functional g defined in (1.5), which we established in [12], see Lemma 3.2 below.

Finally, we address the regularity of the distribution $\zeta \in \partial \mathbb{I}_{\bar{S}_0}(u)$ associated with a weak solution u to (1.6) and given by (1.7)-(1.8). As already mentioned, since $\beta > 0$, equation (1.6) is a fourth-order parabolic variational inequality and, as such, the regularity of $\zeta(t)$ stemming from (1.8) is that it is a distribution in $H^{-2}(D)$ for a.e. $t > 0$. In fact, since the seminal work [5], regularity for the obstacle problem for the biharmonic parabolic equation has received less attention than the same issue for the obstacle problem for second-order parabolic equations. The only regularity result regarding the obstacle problem for the biharmonic parabolic equation we are aware of is [15], whereas [6, 8, 18, 19] are devoted to the elliptic analogue. As in [15], we can prove that $-\zeta(t)$ is actually a non-negative bounded Radon measure on D for a.e. $t > 0$.

Corollary 1.3. *Let the assumptions of Theorem 1.2 be satisfied. If u is a weak solution to (1.6) in the sense of Definition 1.1, then $u \in L_{2,loc}([0, \infty), H^s(D))$ for $s \in (2, 7/2)$ and $-\zeta \in L_{2,loc}([0, \infty), \mathcal{M}_+(D))$, where $\mathcal{M}_+(D)$ is the positive cone of the space $\mathcal{M}(D) = C_0(D)'$ of bounded Radon measures on D .*

Let us finally describe the contents of this paper: in the next section, we state the assumptions on the permittivity σ and the boundary data h_u in (1.3c). In Section 3, we recall the well-posedness of the transmission problem (1.3) and the regularity of its solution established in [12], along with the properties of E_e and g from [12] which are needed for the analysis performed below. We also show in this section the existence of a minimizer u_{n+1}^δ of the functional F_n^δ on \bar{S}_0 for sufficiently small values of the time step $\delta > 0$. After this preparation, we are in a position to prove Theorem 1.2 in Section 4 and Corollary 1.3 in Section 5.

2. ASSUMPTIONS

We provide now the detailed assumptions we put on the permittivity σ and the boundary data h_u occurring in the transmission problem (1.3). As already mentioned, the dielectric properties of

the device are accounted for by the permittivity σ , which is defined by

$$\sigma(x, z) := \begin{cases} \sigma_1(x, z) & \text{for } (x, z) \in \Omega_1, \\ \sigma_2 & \text{for } (x, z) \in D \times (-H, \infty), \end{cases}$$

where

$$\sigma_1 \in C^2(\overline{\Omega}_1) \text{ with } \sigma_1 > 0 \text{ in } \overline{\Omega}_1, \quad \sigma_2 \in (0, \infty). \quad (2.1)$$

In particular, there are $0 < \sigma_{min} < \sigma_{max}$ such that

$$\sigma_{min} \leq \sigma(x, z) \leq \sigma_{max}, \quad (x, z) \in \bar{D} \times [-H, \infty). \quad (2.2)$$

We fix C^2 -functions

$$h_1 : \bar{D} \times [-H - d, -H] \times [-H, \infty) \rightarrow [0, \infty) \quad (2.3a)$$

and

$$h_2 : \bar{D} \times [-H, \infty) \times [-H, \infty) \rightarrow [0, \infty) \quad (2.3b)$$

satisfying

$$h_1(x, -H, w) = h_2(x, -H, w), \quad (x, w) \in D \times [-H, \infty), \quad (2.3c)$$

$$\sigma_1(x, -H) \partial_z h_1(x, -H, w) = \sigma_2 \partial_z h_2(x, -H, w), \quad (x, w) \in D \times [-H, \infty). \quad (2.3d)$$

Moreover, we assume that

$$h_1(x, -H - d, w) = h_2(x, w, w) - V = 0, \quad (x, w) \in \bar{D} \times [-H, \infty), \quad (2.3e)$$

where $V > 0$, and that there are constants $m_i > 0$, $i = 1, 2, 3$, such that

$$|\partial_x h_1(x, z, w)| + |\partial_z h_1(x, z, w)| \leq \sqrt{m_1 + m_2 w^2}, \quad |\partial_w h_1(x, z, w)| \leq \sqrt{m_3}, \quad (2.3f)$$

for $(x, z, w) \in \bar{D} \times [-H - d, -H] \times [-H, \infty)$ and

$$|\partial_x h_2(x, z, w)| + |\partial_z h_2(x, z, w)| \leq \sqrt{\frac{m_1 + m_2 w^2}{H + w}}, \quad |\partial_w h_2(x, z, w)| \leq \sqrt{\frac{m_3}{H + w}}, \quad (2.3g)$$

for $(x, z, w) \in \bar{D} \times [-H, \infty) \times [-H, \infty)$.

A typical example for a function h satisfying the assumptions (2.3) above was given in [12, Example 5.5] which we recall now.

Example 2.1. *Let us consider the situation where σ_1 does not depend on the vertical variable z ; that is, $\sigma_1 = \sigma_1(x)$. In that case, we set*

$$h_1(x, z, w) := V \frac{\sigma_2(H + z + d)}{\sigma_2 d + \sigma_1(x)(H + w)}, \quad (x, z, w) \in \bar{D} \times [-H - d, -H] \times [-H, \infty),$$

and

$$h_2(x, z, w) := V \frac{\sigma_2 d + \sigma_1(x)(H + z)}{\sigma_2 d + \sigma_1(x)(H + w)}, \quad (x, z, w) \in \bar{D} \times [-H, \infty) \times [-H, \infty).$$

Then assumptions (2.3) are easily checked.

For a given function $v \in \bar{S}_0$ we then define

$$h_v(x, z) := \begin{cases} h_{v,1}(x, z) := h_1(x, z, v(x)), & (x, z) \in \overline{\Omega}_1, \\ h_{v,2}(x, z) := h_2(x, z, v(x)), & (x, z) \in \bar{D} \times [-H, \infty). \end{cases} \quad (2.4)$$

Let us point out that assumption (2.3c)-(2.3d) guarantee that h_v defined in (2.4) satisfies the transmission conditions (1.3b), that is,

$$\llbracket h_v \rrbracket = \llbracket \sigma \partial_z h_v \rrbracket = 0 \quad \text{on } \Sigma(v),$$

while assumption (2.3e) along with (1.3c) entails that the electrostatic potential ψ_v equals zero on the bottom plate $D \times \{-H - d\}$ and equals V along the elastic plate $\mathfrak{G}(u)$. Assumptions (2.3f)-(2.3g) are required to guarantee the coercivity of the total energy $E(v)$.

Throughout the paper, c and $(c_i)_{i \geq 1}$ denote positive constants depending only on $D, H, d, \beta, \tau, a, \sigma, (m_i)_{1 \leq i \leq 3}$, and u_0 . The dependence upon additional parameters will be indicated explicitly.

3. AUXILIARY RESULTS

We first recall some results that were derived in [12] and begin with the well-posedness of (1.3).

Lemma 3.1. [12, Theorem 1.1] *Suppose (2.3). For each $v \in \bar{S}_0$, there is a unique variational solution $\psi_v \in h_v + H_0^1(\Omega(v))$ to (1.3). Moreover, $\psi_{v,1} := \psi_v|_{\Omega_1} \in H^2(\Omega_1)$, $\psi_{v,2} := \psi_v|_{\Omega_2(v)} \in H^2(\Omega_2(v))$, and ψ_v is a strong solution to the transmission problem (1.3) satisfying $\sigma \partial_z \psi_v \in H^1(\Omega(v))$.*

The regularity of ψ_v stated in Lemma 3.1 guarantees that $g(v)$ defined in (1.5) is meaningful for $v \in \bar{S}_0$. We collect in the next result some properties of g established in [12].

Lemma 3.2. *Suppose (2.3).*

(a) *If $v \in \bar{S}_0$ and $(v_j)_{j \geq 1} \subset \bar{S}_0$ are such that $v_j \rightharpoonup v$ in $H^2(D)$, then*

$$\lim_{j \rightarrow \infty} \|g(v_j) - g(v)\|_2 = 0 \quad \text{and} \quad \lim_{j \rightarrow \infty} E_e(v_j) = E_e(v).$$

(b) *The mapping $g : \bar{S}_0 \rightarrow L_2(D)$ is continuous and bounded on bounded sets, the set \bar{S}_0 being endowed with the topology of $H^2(D)$.*

Proof. Since weak convergence in $H^2(D)$ implies boundedness in $H^2(D)$ and strong convergence in $H^1(D)$, part (a) is shown in [12, Proposition 3.17 & Corollary 3.12]. As for part (b), the continuity of $g : \bar{S}_0 \rightarrow L_2(D)$ follows from [12, Theorem 1.4], while the boundedness of g on bounded sets is a consequence of [12, Corollary 3.14 & Lemma 3.16] and the continuity of the trace from $H^1(\Omega_1)$ to $L_p(D \times \{-H\})$ for all $p \in [1, \infty)$. \square

We next turn to differentiability properties of E_e . As observed in [12], E_e need not be Fréchet differentiable for all $u \in \bar{S}_0$ but it has always directional derivatives.

Lemma 3.3. [12, Proposition 5.6] *Suppose (2.3). Let $v \in \bar{S}_0$ and $w \in S_0$. Then*

$$\lim_{s \rightarrow 0^+} \frac{1}{s} (E_e(v + s(w - v)) - E_e(v)) = \int_D g(v)(w - v) \, dx.$$

We also derive a lower bound on E .

Lemma 3.4. *Suppose (2.3). There is a constant $c_1 > 0$ such that*

$$E(v) \geq \frac{\beta}{4} \|\partial_x^2 v\|_2^2 - c_1 (1 + \|v\|_2^2), \quad v \in \bar{S}_0.$$

Proof. For the sake of completeness, we recall the proof performed in [12, Theorem 5.1]. Since ψ_v is a variational solution to (1.3), it follows from (2.2), (2.3), and Young's inequality that

$$\begin{aligned} -E_e(v) &= \frac{1}{2} \int_{\Omega(v)} \sigma |\nabla \psi_v|^2 \, d(x, z) \leq \frac{1}{2} \int_{\Omega(v)} \sigma |\nabla h_v|^2 \, d(x, z) \\ &\leq \int_{\Omega(v)} \sigma \left[(\partial_x h(x, z, v(x)))^2 + (\partial_w h(x, z, v(x)))^2 (\partial_x v(x))^2 \right] \, d(x, z) \\ &\quad + \frac{1}{2} \int_{\Omega(v)} \sigma (\partial_z h(x, z, v(x)))^2 \, d(x, z) \\ &\leq (d+1) \sigma_{\max} \int_D \left[\frac{3}{2} (m_1 + m_2 v(x)^2) + m_3 (\partial_x v(x))^2 \right] \, dx. \end{aligned}$$

We next use Poincaré's inequality

$$\|w\|_2 \leq |D| \|\partial_x w\|_2, \quad w \in H_0^1(D), \quad (3.1)$$

and the interpolation inequality

$$\|\partial_x w\|_2^2 \leq \|w\|_2 \|\partial_x^2 w\|_2, \quad w \in H^2(D) \cap H_0^1(D), \quad (3.2)$$

to obtain

$$-E_e(v) \leq c (1 + \|\partial_x v\|_2^2) \leq c (1 + \|v\|_2 \|\partial_x^2 v\|_2).$$

Consequently, Young's inequality and the above upper bound yield

$$E(v) \geq \frac{\beta}{2} \|\partial_x^2 v\|_2^2 - c - c \|v\|_2 \|\partial_x^2 v\|_2 \geq \frac{\beta}{4} \|\partial_x^2 v\|_2^2 - c - c \|v\|_2^2,$$

and the proof is complete. \square

We next provide the basis for the time implicit scheme used later in order to construct a solution to (1.6).

Lemma 3.5. *Set $\delta_0 := \min\{1, (16c_1)^{-1}\} > 0$. Then, for any $\delta \in (0, \delta_0)$ and $f \in \bar{S}_0$, there is $v \in \bar{S}_0$ such that*

$$-\frac{1}{\delta}(v - f) - \beta \partial_x^4 v + (\tau + a \|\partial_x v\|_2^2) \partial_x^2 v - g(v) \in \partial \mathbb{I}_{\bar{S}_0}(v).$$

Moreover,

$$\frac{1}{2\delta} \|v - f\|_2^2 + E(v) \leq E(f).$$

Proof. The proof relies on the direct method of calculus of variations. Consider $\delta \in (0, \delta_0)$ and $f \in \bar{S}_0$ and define

$$\mathcal{F}(v) := \frac{1}{2\delta} \|v - f\|_2^2 + E(v), \quad v \in \bar{S}_0.$$

Then, by Lemma 3.4 and Young's inequality,

$$\begin{aligned} \mathcal{F}(v) &\geq \frac{1}{2\delta} \left(\frac{\|v\|_2^2}{2} - \|f\|_2^2 \right) + \frac{\beta}{4} \|\partial_x^2 v\|_2^2 - c_1 (1 + \|v\|_2^2) \\ &\geq \frac{\beta}{4} \|\partial_x^2 v\|_2^2 + \left(\frac{1}{4\delta} - c_1 \right) \|v\|_2^2 - c_1 - \frac{\|f\|_2^2}{2\delta} \\ &\geq \frac{\beta}{4} \|\partial_x^2 v\|_2^2 + 3c_1 \|v\|_2^2 - c_1 - \frac{\|f\|_2^2}{2\delta} \end{aligned}$$

for all $v \in \bar{S}_0$. Thus, \mathcal{F} is bounded from below on \bar{S}_0 and there is a minimizing sequence $(v_j)_{j \geq 1}$ in \bar{S}_0 satisfying

$$\mu := \inf_{v \in \bar{S}_0} \{\mathcal{F}(v)\} \leq \mathcal{F}(v_j) \leq \mu + \frac{1}{j}, \quad j \geq 1. \quad (3.3)$$

Moreover, the previous lower bound on \mathcal{F} guarantees that $(v_j)_{j \geq 1}$ is bounded in $H_D^2(D)$. Therefore, there is $v \in H_D^2(D)$ such that (up to a subsequence)

$$v_j \rightharpoonup v \text{ in } H^2(D), \quad (3.4a)$$

$$v_j \longrightarrow v \text{ in } H_0^1(D). \quad (3.4b)$$

Clearly, (3.4a) ensures that $v \in \bar{S}_0$ since \bar{S}_0 is closed and convex in $H_D^2(D)$. It then follows from (3.4a) and Lemma 3.2 **(a)** that

$$E_e(v) = \lim_{j \rightarrow \infty} E_e(v_j),$$

while (3.4) and the weak lower semicontinuity of the L_2 -norm readily imply that

$$E_m(v) \leq \liminf_{j \rightarrow \infty} E_m(v_j).$$

Consequently,

$$\mathcal{F}(v) \leq \liminf_{j \rightarrow \infty} \mathcal{F}(v_j) = \mu,$$

and we conclude that $v \in \bar{S}_0$ is a minimizer of \mathcal{F} on \bar{S}_0 . This property, in turn, guarantees that, for $w \in S_0$,

$$0 \leq \liminf_{s \rightarrow 0^+} \frac{1}{s} (\mathcal{F}(v + s(w - v)) - \mathcal{F}(v)).$$

It then follows from Lemma 3.3 that

$$\begin{aligned} 0 \leq & \int_D \left\{ \frac{v - f}{\delta} (w - v) + \beta \partial_x^2 v \partial_x^2 (w - v) - (\tau + a \|\partial_x v\|_2^2) \partial_x v \partial_x (w - v) \right\} dx \\ & + \int_D g(v)(w - v) dx \end{aligned}$$

for all $w \in S_0$. Since S_0 is dense in \bar{S}_0 , this inequality also holds for any $w \in \bar{S}_0$. Therefore,

$$-\frac{1}{\delta}(v - f) - \beta \partial_x^4 v + (\tau + a \|\partial_x v\|_2^2) \partial_x^2 v - g(v) \in \partial \mathbb{I}_{\bar{S}_0}(v).$$

Finally, since $f \in \bar{S}_0$, we have $\mathcal{F}(v) \leq \mathcal{F}(f)$, which completes the proof. \square

4. PROOF OF THEOREM 1.2

Fix $u_0 \in \bar{S}_0$. For $\delta \in (0, \delta_0)$, we set $u_0^\delta := u_0$ and, using Lemma 3.5, we construct by induction a sequence $(u_n^\delta, \zeta_n^\delta)_{n \geq 1}$ in $\bar{S}_0 \times H^{-2}(D)$ such that

$$\zeta_{n+1}^\delta := -A_{n+1}^\delta - \beta \partial_x^4 u_{n+1}^\delta + (\tau + a \|\partial_x u_{n+1}^\delta\|_2^2) \partial_x^2 u_{n+1}^\delta - g(u_{n+1}^\delta) \in \partial \mathbb{I}_{\bar{S}_0}(u_{n+1}^\delta), \quad (4.1)$$

where

$$A_{n+1}^\delta := \frac{1}{\delta} (u_{n+1}^\delta - u_n^\delta) \in H_D^2(D),$$

and

$$\frac{1}{2\delta} \|u_{n+1}^\delta - u_n^\delta\|_2^2 + E(u_{n+1}^\delta) \leq E(u_n^\delta) \quad (4.2)$$

for $n \geq 0$. Let us first note that (4.2) implies

$$\frac{1}{2\delta} \sum_{j=0}^n \|u_{j+1}^\delta - u_j^\delta\|_2^2 + E(u_{n+1}^\delta) \leq E(u_0), \quad n \geq 0. \quad (4.3)$$

A first consequence of (4.3) is an L_2 -estimate on $(u_n^\delta)_{n \geq 1}$, which is adapted from [3, Lemma 3.2.2]. More precisely, it follows from Hölder's and Young's inequalities that, for $n \geq 0$,

$$\begin{aligned} \|u_{n+1}^\delta\|_2^2 - \|u_0\|_2^2 &= \sum_{j=0}^n \left(\|u_{j+1}^\delta\|_2^2 - \|u_j^\delta\|_2^2 \right) \\ &= \sum_{j=0}^n \int_D \left(u_{j+1}^\delta - u_j^\delta \right) \left(u_{j+1}^\delta + u_j^\delta \right) dx \\ &\leq \sum_{j=0}^n \|u_{j+1}^\delta - u_j^\delta\|_2 \|u_{j+1}^\delta + u_j^\delta\|_2 \\ &\leq \frac{1}{4c_1\delta} \sum_{j=0}^n \|u_{j+1}^\delta - u_j^\delta\|_2^2 + c_1\delta \sum_{j=0}^n \|u_{j+1}^\delta + u_j^\delta\|_2^2. \end{aligned}$$

We then infer from Lemma 3.4 and (4.3) that

$$\begin{aligned} \|u_{n+1}^\delta\|_2^2 - \|u_0\|_2^2 &\leq \frac{E(u_0) - E(u_{n+1}^\delta)}{2c_1} + 2c_1\delta \sum_{j=0}^n \left(\|u_{j+1}^\delta\|_2^2 + \|u_j^\delta\|_2^2 \right) \\ &\leq \frac{E(u_0) + c_1 + c_1 \|u_{n+1}^\delta\|_2^2}{2c_1} + 2c_1\delta \|u_0\|_2^2 + 4c_1\delta \sum_{j=1}^{n+1} \|u_j^\delta\|_2^2. \end{aligned}$$

Hence,

$$\|u_{n+1}^\delta\|_2^2 \leq (2 + 4c_1\delta) \|u_0\|_2^2 + 1 + \frac{E(u_0)}{c_1} + 8c_1\delta \sum_{j=1}^{n+1} \|u_j^\delta\|_2^2, \quad n \geq 0.$$

Since $\delta \in (0, \delta_0)$, we have $8c_1\delta < 1/2 < 1$ and we are thus in a position to apply a discrete version of Gronwall's lemma, see [3, Lemma 3.2.4], to conclude that

$$\|u_{n+1}^\delta\|_2^2 \leq \left(6\|u_0\|_2^2 + 2 + \frac{2E(u_0)}{c_1} \right) e^{16c_1(n+1)\delta}, \quad n \geq 0. \quad (4.4)$$

We next use again Lemma 3.4 and (4.3), along with (4.4), to obtain that, for $n \geq 0$,

$$\begin{aligned} &\frac{1}{2\delta} \sum_{j=0}^n \|u_{j+1}^\delta - u_j^\delta\|_2^2 + \frac{\beta}{4} \|\partial_x^2 u_{n+1}^\delta\|_2^2 \\ &\leq \frac{1}{2\delta} \sum_{j=0}^n \|u_{j+1}^\delta - u_j^\delta\|_2^2 + E(u_{n+1}^\delta) + c_1 \left(1 + \|u_{n+1}^\delta\|_2^2 \right) \\ &\leq E(u_0) + c_1 \left(1 + \|u_{n+1}^\delta\|_2^2 \right) \\ &\leq c_2 e^{16c_1 n \delta}. \end{aligned}$$

Owing to the functional inequalities (3.1) and (3.2), we end up with

$$\frac{1}{2\delta} \sum_{j=0}^n \|u_{j+1}^\delta - u_j^\delta\|_2^2 + \|u_{n+1}^\delta\|_{H^2}^2 \leq c_3 e^{16c_1 n \delta}, \quad n \geq 0. \quad (4.5)$$

We next define the functions $u^\delta, A^\delta : [0, \infty) \rightarrow H_D^2(D)$ and $\zeta^\delta : [0, \infty) \rightarrow H^{-2}(D)$ by

$$u^\delta(t) := \sum_{n \geq 0} u_n^\delta \mathbf{1}_{[n\delta, (n+1)\delta)}(t), \quad t \geq 0,$$

$$A^\delta(t) := \sum_{n \geq 1} A_n^\delta \mathbf{1}_{[n\delta, (n+1)\delta)}(t), \quad t \geq 0,$$

and

$$\zeta^\delta(t) := \sum_{n \geq 1} \zeta_n^\delta \mathbf{1}_{[n\delta, (n+1)\delta)}(t), \quad t \geq 0,$$

respectively.

Lemma 4.1. *There are a sequence $\delta_\ell \rightarrow 0$ and*

$$u \in C([0, \infty), H^1(D)) \cap L_\infty((0, \infty), H_D^2(D)) \cap H_{loc}^1([0, \infty), L_2(D))$$

with $u(0) = u_0$ such that

$$u(t) \in \bar{S}_0, \quad t \geq 0, \quad (4.6)$$

and, for all $t > 0$,

$$u^{\delta_\ell}(t) \rightarrow u(t) \text{ in } H^1(D), \quad (4.7a)$$

$$u^{\delta_\ell} \rightarrow u \text{ in } L_2((0, t), H_D^2(D)), \quad (4.7b)$$

$$g(u^{\delta_\ell}) \rightarrow g(u) \text{ in } L_2((0, t), L_2(D)), \quad (4.7c)$$

$$A^{\delta_\ell} \rightarrow \partial_t u \text{ in } L_2((0, t), L_2(D)). \quad (4.7d)$$

In particular,

$$\zeta^{\delta_\ell} \rightarrow \zeta \text{ in } L_2((0, t), H^{-2}(D))$$

for any $t > 0$ and

$$\zeta := -\partial_t u - \beta \partial_x^4 u + (\tau + a \|\partial_x u\|_2^2) \partial_x^2 u - g(u) \in L_{2,loc}([0, \infty), H^{-2}(D)). \quad (4.8)$$

Proof. Given $0 \leq t_1 < t_2$ there are integers $n_1 \leq n_2$ such that $t_i \in [n_i \delta, (n_i + 1) \delta)$, $i = 1, 2$. Either $n_1 = n_2$ and $u^\delta(t_2) = u^\delta(t_1)$. Or $n_1 < n_2$ and we infer from (4.5) and Hölder's inequality that

$$\begin{aligned} \|u^\delta(t_2) - u^\delta(t_1)\|_2 &= \|u_{n_2}^\delta - u_{n_1}^\delta\|_2 \leq \sum_{j=n_1}^{n_2-1} \|u_{j+1}^\delta - u_j^\delta\|_2 \\ &\leq \sqrt{n_2 - n_1} \left(\sum_{j=n_1}^{n_2-1} \|u_{j+1}^\delta - u_j^\delta\|_2^2 \right)^{1/2} \\ &\leq \sqrt{2c_3} \sqrt{n_2 \delta - n_1 \delta} e^{8c_1 n_2 \delta}. \end{aligned}$$

Thus,

$$\|u^\delta(t_2) - u^\delta(t_1)\|_2 \leq \sqrt{2c_3} \sqrt{t_2 - t_1 + \delta} e^{8c_1 t_2}, \quad 0 \leq t_1 < t_2. \quad (4.9)$$

Moreover, for $t > 0$, there is an integer $n \geq 0$ such that $t \in [n\delta, (n+1)\delta)$ and, either $n = 0$ and $u^\delta(t) = u_0$, or, again by (4.5), $\|u^\delta(t)\|_{H^2}^2 \leq c_3 e^{16c_1 t}$. Consequently,

$$\|u^\delta(t)\|_{H^2} \leq c_4 e^{16c_1 t}, \quad t \geq 0. \quad (4.10)$$

Since $H^2(D)$ embeds compactly in $L_2(D)$, we infer from (4.9) and (4.10) that we may apply a variant of the Arzelà-Ascoli theorem, see [3, Proposition 3.3.1], and a diagonal argument to obtain the existence of a sequence $(\delta_\ell)_{\ell \geq 1}$, $\delta_\ell \rightarrow 0$, and

$$u \in C([0, \infty), L_2(D)) \cap L_{\infty, \text{loc}}([0, \infty), H_D^2(D))$$

such that

$$u^{\delta_\ell}(t) \rightarrow u(t) \text{ in } L_2(D), \quad t \geq 0, \quad (4.11)$$

and

$$u^{\delta_\ell} \rightharpoonup u \text{ in } L_2((0, T), H_D^2(D)), \quad T > 0.$$

We have thus proved (4.7b). Next, an interpolation argument, together with (4.10) and (4.11), yields (4.7a) and the stated time continuity of u in $H^1(D)$. Furthermore, combining (4.7a), (4.10), and Lemma 3.2 allows one to apply Lebesgue's theorem to deduce (4.7c). Also, since $u^\delta(0) = u_0$ and $u^\delta(t) \in \bar{S}_0$ for $t \geq 0$ and $\delta \in (0, \delta_0)$, we readily deduce from (4.11) that $u(0) = u_0$ and $u(t) \in \bar{S}_0$ for $t \geq 0$.

Next, for $t > 0$, there is $n_\ell \geq 0$ such that $t \in [n_\ell \delta_\ell, (n_\ell + 1)\delta_\ell)$ and it follows from (4.5) that

$$\int_0^t \|A^{\delta_\ell}(s)\|_2^2 ds \leq \int_0^{(n_\ell+1)\delta_\ell} \|A^{\delta_\ell}(s)\|_2^2 ds = \frac{1}{\delta_\ell} \sum_{j=0}^{n_\ell} \|u_{j+1}^\delta - u_j^\delta\|_2^2 \leq 2c_3 e^{16c_1 t}. \quad (4.12)$$

Since

$$A^\delta(t, x) = \frac{u^\delta(t, x) - u^\delta(t - \delta, x)}{\delta}, \quad (t, x) \in (\delta, \infty) \times D,$$

and $A^\delta(t, x) = 0$ for $(t, x) \in (0, \delta) \times D$, the sequence $(A^{\delta_\ell})_{\ell \geq 1}$ converges to $\partial_t u$ in $\mathcal{D}'((0, \infty) \times D)$ as $\ell \rightarrow \infty$, so that the just established boundedness of $(A^{\delta_\ell})_{\ell \geq 1}$ in $L_{2, \text{loc}}([0, \infty), L_2(D))$ implies that $\partial_t u \in L_{2, \text{loc}}([0, \infty), L_2(D))$ and the convergence (4.7d) (up to a subsequence). The stated convergence of $(\zeta^{\delta_\ell})_{\ell \geq 1}$ and the regularity of ζ are straightforward consequences of the regularity of u and (4.7). \square

We next prove the energy inequality (1.9).

Lemma 4.2. *For $t > 0$,*

$$\frac{1}{2} \int_0^t \|\partial_t u(s)\|_2^2 ds + E(u(t)) \leq E(u_0).$$

Proof. Given $t > 0$ and $\ell \geq 0$, we pick again the integer n_ℓ such that $t \in [n_\ell \delta_\ell, (n_\ell + 1)\delta_\ell)$. Then, by (4.3),

$$\frac{1}{2\delta_\ell} \sum_{j=0}^{n_\ell} \|u_{j+1}^\delta - u_j^\delta\|_2^2 + E(u^{\delta_\ell}(t)) \leq E(u_0). \quad (4.13)$$

Owing to Lemma 3.2 (a), (4.7a), and (4.7b), we have

$$\lim_{\ell \rightarrow \infty} E_e(u^{\delta_\ell}(t)) = E_e(u(t)). \quad (4.14)$$

Since $(u^{\delta_\ell}(t))_{\ell \geq 0}$ is bounded in $H^2(D)$ according to (4.10), we may extract a further subsequence (not relabeled), possibly depending on t , such that $(u^{\delta_\ell}(t))_{\ell \geq 0}$ converges to $u(t)$ weakly in $H^2(D)$ and strongly in $H^1(D)$. Hence

$$E_m(u(t)) \leq \liminf_{\ell \rightarrow \infty} E_m(u^{\delta_\ell}(t)),$$

which gives, together with (4.14),

$$E(u(t)) \leq \liminf_{\ell \rightarrow \infty} E(u^{\delta_\ell}(t)). \quad (4.15)$$

Moreover, due to (4.7d) and (4.12), we have

$$\frac{1}{2} \int_0^t \|\partial_t u(s)\|_2^2 ds \leq \liminf_{\ell \rightarrow \infty} \frac{1}{2\delta_\ell} \sum_{j=0}^{n_\ell} \|u_{j+1}^{\delta_\ell} - u_j^{\delta_\ell}\|_2^2. \quad (4.16)$$

Gathering (4.13), (4.15), and (4.16) completes the proof. \square

Proof of Theorem 1.2. To finish off the proof, we are left with showing that u solves the variational inequality (1.8). To this end, we recall from (4.1) that

$$\begin{aligned} 0 &\leq \int_D \left\{ -A_n^\delta + (\tau + a\|\partial_x u_n^\delta\|_2^2) \partial_x^2 u_n^\delta - g(u_n^\delta) \right\} (u_n^\delta - v) dx \\ &\quad - \beta \int_D \partial_x^2 u_n^\delta \partial_x^2 (u_n^\delta - v) dx \end{aligned} \quad (4.17)$$

for $n \geq 1$, $\delta \in (0, \delta_0)$, and $v \in \bar{S}_0$. Now, consider a non-negative function $\phi \in C_c([0, \infty))$ and $w \in L_{2,loc}([0, \infty), H^2(D))$ such that $w(t) \in \bar{S}_0$ for a.e. $t \in (0, \infty)$. Then, for δ small enough, $\text{supp } \phi \subset (\delta, \infty)$ and, by (4.17),

$$\begin{aligned} &\int_0^\infty \phi(s) \int_D \left\{ -A^\delta(s) + (\tau + a\|\partial_x u^\delta(s)\|_2^2) \partial_x^2 u^\delta(s) - g(u^\delta(s)) \right\} (u^\delta(s) - w(s)) dx ds \\ &\quad - \int_0^\infty \phi(s) \int_D \beta \partial_x^2 u^\delta(s) \partial_x^2 (u^\delta(s) - w(s)) dx ds \\ &= \sum_{n=1}^\infty \int_{n\delta}^{(n+1)\delta} \phi(s) \int_D \left\{ -A_n^\delta + (\tau + a\|\partial_x u_n^\delta\|_2^2) \partial_x^2 u_n^\delta - g(u_n^\delta) \right\} (u_n^\delta - w(s)) dx ds \\ &\quad - \sum_{n=1}^\infty \int_{n\delta}^{(n+1)\delta} \phi(s) \int_D \beta \partial_x^2 u_n^\delta \partial_x^2 (u_n^\delta - w(s)) dx ds \\ &\geq 0. \end{aligned} \quad (4.18)$$

On the one hand, it follows from (4.7) that

$$\begin{aligned} &\lim_{\ell \rightarrow \infty} \int_0^\infty \phi(s) \int_D \left\{ -A^{\delta_\ell}(s) + (\tau + a\|\partial_x u^{\delta_\ell}(s)\|_2^2) \partial_x^2 u^{\delta_\ell}(s) - g(u^{\delta_\ell}(s)) \right\} (u^{\delta_\ell}(s) - w(s)) dx ds \\ &= \int_0^\infty \phi(s) \int_D \left\{ -\partial_t u(s) + (\tau + a\|\partial_x u(s)\|_2^2) \partial_x^2 u(s) - g(u(s)) \right\} (u(s) - w(s)) dx ds. \end{aligned}$$

On the other hand, we infer from (4.7b) and the non-negativity of ϕ that

$$\lim_{\ell \rightarrow \infty} \int_0^\infty \phi(s) \int_D \beta \partial_x^2 u^{\delta_\ell}(s) \partial_x^2 w(s) dx ds = \int_0^\infty \phi(s) \int_D \beta \partial_x^2 u(s) \partial_x^2 w(s) dx ds$$

and

$$\liminf_{\ell \rightarrow \infty} \int_0^\infty \phi(s) \int_D \beta |\partial_x^2 u^{\delta_\ell}(s)|^2 dx ds \geq \int_0^\infty \phi(s) \int_D \beta |\partial_x^2 u(s)|^2 dx ds.$$

Collecting the above identities, taking $\delta = \delta_\ell$ in (4.18), and letting $\ell \rightarrow \infty$, we conclude that

$$\begin{aligned} \int_0^\infty \phi(s) \int_D \{ -\partial_t u(s) + (\tau + a \|\partial_x u(s)\|_2^2) \partial_x^2 u(s) - g(u(s)) \} (u(s) - w(s)) dx ds \\ - \int_0^\infty \phi(s) \int_D \beta \partial_x^2 u(s) \partial_x^2 (u(s) - w(s)) dx ds \geq 0. \end{aligned}$$

That is, recalling the definition (4.8) of ζ ,

$$\int_0^\infty \phi(s) \langle \zeta(s), u(s) - w(s) \rangle_{H_D^2} ds \geq 0$$

for any $w \in L_{2,loc}([0, \infty), H_D^2(D))$ satisfying $w(t) \in \bar{S}_0$ for a.e. $t \in (0, \infty)$ and any non-negative $\phi \in C_c([0, \infty))$. In particular, for all $v \in \bar{S}_0$ and non-negative $\phi \in C_c([0, \infty))$, the choice $w(t) \equiv v$, $t > 0$, in the above inequality gives

$$\int_0^\infty \phi(s) \langle \zeta(s), u(s) - v \rangle_{H_D^2} ds \geq 0,$$

which implies, since \bar{S}_0 is separable, that

$$\zeta(t) \in \partial \mathbb{I}_{\bar{S}_0}(u(t)) \text{ for a.a. } t \geq 0.$$

Finally, since $\partial_t u \in L_{2,loc}([0, \infty), L_2(D))$, it follows from the definition (4.8) of ζ that u solves (1.6) in the sense of Definition 1.1, and the proof of Theorem 1.2 is complete. \square

5. PROOF OF COROLLARY 1.3

We finally derive the additional features enjoyed by ζ as stated in Corollary 1.3.

Proof of Corollary 1.3. Let u be a weak solution to (1.6) in the sense of Definition 1.1 and define ζ by (1.7). We introduce the set

$$\mathcal{Z} := \{t \in (0, \infty) : \zeta(t) \in \partial \mathbb{I}_{\bar{S}_0}(u(t))\}, \quad (5.1)$$

and observe that $|\mathcal{Z}| = 0$ since ζ satisfies (1.8). Moreover, since $u(t) + v$ belongs to \bar{S}_0 for any non-negative $v \in C_c^\infty(D)$, it readily follows from (5.1) that

$$\langle -\zeta(t), v \rangle_{H_D^2} \geq 0, \quad t \in \mathcal{Z}. \quad (5.2)$$

That is, for $t \in \mathcal{Z}$, $-\zeta(t)$ is a non-negative distribution on D and thus a non-negative Radon measure, see, e.g., [7, Proposition 6.6].

Next, let $T > 0$. According to the regularity of u ,

$$K_T := \sup_{t \in [0, T]} \{\|u(t)\|_{H^2}\} < \infty. \quad (5.3)$$

Step 1. Let $t \in [0, T]$ and $x \in [0, L]$. Since $u(t) \in H_D^2(D)$, it follows from (5.3) that

$$\begin{aligned} u(t, x) &= u(t, L) + (x - L)\partial_x u(t, L) + \int_x^L (y - x)\partial_x^2 u(t, y) dy \\ &\geq - \int_x^L (y - x)|\partial_x^2 u(t, y)| dy \geq -\frac{(L - x)^{3/2}}{\sqrt{3}} \|\partial_x^2 u(t)\|_2 \\ &\geq -K_T(L - x)^{3/2}. \end{aligned}$$

Hence, for $x \in [x_T, L]$ with

$$x_T := \max \left\{ L - \left(\frac{H}{2K_T} \right)^{2/3}, \frac{L}{2} \right\} \in [L/2, L),$$

we deduce that $u(t, x) \geq -H/2$. Using the same argument for $x \in [-L, 0]$, we end up with

$$u(t, x) \geq -\frac{H}{2}, \quad x \in [-L, -x_T] \cup [x_T, L], \quad t \in [0, T]. \quad (5.4)$$

Now, consider $t \in [0, T] \cap \mathcal{Z}$ and $v \in C_c^\infty(D)$ such that $\text{supp } v \subset [-L, -x_T] \cup [x_T, L]$. For $\theta \in (0, 1)$ small enough, we infer from (5.4) that $u(t) \pm \theta v$ belongs to \bar{S}_0 , so that (1.8) entails

$$0 \leq \langle \zeta(t), u(t) - u(t) \mp \theta v \rangle_{H_D^2} = \mp \theta \langle \zeta(t), v \rangle_{H_D^2}.$$

Since θ is positive, we conclude that $\langle \zeta(t), v \rangle_{H_D^2} = 0$. Consequently,

$$\text{supp } \zeta(t) \subset [-x_T, x_T], \quad t \in [0, T] \cap \mathcal{Z}. \quad (5.5)$$

Step 2. We now fix a non-negative function $\chi_T \in C_c^\infty(D)$ such that $\chi_T \equiv 1$ on $[-x_T, x_T]$. Then, for $v \in C_c^\infty(D)$ and $t \in [0, T] \cap \mathcal{Z}$, the function $u(t) + \chi_T(\|v\|_\infty \pm v)$ belongs to \bar{S}_0 and it follows from (5.2) and (5.5) that

$$0 \leq \langle -\zeta(t), \chi_T(\|v\|_\infty \pm v) \rangle_{H_D^2} = \|v\|_\infty \langle -\zeta(t), \chi_T \rangle_{H_D^2} \pm \langle -\zeta(t), v \rangle_{H_D^2},$$

the identity $\langle \zeta(t), \chi_T v \rangle_{H_D^2} = \langle \zeta(t), v \rangle_{H_D^2}$ being guaranteed by (5.5) and the properties of χ_T . Thus,

$$\left| \langle -\zeta(t), v \rangle_{H_D^2} \right| \leq \|v\|_\infty \langle -\zeta(t), \chi_T \rangle_{H_D^2},$$

and the density of $C_c^\infty(D)$ in $C_0(D)$ and the already established non-negativity of $-\zeta$ ensure that $-\zeta(t)$ belongs to $\mathcal{M}_+(D)$ with

$$\|-\zeta(t)\|_{\mathcal{M}} \leq \langle -\zeta(t), \chi_T \rangle_{H_D^2}, \quad t \in [0, T] \cap \mathcal{Z}. \quad (5.6)$$

Since $\zeta \in L_2((0, T), H^{-2}(D))$, an immediate consequence of (5.6) is that $\zeta \in L_2((0, T), \mathcal{M}(D))$.

Finally, let $s \in (2, 7/2)$. According to [1, Lemma 4.1 (iii)], $\mathcal{M}(D)$ is continuously embedded in $H_D^{s-4}(D)$, so that the just established regularity of ζ implies that $\zeta \in L_2((0, T), H_D^{s-4}(D))$. Together with the regularity of u , $\partial_t u$, and $g(u)$, this property and (1.7) ensure that $\partial_x^4 u$ belongs to $L_2((0, T), H_D^{s-4}(D))$. Consequently, it follows from elliptic regularity theory that u belongs to $L_2((0, T), H_D^s(D))$. This completes the proof of Corollary 1.3. \square

Remark 5.1. Since $u \in C([0, \infty) \times \bar{D})$ by Theorem 1.2 and since $H^1(D)$ embeds continuously in $C(\bar{D})$, it easily follows from (1.8) (by an argument similar to that leading to (5.5)) that

$$\text{supp } \zeta(t) \subset \mathcal{C}(u(t)) \text{ for a.e. } t \in (0, \infty),$$

the coincidence set $\mathcal{C}(u(t))$ being defined in (1.1).

REFERENCES

- [1] H. AMANN AND P. QUITTNER, *Semilinear parabolic equations involving measures and low regularity data*, Trans. Amer. Math. Soc., 356 (2004), pp. 1045–1119.
- [2] V. R. AMBATI, A. ASHEIM, J. B. VAN DEN BERG, Y. VAN GENNIP, T. GERASIMOV, A. HLOD, B. PLANQUÉ, M. VAN DER SCHANS, S. VAN DER STELT, M. VARGAS RIVERA, AND E. VONDENHOFF, *Some studies on the deformation of the membrane in an RF MEMS switch*, in Proceedings of the 63rd European Study Group Mathematics with Industry, O. Bokhove, J. Hurink, G. Meinsma, C. Stolk, and M. Vellekoop, eds., CWI Syllabus, Netherlands, 1 2008, Centrum voor Wiskunde en Informatica, pp. 65–84. <http://eprints.ewi.utwente.nl/14950>.
- [3] L. AMBROSIO, N. GIGLI, AND G. SAVARÉ, *Gradient flows in metric spaces and in the space of probability measures*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, second ed., 2008.
- [4] D. H. BERNSTEIN AND P. GUIDOTTI, *Modeling and analysis of hysteresis phenomena in electrostatic zipper actuators*, in Proceedings of Modeling and Simulation of Microsystems 2001, Hilton Head Island, SC, 2001, pp. 306–309.
- [5] H. BRÉZIS, *Problèmes unilatéraux*, J. Math. Pures Appl. (9), 51 (1972), pp. 1–168.
- [6] L. A. CAFFARELLI AND A. FRIEDMAN, *The obstacle problem for the biharmonic operator*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 6 (1979), pp. 151–184.
- [7] F. DEMENGEL AND G. DEMENGEL, *Functional spaces for the theory of elliptic partial differential equations*, Universitext, Springer, London; EDP Sciences, Les Ulis, 2012.
- [8] J. FREHSE, *On the regularity of the solution of the biharmonic variational inequality*, Manuscripta Math., 9 (1973), pp. 91–103.
- [9] A. HENROT AND M. PIERRE, *Variation et optimisation de formes*, vol. 48 of Mathématiques & Applications (Berlin) [Mathematics & Applications], Springer, Berlin, 2005.
- [10] ———, *Shape variation and optimization*, vol. 28 of EMS Tracts in Mathematics, European Mathematical Society (EMS), Zürich, 2018.
- [11] PH. LAURENÇOT AND CH. WALKER, *Some singular equations modeling MEMS*, Bull. Amer. Math. Soc. (N.S.), 54 (2017), pp. 437–479.
- [12] ———, *Shape derivative of the Dirichlet energy for a transmission problem*. Preprint, 2019.
- [13] A. E. LINDSAY, J. LEGA, AND K. G. GLASNER, *Regularized model of post-touchdown configurations in electrostatic MEMS: Equilibrium analysis*, Phys. D, 280-281 (2014), pp. 95–108.
- [14] ———, *Regularized model of post-touchdown configurations in electrostatic MEMS: Interface dynamics*, IMA J. Appl. Math., 80 (2015), pp. 1635–1663.
- [15] M. NOVAGA AND S. OKABE, *Regularity of the obstacle problem for the parabolic biharmonic equation*, Math. Ann., 363 (2015), pp. 1147–1186.
- [16] J. A. PELESKO, *Mathematical modeling of electrostatic MEMS with tailored dielectric properties*, SIAM J. Appl. Math., 62 (2001/02), pp. 888–908.
- [17] J. A. PELESKO AND D. H. BERNSTEIN, *Modeling MEMS and NEMS*, Chapman & Hall/CRC, Boca Raton, FL, 2003.
- [18] C. POZZOLINI AND A. LÉGER, *A stability result concerning the obstacle problem for a plate*, J. Math. Pures Appl. (9), 90 (2008), pp. 505–519.
- [19] B. SCHILD, *On the coincidence set in biharmonic variational inequalities with thin obstacles*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 13 (1986), pp. 559–616.

INSTITUT DE MATHÉMATIQUES DE TOULOUSE, UMR 5219, UNIVERSITÉ DE TOULOUSE, CNRS, F-31062 TOULOUSE CEDEX 9, FRANCE

E-mail address: `laurenco@math.univ-toulouse.fr`

LEIBNIZ UNIVERSITÄT HANNOVER, INSTITUT FÜR ANGEWANDTE MATHEMATIK, WELFENGARTEN 1, D-30167 HANNOVER, GERMANY

E-mail address: `walker@ifam.uni-hannover.de`